On Dummett's Verificationist Justification Procedure

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Abstract

We examine the proof-theoretic verificationist justification procedure proposed by Michael Dummett. After some scrutiny, two distinct interpretations with respect to bases are advanced: the independent and the dependent interpretation. We argue that both are unacceptable as a semantics for propositional intuitionistic logic.

1 Introduction

Prawitz [7, 9, 10] and Dummett [1] proposed proof-theoretic inductive definitions of validity for arguments. These definitions assume that introduction rules provide the canonical assertability conditions for complex sentences based on assertability conditions of their constituents. They are an attempt to explain a remark by Gentzen [2, p. 189] to the effect that "the introduction rules are definitions and the eliminations are only their consequences thereof". Broadly conceived, proof-theoretic definitions of validity for logical laws are relevant in the philosophy of language as part of an anti-realist approach to meaning. They are an essential component in a general verificationist theory of meaning (see, for instance, Prawitz [10, § 5]).

Traditionally, proof-theoretic semantics is built on a mix of ideas from proof theory (as expressed in Gentzen's work) and intuitionistic philosophy of mathematics (as expressed in Heyting's BHK interpretation). Indeed, Prawitz [8, 11] conjectured the completeness of intuitionistic logic with respect to proof-theoretic notions of validity, and Dummett [1, p. 270] made the stronger claim that, if no classical canons of reasoning are incorporated

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into the introduction rules, proof-theoretic definitions would validate only intuitionistic logic. However, Sandqvist [12] has advanced a semantics based on constructive ideas and established the validity of classical logic with respect to it. Although based on constructive principles, Sandqvist's semantics has noticeable differences from the proposals from either Dummett [1] or Prawitz [7, 9, 10].

Further results concerning the adequacy of intuitionistic logic with respect to proof-theoretic definitions of validity were published by Sanz et al. [14]. They have shown that a semantics proposed by Prawitz [7, § A.1] validates a classical inference rule, Peirce's rule, in the implicational fragment of propositional logic. Prawitz's semantics in that paper was based on conservative extensions of atomic systems (or *basic systems* in our terminology). This approach however is not explicitly present in Prawitz's most recent works.

Basic systems that discharge hypotheses escape validation of Peirce's rule [14, § 5]. These kinds of basic systems were further explored by Sandqvist [13], who proved completeness for a constructive semantics with basic rules discharging hypotheses. On the other hand, Piecha et al. [6] showed incompleteness of intuitionistic logic with respect to a proof-theoretic characterization with higher-level basic rules. These results are not contradictory because, although both are proof-theoretic in nature, Sandqvist [13] and Piecha et al. [6] employ distinct semantic clauses. Their differences especially concern the clauses for disjunction.

We investigate here the proof-theoretic verificationist justification procedure proposed by Dummett [1, Chapter 11–13]. In Section 2, we present the justification procedure, following Dummett as closely as possible. There follows, in Section 4, a discussion about the rôle played by basic rules in proof-theoretic definitions of validity, especially in the context of Dummett's definitions. Then, in order to provide a formal and precise account of the justification procedure, and also in order to cover the largest possible ground, we work with two possible interpretations: the independent and the dependent. In Section 6, we show that, not only Sandqvist's especially designed semantics, but also a plausible interpretation of Dummett's own semantics leads to a justification of classical logic. Moreover, in Section 7, we show that, even under the dependent (and most favored) interpretation of Dummett's verificationist procedure, propositional intuitionistic logic is not complete.

2 Preliminaries

2.1 Notation and terminology

For the logical constants, we use the symbols $\supset$ (implication), $\vee$ (disjunction), $\wedge$ (conjunction) and $\bot$ (absurdity). Arbitrary sentences are represented with capital latin letters A, B, C and D. Atomic sentences are represented with lowercase Greek letters: α, β, φ, ψ and χ. Trees of sentences are denoted by Π. When necessary, we employ natural numbers as subscripts.

Following Dummett [1, p. 254], arguments are seen as trees composed of sentences. In an argument Π, a *top occurrence* of a sentence is such that there is no other above it. Similarly, the *end occurrence* of a sentence, or conclusion of an argument, is such that there is no other below it. Every sentence A occurring in a path from a top sentence to the end sentence of an argument determines a subargument in the obvious way with A as its conclusion. We adhere to the convention that the occurrence of a sentence A immediately below "Π" means that A is the end occurrence of Π and the occurrence of a list of sentences above "Π" means that those sentences appear as top occurrences of Π. A top occurrence of a sentence A can be *discharged* by the application of some rule, thus yielding a conclusion B which, together with all sentences below B, do not depend on A. Discharge of hypotheses is indicated by enclosing them in square brackets as in "$[A]$". A top sentence occurrence which is not the conclusion of a rule and is not discharged is said to be *open*. An argument with no open top occurrences is *closed*. When convenient, we may ignore the middle steps and use $\langle \Gamma, A\rangle$ to refer to an argument, where Γ is a list of open top occurrences and A stands for the end occurrence, or conclusion. The degree $d(A) = n$ of a sentence A is the number n of logical constants occurring in A.

The natural deduction inference rules for propositional logic are given below:

$$\frac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \supset B}\supset\mathrm{I} \qquad \frac{A \quad A \supset B}{B}\supset\mathrm{E}$$

$$\frac{A \quad B}{A \wedge B}\wedge\mathrm{I} \qquad \frac{A \wedge B}{A}\wedge\mathrm{E} \qquad \frac{A \wedge B}{B}\wedge\mathrm{E}$$

$$\frac{A}{A \vee B}\,\vee\mathrm{I} \qquad \frac{B}{A \vee B}\,\vee\mathrm{I} \qquad \frac{A \vee B \quad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}\,\vee\mathrm{E}$$

The inference rules for the propositional connectives are symmetrically distributed between introduction (I) and elimination (E) rules. They characterize the natural deduction system for Positive Propositional Intuitionistic Logic (NJ^{+}). Propositional Intuitionistic Logic (NJ) is obtained by adding to NJ^{+} the following rule for the absurdity logical constant $\bot$:

$$\frac{\bot}{A}$$

We consider $\bot$ to be a sentential constant. It should not be confused with prime or atomic sentences. Negation is defined as $\neg A \equiv A \supset \bot$.

Fragments of NJ are indicated by a subset of the propositional logical constants as superscripts. For example, $\mathsf{NJ}^{\supset}$ indicate the fragment of NJ with only $\supset$I and $\supset$E.

We also consider *bases*, that is, sets of basic rules for atomic sentences. Bases are discussed in greater detail in Section 4. They can be used to extend NJ or its fragments (NJ^{+}, $\mathsf{NJ}^{\supset,\wedge}$ and etc.). We denote $\mathsf{S}+\mathsf{B}$ the extension of a natural deduction system S with the basis B.

Natural deduction derivations in a system $\mathsf{S}+\mathsf{B}$ (in which B may be empty) are trees of sentences (or arguments, in our terminology) in which any transition in a branch corresponds to an application of a rule from either S or B. Annotations are sometimes employed in our trees to indicate precisely where top occurrences are being discharged.

Example 2.1. Let p, q, r and s be some particular atomic sentences. Let $\mathsf{B}^{\dagger}$ consist of the following rules:

$$\frac{q}{p} \qquad\qquad \frac{s \quad r}{q} \qquad\qquad \frac{}{r}$$

The following argument is a closed derivation in $\mathsf{NJ}+\mathsf{B}^{\dagger}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{[q \wedge s]^{(1)}}{s} \qquad \dfrac{}{r}}{q}}{p}}{(q \wedge s) \supset p}\,(1)$$

2.2 Dummett's verificationist justification procedure

Dummett [1] considers first, second and third grade proof-theoretic justification procedures. Each one is more complex than the previous. A first grade justification of a rule consists in its derivation from a set of primitive rules.

Second grade justifications introduce the main idea of verificationist proof-theoretic semantics: the introduction rules are supposed to fix the meaning of the logical constants they govern. This insight dates back to some remarks of Gentzen to the effect that the introduction rules are definitions and the elimination rules are, in some sense, only their consequences.

In order to make Gentzen's remark more precise, Dummett develops a general justification procedure which, given a set of introduction rules, is supposed to validate all other rules with respect to this set, including the corresponding elimination rules for the logical constants in question. The justification procedure amounts then to a definition of validity *with respect to a given set of introduction rules.* Unless otherwise noted, we consider the set of introduction rules to be composed of the introduction rules of NJ.[1] Humberstone [5, Theorem 4.13.3] proved soundness and completeness of the positive intuitionistic fragment $\mathsf{NJ}^{\wedge,\vee}$ with respect to the second grade justification procedure.

Third grade proof-theoretic justifications are introduced to deal with discharged hypotheses [1, p. 259–260]. In particular, third grade justifications are mandatory whenever the given set of introduction rules contains $\supset$I. Third grade justifications are our main focus.

According to the standard proof-theoretic view, for a complete semantic specification, besides inference rules governing the assertability of complex sentences, criteria for the assertability of atomic sentences are also required. These criteria are given by *bases.* Each rule in a basis enables the inference of atomic sentences from other atomic sentences [1, p. 254]. These rules are also called *boundary rules.* As in Prawitz [7], the literature usually treats basic rules as rules of the following form, where $\alpha_1, \ldots, \alpha_n$ and β are atomic sentences, $n \geq 0$:

$$\frac{\alpha_1 \quad \ldots \quad \alpha_n}{\beta}$$

The form of basic rules and the rôle they play in proof-theoretic semantics has recently been discussed [14] [6]. Since they have an important impact on proof-theoretic definitions of validity, Section 4 contains a more detailed examination of basic rules in the context of Dummett's verificationist justification procedure.

[1]The formulation of an introduction rule for $\perp$ is not necessary for our purposes.

Following intuitionistic standards fixed by Heyting [4, p. 101], Dummett [1, p. 255–256] first considers deductive arguments composed of actual sentences. His definitions apply directly to concrete arguments instead of argument schemata. As a result, validity for schematic arguments (e. g. an inference rule schema) becomes defined only after validity for concrete arguments is established. Since we are focusing on propositional logic, we omit considerations concerning quantification and open sentences.

Although Dummett [1, p. 64] demands an argument to be an actual complete deduction composed of sentences, he claims that his definition "takes no overt account of more than the initial premisses and final conclusion of the argument" [1, p. 264]. This is important because the definition of validity is supposed to be given inductively and, if other sentences in the argument were regarded as relevant, the definition might not be well-founded. In other words, despite being applicable directly to *arguments*, the verificationist concept of validity can also be understood as a general procedure capable of validating one-step *argument schemata* (or inference rules) given a set of introduction rules. Therefore, following Dummett, the concept of argument is used ambiguously either to refer to a tree of inferences (represented by Π) or to n-tuples $\langle A_1, \ldots, A_{n-1}, A_n \rangle$, $n \geq 1$, in which A_n is the conclusion of Π and $A_1, \ldots, A_{n-1}$ are the open top occurrences or premisses of Π. As said before, the n-uples are going to be represented as $\langle \Gamma, A \rangle$.

3 Verificationist Validity

The following definitions are adapted from Dummett's for the case of propositional logic.[2]

Definition 3.0. A sentence occurrence is in the *main stem* of an argument Π if every sentence intervening between it and the conclusion of Π (inclusive) depends only on the open top occurrences of Π. Moreover, a sentence occurrence that is not in the main stem and lies immediately above another one belonging to the main stem is the final conclusion of a *critical subargument* [1, p. 260].

The purpose of the concept of main stem is to keep track of discharged hypotheses as we move up from the conclusion of the argument towards

[2]Definitions are going to be numbered $x.y$, where x indicates the section and y indicates the position inside the section. The numbering system is intended to make a parallelism between the various definitions and characterizations. For instance, Characterization 3.1, Definition 6.1 and Definition 7.1, where $y = 1$, all deal with the notion of canonical argument.

top occurrences. Meanwhile, examining each of the possible branches in turn, we identify subarguments whose conclusion depend on additional open top occurrences. Notice that an application of $\supset$I might not discharge any hypothesis. Thus, whether or not the occurrence of B immediately above $A \supset B$ is in the main stem depends on whether some hypotheses were actually discharged by the application of $\supset$I.

Example 3.1. In the following argument, only the conclusion $(B \wedge C) \supset (A \wedge B)$ is in the main stem. The subtree until that point is a critical subargument.[3]

$$\dfrac{\dfrac{A \qquad \dfrac{[B \wedge C]}{B}}{A \wedge B}}{(B \wedge C) \supset (A \wedge B)}$$

The next characterizations involve the notion of canonical argument and are essential to the whole proof-theoretic justification procedure. As we remarked (Section 2.2), the verificationist procedure aims to justify logical laws based on a given set of introduction rules. However, Dummett's formulation of the notion of canonical argument also mentions basic rules, but he is not explicit about them. Thus, as they stand, these formulations are not precise enough. Nevertheless, they do provide the general framework of the justification procedure even if they should be regarded as provisional. In Section 4, we discuss the rôle played by basic rules and provide two possible ways for interpreting the reference to basic rules in the characterization of canonical arguments.

Provisional Characterization 3.1. An argument is *canonical* if the following three conditions hold [1, p. 260]:

(i) all its open top occurrences are atomic sentences;

(ii) every atomic sentence in the main stem is either an open top occurrence or is derived by a basic rule;

(iii) every complex sentence in the main stem is derived by means of one of the introduction rules.

Provisional Characterization 3.2. A *supplementation* of a given argument is the argument resulting from the addition of a valid canonical argument

[3] When discussing this example, Dummett [1, p. 263] doesn't follow his own definition. He claims that both the premiss A and the final conclusion $(B \wedge C) \supset (A \wedge B)$ are in the main stem. However, since the sentence $A \wedge B$ (which depends on the hypotheses $B \wedge C$) occurs in the path from A to the conclusion, A is not, after all, in the main stem. This causes no further difficulties for understanding his definitions.

for each complex open top occurrence [1, p. 255,261]. The valid canonical arguments added will here be referred to as *supplements*.

Notice that the open top occurrences of the supplementation are exactly those of the supplements plus any open atomic top occurrences already in the original argument. In addition, since the supplements are valid canonical arguments, the notion of a supplementation becomes completely clear only after the characterization of valid canonical arguments.

Example 3.2. For the following inference rule.

$$\frac{B \vee C}{(A \supset B) \vee (A \supset C)}$$

any supplementations will take one of the forms below, where $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_m$ are open atomic top occurrences.

$$\dfrac{\dfrac{\begin{array}{c}\alpha_1, \ldots, \alpha_n \\ \Pi_1 \\ B\end{array}}{B \vee C}}{(A \supset B) \vee (A \supset C)} \qquad\qquad \dfrac{\dfrac{\begin{array}{c}\beta_1, \ldots, \beta_m \\ \Pi_2 \\ C\end{array}}{B \vee C}}{(A \supset B) \vee (A \supset C)}$$

Provisional Characterization 3.3. An argument is *valid* if we can effectively transform any supplementation of it into a valid canonical argument with the same open top occurrences and same conclusion of the supplementation. Furthermore, a canonical argument is *valid* if all its critical subarguments are valid [1, p. 261].[4]

Characterization 3.3 shows us that transformations of supplementations are the main element in the verificationist justification procedure. The fact that supplementations should be obtained by adding valid canonical arguments for complex open top occurrences indicates Dummett's partial adherence to the substitutional point of view concerning open arguments.[5] His commitment to the substitutional point of view is not complete because he allows canonical arguments to contain atomic open top occurrences. As an immediate consequence, for every sentence A, there is a canonical argument

[4]We do not consider instances as in Dummett's original formulation because they are only relevant for predicate logic.

[5]According to the substitutional point of view, hypothetical arguments, i. e. arguments with open assumptions, should be explained in terms of closed arguments by transforming canonical closed arguments for the open assumptions into canonical closed arguments for the conclusion [15, Section 2.2].

for A, built from atomic sentences occurring in A by a series of introduction rules.[6]

As emphasized by Characterization 3.3, transformations should be effective. Dummett does not explicitly elucidate what kinds of transformations are admitted. But, considering that Heyting's BHK clause for the assertion of an implication requires the possession of a construction transforming any construction of the antecedent into a construction of the succedent [4, p. 102], it seems reasonable to assume that Dummett had a similar idea in mind.[7]

Summarizing, an argument should be considered valid when any list of canonical arguments for its complex open top occurrences can be transformed into a canonical argument for the conclusion depending, at most, on atomic open top occurrences already appearing in the original argument or in the supplements.

4 Verificationist validity and basic rules

Characterization 3.1 is the only place where explicit reference to basic rules is made. However, because of their interconnection, the other characterizations that follow also depend on bases. As a result, the concept of verificationist validity can not be properly and unambiguously understood without a careful examination of the rôle played by basic rules. Dummett himself does not offer a detailed discussion of basic rules. His book contains what seem to be conflicting ideas and intuitions on the matter.

4.1 Bases and canonical arguments

The characterizations in Section 3 may be regarded as relative to a previously fixed basis B. In this case, the phrase "in a given basis B" should be added to clause (ii) of Characterization 3.1. Consequently, any reference to canonicity should be understood with respect to such a given basis B. Notice, for instance, that supplementations should be given in B because supplements are canonical arguments.

This interpretation makes the concept of validity dependent on the basis B under consideration. For this reason, we call it the *dependent interpretation.* In a discussion of justification procedures of the second grade, Dummett [1, p. 254] seems to assume the dependent interpretation:

[6] As a limiting case, we have canonical arguments for atomic sentences by an empty series of introduction rules.

[7] Among the transformations that Dummett had envisaged, we think that reduction steps of roundabouts would be included.

> We assume that we are given certain rules of inference, which we recognize as valid, for deriving atomic sentences from one or more other atomic sentences; we may call these 'boundary rules'. We now define a 'canonical argument' to be one in which no initial premiss is a complex sentence (no complex sentence stands at a topmost node) and in which all the transitions are in accordance either with one of the boundary rules [. . .]

Of course, Dummett's primary concern is with the justification of logical laws and it is reasonable to expect that the validity of logical laws should not depend on particular features of bases. Dummett [1, p. 273, our emphasis] says:

> We originally admitted, as occurring within deductive proofs of the kind with which we are concerned, boundary rules allowing the inference of an atomic conclusion from atomic premisses: these were, of necessity, left unspecified. Our original intention was that the boundary rules should be deductively valid. If we now include among them principles of non-deductive (and therefore fallible) inference, this will have the effect that a 'valid' argument, even if canonical, may have true initial premisses but a false final conclusion. *It will obviously not affect the justification procedure, however, as a means of determining the validity of logical laws.*[8]

Two distinct conclusions can be drawn from this passage.

First, in order to define logical validity under the dependent interpretation, some kind of generalization with respect to bases is required. The issue is going to be explored in Section 7.

Second, contrary to dependent interpretation, it might well be the case that we should merely consider what is the general form of a basic rule. It would not even matter if the basic rules considered are deductively correct or not. A basic rule is not recognized as such because it belongs to a basis. It is recognized as basic because it has a certain general form. Hence, the reference to basic rules in the item (ii) of Characterization 3.1 could be interpreted as standing for a generic formulation for rules of basic form. The characterization could thus become independent of any particular basis. We call this approach the *independent interpretation.* It is consistent with the last quotation. In Section 6, the independent interpretation is developed into

[8]This quotation is extracted from a later chapter, after Dummett had already presented his verificationist justification procedure.

a notion of validability which is then straightforwardly used to define logical validity. Logical principles justified in accordance with it do not depend on any specific set of basic rules.

4.2 The general form of basic rules

An important question regarding the general form of basic rules is whether or not they should be allowed to discharge hypotheses. In his first attempt to define a proof-theoretic notion of validity, Prawitz [7] considered the validity of atomic sentences by means of so-called production rules, i. e. basic rules that do not discharge hypotheses. Since then, the standard proof-theoretic approach is to use a system of production rules in order to give the assertability conditions for atomic sentences.

Sanz et al. [14, §,5.1] observe that using basic rules discharging hypotheses is a means to avoid proof-theoretical validation of classical laws in the implicational fragment. Moreover, Sandqvist [13] proposes a proof-theoretic semantics with basic rules discharging hypotheses and proves completeness of propositional intuitionistic logic with respect to it. However, Piecha et al. [6, §,7] raised some objections to basic rules discharging hypotheses. They claim that to allow discharge of atomic open assumptions in basic rules is equivalent to admitting $\supset$ in premisses of basic rules, which will not then look basic. For example, if α and β are atomic, then there is a basic rule equivalent to $\alpha, \alpha \supset \beta \vdash \beta$.

For his part, Dummett [1, p. 255] partially suggests how to think basic rules:

> The need to allow for the application of boundary rules is not as yet apparent but evidently can do no harm: they might be rules governing either non-logical expressions or logical constants not in the given set.

From the quotation, it is not clear if discharge of assumptions can be used in basic rules, but the way seems to be open for it. Nevertheless, Dummett [1, p. 261] seems to tacitly assume that basic rules are production rules. When arguing for the non-circularity of the definition of validity, he assumes that critical subarguments only occur in arguments for complex conclusions:

> It is important to notice that a sentence A standing immediately below the conclusion C of a critical subargument of a canonical argument must be of higher logical complexity than either the conclusion or the premisses of that subargument. This holds

> good of C because A, being a closed sentence in the main stem, must be derived by an application of one of the introduction rules, of which C must accordingly be one of the premisses; by the complexity condition on the introduction rules, A must be of higher logical complexity than any of its premisses. The premisses of the subargument must either be initial premisses of the entire argument, in which case they are atomic, or be hypotheses discharged by the introduction rule, in which case they must again be of lower logical complexity than A.

Notice that, if basic rules are allowed to discharge hypotheses, basic arguments (that is, arguments containing only basic rules) may have critical subarguments. In this case, the non-circularity of Dummett's notion of valid argument becomes problematic. We do not examine this problem here because, for the sake of our arguments, it suffices to consider bases without discharge. However, a great deal of the results below do not depend on the issue of allowing, or not, discharge in basic rules. When this is not so, we make explicit mention of it.

Dummett [1, p. 254, our emphasis] seems to work with a restricted notion of basic rules: they are used "for deriving atomic sentences from *one or more* other atomic sentences". The restricted notion is sufficient because canonical arguments for atomic sentences can be easily obtained by assumption (see Section 3, in particular item (i) of Characterization 3.1). To allow atomic axioms, however, can do no harm. Here, we work with the general notion of basic rules where the set of premisses can be empty.

5 Completeness of intuitionistic logic

We remarked in Section 2.2 that Dummett's definition apply directly to concrete arguments and, consequently, that validity is primarily a property of concrete arguments, i. e. arguments composed of actual sentences as opposed to schematic letters. In contrast with basic rules, the inference rules of NJ are schematic, which means that deducibility in NJ is preserved over any uniform substitution of the sentences. Validity, however, is not defined schematically. As a result, two different notions of completeness can be distinguished. Piecha et al. [6] discusses both notions. We adopt the distinction, but we do not use the same terms.

Definition 5.1. A deductive system is *complete*, or *simply complete*, when, for any set of sentences Γ and any sentence A, if $\langle\Gamma, A\rangle$ is valid ($\Gamma \models A$) then $\langle\Gamma, A\rangle$ is deducible ($\Gamma \vdash A$). Furthermore, a deductive system is *structurally* *complete* when, for all formulas Γ and A, if all sentential instances Γ', A' (of Γ, A) are valid ($\Gamma' \models A'$), then $\langle \Gamma, A \rangle$ is deducible ($\Gamma \vdash A$).

Piecha et al. [6] showed that, for a constructivist semantics monotonic over basis extensions, propositional intuitionistic logic is neither complete nor structurally complete if basic rules are allowed to discharge atomic rules. However, as we shall see in Section 7, Dummett's verificationist procedure has its own idiosyncrasies. In particular, monotonicity over basis extensions does not hold (Corollary 7.2).[9]

Definition 5.2. We say that a verificationist notion of logical validity is *normal* if it has the following two properties:

(i) it validates (justifies) the standard elimination rules with respect to its corresponding introduction rules;

(ii) every argument obtained by composition of valid inferences is also valid.

Normal proof-theoretic notions of logical validity satisfy the transitivity of the consequence relation (as defined trough the validity of inferences) and fullfill Gentzen's insight to the effect that the elimination rules are, in some sense, consequences of the introduction rules. It is clear that any adequate proof-theoretic definition of logical validity based on introduction rules must be normal.

6 Independent interpretation

In the independent interpretation, basic rules are recognized by their general form. However, a restriction must be imposed on the transformations described in Characterization 3.3: they can not add new basic rules. The rationale is that, whatever basic rules appear in the supplements, a transformation should not use more than those rules in order to construct the canonical argument for the conclusion. The restriction is similar to the one Dummett already imposes on open top occurrences.

By adopting the independent interpretation, validity of basic arguments may be left unspecified. The notion characterized under the independent interpretation is going to be called *validability*, instead of validity. We write the definitions accordingly.

Definition 6.1. In the context of the independent interpretation, an argument is *I-canonical*, if the following three conditions hold:

[9] This remark refers to the dependent interpretation.

(i) all its open top occurrences are atomic sentences;

(ii) every atomic sentence in the main stem is either an open top occurrence or is derived by a rule of form

$$\frac{\alpha_1 \quad \dots \quad \alpha_n}{\beta}$$

where $\alpha_1, \dots, \alpha_n$ and β are atomic sentences and $\alpha_1, \dots, \alpha_n$ can be empty; we call every application of a rule of this form an application of a *basic rule*;

(iii) every complex sentence in the main stem is derived by means of one of the introduction rules.

Lemma 6.1. *I-canonical arguments have the following properties.*

1. *I-canonical arguments may have atomic open top occurrences;*

2. *I-canonical arguments whose conclusion are atomic sentences proceed only by basic rules from atomic top occurrences, if any;*

3. *I-canonical arguments always have an introduction rule as its last step when the conclusion is a complex sentence.*

Proof. Each item follows from the corresponding item in Definition 6.1. □

Definition 6.2. An *I-supplementation* of a given argument is the argument resulting from the addition of a validable I-canonical argument for each complex open top occurrence. The validable I-canonical arguments added will here be referred to as *I-supplements*.

Definition 6.3. An argument is *validable* if we can effectively transform any I-supplementation (with any kind of basic rules) into a validable I-canonical argument for the conclusion containing no additional open top occurrences and no additional basic rules. Furthermore, an I-canonical argument is *validable* if all its critical subarguments are validable.

All four definitions are interconnected. In particular, Definition 6.2 and Definition 6.3 simultaneously define the concepts of *I-supplementation*, *validable I-canonical argument* and *validable argument*. As a result, a double induction is involved. This fact might raise doubts concerning well-foundedness of the notion of validability.

Definition 6.4. The degree of an argument $\langle \Gamma, A \rangle$ is the maximum of the degrees of the sentences in Γ and the conclusion A.

Theorem 6.1. *The definition of validable argument is well-founded.*[10]

Proof. Let $\langle \Gamma, A \rangle$ be an argument of degree n. First, we work the case for $n > 0$. If $\langle \Gamma, A \rangle$ is a I-canonical argument, then its validability depends only on the validability of its critical subarguments, which, by Definition 6.3, are all of lower degree.[11] If $\langle \Gamma, A \rangle$ is not I-canonical, then, according to the same definition, in order to judge its validability, we have to consider transformations of I-supplements for the sentences in Γ into validable I-canonical arguments for A. By definition, all these I-canonical arguments are of degree n, at most. That is, validability for I-canonical arguments of degree n has to be defined beforehand, which is indeed the case. Now, if $\langle \Gamma, A \rangle$ is of degree $n = 0$ and it is not I-canonical, then its validability depends on the validability of I-canonical arguments of degree, at most, $n = 0$. By item (ii) of Lemma 6.1 and Definition 6.3, all I-canonical arguments of degree $n = 0$ are validable. □

Definition 6.5. An argument is *logically valid under the independent interpretation* (*LI-valid*) when it is validable and it contains no applications of basic rules.

Definition 6.6. An argument is *valid in a basis* B under the independent interpretation when it is validable and all the basic rules used in the argument belong to B.

Theorem 6.2. *Let φ and ψ be atomic sentences. Then atomic Peirce's rule*

$$\frac{(\varphi \supset \psi) \supset \varphi}{\varphi}$$

is LI-valid.

Proof. We show that any I-supplementation can be effectively transformed into a validable I-canonical argument for φ depending on the same premisses and no additional basic rule. Suppose Π_1 is an I-supplementation depending on top occurrences $\alpha_1, \ldots, \alpha_n$ as specified by Definition 6.2. Thus, the penultimate step in Π_1 is an application of $\supset$I as shown below.

$$\frac{\dfrac{\begin{matrix}\alpha_1, \ldots, \alpha_n \\ \Pi_2 \\ \varphi\end{matrix}}{(\varphi \supset \psi) \supset \varphi}}{\varphi} \qquad (\Pi_1)$$

[10] Our proof is essentially the same one given by Dummett [1, p. 263].

[11] Here, we assume that the introduction rules comply with a complexity condition, as formulated by Dummett [1, p. 258]. The introduction rules of NJ are examples of such rules.

There are two possibilities: either (1) the penultimate occurrence of φ is in the main stem and we already have a validable I-canonical argument for φ from the same open top occurrences, by item (ii) of Definition 6.1, or (2) φ is not in the main stem and we have a critical subargument Π_2 with φ as conclusion (by Definition 3.0). In case (2), φ depends on additional open top occurrences, besides $\alpha_1, \ldots, \alpha_n$ (again, by Definition 3.0). Considering that these top occurrences were later discharged by an application of $\supset$I whose conclusion is $(\varphi \supset \psi) \supset \varphi$, then they can only be of the form $\varphi \supset \psi$.

$$\begin{array}{c} \varphi \supset \psi, \alpha_1, \ldots, \alpha_n \\ \Pi_2 \\ \varphi \end{array}$$

By Definition 6.3, the critical subargument Π_2 is a *validable* argument. From the validability of Π_2, we show how to obtain a validable I-canonical argument for φ from atomic open top occurrences $\alpha_1, \ldots, \alpha_n$ and no additional basic rules. Because Π_2 is validable, we have a procedure to effectively transform *any* I-supplementation Π_3 into a validable I-canonical argument for φ from open top occurrences $\alpha_1, \ldots, \alpha_n$,$\beta_1, \ldots, \beta_m$.

$$\begin{array}{c} \dfrac{\begin{array}{c}\beta_1, \ldots, \beta_m \\ \Pi_4 \\ \psi\end{array}}{\varphi \supset \psi,} \quad \alpha_1, \ldots, \alpha_n \\ \Pi_2 \\ \varphi \end{array} \tag{Π_3}$$

In particular, consider the following I-supplementation obtained by substitution of I-supplements for the open occurrences of $\varphi \supset \psi$.

$$\begin{array}{c} \dfrac{\dfrac{[\varphi]}{\psi}}{\varphi \supset \psi}, \alpha_1, \ldots, \alpha_n \\ \Pi_2 \\ \varphi \end{array} \tag{$\Pi_{3_{\varphi/\psi}}$}$$

This I-supplementation (containing φ/ψ as additional rule) is then transformed into a validable I-canonical argument Π_7 from open top occurrences $\alpha_1, \ldots, \alpha_n$ only. By Lemma 6.1, this validable I-canonical argument proceeds solely by basic rules. Examining Π_7, if the rule φ/ψ is not used, then we have in fact a validable I-canonical argument from open top occurrences $\alpha_1, \ldots, \alpha_n$ without additional rules and our proof is complete. Otherwise, if the rule is used, we take its first application as depicted below.

$$\begin{array}{c} \alpha_1, \ldots, \alpha_n \\ \Pi_5 \\ \dfrac{\varphi}{\psi} \\ \Pi_6 \\ \varphi \end{array} \tag{Π_7}$$

Since the rule φ/ψ does not occur in the subargument Π_5, we obtain the required valid I-canonical argument for φ from open top occurrences $\alpha_1, \ldots, \alpha_n$ and no additional basic rules. Finally, given that Peirce's rule is a one-step argument and contains no application of basic rules, by Definition 6.5, its validability implies its LI-validity.[12] □

Corollary 6.1. NJ *is not complete under the independent interpretation.*

Once all atomic instances of Peirce's rule are shown to be valid, it is possible to generalize the result for a fragment of the language without disjunction. This fragment is powerful enough to account for all valid propositional classical reasonings (with the other constants being defined in terms of implication, conjunction, and negation). Thus, Theorem 6.3 below certainly frustrates the expectation that proof-theoretic validity, as defined by Dummett, provides justification only for constructive reasonings.

Theorem 6.3. *Let A and B be any sentences. If LI-validity is a normal notion of logical validity, then Peirce's rule*

$$\frac{(A \supset B) \supset A}{A}$$

is LI-valid in the fragment $\{\supset, \wedge, \bot\}$.

Proof. First, by induction on the degree of B, using Theorem 6.2. We only show the case for $\wedge$, where $B = D \wedge E$. The case for $\supset$ is analogous.

$$\cfrac{\cfrac{\cfrac{\cfrac{(A \supset (D \wedge E)) \supset A \qquad \cfrac{\cfrac{\cfrac{[A \supset D]^{(2)} \quad [A]^{(3)}}{D} \qquad \cfrac{[A \supset E]^{(1)} \quad [A]^{(3)}}{E}}{D \wedge E}}{A \supset (D \wedge E)}\ {\scriptstyle (3)}}{\cfrac{A}{(A \supset D) \supset A}\ {\scriptstyle (2)}}}{\cfrac{A}{(A \supset E) \supset A}\ {\scriptstyle (1)}}\ \text{Peirce's rule}}{A}\ \text{Peirce's rule}$$

[12] The proof depends on the restriction to basic rules without discharge. In particular, it depends on the absence of discharges among the rules in Π_6. We thank an anonymous referee for pointing this out.

Next, we apply induction on the degree of A. We show the case for $\supset$, where $A = D \supset E$. Again, the other cases are similar.

$$
\begin{prooftree}
\AxiomC{$((D\supset E)\supset B)\supset(D\supset E)$}
\AxiomC{$[E\supset B]^{(2)}$}
\AxiomC{$[D\supset E]^{(3)}$}
\AxiomC{$[D]^{(1)}$}
\BinaryInfC{E}
\BinaryInfC{B}
\RightLabel{(3)}
\UnaryInfC{$(D\supset E)\supset B$}
\BinaryInfC{$D\supset E$}
\AxiomC{$[D]^{(1)}$}
\BinaryInfC{E}
\RightLabel{(2)}
\UnaryInfC{$(E\supset B)\supset E$}
\RightLabel{Peirce's rule}
\UnaryInfC{E}
\RightLabel{(1)}
\UnaryInfC{$D\supset E$}
\end{prooftree}
$$

□

Corollary 6.2. $\mathsf{NJ}^{\supset,\wedge,\bot}$ *is not structurally complete under the independent interpretation, if LI-validity is a normal notion.*

For a last remark concerning the independent interpretation, consider what would happen if I-canonical arguments were restricted to be closed instead of allowing atomic open top occurrences. Such a definition would be fully substitutional, that is, a valid argument would transform closed arguments of the premisses into a closed argument of the conclusion. As a consequence, for some sentences there may be no closed argument in a given basis B but only in extensions of B. As can be easily verified, Theorem 6.2 still holds when there are no open top occurrences $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$, if bases are still assumed to be given by rules with no discharge.

7 Dependent interpretation

Under the dependent interpretation, the primary notion of validity becomes relative to a fixed basis B. In other words, the supplementation of arguments and the valid canonical arguments are all given in a fixed basis B. We give definitions that correspond to the characterizations in Section 3, making explicit reference to a basis B.

Definition 7.1. An argument is *canonical in* B if the following three conditions hold:

(i) all its open top occurrences are atomic sentences;

(ii) every atomic sentence in the main stem is either an open top occurrence or is derived by a rule in B;

(iii) every complex sentence in the main stem is derived by means of one of the introduction rules.

Lemma 7.1. *Canonical arguments have the following properties.*

1. *Canonical arguments may have atomic open top occurrences;*

2. *Canonical arguments in* B *whose conclusion are atomic sentences proceed only by rules in* B *from atomic top occurrences, if any;*

3. *Canonical arguments always have an introduction rule as its last step when the conclusion is a complex sentence.*

Proof. Each item follows from the corresponding item in Definition 7.1. □

Definition 7.2. A *supplementation in* B of an argument is the argument resulting from the addition of a valid canonical argument in B for each complex open top occurrence. The valid canonical arguments added will here be referred to as *supplements.*

Definition 7.3. An argument is *valid in a basis* B if we can effectively transform any supplementation of it in B into a valid canonical argument in B with the same open top occurrences and the same conclusion of the supplementation. Furthermore, a canonical argument is *valid in* B if all its critical subarguments are valid in B.

Theorem 7.1. *The definition of valid argument in* B *is well-founded.*

Proof. Similar to that of Theorem 6.1. For the dependent interpretation with its notion of validity in B, the validity of canonical arguments of degree $n = 0$ are completely determined by the basis B. □

Two distinct ways for defining logical validity under the dependent interpretation might be considered. Either logical validity is defined as validity in the empty basis or, instead, it is defined as validity in all bases. These two alternatives are different. As Dummett [1, p. 273] points out, the notion of logical validity has to maintain a certain independence of bases and basic rules, which then makes it natural to consider the case of the empty basis. However, against what might be expected, logical validity defined as validity in the empty basis raises problems.

Theorem 7.2. *Let* φ *and* ψ *be distinct atomic sentences. The argument*

$$\frac{\varphi \supset \psi}{\psi} \tag{1}$$

is valid in $\mathsf{B} = \emptyset$.

Proof. By Definition 7.3, (1) is valid if any supplementation of $\varphi \supset \psi$ can be transformed into a valid canonical argument for ψ depending on the same open top occurrences. If $\mathsf{B} = \emptyset$, every possible supplementation involves the assumption of ψ. Among them, the simplest one is:

$$\cfrac{\cfrac{\psi}{\varphi \supset \psi}}{\psi}$$

By Definition 7.1, the assumption ψ alone is a valid canonical argument. $\square$

Corollary 7.1. *For φ and ψ distinct atomic sentences, $(\varphi \supset \psi) \supset \psi$ is valid in the empty basis.*

Notice that $(\varphi \supset \psi) \supset \psi$ is not classically valid. Curiously, intuitionistic logic would be structurally complete if logical validity were defined as validity in the empty basis because Corollary 7.1 cannot be generalized for any two sentences. However, it seems intuitively impossible to justify the fact that $(0 = 1 \supset 1 = 2) \supset 1 = 2$ would have to be logically valid.

Apparently, we are obliged to consider logical validity under the dependent interpretation as validity in all bases. As a consequence, argument (1) becomes logically invalid.

Definition 7.4. A basis C *extends* a basis B whenever all basic rules belonging to B also belong to C. It is clear that the set of all bases constitutes a lattice by this relation of extension.

Corollary 7.2. *Validity of arguments is not preserved by basis extension.*[13]

Proof. According to Theorem 7.2, $\langle \varphi \supset \psi, \psi \rangle$ is valid in the empty basis when φ and ψ are distinct atomic sentences. Clearly, it is not valid in the immediate extension of the empty basis containing φ/ψ. $\square$

Now, the only way to establish logical validity of a rule schema requires the construction of a general procedure showing how to validate every substitutional instance in every basis. This seems to be in accordance with the constructivist point of view. On the other hand, the invalidity of an inference schema can be established by a counterexample, i. e., an invalid instance over a specific basis.

Definition 7.5. An argument is *logically valid under the dependent interpretation* (*LD-valid*) if it is valid in all bases.

[13]This result is due to Goldfarb [3].

Example 7.1. If LD-validity is a normal notion (see Definition 5.2), the following rule schema

$$\frac{A \supset (B \supset C)}{(A \supset B) \supset (A \supset C)}$$

is LD-valid. A canonical argument for the premiss in any basis B, i. e., a supplement, is such that its immediate subargument Π_1 shows that $\langle A, B \supset C\rangle$ is valid in B. The following argument must be a valid canonical argument, as required:

$$\dfrac{\dfrac{\dfrac{\dfrac{[A]^{(1)} \quad [A \supset B]^{(2)}}{B} \qquad \begin{matrix}[A]^{(1)} \\ \Pi_1 \\ B \supset C\end{matrix}}{C}}{A \supset C}\,(1)}{(A \supset B) \supset (A \supset C)}\,(2)$$

At first sight, it may seem strange that validity is not preserved under basis extension (Corollary 7.2). One common interpretation is that a basis represents a state of knowledge. Monotonicity of validity over basis extension would, then, represent stability of validity when knowledge is expanded. Historically, a technical reason for considering stability over basis extension was the intent to avoid vacuous validation of implications [7, § A.1] which would then validate negations of any non-valid sentence. We think that Dummett was trying to solve this problem when he admitted atomic open assumptions in canonical arguments.

Although unusual, non-monotonicity of validity over basis extensions enables us to show invalidity for some non-intuitionistic rule schemata.

Example 7.2. Peirce's rule is not valid in basis $\mathsf{B} = \{\psi/\varphi\}$. There is no way of transforming the following valid canonical argument into a valid canonical argument for φ depending on the same (empty) set of open atomic top occurrences:

$$\dfrac{\dfrac{\dfrac{[\varphi \supset \psi]}{\psi}\ \text{valid in } \mathsf{B}}{\varphi}}{(\varphi \supset \psi) \supset \varphi}$$

As pointed out, Dummett's verificationist procedure corresponds only partially to the intuitionistic background settled by Heyting because canonical arguments admit open atomic top occurrences. Therefore, the concept of canonical argument does not correspond to the concept of a categorical proof. Consequently, hypothetical proofs are not reduced to categorical proofs. Instead, the verificationist procedure achieves, at best, a reduction of the general concept of hypothesis to the concept of atomic hypothesis. Futhermore, the fact that, for any two atomic sentences φ and ψ, $(\varphi \supset \psi) \supset \psi$ is validated in all bases in which ψ is not derivable from φ, does lack a constructivist defense. The problem here seems to be located in the definition of canonical arguments with open atomic top occurrences.

We do not claim that Dummett's intention was to reduce the general concept of hypothesis to the concept of atomic hypothesis. It seems more plausible that, trying to solve the problem of vacuous validation of implications, he adopted a concept of construction that seems somehow immanent to constructivism: Introduction rules are the canonical means used for constructing complex sentences starting from atomic sentences. In other words, there is at least one way to construct a sentence starting with its component atomic sentences.

In addition to the idiosyncrasies already discussed, other problems can be found in the dependent interpretation.

Theorem 7.3. *For any atomic φ,*

$$\frac{\varphi \supset (B \vee C)}{(\varphi \supset B) \vee (\varphi \supset C)}$$

is LD-valid.

Proof. A supplementation of $\varphi \supset (B \vee C)$ from assumptions $\alpha_1, \dots, \alpha_n$, with $B \vee C$ not in the main stem,[14] requires the critical subargument Π_1 to be valid in a basis B, by Definition 7.2.

$$\begin{array}{c} \varphi, \alpha_1, \dots, \alpha_n \\ \Pi_1 \\ B \vee C \end{array}$$

Since all assumptions are atomic, we obtain a canonical argument for $B \vee C$ depending solely on φ and $\alpha_1, \dots, \alpha_n$. The last step on this canonical argument is $\vee\mathsf{I}$ from either B or C. In both cases, we obtain a canonical argument for $(\varphi \supset B) \vee (\varphi \supset C)$ in basis B. □

Corollary 7.3. NJ *is not complete under the dependent interpretation.*

If the definition of canonical argument were changed in order to eliminate open atomic top occurrences, then, in every non-trivial basis, there would be no valid canonical argument for some sentences. As a result, when there is no canonical argument for A in basis B, $\langle A, B \rangle$ becomes valid in B, irrespective of B. As can be easily verified, Theorem 7.3 still holds.

[14]The case with $B \vee C$ in the main stem is trivial.

8 Concluding Remarks

The definition of validity based on introduction rules proposed by Dummett [1] has noteworthy differences from other proof-theoretic definitions in the literature. For instance, Dummett's verificationist procedure accepts atomic top occurrences of sentences to remain open in valid canonical arguments. His characterizations, however, are not precise enough when it comes to the rôle played by basic rules.

Based on some passages from his work, we proposed two interpretations of Dummett's verificationist justification procedure with respect to bases. We argued that, no matter what interpretation we choose, dependent or independent, and no matter what definition of canonical argument we choose, closed or with open atomic assumptions, the verificationist justification procedure is not adequate as a semantics for propositional intuitionistic logic.

In particular, we showed (Corollary 6.2) that, according to a plausible interpretation of Dummett's procedure, the independent interpretation, Peirce's rule is valid. Unlike Sandqvist [12], we established the validity of classical logic under an interpretation of Dummett's own semantic characterizations. Thus, if the interpretation we proposed in Section 6 is accepted, Theorem 6.3 frustrates the expectation, expressed by Dummett [1, p. 270] himself, that proof-theoretic validity provides justification only for constructive reasonings.

In Section 7, we advanced a dependent interpretation. The dependent notion of validity has the interesting and surprising property that it is not conservative over extensions of bases (Corollary 7.2). We saw that the best approach to obtain logical validity under the dependent interpretation was to define it as validity in all bases. We also showed the invalidity of Peirce's rule by means of a counterexample in a specific basis. Finally, with Theorem 7.3, we established that propositional intuitionistic logic is not complete even under the dependent interpretation.

The dependent interpretation could still be defended as adequate by the adoption of a more permissive notion of completness: structural completness. However, from the point of view of structural completeness, the question of which definition of logical validity to adopt arises: validity in all bases or validity in the empty basis? They are not equivalent.

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References

[1] Michael Dummett. *The Logical Basis Of Metaphysics*. Harvard University Press, Cambridge, Massachusetts, 1991.

[2] Gerhard Gentzen. Untersuchungen über das logische schließen I. *Mathematische Zeitschrift*, 39(1):176–210, 1935.

[3] Warren Goldfarb. On Dummett's proof-theoretic justification of logical laws. In Thomas Piecha and Peter Schroeder-Heister, editors, *Advances in Proof-Theoretic Semantics*, volume 43 of *Trends in Logic*, pages 195–210. Springer, 2016.

[4] Arend Heyting. *Intuitionism: An Introduction*. North-Holland Publishing Company, Amsterdam, 3rd edition, 1971.

[5] Lloyd Humberstone. *The Connectives*. MIT Press, Cambridge, Massachusetts, 2011.

[6] Thomas Piecha, Wagner de Campos Sanz, and Peter Schroeder-Heister. Failure of completeness in proof-theoretic semantics. *Journal of Philosophical Logic*, 44(3):321–335, 2015.

[7] Dag Prawitz. Ideas and results in proof theory. *Studies in Logic and the Foundations of Mathematics*, 66:235–307, 1971.

[8] Dag Prawitz. Towards a foundation of a general proof theory. In Patrick Suppes, Leon Henkin, Athanase Joja, and G. C. Moisil, editors, *Logic, Methodology and Philosophy of Science IV*, volume 74 of *Studies in Logic and the Foundations of Mathematics*, pages 225–250, Amsterdam, 1973. North Holland.

[9] Dag Prawitz. On the ideia of a general proof theory. *Synthese*, 27(1): 63–77, 1974.

[10] Dag Prawitz. Meaning approached via proofs. *Synthese*, 148(3):507–524, 2006.

[11] Dag Prawitz. An approach to general proof theory and a conjecture of a kind of completeness of intuitionistic logic revisited. In Luiz Carlos Pereira, Edward Hermann Haeusler, and Valéria de Paiva, editors,

Advances in Natural Deduction, volume 39 of *Trends in Logic*, pages 269–279. Springer, Dordrecht, 2014.

[12] Tor Sandqvist. Classical logic without bivalence. *Analisys*, 69(2):211–218, April 2009.

[13] Tor Sandqvist. Base-extension semantics for intuitionistic sentential logic. *Logic Journal of the IGPL*, 23(5):719–731, 2015.

[14] Wagner de Campos Sanz, Thomas Piecha, and Peter Schroeder-Heister. Constructive semantics, admissibility of rules and the validity of Peirce's law. *Logic Journal of the IGPL*, 22(2):297–308, 2014.

[15] Peter Schroeder-Heister. Proof-theoretic semantics. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy.* The Metaphysics Research Lab, Stanford, California, 2012.