Hermann Grassmann and the Prehistory of Universal Algebra *

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I was brought to Universal Algebra against my will, as it were by Hermann Grassmann, and the main point of this paper is to describe a piece of Grassmann's work and to ask those who know the subject better than I do whether it may be said to anticipate Universal Algebra.

Before doing that, though, I would like to set the scene by first briefly describing a typical piece of Universal Algebra, and then giving a rapid sketch of the prehistory of Universal Algebra by which I mean simply those developments that predate the definitive formulation of the subject but which, with hindsight, may be seen to have anticipated or influenced it.

1 Universal Algebra

I am not going to assume familiarity with Universal Algebra the few remarks that I will make now should suffice to make what follows intelligible. To save argument I will define the subject as comprising what is to be found in modern books bearing the title *Universal Algebra*, such as the ones by Cohn [5] and Grätzer [8].

Let me outline some terminology and a typical construction.

A *universal algebra* is a set G together with a system of n-ary operations for G; here n may vary and the number of operations may be infinite. (An n-ary operation is simply a function $G^n \to G$.)

These are the objects of study of the subject, Universal Algebra. They include, for example, groups, rings, linear spaces and lattices (and, with slight modification of the definition, even fields, projective planes, etc.).

In the case where there is just a single binary operation (denoted here by juxtaposition) we have a *groupoid*. Given a set of symbols $S = \{x_1, x_2, ..., x_n\}$, the set

$$G = \{x_1, x_2, ..., x_n, (x_1x_1), (x_1x_2), ..., (x_nx_n), (x_1(x_1x_1)), (x_1(xx_1x_2)), ...$$
$$(x_1(x_nx_n)), (x_2(x_1x_1)), ..., (x_n(x_nx_n)), ((x_1x_1)x_1), ((x_1x_1)x_2), ...\}$$

obtained by repeated juxtaposition of the symbols already written down, forms a groupoid in a natural way (the binary operation being juxtaposition). This is the *free groupoid* on S.

Now let A be the set of all (formal, finite) linear combinations of elements of G (with coefficients from a field R):

$$A = \{\Sigma\alpha_jX_j : \alpha_j \in R, X_j \in G\}.$$

*This article appeared in the American Mathematical Monthly 89 (1982).

A becomes a linear space if we define addition and multiplication by scalars in the natural way, and indeed it becomes a (nonassociative) linear algebra over R if we define

$$(\Sigma\alpha_i X_i)(\Sigma\beta_j X_j) = \Sigma\alpha_i\beta_j (X_i X_j),$$

where we need an obvious convention to cope with the fact that the indices i and j may range over different finite sets of natural numbers. This is the *free linear algebra* on S over R.

The free algebra A may be put to work in the following way. A linear algebra B is said to satisfy the *law* (or *universal relation*)

$$\Sigma\alpha_j X_j = 0,$$

where $\Sigma\alpha_j X_j \in A$, if a valid equation is obtained whenever all the free algebra generators $x_j \in S$ on the left are replaced by elements of B. The algebras satisfying a fixed set of laws form what is called a *variety* of algebras. A typical law of degree 3, for example, is

$$x_1(x_2x_3) = (x_1x_2)x_3;$$

it determines the variety of associative algebras. (Here, as is usual, we have omitted the outermost brackets.)

2 A Sketch of the Prehistory of Universal Algebra

There are some branches of mathematics to which one may arguably assign a clearly defined starting point. In the case of Universal Algebra what suggests itself as the beginning, at least in a narrow sense, is Garrett Birkhoff's paper [1] *On the Structure of Abstract Algebras* which appeared in the Proceedings of the Cambridge Philosophical Society in 1935. Here the definition of a Universal Algebra appears for the first time and the broad outlines of the subject may already be discerned. In particular "abstract algebras are divided by a very simple scheme into self-contained 'species'. Within each species a perfect duality is found between families of formal laws and the families of algebras satisfying them". Here the term "family of algebras of a given species" is used in a technical sense meaning a class closed under taking subalgebras, homomorphic images and direct products or what is nowadays called a *variety*.

This said, one must point out that undoubtedly the most influential figure in the movement towards abstraction and generality in algebra which culminated in Universal Algebra was Emmy Noether, who died in 1935. In the twenties her school had investigated the notion of a group with operators, which is already very general, including, for example, groups, rings and linear spaces. As Birkhoff himself said in 1946 [2] "they had developed many of the most important ideas of Universal Algebra" for example, the three fundamental isomorphism theorems, which are now recognized as theorems of Universal Algebra, were given for groups with operators in van der Waerden's *Moderne Algebra* [13] in 1931 and van der Waerden was of course a protegé of Noether. As Birkhoff in the 1935 paper refers to van der Waerden and also, to Hasse's *Höhere Algebra* [9], there is no doubt that he was familiar with the work of the Noether school. Birkhoff's first papers were on lattices and it is not surprising that he should have sought a notion more general than that of a group with operators to express these structures as well.

While it is certainly the sequence Dedekind-Noether-Birkhoff which is the main line of influence in the prehistory of Universal Algebra, I was to consider another very different family of ideas. In 1898 Cambridge University Press published a large and impressive-looking tome [14] bearing the title *Universal Algebra, Volume I* by A.N. Whitehead (1861-1947). It was Whitehead's first book, and five years after it appeared he was elected to the Royal Society. Birkhoff has said (in [3]) that it was from this book that he borrowed the name "Universal Algebra". Whitehead was Professor of Philosophy at Harvard from 1924 to 1937, a period which includes the years in which Birkhoff was a student there (and his father Professor of Mathematics).

To give some idea of what is in Whitehead's book, let me quote from the preface:

> After the general principles of the whole subject have been discussed in Book I of this volume, the remaining books of the volume are devoted to the separate study of the Algebra of Symbolic Logic and of Grassmann's Calculus of Extension.

In fact, there are seven books in all, of which only one considers symbolic logic; it turns out that even Book I, called *The Principles of Algebraic Symbolism*, is based on Grassmann.

Volume II of Whitehead's book never appeared, perhaps because it was around 1898 that he began his collaboration with a greater mind, Bertrand Russell. It was intended that it would deal with the theory of associative linear algebras which had begun with Benjamin Peirce's famous *Memoir* [12] of 1870 (which only appeared in print in 1881).

Perhaps the main thing that one might expect of such a work is unification, and that is what Whitehead aimed for and what one may fairly say, I think, he failed to achieve.

> Universal Algebra [he says] is the name applied to that calculus which symbolizes general operations, defined later, which are called Addition and Multiplication. There are certain general definitions which hold for any process of addition and others which hold for any process of multiplication. These are the general principles of any branch of Universal Algebra.

Briefly these principles amount to commutativity and associativity of addition and distributivity (left and right) of multiplication over addition. These ideas are taken without improvement straight from the first chapter of Grassmann's *Ausdehnungslehre* [6] of 1844. Even the idea of considering in one book the algebras of Boole and of Grassmann is not original. Peano had done the same thing in 1888 (in a book [11] of which Whitehead knew), and had written in his preface as follows:

> The geometric calculus is preceded by an introduction which treats the operations of deductive logic; these present great analogies with those of algebra and of the geometric calculus.

After distingujishing the algebras of Boole and of Grassmann as being respectively of the nonnumerical genus (meaning that $a + a = a$) and of the numerical genus (meaning that $a + a = 2a \neq a$), Whitehead goes on to consider the two topics in isolation from one another. One need say no more of this first *Universal Algebra*   not really the first even, since Sylvester had previously used the titel for a paper on matrices   except to remark that Whitehead (along with all Grassmann's expositors that I know of) offers a presentation of Grassmann's ideas which is inferior to the original.

3 Grassmann and Boole

Let us step back now to the middle of the nineteenth century and the work of Boole (1815-1864) and Grassmann (1809-1877). I do not want to spend much time on the origins of Boole's algebra of logic which first appeared in 1847. It is perhaps worth noting, though, some similarities in the lives and personalities of these two highly original individuals. Both were fascinated by languages, philosophy, and theology, and came to mathematics late (around the age of twenty) and both had the good fortune never to take a university course in mathematics. Both were interested in the teaching of elementary mathematics, and, perhaps most significantly, both fell under the influence of two great mathematical works, the *Mécanique Analytique* (1788) of Lagrange and the *Mécanique Céleste* (1799-1825) of Laplace. As is well known, there was a very considerable difference in the reception which their ideas received.

Boole became part of an English school of mathematicians who were investigating in piecemeal fashion the so-called "laws of symbolic algebra" and one may say, I think, that had he not discovered his algebra of logic someone else would soon have done so. Grassmann's case is different   as is evidenced by the fact that so many of his ideas were rediscovered many decades later on (and by the blank reception they received initially).

Though both Boole and Grassmann were motivated by what we would call modelling problems, they both came early to the view that mathematics deals with formal structures and that its truth does not reside in any interpretation of its symbols; in this they were pioneers. Thus in 1847 in *The Mathematical Analysis of Logic* [4] Boole writes that

> ... the validity of the processes of analysis does not depend upon the interpretation of the symbols which are employed, but solely upon the laws of their combination. Every system of interpretation which does not affect the truth of the relations supposed is equally admissible.

And three years earlier in his introduction to the first *Ausdehnungslehre* we find Grassmann expressing the opinion that

> ... geometry can in no way be viewed, like arithmetic or the theory of combinations, as a branch of mathematics; instead, geometry relates to something already given in nature, namely, space. I also had realised that there must be a branch of mathematics which yields in a purely abstract way laws similar to those of geometry, which is limited to space. By means of the new analysis it is possible

> to form such a purely abstract branch of mathematics; indeed this new analysis, developed without assuming any principles established outside its own domain and proceeding purely by abstraction, was itself this science.

A final link between them is the fact that both were in a sense anticipated by Leibniz (1646-1716) who had sought both a "calculus ratiocinator" (or algebra of logic) and a "characteristica geometrica" (or algebra of geometry), and indeed had developed attempts at both, albeit somewhat more successfully with the logical algebra than with the geometric. In Leibniz's vision there was to be a "characteristica universalis" which would embrace both these algebras; this has indeed come to pass, and one may say perhaps that in a very wide sense it is Leibniz who is the father (or, more accurately, the prophet) of Universal Algebra. Most of these ideas of Leibniz were unpublished until around his bicentenary in 1846, and the question of their influence on Grassmann and Boole is a delicate one which I will not go into.

The fame of Hermann Grassmann today rests on his creation of exterior algebra. What should be realized, though, is that he is in fact the main creator of linear algebra in the modern sense, and that while it is true that geometric considerations motivated his work, he wished to be seen as being an abstract algebraist and that is what he was. In the *Ausdehnungslehre* of 1844 Grassmann plainly wanted to develop his theory in axiomatic "modern algebra" style, but this he was unable to do. To find a modern parallel to the constructive approach which he adopted in the second *Ausdehnungslehre* one must go, I thin, to a text in Universal Algebra.

4 Grassmann's Products

I come now to the main part of this paper. It concerns Chapter 2 of Grassmann's *Ausdehnungslehre* of 1862 (which, though it bears the date 1862 on the title page, actually appeared in 1861).

As background we need to know that in the first chapter, which is called *Addition and subtraction of extensive quantities, and their multiplication and division by numbers*, Grassmann had developed in detail, essentially as it is done today, the theory of basis and dimension for finite-dimensional linear spaces. The arena in which this is done is a finite-dimensional real linear space of *extensive quantities*; this is called a *region* and a basis for it, $e_1, e_2, ..., e_n$, is called a *system of units*. Although attention is focused on a well-defined space in this way, one must realize that not all extensive quantities are spanned by $e_1, e_2, ..., e_n$. One would like to say, in modern fashion, that the extensive quantities form a linear space of which the region spanned by $e_1, e_2, ..., e_n$ is a subspace, but Grassmann does not pin down an ambient space in this way.

Grassmann begins Chapter 2 by defining the product of quantities $a = \Sigma\alpha_i e_i$ and $b = \Sigma\beta_j e_j$ from the region determined by $e_1, e_2, ..., e_n$ to be

$$ab = \Sigma\alpha_i\beta_j\,[e_i e_j]\,.$$

After remarking in passing that, being an extensive quantity, this product must itself be a linear combination of a system of units, and that particular kinds of "product structure" will be singled out when one specifies what this system of units is to be and how the products $[e_i e_j]$ are generated by them, he indicates that he will, for the present, "deal only with laws which follow from the general definition of product (above), and which therefore hold for every kind of product". He immediately proves, among other things, that for extensive quantities $a = \Sigma\alpha_i e_i,\ b = \Sigma\beta_j e_j,\ \ c = \Sigma\gamma_j e_j$ and a real number α one has

$$(a+b)\,c = ac + bc, \quad c\,(a+b) = ca + cb,$$

$$\alpha\,(ab) = (\alpha a)\,b \text{ and } \mathrm{b}\,(\alpha a) = \alpha\,(ba)\,.$$

In the second part of Chapter 2, called *Products of several quantities*, higher order products like $(ab)\,((cd)\,e)$ are considered. They are obtained by iteration of multiplication (justaposition) of pairs of extensive quantities; it is clear that there is no assumption of associativity. Since any three quantities a, b and c necessarily belong to some region, the above laws must hold for arbitrary quantities. At this point one may reasonably say that Grassmann's quantities form a free (nonassociative) linear algebra; but again it must be emphasized that this ambient space is not seen as a whole but rather is explored by means of local investigations confined to finitely-generated subalgebras.

The third part of Chapter 2 begins with a definition:

> If a product structure is determined by the fact that some of the products of units are dependent, then I call each equation expressing such a dependence a determining equation for that type of product structure. A set of p determining equations, none of which is derived from the others, and such that there is no other equation expressing dependence among the products, is called a system of determining equations associated with that product structure.

I find it helpful here to think in terms of a multiplication table. In a linear algebra generated by $e_1, e_2, ..., e_n$ the multiplicative structure is determined by listing all products of pairs of the linear space generators

$$e_1, e_2, ..., e_n, (e_1 e_1)\,, (e_1 e_2)\,, ..., (e_n e_n)\,, e_1\,(e_1 e_1)\,, e_1\,(e_1 e_2)\,, ...\ \ .$$

In the free case these products are obtained simply by juxtaposition. The effect of a determining equation

$$\Sigma\alpha_j E_j = 0$$

(where the E_j are elements of this sequence of linear space generators) is to allow us to eliminate one element, say E_1, from this list, and indeed also to eliminate any other element which has E_1 as a factor; for example, if $e_2 e_1 = -e_1 e_2$ is a determining equation, then after deleting the element $(e_3\,(e_2 e_1))\,(e_4 e_5)$ from our list (since it equals $(e_3\,(e_1 e_2))\,(e_4 e_5)$ we still have a list which spans our algebra. It appears that Grassmann has here delineated, albeit inelegantly, the idea of presenting an algebra by means of generators $(e_1, e_2, ..., e_n)$ and relations $(\Sigma\alpha_j E_j = 0)$. (Nowadays we regard this algebra as the quotient of the free algebra by the ideal generated by the $\Sigma\alpha_j E_j$.) Any finitely-generated linear algebra may be obtained in this way.

We now come to a key definition:

> A product structure whose determining equations remain valid when the units occurring in them are replaced by arbitrary quantities spanned by the units is called a linear product structure.

For example (confining attention as usual to products of elements of the region spanned by $e_1, e_2, ..., e_n$ what it means for the determining equation

$$e_e e_1 = e_1 e_2$$

to be linear is that

$$ba = ab$$

for every quantity $a = \Sigma\alpha_i e_i$ and $b = \Sigma\beta_j e_j$. We are here close to the general concept of a law; indeed the commutative law $x_2 x_1 = x_1 x_2$ will be satisfied by arbitrary quantities if and only if this linear determining equation holds on very region.

Let me now quote from P.M. Cohn's *Universal Algebra* [5]:

> Any systematic study of linear K-algebras would proceed by considering the possible sets of laws.

This is precisely what Grassmann does. He immediately proves the following theorem:

> For products of two factors there are, apart from the product structure with no determining equations, and the one in which all products are zero, only two types of linear product, namely the one whose system of determining equations has the form
>
> $$e_i e_j + e_j e_i = 0$$
>
> and the one for which it has the form
>
> $$e_i e_j = e_j e_i,$$
>
> where both i and j may take values from 1 to n, and $e_1, e_2, ..., e_n$ are units.

What this means is that if an equation

$$\Sigma\alpha_{ij} x_i x_j = 0$$

is satisfied when the x_i are replaced by arbitrary elements $a = \Sigma\alpha_i e_i$ of the region spanned by $e_1, e_2, ..., e_n$, then it must be the case that for all such $a = \Sigma\alpha_i e_i$ and $b = \Sigma\beta_j e_j$, either $ab = 0$ or $ba = ab$ or $ba = -ab$. I would like to consider the relationship of this result to the following theorem which was apparently first stated explicitly (in [10]) in 1950:

> Any law for linear algebras that is homogeneous and of degree 2 and does not hold trivially is equivalent to either $xy = 0$ or $yx = xy$ or $yx = -xy$.

I claim that this is an immediate consequence of Grassmann's result. Indeed, suppose that we have an algebra in which such a law holds. Grassmann's theorem then entails that for each region the multiplication of pairs of elements is governed by one of the three formulae $xy = 0, yx = xy$ or $yx = -xy$. To prove the stated theorem it will suffice to show that for any two regions the governing formula must be the same. The only way that it could happen that none of the three holds identically would be if $yx = xy$ held nontrivially on some region and $yx = -xy$ on some other; and that cannot occur since a single law must hold on the join of the two regions.

One may fairly say, I think, that had Grassmann been able to break the spell which restricted his vision to local aspects of the algebra of quantities, he would have proved this theorem which is evidently fundamental in the study of linear algebras.

It may be argued that the proper job of the historian is to see ideas in the context of their time, but, in their time, Grassmann's ideas were not comprehended. To do justice to this great mathematician (as a creator rather than as an influence) one must, I believe, see his work in the light of twentieth century developments in algebra. In a real sense, he is our contemporary.

I would like to thank Barry Gardner for many enlightening conversations about Universal Algebra, and Mike Newman for drawing my attention to reference [3].

References

[1] Garrett Birkhoff, On the structure of abstract algebras, Proc. Cambridge Philos. Soc., 31 (1935) 433-454.

[2] Garrett Birkhoff, Universal algebra, Canad. Math. Congress Proc., 1946.

[3] Garrett Birkhoff, The rise of modern algebra to 1936, Graduate Studies, no.13, Texas Tech. University, 1976.

[4] George Boole, The Mathematical Analysis of Logic, Cambridge, 1847.

[5] P.M. Cohn, Universal Algebra, Harper and Row, New York, 1965.

[6] Hermann Grassmann, Die Lineale Ausdehnungslehre, Leipzig, 1844.

[7] Hermann Grassmann, Die Ausdehnungslehre, Leipzig, 1862.

[8] George Grätzer, Universal Algebra, Van Nostrand, Princeton, N.J. 1968.

[9] H. Hasse, Höhere Algebra, Berlin, 1927.

[10] A.I. Malcev, On algebras with identical defining relations, Mat. Sb. 26 (68) (1950) 19-33.

[11] G. Peano, Calcolo Geometrico secondo l'Ausdehnungslehre di H. Grassmann, Torino, 1888.

[12] B. Peirce, Linear Associative Algebra, Amer. J. Math., 4 (1881) 97-229.

[13] B.L. van der Waerden, Moderne Algebra, 2 vols., Berlin, 1931.

[14] A.N. Whitehead, Universal Algebra, vol.1, Cambridge, 1898.