Logic Journal of the IGPL 30:1 (2022) 71—77

‘Sometime a paradox’, now proof: Yablo is not first order

SAEED SALEHI*, *Research Institute for Fundamental Sciences, University of Tabriz, PO Box 51666-16471, Tabriz, Iran; and School of Mathematics, Institute for Research in Fundamental Sciences, PO Box 19395-5746, Tehran, Iran.*

Abstract

Interesting as they are by themselves in philosophy and mathematics, paradoxes can be made even more fascinating when turned into proofs and theorems. For example, Russell’s paradox, which overthrew Frege’s logical edifice, is now a classical theorem in set theory, to the effect that no set contains all sets. Paradoxes can be used in proofs of some other theorems—thus Liar’s paradox has been used in the classical proof of Tarski’s theorem on the undefinability of truth in sufficiently rich languages. This paradox (as well as Richard’s paradox) appears implicitly in Gödel’s proof of his celebrated first incompleteness theorem. In this paper, we study Yablo’s paradox from the viewpoint of first- and second-order logics. We prove that a formalization of Yablo’s paradox (which is second order in nature) is non-first-orderizable in the sense of George Boolos (1984).

This was sometime a paradox, but now the time gives it proof.
—WILLIAM SHAKESPEARE (*Hamlet*, Act 3, Scene 1).

Keywords: Yablo’s paradox, non-first-orderizability.

1 Introduction

Priest [15, p. 160] tells us that ‘the programme of solving the paradoxes is doomed to failure’. Presumably, mathematicians and philosophers know this, at least implicitly, and have learned to live with paradoxes—at least as long as the paradoxes do not crumble the foundations of our logical systems. Paradoxes have turned out to be more than puzzles or destructive contradictions; indeed they have been used in proofs of some fundamental logico-mathematical theorems. Let us take the most well-known, and perhaps the oldest, paradox: the Liar’s paradox. When translated into the language of logic, this paradox seems to claim the existence of a sentence λ such that $\lambda \leftrightarrow \neg\lambda$ holds. However, Liar’s paradox can be moulded into a propositional tautology: $\neg(\mathsf{p} \leftrightarrow \neg\mathsf{p})$. In fact, when trying to convince oneself about the truth of $\neg(\mathsf{p} \leftrightarrow \neg\mathsf{p})$, one can see that the supposed argument is not that much different from the argument of Liar’s paradox. One can clearly see that the paradox becomes a (semantic) proof for that tautology, hence the title of the article. It is worth noting that Gödel mentions in [9, p. 149], after informally presenting his proof of the first incompleteness theorem in Section 1, that: ‘The analogy of this argument with the Richard antinomy leaps to the eye. It is closely related to the ‘Liar’ too;[14]...’where in the footnote 14 he writes that ‘[14]Any epistemological antinomy could be used for a similar proof of the existence of undecidable propositions.’

Let us take a second example, Russell’s paradox. If there existed a set r such that $\forall x(x \in \mathsf{r} \leftrightarrow x \notin x)$, then we would have a contradiction. So, the sentence $\neg\exists y \forall x(x \in y \leftrightarrow x \notin x)$ is a theorem in the theory of sets, whose proof is nothing more than the argument of Russell’s paradox. Going deeper into the

*E-mail: root@saeedsalehi.ir

Advance Access published 3 November 2020 https://doi.org/10.1093/jigpal/jzaa051

proof (or the paradox), one can see that no real set-theoretic properties of the membership relation ($\in$) is used. Namely, for an arbitrary binary relation $\mathtt{s}$, the sentence $\neg\exists y\forall x[\mathtt{s}(y,x) \leftrightarrow \neg\mathtt{s}(x,x)]$ is a first-order logical tautology (see [16, Exercise 12, p. 76]). Now, if we interpret the predicate $\mathtt{s}(y,x)$ as ‘y shaves x’, then we get Barber’s paradox (due to Russell again). More generally, for any formula $\varphi(x,y)$ with the only free variables x and y, the sentence $\neg\exists y\forall x[\varphi(x,y)\leftrightarrow\neg\varphi(x,x)]$ is a first-order logical tautology whose semantic proof is very similar to the argument of Russell’s or Barber’s paradox. In a similar way, $\neg\exists X^{(2)}\exists y\forall x[X^{(2)}(x,y)\leftrightarrow\neg X^{(2)}(x,x)]$ is a second-order tautology.

In this paper, we are mainly interested in Yablo’s paradox [18–20].[1] Several papers (not all of them cited here) and one book [6] have been written on different aspects of this paradox. Yablo’s paradox says that there can be no sequence $\{Y_n\}_{n\in\mathbb{N}}$ such that for all $n\in\mathbb{N}$, Y_n is true if and only if Y_k is untrue for all $k>n$. The reason is that none of those sentences can have a truth value (the sentences Y_n would be neither true nor false). Yablo himself humbly calls this paradox ‘the ω-Liar paradox’. The paradoxicality of the sequence $\{Y_n\}_{n\in\mathbb{N}}$ of sentences with the above-mentioned property follows from the easily observed fact that if Y_m is true for some m, then Y_{m+1}, as well as Y_ks for $k>m+1$, should be untrue. So, by the falsity of Y_{m+1}, there should exist some $j>m+1$ such that Y_j is true, a contradiction. Whence, all Y_ms should be untrue, and so Y_0 must be true, another contradiction!

2 Yablo’s paradox in second-order logic

In order to formalize Yablo’s paradox in a first- or second-order language, we first abstract away even the order relation that appears in the paradox and replace it with an arbitrary binary relation symbol R. A non-arithmetical formulation of Yablo’s paradox in first-order setting can be found in [10], and an arithmetical treatment of Yablo’s paradox in second-order setting can be found in [14].

Let us take $\mathcal{Y}_1$ to be the first-order scheme

$$\neg\forall x\big(\varphi(x)\longleftrightarrow\forall y[xRy\rightarrow\neg\varphi(y)]\big),$$

where $\varphi(x)$ is an arbitrary first-order formula with the only free variable x. Here, the sentences Y_n are represented by $\varphi(n)$ and quantifiers of the form $\forall k>n\cdots$ are represented by $\forall k(nRk\rightarrow\cdots)$.

NOTATION 2.1 ($\mathcal{Y}$: Yablo’s paradox in second-order logic).
By $\mathcal{Y}$, we mean the following second-order sentence:

$$\neg\exists X^{(1)}\forall x\big(X^{(1)}(x)\longleftrightarrow\forall y[xRy\rightarrow\neg X^{(1)}(y)]\big),$$

where R is a fixed binary relation symbol.

Some sufficient conditions for proving $\mathcal{Y}_1$ and $\mathcal{Y}$ are the *seriality* (having no maximal element) and *transitivity* of the binary relation R:

$(\mathcal{S})$: $\forall x\exists y(xRy)$, and $(\mathcal{T})$: $\forall x,y,z\,(xRy\wedge yRz\rightarrow xRz)$.

That is to say, $\mathcal{S}\wedge\mathcal{T}\rightarrow\mathcal{Y}_1$ is a first-order tautology, and $\mathcal{S}\wedge\mathcal{T}\rightarrow\mathcal{Y}$ is a second-order tautology; see [10, 13]. None of these conditions are necessary for $\mathcal{Y}$. For example, in the digraph $\langle D;R_0\rangle$ with $D=\{a,b,c\}$ and $R_0=\{(a,a),(a,b),(b,b),(b,c)\}$, neither $\mathcal{S}$ nor $\mathcal{T}$ holds. However, $\mathcal{Y}$ holds since if

[1] A closely related paradox is Visser’s [17], which we do not study here.

for some $X \subseteq D$ we had $x \in X \leftrightarrow \forall y(xR_0y \rightarrow y \notin X)$ for every $x \in D$, then by aR_0a and bR_0b we had $a, b \notin X$ and so there should existed some $y \in X$ such that aRy, a contradiction. For $R_1 = R_0 \cup \{(c,c)\}$, the digraph $\langle D; R_1\rangle$ satisfies $\mathcal{Y}$ and $\mathcal{S}$ but not $\mathcal{T}$; and for $R_2 = (R_0 - \{(a,b)\}) \cup \{(b,a)\}$, the digraph $\langle D; R_2\rangle$ satisfies $\mathcal{Y}$ and $\mathcal{T}$ but not $\mathcal{S}$. This is a depiction of R_0, R_1, R_2:

$$_{(\mathcal{Y},\neg\mathcal{S},\neg\mathcal{T})}R_0\text{: } {}^{\circlearrowleft}a \rightarrow {}^{\circlearrowleft}b \rightarrow c \quad _{(\mathcal{Y},\mathcal{S},\neg\mathcal{T})}R_1\text{: } {}^{\circlearrowleft}a \rightarrow {}^{\circlearrowleft}b \rightarrow {}^{\circlearrowleft}c \quad _{(\mathcal{Y},\neg\mathcal{S},\mathcal{T})}R_2\text{: } {}^{\circlearrowleft}a \leftarrow {}^{\circlearrowleft}b \rightarrow c$$

As a matter of fact, some conditions weaker than $\mathcal{S} \wedge \mathcal{T}$ can prove $\mathcal{Y}_1$ and $\mathcal{Y}$. For example, the sentence

$(\mathcal{A})$: $\forall x \exists y(xRy \wedge \forall z[yRz \rightarrow xRz])$,

does so (see Theorem 2.3 below). To see that $\mathcal{A}$ is really weaker than $\mathcal{S} \wedge \mathcal{T}$, consider the non-transitive digraph $\langle D; R_1\rangle$ above; it satisfies $\mathcal{A}$ since for every $x \in D$ we have $xR_1x \wedge \forall z[xR_1z \rightarrow xR_1z]$. Even some weaker conditions than $\mathcal{A}$ prove ($\mathcal{Y}_1$ and) $\mathcal{Y}$.

DEFINITION 2.2 (Some Sufficient Conditions for Proving $\mathcal{Y}$).
Let $\theta_0(x)$ be the formula $\exists y(xRy \wedge \forall z[yRz \rightarrow xRz])$. For every $n \in \mathbb{N}$, if $\theta_n(x)$ is defined, then let $\theta_{n+1}(x) = \exists y(xRy \wedge \forall z[yRz \rightarrow \theta_n(z)])$.

We now show that $\{\forall x\, \theta_n(x)\}_{n\in\mathbb{N}}$ is a decreasing sequence of sentences each of which implies $\mathcal{Y}_1$ and $\mathcal{Y}$ (where by *decreasing* we mean that every sentence implies its successor but not vice versa). Note that $\mathcal{A} = \forall x\, \theta_0(x)$.

THEOREM 2.3 ($\bigwedge\!\!\!\bigwedge_{n\in\mathbb{N}}\ \forall x\, \theta_{n+1}(x) \not\models \forall x\, \theta_n(x) \models \forall x\, \theta_{n+1}(x) \models \mathcal{Y}$).
For every $n \in \mathbb{N}$, we have
(1) $\forall x\, \theta_n(x) \models \mathcal{Y}$; (2) $\forall x\, \theta_n(x) \models \forall x\, \theta_{n+1}(x)$; (3) $\forall x\, \theta_{n+1}(x) \not\models \forall x\, \theta_n(x)$.

PROOF
(1) By induction on n. For $n = 0$, assume that $\forall x\, \theta_0(x)$ holds in a directed graph $\langle D; R\rangle$ and that for a subset $X \subseteq D$ we have $\forall x(x \in X \leftrightarrow \forall y[xRy \rightarrow y \notin X])$. For every $a \in D$ there exists, by $\forall x\, \theta_0(x)$, some $b \in D$ such that aRb and $\forall z[bRz \rightarrow aRz]$. Now, if $a \in X$, then $b \notin X$, and so there should exist some $c \in D$ such that bRc and $c \notin X$. Also, aRc should hold, which contradicts $a \in X$. Thus, $X = \emptyset$. However, then for any $a \in D$ there should exist some $b \in D$ with aRb and $b \in X$, and so $X \neq \emptyset$, a contradiction. Thus, there exists no such $X \subseteq D$, whence $\forall x\, \theta_0(x) \models \mathcal{Y}$.

Now, suppose that $\forall x\, \theta_n(x) \models \mathcal{Y}$ holds. Take a directed graph $\langle D; R\rangle$ and assume that $\forall x\, \theta_{n+1}(x)$ holds in it. If for some $X \subseteq D$ we have $\forall x(x \in X \leftrightarrow \forall y[xRy \rightarrow y \notin X])$, then for every $a \in D$ there exists some $b \in D$ such that aRb and we have $\theta_n(z)$ for every z in the set $D_b = \{z \in D \mid bRz\}$. If $D_b \neq \emptyset$, then the directed graph $\langle D_b, R \cap D_b^2\rangle$ satisfies $\forall x\, \theta_n(x)$, and so, by the induction hypothesis, the set $X \cap D_b$ cannot exist. So, we necessarily have $D_b = \emptyset$. If $a \in X$ holds, then we should have $b \notin X$ and so there should exists some $c \in D_b$ with $c \notin X$, a contradiction. Thus, $X = \emptyset$. Then, for every $a \in D$, since $a \notin X$, there should exist some b with aRb and $b \in X$, another contradiction. This shows that $\forall x\, \theta_{n+1}(x) \models \mathcal{Y}$.

(2) Suppose that $\forall x\, \theta_n(x)$, and fix an x. There is some y such that xRy; also for every z with yRz we have $\theta_n(z)$ by the assumption $\forall x\, \theta_n(x)$. So, $\forall x\, \theta_{n+1}(x)$ holds.

(3) Consider $\langle D; R\rangle$, with $R = \{(a_i, a_{i+1}) \mid 0 \leqslant i < 2n\} \cup \{(a_{2n}, a_{2n})\}$ over the set $D = \{a_0, a_1, \ldots, a_{2n}\}$. In the directed graph $\langle D; R\rangle$, obviously, $\forall x\, \theta_{n+1}(x)$ holds, but $\forall x\, \theta_n(x)$ does not hold, since we have $a_{2n-2}Ra_{2n-1}Ra_{2n}$ but $\neg(a_{2n-2}Ra_{2n})$. □

As a result, $\mathcal{Y}$ does not imply $\forall x\, \theta_n(x)$ for any $n \in \mathbb{N}$.

REMARK 2.4 (Another decreasing sequence of sentences implying Yablo).
All the sentences $\forall x\,\theta_n(x)$ above imply the seriality of the relation R but do not imply its transitivity. Another condition[2] that is weaker than transitivity but its conjunction with seriality implies $\mathcal{Y}$ was constructed by Cook [5, 6]. Let us say that the relation R is *n-transitive* when for every $\{a_i\}_{i\leqslant n}$ if a_iRa_{i+1} holds for each $i\leqslant n$, then a_0Ra_n. Now [5, Theorem 2.7] (and [6, Theorem 1.3.2]) says that

(1) For even n, seriality and n-transitivity together imply $\mathcal{Y}$.
Next, we note that

(2) For every n, n-transitivity implies n^2-transitivity.
Indeed, n-transitivity implies $(n+k[n-1])$-transitivity for every k (see [6, p. 47]).
Finally, we note that

(3) For every n, n-transitivity does not imply m-transitivity for any $m<n$.

Since the relation $\{(i,j) \mid (n\leqslant i<j)\vee(i<n\wedge[j=i+1\vee n\leqslant j])\}\subseteq\mathbb{N}^2$ is n-transitive, but not m-transitive when $m<n$.

Thus, for every even $\ell>1$, if τ_n says 'R is ℓ^{2^n}-transitive', then $\{\mathcal{S}\wedge\tau_n\}_n$ is a decreasing sequence of sentences each of which implies $\mathcal{Y}$. $\lhd$

We saw that Theorem 2.3(1) turns Yablo's paradox into a theorem (under some conditions on R). As another instance of turning Yablo's paradox into a proof we can mention [7, Theorem 1]. One other instance[3] is in [1] (Lemma 9.8) where Beklemishev and Pakhomov note in Remark 9.9 that

> 'The construction of Y in the proof of Lemma 9.8 essentially is a variant of Yablo-Visser paradox [17, 18]. An interesting feature of the proof is that it did not require the use of any kind of fixed points. This contrasts with Visser's construction [17], where he used the Diagonal Lemma to show that the paradox is applicable to descending sequence of truth predicates.'

In the next section, we show that no first-order sentence in the language of $\langle R\rangle$ is equivalent to the second-order sentence $\mathcal{Y}$. So, neither the sentence $\mathcal{Y}$ nor its negation $\neg\mathcal{Y}$ is first-orderizable (see [3, 4]). Not only the second-order sentence $\mathcal{Y}$ is non-equivalent to any first-order sentence but also it is not equivalent to any first-order theory (not even to theories with infinitely many axioms). Actually, $\neg\mathcal{Y}$ is equivalent to the existence of a kernel in a directed graph $\langle D;R\rangle$; see e.g. [2, 6]. So, by our result, the existence or non-existence of a kernel in a directed graph is not equivalent to any first-order sentence in the language of directed graphs. Whence, Yablo's paradox, formalized as $\mathcal{Y}_1$ or $\mathcal{Y}$ in the Notation of 2.1, does not turn by itself into a first- or second-order tautology, and some conditions should be put on R to make it a theorem. As argued in [11, 12], however, Yablo's paradox can be nicely translated into some theorems in linear temporal logic or in modal logic (see [8]).

3 Non-first-orderizability of Yablo's paradox

Consider the language $\langle\mathfrak{s}\rangle$, where $\mathfrak{s}$ is a unary function symbol. A standard structure for this language is $\langle\mathbb{N};\mathfrak{s}\rangle$, where $\mathfrak{s}$ is interpreted as the successor function: $\mathfrak{s}(n)=n+1$ for all natural numbers $n\in\mathbb{N}$.

---

[2] This was kindly suggested by an anonymous referee of this journal.

[3] Both of these instances were kindly pointed out by another anonymous referee of this journal, who also noted that 'the main result of [7]' (which is its Theorem 1) 'essentially is an instance of Yablo's paradox (as a theorem of second-order logic)' even though the authors of [7] had not noticed that.

DEFINITION 3.1 (Theory of successor, and kernel of a directed graph).
Let $\mathscr{S}$ be the theory in the language $\langle\mathfrak{s}\rangle$ consisting of the following axiom:

$$\forall x,y\,\big(\mathfrak{s}(x)=\mathfrak{s}(y)\rightarrow x=y\big).$$

To every structure $\langle M;\mathfrak{s}\rangle$, a directed graph $\langle M;R\rangle$ is associated, where R is defined by $xRy \iff y=\mathfrak{s}(x)$, for all $x,y\in M$.

For a directed graph $\langle D;R\rangle$, a subset $K\subseteq D$ is called a kernel when it has the following property: $\forall x\big(x\in K \leftrightarrow \forall y[xRy\rightarrow y\notin K]\big)$.

For a formula φ over the language $\langle R\rangle$, let $\varphi^{\mathfrak{s}}$ be the result of replacing each uRv in φ by $\mathfrak{s}(u)=v$ for variables u,v. The formula $\varphi^{\mathfrak{s}}$ is thus over the language $\langle\mathfrak{s}\rangle$.

So, the second-order sentence $\mathcal{Y}$ asserts the non-existence of a kernel in a directed graph with relation R, and the $\mathfrak{s}$-translation of Yablo's paradox $\mathcal{Y}^{\mathfrak{s}}$ is equivalent to the second-order sentence $\neg\exists X^{(1)}\forall x\big[X^{(1)}(x) \longleftrightarrow \neg X^{(1)}\big(\mathfrak{s}(x)\big)\big]$. Each structure $\langle M;\mathfrak{s}\rangle$ that satisfies $\mathscr{S}$ may contain some copies of

$$\mathbb{N}\approx\{a_0,a_1,a_2,\ldots\}$$

with $a_{n+1}=\mathfrak{s}(a_n)$ for all $n\in\mathbb{N}$, such that there is no $a\in M$ with $\mathfrak{s}(a)=a_0$. It may also have some copies of

$$\mathbb{Z}\approx\{\ldots,a_{-2},a_{-1},a_0,a_1,a_2,\ldots\},$$

in which $a_{m+1}=\mathfrak{s}(a_m)$ for all $m\in\mathbb{Z}$. There could be also some finite cycles

$$\mathbb{Z}_m\approx\{a,\mathfrak{s}(a),\mathfrak{s}^2(a),\ldots,\mathfrak{s}^{m-1}(a)\}\ \text{with}\ \mathfrak{s}^m(a)=a,$$

for some $m>0$. Let us note that, by the axiom $\mathscr{S}$, no two copies of $\mathbb{N}$ or $\mathbb{Z}$ or a finite cycle can intersect. Indeed, no model $\langle M;\mathfrak{s}\rangle$ of $\mathscr{S}$ can contain anything but these.

LEMMA 3.2 (Axiomatizability of $\neg\mathcal{Y}^{\mathfrak{s}}+\mathscr{S}$).
The associated directed graph $\langle D;R\rangle$ of a model $\langle M;\mathfrak{s}\rangle$ of $\mathscr{S}$ satisfies $\neg\mathcal{Y}$, if and only if $\langle M;\mathfrak{s}\rangle$ has no odd cycles, if and only if the structure $\langle M;\mathfrak{s}\rangle$ satisfies the axioms $\neg\exists x(\mathfrak{s}^{2n+1}(x)=x)$ for each $n\in\mathbb{N}$.

PROOF
The second equivalence is straightforward; so, we prove the first one only.

($\Longrightarrow$): suppose that the directed graph $\langle M;R\rangle$ associated to $\langle M;\mathfrak{s}\rangle$ has a kernel K, and also (for the sake of a contradiction) that $\langle M;\mathfrak{s}\rangle$ has an odd cycle such as $\{a,\mathfrak{s}(a),\mathfrak{s}^2(a),\ldots,\mathfrak{s}^{2m+1}(a)=a\}$, for some $m>0$. Then, if $a\in K$, then $\mathfrak{s}(a)\notin K$, then $\ldots$ $\mathfrak{s}^{2m}(a)\in K$, and so $a=\mathfrak{s}^{2m+1}(a)\notin K$, a contradiction. Also, if $a\notin K$, then $\mathfrak{s}(a)\in K$, then $\ldots$ $\mathfrak{s}^{2m}(a)\notin K$, and so $a=\mathfrak{s}^{2m+1}(a)\in K$, a contradiction again. Therefore, if $\langle M;R\rangle$ has a kernel, then $\langle M;\mathfrak{s}\rangle$ can have no odd cycle.

($\Longleftarrow$): suppose that $\langle M;\mathfrak{s}\rangle$ has no odd cycles. Let $K\subseteq M$ consist of the even-indexed members of M, in the sense that if M contains copies of $\mathbb{N}$ or $\mathbb{Z}$ then K contains only the even numbers from those copies, and if M contains a finite even cycle like $\{a,\mathfrak{s}(a),\mathfrak{s}^2(a),\cdots,\mathfrak{s}^{2m+2}(a)=a\}$ then K contains only the elements $\{a,\mathfrak{s}^2(a),\mathfrak{s}^4(a),\cdots,\mathfrak{s}^{2m}(a)\}$ from that cycle. Then the set K is a kernel of $\langle M;R\rangle$, since an element of M is in K, if and only if it is even-indexed, if and only if its successor is odd-indexed, if and only if its successor is not in K. Thus, the digraph $\langle M;R\rangle$ satisfies $\neg\mathcal{Y}$.

This would have not been possible had there been an odd cycle, i.e. an element α such that $\mathfrak{s}^{2m+1}(\alpha)=\alpha$ for some $m>0$, since α would had been odd- and even-indexed at the same time. □

So, the second-order theory $\neg\mathcal{Y}^{\mathfrak{s}}+\mathscr{S}$ is first-order axiomatizable over $\langle\mathfrak{s}\rangle$. However, we show below that it is not finitely axiomatizable. We will use two applications of the compactness theorem of first-order logic (see e.g. [16, Theorem 4.2.1]):

(a) If a class $\mathscr{K}$ of structures is axiomatized by a theory T, then $\mathscr{K}$ is finitely axiomatizable if and only if $\mathscr{K}$ is axiomatizable by a finite subset of T (see [16, Lemma 4.2.9]).
(b) A class $\mathscr{K}$ of structures is finitely axiomatizable if and only if $\mathscr{K}$ and $\mathscr{K}^{\complement}$ (the complement of $\mathscr{K}$) are both axiomatizable (see [16, Lemma 4.2.10]).

THEOREM 3.3 (Non-first-orderizability of $\mathcal{Y}$ and $\neg\mathcal{Y}$).
The second-order sentence $\mathcal{Y}$ is not equivalent to any first-order sentence.

PROOF
If there is a first-order sentence in the language $\langle R\rangle$ equivalent to $\mathcal{Y}$, then by Lemma 3.2, $\mathscr{S}'=\mathscr{S}\cup\{\neg\exists x(\mathfrak{s}^{2n+1}(x)=x)\mid n\in\mathbb{N}\}$ should be finitely axiomatizable. However, this is not true since for any finite subset of this theory there exists a structure that satisfies that finite sub-theory but is not a model of the whole theory $\mathscr{S}'$ (see (a) above): it suffices to take a sufficiently large odd cycle. □

Thus, there can exist no first-order sentence η such that the second-order sentence $\eta\leftrightarrow\mathcal{Y}$ is a tautology. As a result, the proposed formalization $\mathcal{Y}$ of Yablo's paradox in 2.1, being second order in nature, is not equivalent to any first-order sentence.

We end the paper with a stronger result: there cannot exist any first-order theory that is equivalent to $\mathcal{Y}$. So, Yablo's paradox is not even infinitely first-order (i.e. it is even non-equivalent to every first-order theory consisting of infinitely many sentences).

THEOREM 3.4 (Non-equivalence of $\mathcal{Y}$ to first-order theories).
The second-order sentence $\mathcal{Y}$ is not equivalent to any first-order theory.

PROOF
By Lemma 3.2, the theory $\neg\mathcal{Y}^{\mathfrak{s}}+\mathscr{S}$ is first-order axiomatizable over $\langle\mathfrak{s}\rangle$; if $\mathcal{Y}^{\mathfrak{s}}$ is first-order axiomatizable, then by (b) above the theory $\neg\mathcal{Y}^{\mathfrak{s}}+\mathscr{S}$ should be finitely first-order axiomatizable. However, this contradicts Theorem 3.3. So, $\mathcal{Y}^{\mathfrak{s}}$ is not first-order axiomatizable; hence, $\mathcal{Y}$ is not equivalent to any first-order theory. □

We conjecture that the second-order sentence $\neg\mathcal{Y}$, too, is not equivalent to any first-order theory. This does not concern the main topic of this article, since the sentence $\neg\mathcal{Y}$ does not express Yablo's paradox, while $\mathcal{Y}$ does that.

Acknowledgements

This research was partially supported by a grant (№ 95030033) from the Institute for Research in Fundamental Sciences, $\mathbb{IPM}$, Tehran, Iran.
My warmest thanks go to Kaave Lajevardi for thoroughly revising the prose and beats of the article and for drawing my attention to the quoted Shakespearean epigram.
I am also grateful to two anonymous referees of this journal for carefully reading the paper and for their most valuable comments and suggestions.

References

[1] L. D. Beklemishev and F. N. Pakhomov. Reflection algebras and conservation results for theories of iterated truth. arXiv:1908.10302v2, 1 –49, 2019.

[2] C. Berge. *Graphs*, 3rd edn. North Holland, 1991.

[3] G. Boolos. To be is to be a value of a variable (or to be some values of some variables). *The Journal of Philosophy*, **81**, 430–449, 1984. doi: 10.2307/2026308. Reprinted in: G. Boolos. *Logic, Logic and Logic*, pp. 54–72. Harvard University Press, 1998.

[4] G. Boolos. Nonfirstorderizability again. *Linguistic Inquiry*, **15**, 343, 1984. https://www.jstor.org/stable/4178386.

[5] R. T. Cook. Patterns of paradox. *The Journal of Symbolic Logic*, **69**, 767–774, 2004. doi: 10.2178/jsl/1096901765.

[6] R. T. Cook. *The Yablo Paradox: An Essay on Circularity*. Oxford University Press, 2014.

[7] D. Flumini and K. Sato. From hierarchies to well-Foundedness. *Archive for Mathematical Logic*, **53**, 855–863, 2014. doi: 10.1007/s00153-014-0392-9.

[8] T. Forster and R. Goré. Yablo's paradox as a theorem of modal logic. *Logique et Analyse*, **59**, 283–300, 2016. doi: 10.2143/LEA.235.0.3170108.

[9] K. Gödel. On formally undecidable propositions of principia mathematica and related systems. I (1931). In *Kurt Gödel Collected Works, Volume I: Publications 1929–1936*, S. Feferman *et al.*, eds, pp. 135–152. Oxford University Press, 1986.

[10] V. Halbach and S. Zhang. Yablo without Gödel. *Analysis*, **77**, 53–59, 2017. doi: 10.1093/analys/anw062.

[11] A. Karimi and S. Salehi. Theoremizing Yablo's paradox. arXiv:1406.0134, 1–7, 2014.

[12] A. Karimi and S. Salehi. Diagonal arguments and fixed points. *Bulletin of the Iranian Mathematical Society*, **43**, 1073–1088, 2017. https://bit.ly/31pQVYF.

[13] J. Ketland. Yablo's paradox and ω-inconsistency. *Synthese*, **145**, 295–302, 2005. doi: 10.1007/s11229-005-6201-6.

[14] L. M. Picollo. Yablo's paradox in second-order languages: consistency and unsatisfiability. *Studia Logica*, **101**, 601–617, 2013. doi: 10.1007/s11225-012-9399-6.

[15] G. Priest. Unstable solutions to the liar paradox, In *Self-Reference: Reflections on Reflexivity*, S. J. Bartlett and P. Suber, eds, pp. 145–175. Martinus Nijhoff Publishers, 1987. doi: 10.1007/978-94-009-3551-8_9.

[16] D. van Dalen. *Logic and Structure*, 5th edn. Springer, 2013.

[17] A. Visser. Semantics and the liar paradox. In *Handbook of Philosophical Logic, Volume IV: Topics in the Philosophy of Language*, D. Gabbay & F. Günthner, eds, pp. 617–706. Reidel, 1989. doi: 10.1007/978-94-009-1171-0_10.

[18] S. Yablo. Truth and reflection. *Journal of Philosophical Logic*, **14**, 297–349, 1985. doi: 10.1007/BF00249368.

[19] S. Yablo. Paradox without self-reference. *Analysis*, **53**, 251–252, 1993. doi: 10.1093/analys/53.4.251.

[20] S. Yablo. Circularity and paradox. In *Self-Reference*, T. Bolander *et al.*, eds, pp. 139–157. Center for the Study of Language and Information, 2006.

Received 1 July 2020