South American Journal of Logic
Vol. 3, n. 1, pp. 59–86, 2017
ISSN: 2446-6719

A 4-valued Logic of Strong Conditional

Fabien Schang

Abstract

How to say no less, no more about conditional than what is needed? From a logical analysis of necessary and sufficient conditions (Section 1), we argue that a stronger account of conditional can be obtained in two steps: firstly, by reminding its historical roots inside modal logic and set-theory (Section 2); secondly, by revising the meaning of logical values, thereby getting rid of the paradoxes of material implication whilst showing the bivalent roots of conditional as a speech-act based on affirmations and rejections (Section 3). Finally, the two main inference rules for conditional, viz. Modus Ponens and Modus Tollens, are reassessed through a broader definition of logical consequence that encompasses both a normal relation of truth propagation and a weaker relation of falsity non-propagation from premises to conclusion (Section 3).

Keywords: Bilateralism, Conditional, Connection, Modi, Relevance.

Introduction: Elementary, my dear Wason?

Undergraduate students are currently introduced to logical conditional –or implication[1], through an explanation in terms of truth-tables. Hence their surprised faces, when confronted to the truth of a conditional sentence whenever its antecedent is false. But this logical method is neither the only one nor the best one. The literature on logical conditional is tremendous and some of its most famous developments intended to achieve the same purpose as ours, namely: to give a better definition of logical conditional, by getting rid of its undesired properties in classical logic.
Among the well-known logical systems which arose in this respect, let us quote relevant logic and connexive logic [1],[29]. In the former, the notion of relevance

[1] Strictly speaking, 'implication' is valid conditional. But both concepts of implication and conditional are currently made equivalent in the literature, which is not without some effect on the general confusion concerning what this logical connective is intended to mean.

has been made an essential feature of conditional, especially with the pioneer works of Anderson & Belnap [1] and the semantic development of Routley [19]. These were meant to face with the problem of topic-neutrality: how can an conditional be said true, whenever the antecedent and the consequent have nothing in common? In the latter, the properties of consistency and compatibility are preconditions for a proper characterization of conditional. We do not enter into the details of these systems, in the present paper. Although there is no ideal solution to the logical challenge of relevance, our aim is exactly the same: to make better sense of implication in a logical system, thereby accounting for the intuitive preconditions of relevance and connection between the antecedent and the consequent. It is in the nature of what makes conditional both relevance- and connection-friendly that we are going to depart from both relevantists and connectivists.

Just as the Ramsey's Test is used to be, Wason's Four-Card Selection Problem [30] –better known as the 'Selection Task Problem', is another good way to exemplify what logical conditional is supposed to mean. It consists in applying a rule stated in the following indicative words: *If one card has a D on one face, then it has a 5 on the other side.* Let us symbolize it by an ordered pair $\langle D, 5\rangle$, given that the order of appearance matters. The player is supposed to rely upon this rule in order to make the right choice, after picking the first card. The rule can be generalized in this logical abstract statement: *If* φ *then* ψ, the variables φ and ψ standing for any two atomic sentences connected by our conditional relation "if-then". Because of its game-liked nature, Wason's selection problem assumes that there is *good* and a *bad* way of playing the game of selecting cards.

At least two questions are in order, in this respect. First, how an application of the above rule is to be checked after selecting the first card. Second, how many sides need to be checked in order for the player to win the game. Our answer is that at least two ordered pairs correspond to *relevant* situations and need to be checked accordingly: $\langle D, ?\rangle$ and $\langle 7, ?\rangle$. This means that the player has still a reason to pick a second card only if their first selection concerns the clause of the rule. Thus, there are two relevant forms of reasoning or inferences that help to test the validation of the above rule from the first to the second selection of a card. The first reasoning runs as follows: if (φ) The card has D on one face, then either (ψ) The card has 5 on the other side, or ($\neg\psi$) the card has 7 on the other side. The first sequence (φ)–(ψ) is sound whilst the second (φ)–($\neg\psi$) is not, because of its violating the basic and unique rule of the game. The inference behind the first sequence is better known as *Modus (Ponendo) Ponens*, i.e., the mode that affirms (by affirming). To apply this rule of inference consists in inferring the *truth* of ψ from the *truth* of φ after stating that ψ follows from φ. Let us represent the whole sequence in three different forms, viz. literally, syntactically, and semantically.

1. If D, then 5	1. $\varphi \to \psi$	1. $v(\varphi \to \psi) = T$
2. Now D	2. φ	2. $v(\varphi) = T$
3. Therefore 5	3. $\therefore \psi$	3. $v(\psi) = T$

The second reasoning starts from another initial drawing: if ($\neg\psi$) the card has 7 on one face, then either ($\neg\varphi$) the card has K on the other side or (φ) the card has D on the other side. Only the first sequence ($\neg\psi$)–($\neg\varphi$) is sound and exemplifies the second rule of inference for conditional: *Modus (Tollendo) Tollens*, which is the mode that denies (by denying) and consists in avoiding prohibited cases by inferring the *falsity* of ψ from the *falsity* of φ.

1. If D, then 5	1. $\varphi \to \psi$	1. $v(\varphi \to \psi) = T$
2. Now 7	2. $\neg\psi$	2. $v(\psi) = F$
3. Therefore K	3. $\therefore \neg\varphi$	3. $v(\neg\varphi) = T$

Only these first two sequences are valid modes of reasoning leading to a unique valid conclusion, whereas the following two ones equally fail. In the third form of reasoning, ($\neg\varphi$) the card has K on one face and either (ψ) the card has 5 on the other side or ($\neg\psi$) the card has 7 on the other side. No inference rule is correctly applied in the above two sequences, since none applies the stated rule about having the pair {D,5}.

1. If D, then 5	1. $\varphi \to \psi$	1. $v(\varphi \to \psi) = T$
2. Now K	2. $\neg\varphi$	2. $v(\varphi) = F$
3. Therefore 5 or 7	3. $\therefore \psi$ or $\neg\psi$	3. $v(\neg\psi) = T$ or F

In the fourth and last sequence, (ψ) the card has 5 on one face and either (φ) the card has D on the other side or ($\neg\varphi$) the card has K on the other side. Again, the second drawing card has no relevance because the first drawing has no effect on the general rule: 5 must be drawn *once* D has been drawn initially, whilst no such obligation is applied to the converse sequence $\langle 5, D \rangle$ of the rule.

1. If D, then 5	1. $\varphi \to \psi$	1. $v(\varphi \to \psi) = T$
2. Now 5	2. ψ	2. $v(\psi) = T$
3. Therefore D or K	3. $\therefore \varphi$ or $\neg\varphi$	3. $v(\neg\varphi) = T$ or F

To summarize the situation, there are three kinds of results in the selection problem: *good*, *bad*, or *none* (neither good nor bad). We can also express this threefold distinction with modal properties like right-wrong-none, mandatoryprohibited-allowed, or even necessary-impossible-contingent. Whether the rule is of a deontic, epistemic or alethic nature is not the point of the following. If so, then let us symbolize the three sorts of drawings by three sorts of emoticons: satisfied (in symbols: ☺), unsatisfied (in symbols: ☹), and none (in symbols: –).

	letter	number	
1	D	5	☺
2	D	7	☹
3	K	5	–
4	K	7	–

Rescher [17] and, more recently, Pailos & Rosenblatt [13] proposed a non-truth-functional analysis of what corresponds to the common-language notion of 'implies', thus accepting valuations in which there may be more than one output value. We do not want to follow this non-deterministic way of dealing with implication, however, due to our precise focus on what a truth-value is intended to mean from our perspective. Anyway, any standard, truth-functional presentation of implication makes no difference between satisfaction and indifference, as witnessed by the last two columns of the following classical truth-table:

	φ	ψ	$\varphi \to \psi$
1	T	T	T
2	T	F	F
3	F	T	T
4	F	T	T

There is one clear difference between these two tables: in the second table, the satisfaction of the rows 3 and 4 embeds the two paradoxes of material (or 'Philonian') implication by resulting in unnatural cases of truth; in the first table, the more pragmatic meaning of the emoticons makes a clear difference between three sorts of result: satisfied (by truth), dissatisfied (by falsity), and not satisfied (by either). Where is the right pattern of implication, accordingly? Our main statement is twofold, namely: that there a crucial logical difference between satisfaction and dissatisfaction, on the one hand; that the introduction of a third truth-value cannot explain what is wrong with implication on its own, on the other hand. A technical proof that any three-valued analysis of implication turns out to fail has been given by Vidal [24]. But while the latter favored an analysis in terms of possible-world semantics and strict implication, our claim is that bilateralism represents an alternative solution to the problem that surrounds the logical analysis of conditional reasoning, and that a number of papers in the relevant literature lead one astray about the nature of the problem.

1 Necessity and sufficiency

Given that a merely truth-functional depiction of logical implication is misleading, another way to make sense of the latter consists in defining it in terms of necessary and sufficient conditions.

1.1 Sufficient condition

A *sufficient* condition is lexicalized by the grammatical conjunction 'If' and is to be expressed by the phrase 'ψ if φ' in the logical sentence $\varphi \to \psi$. The sufficient condition is φ, viz. the *antecedent* of a conditional.
However, the meaning of such a sufficiency is unclear and leads to two additional questions. To the first question: *what* is sufficient in an conditional, the answer is: the antecedent, φ. To the second question: *what* φ is sufficient *for*, the answer is not that easy. On the one hand, the truth of φ is not sufficient in itself for affirming the truth of the consequent ψ, insofar as the truth of φ does not mean that the truth of ψ is thereby implied by it without further ado. To do this would be committing a big confusion between 'material' and 'formal' implication, the latter being an implicative form of *tautology*. On the other hand, the truth of φ is not sufficient for affirming the whole conditional $\varphi \to \psi$, either, given that $\varphi \to \psi$ may be false when φ is true –in the special case when ψ is false.
Another way to account for sufficiency is to equate it with the notion of *irrelevance*, however opposite these may appear on the surface. Indeed: the *falsity* of φ is 'sufficient' for having $\varphi \to \psi$, precisely because of its irrelevance with respect to its truth; at the same time, the truth of φ is not sufficient to ensure the truth of $\varphi \to \psi$.

1.2 Necessary condition

A *necessary* condition, to be read 'only if', is said to be the converse of a sufficient condition: if φ is a sufficient condition for ψ, then ψ is a necessary condition for φ and is to be read 'φ only if ψ'.[2] The necessary condition plays the role of *consequent* in the conditional.
The truth of ψ is necessary for affirming the antecedent φ, in the sense that φ cannot be affirmed without affirming ψ at once. The truth of φ is not necessary for affirming the whole implication, however, given that $\varphi \to \psi$ may be true

[2]For an objection to the converse relation between necessary and sufficient conditions, and a reply to it, see, e.g., [6]. The various arguments mentioned therein rely on subjunctive conditionals and interpretations of 'it' in causal and temporal senses. The present paper is strictly concerned with indicative conditional, however, for want of sufficient conviction about a proper logical treatment of counterfactuals.

while ψ is false (i.e., when φ is false, too). Likewise, the concept of relevance seems a better way to make sense of such a necessity: ψ is necessary in that its truth is relevant to the truth of $\varphi \to \psi$ whenever φ is also true.

In other words, 'sufficient' and 'necessary' should better be rendered in terms of 'irrelevant' and 'relevant' truth-values of a conditional: the falsity of the antecedent is sufficient to ensure the truth of the whole conditional; the truth of the consequent is necessary to ensure this truth, but only when the antecedent is already given true. Not only are these explanations not symmetrical with each other, insofar as the necessary condition requires an additional precondition for its definition (i.e., that the antecedent be true). But also, the meaning of such definienda is to be clarified in other terms than mere truth-value assignments (as in the below figure).

	sufficient condition	necessary condition
what?	$v(\varphi) = F$	$v(\psi) = T$
what for?	$v(\varphi \to \psi) = T$	$v(\varphi \to \psi) = T$ when $v(\varphi) = T$

The main problem lies in the fact that the mainstream definition seems to say more than is intended by a properly implicative relation; the paradoxes of material implication witness such an over-explanation, especially when the antecedent is false and turns out to be irrelevant for the truth of the whole sentence.

2 Inclusion and ordering

True, whoever thinks of logical constants in purely terms of truth-functions and without intuitive assumptions may contend with the below characterization of conditional in the fourth column.

φ	ψ	1	2	3	**4**	5	6	7	8	9	10	11	12	13	14	15	16
T	T	T	T	T	$\boldsymbol{T}$	F	T	T	F	T	F	F	F	F	F	T	F
T	F	T	T	T	$\boldsymbol{F}$	T	T	F	F	F	T	T	F	F	T	F	F
F	T	T	T	F	$\boldsymbol{T}$	T	F	F	T	T	F	T	F	T	F	F	F
F	F	T	F	T	$\boldsymbol{T}$	T	F	T	T	F	F	F	T	F	F	F	F

Now although Quine has been known as one of the most convinced champions of bivalence, he clearly acknowledged the artificiality of the definition of implication by means of his truth-function theory:

The mode of composition described in the table constitutes the *nearest* truth-functional approximation to the conditional of ordinary discourse.[3]

[3] [15]: 15 (our italics).

Albeit approximatively right, such a definition of conditional remains the cause of the troublesome paradoxes of material implication. If the problem posed by any scientific theory is the price to pay for its theoretical simplicity, what of these paradoxes?
One intuitive characterization of conditional abandons the simplicity of truth-tables and assumes a *set-theoretical* reading. It explains the current assimilation of 'conditional' and 'implication' although, strictly speaking, the latter corresponds to logical implication: it is a metalinguistic property having to do with logical consequence. And yet, the Quinean distinction between the mention of a logical constant in the metalanguage and its use in object language cannot ignore the blatant similarity of conditional as mere implication, implication as logical implication, consequence, and entailment. For this reason, the conditional sentence 'If φ then ψ' is commonly put on a par with the set-theoretical definition of entailment as an inclusive relation of *containment*. That is, the formula 'φ implies ψ' means that whatever makes φ true also does this with ψ because the latter is *included into* the former. Albeit intuitive as a reading of implication, truth-functionality and topic-neutrality are lost by doing so: truth-tables cannot be explained in terms of an inclusive relation, on the one hand; containment assumes that the antecedent and consequent have something common in their sentential contents, on the other hand.
Nevertheless, the heavy price of paradoxicality motivated the introduction of modal notions towards a better definition of implication.
For instance, MacColl [12] stated that implication is a sentential relation the negation of which is impossible (i.e., necessarily false).[4] Such a negative definition matches with the classical $(\varphi \rightarrow \psi) =_{def} \neg(\varphi \wedge \neg\psi)$. Only one half of MacColl's definition matches with the classical one, with respect to the truth-conditions of implication: both antecedent and consequent must be necessarily true, in which case the conjunction of either and its negation is impossibly true (i.e., false). But, of course, conjunction and implication are not on a par with respect to their falsity-conditions: the occurrence of at least one false conjunct makes conjunction false, whereas the falsity-condition of implication is more demanding by requiring one true antecedent and one false consequent.

Do the modal notions of necessity and impossibility improve the definition of implication? Possible world semantics has been especially used for subjunctive conditionals or counterfactuals, as recalled by Vidal [28]; now such a semantic device can hardly make better sense of our classical, indicative conditional so long as 'worlds' are not explained themselves.[5]

[4]MacColl made use of another symbolism, namely: (A:B) = $(\mathrm{AB}')^{\eta}$, which corresponds to the modern formulation of strict conditional $(\varphi \dashv \psi) =_{def} \neg\Diamond(\varphi \wedge \neg\psi)$.

[5]See especially [9] and [24], by reference to a possible-world semantic treatment of implica-

And yet, the criterion of inclusion for defining implication was well rendered by MacColl's symbolic logic, in the sense that every case of certainty (or necessity) was included there as a special subcase of truth. His negative definition of implication was motivated by an objection to Schröder's implication, following which the class of impossibilities was included into the class of necessities. In this respect, algebraic semantics seems to give a better account by characterizing implication as an ordering relation between truth-values: $\varphi \rightarrow \psi$ means that $v(\varphi) \leq v(\psi)$. For assuming that truth is greater than falsity in a lattice, the analogy with integers is such that T and F are currently replaced by 1 and 0 in algebraic semantics: $0 \leq 1$ and, accordingly, any sentence whose truth-value is lower than or equal with the truth-value of another sentence can play the role of its antecedent. Hence the validity of valuations like $F \rightarrow \psi$ and $\varphi \rightarrow T$ as well as the sole invalidity of $T \rightarrow F$, for any value of the consequent ψ and the antecedent φ.

However useful this algebraic criterion of ordering may appear, it is still intuitively queer and differs from the relation of inclusion. Firstly, the antecedent has a lower truth-value than the consequent although the latter is supposed to be conditioned by the former; how can a condition be false, if its proper role is to make sense of the whole relation between a premise and its conclusion? Second, paradoxicality is not cancelled by this algebraic trick since, again, the cases $F \rightarrow \psi$ and $\varphi \rightarrow T$ are paradox-makers and stem from abstractly disconnected truth-values. The *connection*, or relevant link, between both terms is what should be explained by a satisfactory definition of implication.

We argue that the real point of implication is the following: to make impossible the truth of the *whole conditional* whenever the antecedent is not true, which is a stronger requirement than MacColl's modal definition and clearly discards the undesired case of true implication with a false antecedent. For that purpose, let us reconsider Frege's theory of judgment and see what Dummett proposed about the meaning of implication. We take both to be guiding authors for an intuitively and formally sound solution to our problem, namely: to afford a definition of conditional, and only that one. Nothing more, and nothing less than it.

3 Strong conditional

In the following, we propose an alternative logic for two kinds of independent judgments, namely: affirmation, and denial. Then a corresponding logical system is devised to improve the definition of conditional without losing the

tions. Moreover, a tricky debate occurred about what "closest" possible worlds are supposed to mean in order to determine the truth-conditions of counterfactuals.

property of compositionality.

3.1 A question-answer game

In his logical considerations, Frege depicted logical reasoning as a sequence of judgments made by means of sentential contents. Returning to the Fregean distinction between 'judgeable content' (the sense of sentence, i.e., its *Proposition*) and judgment (the reference of a sentence, i.e., its *truth-value*), a sentence occurs in a language as a form of words endowed with a specific role. The aim of this sentence is formulated by the initial question of the speaker, as witnessed by Frege in these words:

A propositional question contains a demand that we should either acknowledge the truth of a thought, or reject it as false.[6]

In other words, a given sentence φ is said to be true whenever the question asked by a speaker is answered positively by the hearer; otherwise, φ is said to be false and the hearer answers negatively to the same question about whether φ is the case. A corollary of this theory is that falsity is nothing but another expression of truth through a negative sentence: the truth of φ entails the falsity of its negation $\neg\varphi$, and this nicely coheres with our linguistic intuitions concerning negation. But what if a negative answer is first given by the hearer, i.e., (i) 'No, φ is not true!'? According to Frege, there is no logical difference at all between this answer and (ii) 'Yes, φ is false!' or, equivalently, (iii) 'Yes, $\neg\varphi$ is true!'. We agree with Frege about the latter equivalence between (ii) and (iii), thereby defining falsity as a by-product of truth and sentential negation:

Falsity. For every sentence φ, φ is false if and only if its sentential negation $\neg\varphi$ is true.

At the same time, we disagree with him concerning the first alleged equivalence between (i) and (ii). After all, the answerer can express doubt in giving a negative answer. Now such a circumstance is overtly discarded by Frege [4], thereby arguing for his one-sided theory of judgment and the ensuing principle of bivalence that frames his theory of judgment. In the following, we propose an alternative logic for two kinds of independent judgments, namely: affirmation, and denial. An *affirmation* is an expression of acceptance by the speaker; it is not only a truth-, but also a falsity-claim and generally means any speech-act committing the speaker in the truth-value of the sentential content. A *denial* is an expression of rejection by the speaker; it may be a merely non-committal

[6] [4]: 117.

act expressed by a no-answer; although denying is currently used as a synonym for a falsity-claim, the latter is just a special case of denial where the speaker commits in the falsity and does not commit in the truth.

3.2 A logic of acceptance and rejection

An alternative logic for conditional is displayed in the following 4-valued logic of affirmation and rejection $\mathbf{AR}_4 = \langle \mathfrak{L}, \mathbf{4}, f_c, D \rangle$.[7] It consists of a sentential language $\mathfrak{L}$; a set of four truth-values $\mathbf{4} = \{11,10,01,00\}$ –truth and falsity being independent from each other; a set of logical connectives $f_c = \{\neg, \wedge, \vee, \twoheadrightarrow\}$; and a subset of designated values $D = \{11, 10\}$, defining logical consequence as a relation of truth-preservation. A designated value is whatever assigns the value of truth to a sentence. Importantly, our illocutionary-minded approach to logic entails that the truth-value bearers of $\mathbf{AR}_4$ are judgments or statements (which are answers to corresponding questions), rather than sentences (expressed by means of questions). Two sorts of answer correspond to two sorts of independent judgment in $\mathbf{AR}_4$, unlike Frege's unilateral view of judgment according to which a truth-claim is either an affirmative sentence φ or a negative sentence $\neg\varphi$.

Letting $\mathbf{Q}(\varphi)$ be the sentential *question* about φ, then $\mathbf{A}(\varphi)$ is the corresponding answer to whether φ is the case, or not. Any yes-answer –or affirmation, is symbolized by 1; any no-answer –or denial, is symbolized by 0. In order to make sense of doubt, we need to mark out a relevant difference between denying truth and affirming falsity; this entails the independence of affirmative and negative judgments, by contradistinction to Frege's policy. In the light of our bilateralist view of judgments as either affirmative or negative, this results in two main functions in $\mathbf{AR}_4$.

The first function is an interrogative function: $\mathbf{Q}(\varphi) = \langle \mathbf{q}_1(\varphi), \mathbf{q}_2(\varphi) \rangle$. It includes two basic questions:

$\mathbf{q}_1(\varphi)$: 'Is it the case that φ? (i.e., 'Is φ true?')
$\mathbf{q}_2(\varphi)$: 'Is it the case that $\neg\varphi$? (i.e., 'Is $\neg\varphi$ true?' or, equivalently, 'Is φ false?')

Again, the second question does not mean the same as 'Is it not the case that φ?': the former entails the latter, but the converse does not hold.

The second function is an answerhood function: $\mathbf{A}(\varphi) = \langle \mathbf{a}_1(\varphi), \mathbf{a}_2(\varphi) \rangle$, in which yes- or no-answers $\mathbf{a}_i(\varphi) \mapsto \{1, 0\}$ are related to the above two corresponding questions.

[7] For more details about the philosophical motivations of this system and its connection with Dunn & Belnap's logic of First Degree Entailment, FDE, see Schang & Costa-Leite [21].

Our presentation follows and departs at once from Frege's philosophy of logic: it follows his theory of sense and reference, insofar as questions make sense of a sentential content whilst ordered answers assign truth-values as its reference. At the same time, our truth-values are not single properties of sentences but structured answers to sentential questions. The independence of affirmative and negative judgments leads to four truth-values, which clearly departs from the bivalent framework of Frege's logic.
For any value of a sentence φ endowed with a logical value $\mathbf{A}(\varphi) = \langle \mathbf{a}_1(\varphi), \mathbf{a}_2(\varphi) \rangle$, we have in $\mathbf{AR}_4$ the following definitions of negation $\neg$, conjunction $\wedge$, disjunction $\vee$, and a stronger version of implication $\twoheadrightarrow$. Each of the connectives or binary operators are defined in terms of minimal and maximal functions $\cap$ and $\cup$ on single values $\mathbf{a}_i(\varphi)$.

Negation
$\mathbf{A}(\neg\varphi) = \langle \mathbf{a}_2(\varphi), \mathbf{a}_1(\varphi) \rangle$

φ	$\neg\varphi$
11	11
10	01
01	10
00	00

Conjunction
$\mathbf{A}(\varphi \wedge \psi) = \langle \mathbf{a}_1(\varphi) \cap \mathbf{a}_1(\psi), \mathbf{a}_2(\varphi) \cup \mathbf{a}_2(\psi) \rangle$

$\wedge$	11	10	01	00
11	11	11	01	01
10	11	10	01	00
01	01	01	01	01
00	01	00	01	00

Disjunction
$\mathbf{A}(\varphi \vee \psi) = \langle \mathbf{a}_1(\varphi) \cup \mathbf{a}_1(\psi), \mathbf{a}_2(\varphi) \cap \mathbf{a}_2(\psi) \rangle$

$\vee$	11	10	01	00
11	11	10	11	10
10	10	10	10	10
01	11	10	01	00
00	10	10	00	00

Strong conditional
$\mathbf{A}(\varphi \twoheadrightarrow \psi) = \langle \mathbf{a}_1(\varphi) \cap \mathbf{a}_1(\psi), \mathbf{a}_1(\varphi) \cap \mathbf{a}_2(\psi) \rangle$

$\twoheadrightarrow$	11	10	01	00
11	11	10	01	00
10	11	10	01	00
01	00	00	00	00
00	00	00	00	00

The above matrix shows that an implication can never be affirmed successfully whenever its antecedent is not so: the whole relation between the antecedent and the consequence is therefore made irrelevant, doing justice to what Strawson argued in [25] about *entailment* and *presupposition.* Against Russell's logical analysis of negative referential sentences, Strawson argued that such a sentence as 'The present king of France is bald' is neither true nor false whenever there is no present king of France. In the same vein, our defective account of implication is to the effect that the latter is neither told true nor told false once its antecedent is not affirmed.

Strong conditional matches with classical conditional with respect to its falsity-conditions only, differing from it with respect to its truth-conditions. Furthermore, our definition of it strengthens implication without making it collapse with conjunction. For both the latter and strong conditional have the same *truth*-conditions, i.e., $\mathbf{a}_1(\varphi \wedge \psi) = \mathbf{a}_1(\varphi \twoheadrightarrow \psi) = \mathbf{a}_1(\varphi) \cap \mathbf{a}_1(\psi)$. But, they differ in their *falsity*-conditions: $\mathbf{a}_2(\varphi \wedge \psi) = \mathbf{a}_2(\varphi) \cup \mathbf{a}_2(\psi)$, whereas $\mathbf{a}_2(\varphi \twoheadrightarrow \psi) = \mathbf{a}_1(\varphi) \cap \mathbf{a}_2(\psi)$. Now given that the identity of a logical constant is provided by these two independent conditions from our bilateral perspective, the occurrence of two separate judgments is both crucial and useful to make sense of the related notions of relevance and connection with respect to conditional.

In this respect, strong conditional fulfills some expectations of connexive logic such as *consistency* and *compatibility.* About consistency, Aristotle claimed in his *Prior Analytics* (57b14) that

> It is impossible that if [φ], then not-[φ].

Angell [2] called this requirement the 'conditional principle of non-contradiction', which corresponds to the so-called Aristotle's Theses of connexive logic, $\neg(\varphi \to \neg\varphi)$, and $\neg(\neg\varphi \to \varphi)$. Strong conditional matches with this, in that it prevents any antecedent to imply its own negation.

About compatibility, it has been recalled in [29] that, according to Sextus Empiricus,

> Those who introduce the notion of connection say that a conditional is sound when the contradictory of its consequent is incompatible with its antecedent.

Unlike Aristotle's Theses, connection refers to logical relations between two different sentences and corresponds to the two Boethian Theses of connexive logic: $(\varphi \rightarrow \psi) \rightarrow \neg(\varphi \rightarrow \neg\psi)$, and $(\varphi \rightarrow \neg\psi) \rightarrow \neg(\varphi \rightarrow \psi)$. It turns out that Aristotle's and Boethius' theses are partly satisfied into $\mathbf{AR}_4$ by *invalidating* any conditional of the form $\Phi \twoheadrightarrow \Psi$ whose antecedent Φ is false. At the same time, the theses of connexive logics demand more by requiring the *validity* of their negative form $\neg(\Phi \twoheadrightarrow \Psi)$. This cannot work in $\mathbf{AR}_4$ insofar as the negation of a neither-true-nor-false sentence does not turn into something true. In this sense, the so-called 'theses' of connexive logic are statements whose requirements about consistency and compatibility are taken into account in $\mathbf{AR}_4$ without being 'theses' in the logical sense of being true in every model. Even if such a technical requirement can be satisfied by our strong conditional *mutatis mutandis* [8], our point is that some misunderstanding may occur between the *syntactic* and *semantic* readings of connexive logic. From a deflationist point of view, sentences like φ and $\neg\varphi$ are made on a par with the idea that φ is true or φ is false, respectively. If so, then the theses of connexive logics are not valid in $\mathbf{AR}_4$; and yet, the views that no falsity can be derived from a truth and no truth can be derived from a falsity are satisfied by the truth- and falsity-conditions of strong conditional.

With respect to the case of false antecedents, our definition of implication intends to give a proper formal version of what Quine took to be a proper informal reading of conditionality:

> An affirmation of the form 'If p then q' is commonly felt less as an affirmation of a conditional than as a conditional affirmation of the consequent. If, after we have made such an affirmation, the antecedent turns out to be true, then we consider ourselves committed to the consequent, and are ready to acknowledge error if it proves false. If on the other hand the antecedent turns out to have been false, our conditional affirmation is *as if it had never been made.*[9]

Here is an ensuing list of strikingly valid formulas (1)-(9) and invalid formulas (10)-(17) in $\mathbf{AR}_4$:

(1) $\varphi \models \neg\neg\varphi$,	(2) $\varphi \wedge \psi \models \varphi$
(3) $\varphi \models \varphi \vee \psi$,	(4) $\varphi, \varphi \twoheadrightarrow \psi \models \psi$
(5) $\varphi \twoheadrightarrow \psi, \psi \twoheadrightarrow \gamma \models \varphi \twoheadrightarrow \gamma$	
(6.1) $\neg(\varphi \wedge \psi) \models \neg\varphi \vee \neg\psi$,	(6.2) $\neg\varphi \vee \neg\psi \models \neg(\varphi \wedge \psi)$

[8] See Schang, F. "4-valued strong connexivity" (draft).
[9] [16]: 21.

$$
\begin{array}{ll}
(7.1)\ \neg(\varphi \vee \psi) \models \neg\varphi \wedge \neg\psi, & (7.2)\ \neg\varphi \wedge \neg\psi \models \neg(\varphi \vee \psi) \\
(8)\ \varphi \twoheadrightarrow \psi \models \varphi, & (9)\ \varphi \twoheadrightarrow \psi \models \psi \\
(10)\ \neg\neg\varphi \not\models \varphi, & (11)\ \psi \not\models \varphi \vee \neg\varphi \\
(12)\ \varphi \wedge \neg\varphi \not\models \psi, & (13)\ \varphi \vee \psi, \neg\psi \not\models \neg\varphi \\
(14)\ \varphi \twoheadrightarrow \psi \not\models \neg\psi \twoheadrightarrow \neg\varphi, & (15)\ \varphi \not\models \psi \twoheadrightarrow \varphi \\
(16)\ \neg\varphi \not\models \varphi \twoheadrightarrow \psi, & (17)\ \varphi \not\models \psi \twoheadrightarrow \psi
\end{array}
$$

As there is no tautology in $\mathbf{AR}_4$, the classical inference rules are maintained only if material implication is replaced by the metalogical implication $\models$ from premises to conclusion. For example, the law of double negation elimination is validated in its form (1) and invalidated in its form (9). Excluded middle $\varphi \vee \neg\varphi$ is shown invalid in (11), through the failure of the Principle of Implosion; a similar result occurs in (12) with respect to the invalidity of non-contradiction $\neg(\varphi \wedge \neg\varphi)$, thereby showing the paracomplete and paraconsistent nature of $\mathbf{AR}_4$. At the same time, the intuitive properties of conjunction, disjunction and conditional hold in (2)-(5), including the rule of Modus Ponens in (4), and the Morganian behavior of negation in $\mathbf{AR}_4$ is marked by the formulas (6)-(7). Although the failure of Modus Tollens occurs in (14), the Section 3.4 will propose another formulation of it to preserve its logical significance for want of its validity. The same account can be done with respect to the valid formulas (8)-(9),[10] following which the truth of strong conditional entails the *separate* truth of both the antecedent and the consequent. Although these disconnected truths seem to contradict the expected connection from antecedent and consequent, the strengthened truth-condition of strong conditional may motivate this point together with our largest definition of logical consequence, according to which preserving truth is not the only way to define logical constants –the connection between antecedent and consequent actually occurs whenever the consequent is false, as will shown later on. Finally, the usually counterintuitive features of implication are made invalid in $\mathbf{AR}_4$, as witnessed by the formulas (15)-(17).

The above formulas are taken to be intuitive features of conditional, in the sense that they do justice to the underlying relation of *implication* from antecedent to consequent. Are they really so, and to what extent does strong conditional represent a more 'natural' account of conditional?

3.3 'Natural' implication

To recapitulate the situation thus far, we have introduced 4-valuedness in order to improve the definition of conditional in a pragmatic sense of connection

[10] I am grateful to Peter Verdée for this note about the seemingly "irrelevant" validity of (8)-(9).

between speech-acts. Now our central problem is to catch what is to be meant by a 'natural' definition of implication. Tomova [26] directly tackled this problem by proposing a set of four conditions for naturalness within three-valued systems. Thus, an implication $\rightarrow$ is said to be 'natural' whenever:

1. Its restrictions to the subset of truth-values {T,F} coincide with the classical valuations of implication.
2. If $\varphi \rightarrow \psi \in D$ and $\varphi \in D$, then $\psi \in D$, i.e. matrices for implication verify Modus Ponens.
3. Assuming $v(\varphi) \leq v(\psi)$, then $\varphi \rightarrow \psi \in D$.
4. $\varphi \rightarrow \psi \notin D$, in other cases.

Only the second condition is satisfied by strong conditional $\twoheadrightarrow$ in $\mathbf{AR}_4$. The first condition is not satisfied once the antecedent is not affirmed, i.e. whenever $\mathbf{a}_1(\varphi) = 0$ in $\mathbf{A}(\varphi \twoheadrightarrow \psi)$; unlike classical valuation, the whole implication is then neither true nor false in such a case, and we have taken this to be a precondition of a proper implication. The third condition appears to be an unnatural clause for implication in most of the algebraic systems, including the implicative twist-structure proposed by Rivieccio [18]. If the only reason to satisfy this third condition is to make an algebraic difference between the definitions of implication and conjunction, then $\mathbf{AR}_4$ does suffice for this purpose whilst avoiding the ensuing paradox of material implication.
Is our strong conditional not properly natural for all? To the contrary, it seems that the above conditions lead to paradoxical valuations and do not avoid those overcome by $\twoheadrightarrow$. In order to give a natural definition of $\varphi \twoheadrightarrow \psi$, we take the following three conditions to be essential between φ and ψ, namely: compatibility, connection, and ordering. *Compatibility* is expressed in $\mathbf{AR}_4$ by the falsity-condition of implication, such that any failure of compatibility results in a case of falsity. *Connection* is rendered by the truth-condition, such that any failure of connection results in a weaker case of non-truth. *Ordering* corresponds to the relevant feature that makes implication false, given that having a true antecedent and a false consequent is not the same as having a false antecedent and a true consequent. Having these three preconditions seems to be a better way of having a 'natural' implication than accepting paradoxes, at any rate.
Moreover, the validity of (MP) with strong conditional goes together with the validity of Deduction Theorem (DT):

(DT) $\Gamma, \varphi \models \psi$ iff $\Gamma \models \varphi \twoheadrightarrow \psi$

Such a property of equating semantic consequence with the validity of conditional still holds in $\mathbf{AR}_4$, due to the clause of validity included into (DT): if a strong conditional $\varphi \twoheadrightarrow \psi$ is *valid* then it can be turned into a semantic consequence $\varphi \models \psi$, and conversely. That strong conditional invalidates any formula including a false antecedent does not infringe (DT), accordingly, given that conditionals with false antecedents are ruled out by the above clause of validity. And yet, another problem remains with our stronger implication. It concerns the link between inference and validity, especially with respect to the definition of Modus Tollens. Given that the latter is an essential feature of implication, $\mathbf{AR}_4$ should be refined in order to overcome the following other objection to strong implication.

3.4 Logical meaning beyond validity

The combination of bilateralism and many-valuedness gives rise to a more comprehensive theory of judgments and a more intuitive definition of conditional. However, there may be cons about its characterization. Not only is the inter-definition between conditional, conjunction and disjunction, lost: the formulas $\neg\varphi \vee \psi$ and $\neg(\varphi \wedge \neg\psi)$ are no more logically equivalent with $\varphi \twoheadrightarrow \psi$ in $\mathbf{AR}_4$. But also, and more importantly, the property of Modus Tollens (MT) seems also to be lost by our stronger version of conditional:

(MT) $\varphi \twoheadrightarrow \psi, \neg\psi \models \neg\varphi$

If the above consequence relation did hold, then the truth of the two premises $\varphi \twoheadrightarrow \psi$ and $\neg\psi$ would entail the truth of the conclusion $\neg\varphi$. But it cannot be so in $\mathbf{AR}_4$: if, e.g., $\mathbf{A}(\varphi) = \mathbf{A}(\psi) = 10$, the first premise $\varphi \twoheadrightarrow \psi$ is true whilst the second ψ is not. The only way for the latter to be also true is then to have $\mathbf{A}(\psi) = 11$. Then the valuations $\mathbf{A}(\varphi) = 10, \mathbf{A}(\psi) = 11$ make the premise true, but the resulting conclusion $\mathbf{A}(\neg\psi) = 01$ makes (MT) invalid. The situation is even worse with a bivalentist, who discards our non-standard values 00 and 11 from the outset. End of the game, accordingly?
We do not think so, because we assume a definition of logic that does not reduce itself to a list of rules making formulas valid. Moroever, there is no tautology in $\mathbf{AR}_4$ since there is no statement φ in $\mathbf{AR}_4$ such that the speaker would be sure to tell the truth by affirming it. This crucially depends upon his own speech-acts, depending upon whether (s)he affirms or denies some components in φ. More generally, our point is that *logic is not just a question of validity.* Rather, the meaning of a logical constant is afforded by a game with different moves and several purposes, and the search for winning strategies is only but one among different ways of playing a game. This is obviously reminiscent of

dialogical logic [10] and Hintikka's Game-Theoretical Semantics [7]; but, our algebraic 4-valued semantics wants to insist on the four valuations to show the way in which the speech-acts of affirmation and denial may redefine inference rules beyond the sole game of validity as truth-preservation. So let us explain how this game makes sense of strong conditional in particular.

Telling the truth is the major aim of logic, and the meaning of a logical constant consists in describing the ways to achieve it. Hence our present question: how to tell the truth with a conditional? The problem with classical logic is that the conditions to win this game of truth-telling are not sufficiently stringent: there are too many ways of winning the game classically, so that the inference rules for conditional have to be refined.

Now what if the speaker is not in position to tell the truth, for want of sufficient evidence for, say, the antecedent of a conditional? From a unilateralist view of judgment, whoever does not tell something true is thereby led to say something false. This argument can be used against our 4-valued characterization of logical constants in $\mathbf{AR}_4$, if non-truth comes to be reduced to a non-designated value and, hence, to logical falsity. This is too a reductive, black-and-white picture of the game: a speaker who does not affirm the antecedent of a conditional should be put 'off the game', as suggested by Dummett [3] in his comparison of implication as a bet. Therefore, matrices are not a sufficient way to afford the meaning of a logical constant; while these can characterize strategies to tell the truth, the following wants to show that (MT) does not concern such *winning* strategies.

Some game-theoretical considerations may help to clarify the use of some concepts like truth and falsity, or assertion and rejection. Showing the truth of a sentence is winning the game of logic; showing its falsity is defeating an Opponent, thereby winning in an indirect way. Otherwise, the game ends with a draw: rejection has the last word, for want of conclusive assertion for or against the thesis.

We may even attempt some comparisons with another game: football [20]. Thus, the values assigned to φ,ψ are single moves intended to reach a final attitude about the value of $\varphi \circ \psi$ (where $\circ$ is any binary logical constant): assertion is an offensive move, whether affirmative or negative; weak rejection is a defensive move, whereas strong rejection is an attack on its own. The defective feature of conditional has to do with offside position: when the antecedent is not affirmed the situation is as if the team action has been aborted, borrowing from Quine's preceding account on what conditional means informally.

Being offside does not enable to win the game of truth-telling, as the case turns out to be with the looser truth-conditions of classical implication. But it does not make lose one, either: a bivalent reading would present offside as a situation leading to a sanction like own-goal, assuming that any move is either to score or to be scored in the end. As for the case in which the speaker rejects the truth of the consequent, the situation is more awkward since the player does not stand 'off' the game by doing so: (s)he can lose it, in case (s)he then affirms the truth of the antecedent. Truth counts above all, admittedly: in football, the best way not to lose is to score more goals than the opposite team in order to win the match; but also, a defensive strategy can be viewed as a complementary strategy in order not to be scored, that is, not to lose the match. We can compare assertions with an offensive strategy, that of scoring goals. Now an assertion can be affirmative ($\mathbf{a}_1(\varphi) = 1$), or negative ($\mathbf{a}_2(\varphi) = 1$). In the latter case, the search for falsity-claims might appear contrary to the logical purpose of preserving truth, just as it may seem irrational for a football team to play by intending to score own goals. And yet, we can even imagine such queer games in which moves that help to win in one game are moves that make one lose in another version of this game. Rather than losing a game, (MT) can be viewed as a rule that helps *not to lose* by playing logic in a reasoning including conditional.

By extension, we can redefine the following four medieval modes of judgment (M1)-(M4) in the light of our bilateralist theory. The classical versions are the Fregean-minded readings where denial was systematically rendered as the affirmation of a sentential negation, that is, as a strong rejection; they have been replaced here below by the weaker reading of denial as a mere no-answer, especially in (M2): the stronger version unduly states that the conclusion holds, according to our account of (MT) as a non-losing inference rule.

More generally, let $\circ$ be one of the binary operators of $\mathbf{AR}_4$. Then each of the *Modi ...do Ponens* are inference rules used as winning or *affirmative* strategies, by affirming something in order not to satisfy the wanted truth-conditions $\mathbf{a}_1(\varphi \circ \psi)$; and each of the *Modi ...do Tollens* are inference rules used as non-losing or *negative* strategies, by denying something in order not to satisfy the unwanted falsity-conditions $\mathbf{a}_2(\varphi \circ \psi)$. In every such case, a 'Mode that X-s by Y-ing' is a mode where the speaker performs the speech-act X in the conclusion after performing the speech-act Y in the second premise.

(M1) Modus Ponendo Ponens ('Mode that affirms by affirming')
If a conditional is affirmed together with its antecedent, then its consequent is affirmed.
1. $\mathbf{a}_1(\varphi \twoheadrightarrow \psi) = 1$
2. $\mathbf{a}_1(\varphi) = 1$
3. $\mathbf{a}_1(\psi) = 1$ (by $\mathbf{a}_1(\varphi \twoheadrightarrow \psi)$)

(M2) Modus Tollendo Tollens ('Mode that denies by denying')
If a conditional is affirmed and its consequent is denied, then its antecedent is

also denied.
1. $\mathbf{a}_1(\varphi \twoheadrightarrow \psi) = 1$
2. $\mathbf{a}_1(\psi) = 0$
3. $\mathbf{a}_1(\varphi) = 0$ (by $\mathbf{a}_2(\varphi \twoheadrightarrow \psi)$)

(M3) Modus Ponendo Tollens ('Mode that denies by affirming')
If the conjunction is denied and one of its conjuncts is affirmed, then its other conjunct is denied.
1. $\mathbf{a}_1(\varphi \wedge \psi) = 0$
2. $\mathbf{a}_1(\varphi) = 1$
3. $\mathbf{a}_1(\psi) = 0$ (by $\mathbf{a}_2(\varphi \wedge \psi)$)

(M4) Modus Tollendo Ponens ('Mode that affirms by denying')
If a disjunction is affirmed and one of its disjuncts is denied, then the other conjunct is affirmed.
1. $\mathbf{a}_1(\varphi \vee \psi) = 1$
2. $\mathbf{a}_1(\varphi) = 0$
3. $\mathbf{a}_1(\psi) = 1$ (by $\mathbf{a}_1(\varphi \vee \psi)$)

By means of this extended, both game-theoretical and algebraic description of logic, we are also in position to reconsider one of the trickiest troubles around the definition of conditional, namely: the well-known Frege-Geach Problem.

3.5 The Frege-Geach Problem

We do not want to enter in depth into the philosophical debate between cognitivism and expressivism. But we cannot beg it out, insofar as it directly concerns our illocutionary definition of implication.
In a nutshell, expressivists take sentences to express mental states, whilst cognitivists do not. An objection made by cognitivists to the expressivists is that their interpretation of normative sentences should hold in every context of discourse, including conditional statements. But assuming that the components of a sentence are discharged from any commitment by the speaker, their explanation should fail. Applying this problem to our illocutionary treatment of logic as a game of statements (Frege's judgments) rather than mere sentences (Frege's judgeable contents), our analysis of conditional should equally fail because we, as the expressivists, treat judgments as the primary vehicles of meaning.[11] But if so, then our account should break down once the speaker

[11] A standard example of normative sentence, to exemplify the debate around the Frege-Geach Problem, is something like 'X is good' rather than 'φ is true'. But, our illocutionary approach to logic equally means that speakers *do* commit themselves by making speech-acts.

does not commit in the truth or falsity of the components φ and ψ in $\varphi \twoheadrightarrow \psi$. Our reply to this problem is that not every judgment is committal *upon the sentential content*, albeit committal upon the logical behavior of the speaker. Take the act of denial: a bilateralist view of judgments makes the latter different from negative assertion, thereby showing that (MT) can be explained as a combination of assertion and denial in its inferential process. To be more precise, by denying a sentential content the speaker does *refuse* to claim either is being true or false. Searle defined denial with this weak sense of rejection in mind:

If F is a sign indicating performance of a certain speech act, then the effect of 'not' on that sign is to produce a new sign which indicates (but does not state) that the original speech act is under consideration but the speaker is *not yet prepared* to accept the commitments involved in performing it.[12]

But there is still a problem with Modus Ponens (MP):

(MP) $\varphi \twoheadrightarrow \psi, \varphi \models \psi$

since the Frege-Geach Problem equally concerns the commitment of speakers in (MP) and (MT). According to us, the complete meaning of a logical constant is not given by its valid but, rather, *relevant* inferences; irrelevant truths do not matter, as the paradoxes of material implication are, and an inference is said to be relevant if it contributes to fulfill the speaker's purpose. Now there are at least two problems with bilateralism. For one thing, one can affirm a whole implication without affirming anything about either of its components. Frege [5] noticed it to make the difference between a whole implication thought and its components in the process of inference:

Of course we cannot infer anything from a false thought: but the false thought may be part of a true thought, from which something can be inferred. The thought contained in 'If the accused was in Rome at the time at the deed, he did not commit the murder' may be acknowledged to be true by someone who does not know if the accused was in Rome at the time of the deed nor if he committed the murder. Of the two component thoughts contained in the whole, neither the antecedent nor the consequent is being

[12][23]: 58 (our italics); also cited in [6]: 107. The italicized passage helps to make an essential difference between non- committal judgments (questions with no positive answer, in $\mathbf{AR}_4$) and plain non-judgments (no question at all, in $\mathbf{AR}_4$): whoever entertains a sentential content may refuse to assign any truth-value to it; but by doing so, he does something after all by denying this content. This case of denial is that overtly neglected by the Fregean theory of judgment; it can be viewed as a sort of 'non-committal commitment', just as inaction is said to be a voluntary action of doing nothing once performed knowingly.

uttered assertively when the whole is presented as true. We then have only a single judgment, but three thoughts, viz. the whole thought, the antecedent, and the consequent.[13]

Frege also made use of this explanation to disqualify the relevance of properly negative judgments. For if no particular judgment is made inside an implication, then denial can be reduced to an expression of sentential negation. We agree about the non-committal use of antecedent and consequent, whenever the speaker 'does not know enough' to affirm anything about their truth or falsity. And yet, we disagree with Frege in two respects: on his first assumption that any judgment is a committal speech-act, given our account of denial as a *non-committal* speech-act; on his second assumption that negative judgment has no relevant role to play in the game of logic.
Let us review the two kinds of conditional inference quoted by Frege [4]. In the first one,

(a) If the accused was in Rome at the time of the deed, he did *not* commit the murder. He was in Rome at this time. Therefore he did *not* commit the crime.

(MP) is performed by means of an affirmative antecedent φ and a negative consequent ψ. The three-part inferential process can be reconstructed in $\mathbf{AR}_4$ as follows:

1. If φ, then $\neg\psi$	1. $v(\varphi \twoheadrightarrow \neg\psi) = T$	1. $\mathbf{a}_1(\varphi \twoheadrightarrow \neg\psi) = 1$
2. φ	2. $v(\varphi) = T$	2. $\mathbf{a}_1(\varphi) = 1$
3. $\therefore \neg\psi$	3. $v(\neg\psi) = T$	3. $\mathbf{a}_1(\neg\psi) = 1$

The second example is an instance of (MT), where the second premise related to the consequent rather than the antecedent:

(b) If the accused was in Rome at the time of the deed, he did *not* commit the murder. He did commit the murder. Therefore he was *not* in Rome at this time.

1. If φ, then $\neg\psi$	1. $v(\varphi \twoheadrightarrow \neg\psi) = T$	1. $\mathbf{a}_1(\varphi \twoheadrightarrow \neg\psi) = 1$
2. ψ	2. $v(\neg\psi) = F$	2. $\mathbf{a}_2(\neg\psi) = 1$
3. $\therefore \varphi$	3. $v(\varphi) = F$	3. $\mathbf{a}_2(\varphi) = 1$

A serious problem seems to arise with (MT) for our illocutionary account of logical constants: how can our speaker affirm the first two sentences consistently,

[13] [4]: 119-120.

given our strengthened truth-conditions of conditional? For if the speaker can affirm a strong conditional only by affirming its two components, as the case is with the truth-conditions of conjunction, then our speaker seems condemned to affirm and reject at once the consequent $\neg\psi$ in the two premises. Our reply to this objection is that it stems from a deep misunderstanding about what a conditional assertion should really mean: whoever asserts the truth of an implication in $\mathbf{AR}_4$ does not thereby assert its two components; rather, whoever does that means that the truth of either component is *connected* to the truth of the other. Now the relevance of their connected truth does not go on a par with their actual truth.

For one thing, Dummett [3] endorsed Quine's view on implication by comparing the assignment of truth-values with a bet: the latter can be aborted because

> there may be a gap between the winning of a bet and the losing of it, as with a conditional bet.

Dummett similarly claimed that an implication is 'neither true nor false' whenever affirming the truth of the antecedent is not in the speaker's power. Hence the ensuing gap between truth and falsity, that is, between winning and losing in a bet, in the sense that

> to determine what is to involve one of these is not yet to determine completely what is to involve the other".[14]

Such a gap is betrayed by the *contrary* relationship between the truth- and falsity-conditions of our stronger conditional: there are circumstances in which the whole statement is neither true nor false, unlike the other logical constants of disjunction and conjunction which always led to the complementary result of winning or losing statements.

And yet, Dummett went on saying that there are two possible interpretations of conditional:

> There is a distinction between a conditional bet and a bet on a truth of a material conditional; if the antecedent is unfulfilled, in the first case the bet is off –it is just as if no bet had been made –but in the second case the bet is won.[15]

Vanderveken [27] corroborates Dummett's view on the twofold reading of implication:

[14][3]: 8-9.
[15][3]: 8.

As Dummett pointed out, the different is striking in the case of a conditional bet of the form 'If A then I bet 5$ that B' and of a bet of the form 'I bet 5$ that if A then B'. In the first case, if the antecedent is false, there is no winner no loser, but in the second case there is always a winner and a loser when the bet is accepted, since the conditional expression is either true or false in the world of the utterance.[16]

Despite our great sympathy for Vanderveken's illocutionary logic, we clearly disagree with the above Dummettian distinction between two uses of conditional. Borrowing from the symbolism of Searle [23], let $F(\varphi)$ be the logical form of any speech-act where F is an illocutionary force attached to a sentential content. Let assertion and rejection be two such forces.[17] Then what Vanderveken means here above is that 'bet on a conditional' and 'conditional bet' differ: the former has the logical form $\varphi \twoheadrightarrow F(\psi)$, whereas the second is $F(\varphi \twoheadrightarrow \psi)$. According to the author, the latter always comes to a winning case: the gambler wins whenever φ is false or ψ is true, and he loses only if φ is true and ψ is false. In other words, Vanderveken sticks to the truth- and falsity-conditions of material implication inside the illocutionary force F. We do not. After all, what difference in use is there between betting in both cases whenever the antecedent is false?
The situation can be also compared with what Łukasiewicz [11] said about the necessity of syllogisms, whether in an absolute or relative sense of the word. Just as with Frege's above examples, a syllogism is made of two premises leading to a conclusion. A conclusion is said to be *relatively* necessary if the truth of the conclusion depends upon that of the premises. By analogy with betting, Vanderveken seems to say that there is a logical difference between the expressions 'If the premises are true, then the conclusion is necessarily true' and 'Necessarily, if the premises are true then so is the conclusion'. We do not see any logical difference in these whenever premises are not true, just as we do not equate our bilateralist redefinition of implication with a case of absolute necessity.

Actually, our bilateralist definition of conditional would better be compared with the property of modal distribution over conditional in the modal K-system.[18] What is the meaning of $\Box(\varphi \twoheadrightarrow \psi) \models (\Box\varphi \twoheadrightarrow \Box\psi)$? In terms of possible-world semantics, this inferential property means that, assuming that a given implication is true in every possible world, then the alleged truth of the antecedent in every possible world entails the same situation for the consequent. Compare $\Box$ with F, or with assertion. Thus, our strengthened condi-

[16] [24]: 25.
[17] About the objections to the status of illocutionary negation, or rejection, see, e.g., [5].
[18] About this analogy between the distributions of assertion and necessity, see [22].

tional means that every situation in which the conditional is asserted is such that an assertion of the antecedent entails an assertion of the consequent. Admittedly, this does not preclude some situation in which the antecedent would be rejected, and the game cannot be won in such a case. But this does not alter anything in our definition of the success-conditions of conditional, i.e., its truth- or, better, truth-telling conditions by the speaker.
Actually, the second inference (b) given by Frege [4] can be read in a twofold relative sense of 'conditionality'. In the first reading (b_1), the speaker commits in the falsity of the consequent according to the second premise. This is the strong version of rejection, as depicted by the following pattern:

(b_1) If the accused was in Rome at the time of the deed, he did *not* commit the murder. He did commit the murder. Therefore he was *not* in Rome at this time.

1. If φ, then $\neg\psi$	1. $v(\varphi \twoheadrightarrow \neg\psi) = T$	1. $\mathbf{a}_1(\varphi \twoheadrightarrow \neg\psi) = 1$
2. ψ	2. $v(\psi) = T$	2. $\mathbf{a}_1(\psi) = 1$
3. $\therefore \neg\varphi$	3. $v(\neg\varphi) = T$	3. $\mathbf{a}_1(\varphi) = 1$

In the second premise, the speaker commits in the falsity of the consequent $\neg\psi$ by affirming the truth of ψ; therefore, (s)he cannot reject this consequent without also rejecting the antecedent φ, under the penalty of accepting the falsity of his own initial thesis. Therefore, the speaker rejects φ in order not to lose the game. At the same time, has he won by doing so? We do not think so, due to our strengthened definition of conditional which admits of only one way of winning the game (i.e., telling the truth) with a conditional sentence.
There is still a second, weaker version of (b_2) where the speaker does not know anything about whether the consequent is the case or not:

(b_2) If the accused was in Rome at the time of the deed, he did *not* commit the murder. I *do not say* that he did *not* commit the murder. Therefore I *do no say* that he was in Rome at this time.

1. If φ, then $\neg\psi$	1. $v(\varphi \twoheadrightarrow \neg\psi) = T$	1. $\mathbf{a}_1(\varphi \twoheadrightarrow \neg\psi) = 1$
2. not $\neg\psi$	2. $v(\neg\psi) \neq T$	2. $\mathbf{a}_1(\neg\psi) = 0$
3. $\therefore$ not $\neg\varphi$	3. $v(\neg\varphi) \neq T$	3. $\mathbf{a}_1(\varphi) = 0$

Illocutionary negation, or rejection, is brought out in (b_2) by the verbal expression 'I do not say that', unlike the shorter and committal internal negation 'not' occurring in (b_1). In $\mathbf{AR}_4$, the difference between locutionary and illocutionary negation is a difference between the sentential operator $\neg$ and the statemental or judicative no-answer 0. The logical relation between the stronger and weaker versions of rejection is a relation of species and genus: whoever asserts the falsity of a sentence thereby rejects it, whereas the converse need not hold. It does not hold in the above sample (b_2), but the logical result of the two inferences (b_1)–(b_2) is the same: loss is avoided by rejecting the conditions under which it would obtain.

Finally, note that Frege never referred to *irrelevant* situations in his exemplification of (MP) and (MT), i.e., cases where the second premise corresponds to a false antecedent or a true consequent. Far from condemning the relevance of negative judgments, this passage of Frege [4] enabled to show the unavoidable connection between conditional and inference, and to recall our dialogical view of conditional in terms of relevant moves motivated by two complementary strategies: winning the game by obeying the rule (MP), if the speaker is entitled to affirm what he needs for this purpose; not losing the game by obeying the rule (MT), as far as the speaker can avoid any situation in which the falsity-conditions of his (her) thesis would obtain.

Conclusion: Elementary, my dear Wason!

We proposed a strengthened version of conditional, and the result has been defended by means of an algebraic and game-theoretical account of logic. Apart a probabilistic or non-compositional approach, we presented conditional as an illocutionary operation between two possible speech-acts: assertion, or rejection. Boolean logic has been presupposed throughout the paper, using the two basic values 1 and 0 to express the yes- and no-answers of affirmation and denial. At the same time, our logical system $\mathbf{AR}_4$ is many-valued by making use of truth-values as structured and ordered answers to sentential questions.

By doing so, we did justice to Frege's theory of judgment in his depiction of logic as a game of scientific investigation. But we departed from him by assigning a crucial role to a bilateralist theory of judgment: the independence of assertion and rejection helps to give a sound definition of implication, both endorsing the necessary condition and discarding the sufficient conditions afforded by the classical or material implication.

In conclusion, the two main properties of conditional have been redefined through a more comprehensive view of logic. Just as Wason alluded to with his selection problem, conditional is closely related to rules about obligatory or forbidden actions in a game: (MP) says what is obligatory to do in order to win the game ; (MT) says what is forbidden to do, i.e., what is obligatory to do in order not to lose the game.

All of this cannot be made obvious, so long as logic is reduced to the one-sided activity of winning or telling the truth. Being a good logician is not just an affair of winning a game; it is also a question of acting rationally and taking the right decision in any given situation. Some of these decisions can be irrelevant, as in a card game or a football match. If so, they cannot pave the way to a winning strategy. This truth is very trivial or elementary, assuming a bilateralist view of logic and the logical independence of two basic speech-acts: assertion, and rejection.

Acknowledgements. Thanks to Peter Verdée, Aleksandra Barbara Samonek, Laurent Dubois, Bruno Leclercq and an anonymous referee for their comments on this paper.

References

[1] Anderson, A.R. & Belnap, N.D. *Entailment: The Logic of Relevance and Necessity.* Princeton, Princeton University Press (1975).

[2] Angell, R. "Three Logics of Subjunctive Conditionals", *Journal of Symbolic Logic*, Vol. 32(1967): 556-557.

[3] Dummett, M. "Truth", *Proceedings of the Aristotelian Society*, Vol. 59(1959): 141-62.

[4] Frege, G. "Negation", in Geach, P . & Black, M. (eds.), *Transcriptions from the Philosophical Writings of Gottlob Frege*, Basil Blackwell, Oxford (1960): 117-36.

[5] Garner, R.T. "Some doubts about illocutionary negation", *Analysis*, Vol. 31(1971): 106-12.

[6] Gomes, G. "Are necessary and sufficient conditions converse relations?", *Australasian Journal of Philosophy*, Vol. 87(2009): 375-87.

[7] Hintikka, J. *The Semantics of Questions and the Questions of Semantics: Case Studies in the Interrelations of Logic, Semantics, and Syntax*, Holland Publishing Company (1976).

[8] Johnson-Laird, P.N. *Mental Models: Towards a Cognitive Science of Language, Inference, and Consciousness.* Cambridge: Cambridge University Press.

[9] Lewis, D. *Counterfactuals.* Oxford: Blackwell Publishers and Cambridge, MA: Harvard University Press (1973).

[10] Lorenzen, P. & Lorenz, K.: *Dialogische Logik.* Darmstadt: Wissenschaftliche Buchgesellschaft (1978).

[11] Łukasiewicz, J. *Aristotle's Syllogistic from the Standpoint of Formal Logic*, Clarendon Press, Oxford (1951).

[12] MacColl, H. *Symbolic Logic and its Applications*, London: Longmans (1906).

[13] Pailos, F. & Rosenblatt, L. "Non-deterministic Conditionals and Transparent Truth", *Studia Logica*, Vol. 103(2015): 579-598.

[14] Parsons, C. "Assertion, Denial, and the Liar Paradox", *Journal of Philosophical Logic*, Vol. 13(1984): 137-152.

[15] Quine, W.V.O. *Mathematical Logic*, Norton (1940).

[16] Quine, W.V.O. *Methods of Logic*, Holt, Rinehart and Winston (1982).

[17] Rescher, N. "Quasi-truth-functional systems of propositional logic", *The Journal of Symbolic Logic*, Vol. 27(1963): 1-10.

[18] Rivieccio, U. "Implicative twist-structures", *Algebra Universalis*, Vol. 71(2014): 155-186.

[19] Routley, R. *Relevant Logics and their Rivals*, Ridgeview Pub Co (1983).

[20] Schang, F. "The Football of Logic", *Studia Humana*, Vol. 6(2017): 50-60.

[21] Schang, F. & Costa-Leite, A. "Une sémantique générale des croyances justifiées, *CLE-Prints*, Vol. 16 (2016): 1-24.

[22] Schang, F. & Trafford, J. "Is 'no' a force-indicator? Yes, sooner or later!", *Logical Universalis*, Vol. 11 (2017): 225-251.

[23] Searle, J. *Speech Acts (An Essay in the Philosophy of Language)*, Cambridge University Press (1969).

[24] Stalnaker, R. "A Theory of Conditionals", *Studies in Logical Theory, American Philosophical Quarterly*, Monograph, Vol. 2(1968): 98-112.

[25] Strawson, P. "On referring", *Mind*, Vol. 59(1950): 320-44.

[26] Tomova, N. "A Lattice of Implicative Extensions of Regular Kleene's Logics", *Reports on Mathematical Logic*, Vol. 47(2012): 173-182.

[27] Vanderveken, D. *Meaning and Speech Acts (Volume I: Principles of Language Use)*, Cambridge University Press (1990).

[28] Vidal, M. "The defective conditional in mathematics", *Journal of Applied Non-Classical Logics*, Vol. 24(2014): 169-179.

[29] Wansing, H. "Connexive Logics", *Stanford Encyclopedia* (2010), Fall.

[30] Wason, P. C. "Reasoning about a rule". *Quarterly Journal of Experimental Psychology*, Vol. 20(1968): 273-281.

Fabien Schang
Department of Philosophy
Federal University of Goiás (UFG)
Av Esperança, SN, Campus Samambaia, Goiânia - GO, Brazil
E-mail: schangfabien@gmail.com