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Negation and Dichotomy. Do They Refer to the Same 'Thing'? On the Identity of Logical Negation from a Semantic Perspective*

Properties of classical negation '~'

Reductio ad Absurdum

RA1 If $\varphi \Rightarrow \psi$ and $\varphi \Rightarrow \sim\psi$, then $\sim\varphi$
RA2 If $\sim\varphi \Rightarrow \psi$ and $\sim\varphi \Rightarrow \sim\psi$, then φ
RA3 If $\varphi \Rightarrow \sim\varphi$, then $\sim\varphi$
RA4 If $\sim\varphi \Rightarrow \varphi$, then φ

Contraposition

C1 If $\varphi \Rightarrow \psi$, then $\sim\psi \Rightarrow \sim\varphi$
C2 If $\sim\varphi \Rightarrow \sim\psi$, then $\psi \Rightarrow \varphi$
C3 If $\sim\varphi \Rightarrow \psi$, then $\sim\psi$
C4 If $\varphi \Rightarrow \sim\psi$, then $\psi \Rightarrow \sim\varphi$

Double negation

DN1 $\varphi \Rightarrow \sim\sim\varphi$
DN2 $\sim\sim\varphi \Rightarrow \varphi$

The two 'laws'

LC $(\sim(\varphi \wedge \sim\varphi)) \in \{T\}$
LEM $(\varphi \vee \sim\varphi) \in \{T\}$

De Morgan laws

Conjunction

DM∧1 $\sim(\varphi \wedge \psi) \Leftrightarrow \sim\varphi \vee \sim\psi$
DM∧2 $\sim(\sim\varphi \wedge \psi) \Leftrightarrow \varphi \vee \sim\psi$
DM∧3 $\sim(\varphi \wedge \sim\psi) \Leftrightarrow \sim\varphi \vee \psi$
DM∧4 $\sim(\sim\varphi \wedge \sim\psi) \Leftrightarrow \varphi \vee \psi$

Disjunction

DM∨1 $\sim(\varphi \vee \psi) \Leftrightarrow \sim\varphi \wedge \sim\psi$

DM∨2 $\sim(\sim\varphi \vee \psi) \Leftrightarrow \varphi \wedge \sim\psi$

DM∨3 $\sim(\varphi \vee \sim\psi) \Leftrightarrow \sim\varphi \wedge \psi$

DM∨4 $\sim(\sim\varphi \wedge \sim\psi) \Leftrightarrow \varphi \vee \psi$

Conditional

DM→1 $\varphi \rightarrow \psi \Leftrightarrow \sim\varphi \vee \psi$

DM→2 $\sim\varphi \rightarrow \psi \Leftrightarrow \varphi \vee \psi$

DM→3 $\varphi \rightarrow \sim\psi \Leftrightarrow \sim\varphi \vee \sim\psi$

DM→4 $\sim\varphi \rightarrow \sim\psi \Leftrightarrow \varphi \vee \sim\psi$

1. Negation from an algebraic viewpoint

According to a Fregean or *referential* view of semantics, each formula from an interpreted language names and is associated with a reference, viz. a *truth-value*. Thus, such a sentence like ‘Socrates is a philosopher’ is taken to be a name for one logical object among two possible ones, namely: *truth* or *falsity*, depending upon whether Socrates *is* a philosopher or *not*.

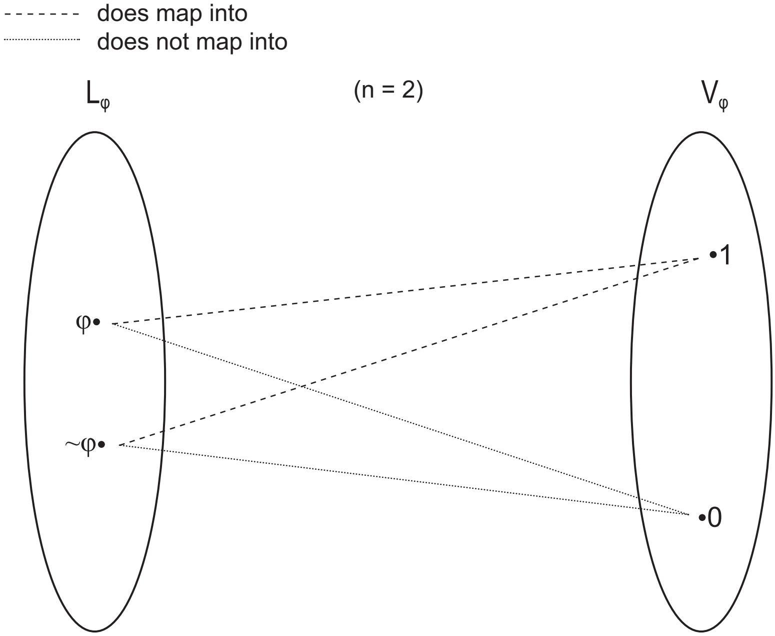
Each formula φ is interpreted algebraically by a mapping (valuation) from a set of formulas L_φ to a set of references or truth-values V_φ, to be symbolized as follows: $L_\varphi \rightarrow V_\varphi$ (see **Figure 1**). Each non-atomic, complex formula $\oplus_n$ with n components is interpreted by a mapping from a set of input values in V_φ to the set of output values in V_φ, to be symbolized as follows: $V_\varphi \rightarrow V_\varphi$. For example, if the constructor-sign ⊕ stands for the unary operator of *classical* negation '~' then '~φ' is a complex formula, to be interpreted by a mapping from the reference of an atomic formula φ to the reference of its negated form ~φ. In classical logic, such a mapping will proceed from truth to falsity or from falsity to truth exclusively, depending upon the input value of φ. Let us use '~' as a symbol for classical negation, *i.e.* the operator to be attached only to sentences in a *bivalent* frame; any assignment of truth or falsity to ordinary sentences turns them into propositions.

If we accept such a referential definition of semantic interpretation, each formula appears as a sort of definite description the reference of which may vary: it may name just one reference or truth-value (the mapping is a *total* function, that is a one-to-one or bijective relation between two sets), several truth-values (the mapping is a one-to-many or surjective relation between two sets), or no truth-value at all (the map is a *partial* function). The number of truth-values in V can be discussed and will result in different logical systems.

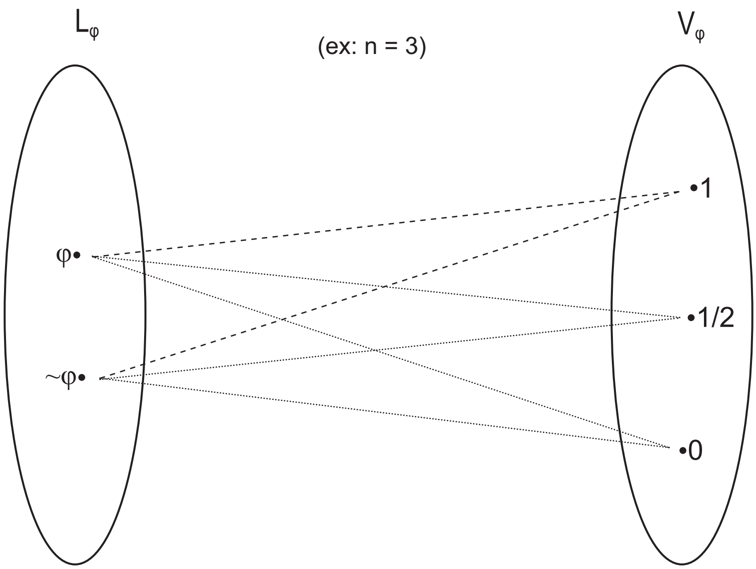
How many truth-values are to be contained in a semantic set: two, three, or infinitely many values? Some philosophers like Quine claimed that any

Figure 1

Classical valuation as a one-to-one (bijective) bivaluation



Non-Classical valuation as a one-to-one (bijective) n-valuation



semantic set must count only two elements, *i.e.* the subsets of true and false formulas.[1] Other logicians hold some similar but more complex position, namely Roman Suszko and his plea for two-valuedness;[2] we will return to Suszko's position later, since it will be used as a central argument for restoring some commonsensical logical laws as opposed to some scientific (many-valued) ones.

As a pioneer of many-valued logics in their application to philosophical problems, Jan Łukasiewicz once suggested two different kinds of many-valued sets, namely: a set of *three* truth-values or an *infinity* of them, depending upon the meaning to be assigned to the third value of 'possibility' besides truth and falsity.[3] The preceding quotation by Suszko has shown that he was clearly opposed to Łukasiewicz's many-valued sets, hence his distinction between *algebraic* and *logical* values.

Each of these semantic sets are to be found in modern logical systems: a case for bivalence (with n = 2 truth-values) is Classical Logic (hereafter: **CL**); a case for trivalence (with n = 3 truth-values) is exemplified by several three-valued systems with a specific meaning of the third truth-value as 'neither ... nor'. There is Łukasiewicz's $\mathbf{Ł}_3$ for contingent events with 'true', 'false', and 'indeterminate' as truth-values; Kleene's $\mathbf{K}_3$ for mathematical statements with 'true', 'false', and 'undecided' as truth-values; Bochvar's $\mathbf{B}_3$ for paradoxical statements with 'true', 'false' and 'senseless' as truth-values.

In spite of Łukasiewicz's preceding plea for either three or infinitely many truth-values, let us notice that he himself recognized afterwards (in 1953) the inappropriateness of his three-valued system and suggested instead a *four-valued* system of modal logic in order to avoid some unpleasant result in $\mathbf{Ł}_3$, namely: the Law of Non-Contradiction didn't hold in it, whereas Łukasiewicz wanted to invalidate the Law of Excluded Middle only. Four-valued logics appear as a specific case of semantic sets in which each truth-value is a combined element from the powerset P(V). In a set V = $\{\{0\},\{1\}\}$ with n = 2 elements, that is, $\{0\}$ as the subset of *only* false formulas and $\{1\}$ as the subset of *only* true formulas, the powerset P(V) = $\{\{\emptyset\},\{0\},\{1\},\{1,0\}\}$ includes $2^n = 2^2 = 4$ subsets with two additional cases, namely: $\{\emptyset\}$ as the set of neither-true-nor-false formulas, and $\{1,0\}$ as the set of both-true-and-false formulas. Three samples of four-valued logics (*inter alia*) are:

- *Directional Logic* (hereafter: **DL**), by Leonard Slawomir Rogowski (in 1961). In this logical system, the classical sets $\{1\}$ and $\{0\}$ are reinterpreted by 't' as strictly true and by 'f' as strictly false, respectively. In addition to these two classical values, Rogowski adds the 'subtrue' set $\{u\}$ as coming to be false and the 'subfalse' set $\{i\}$ as coming to be true.

- *Relevance Logic*, by Nuel Belnap (1977) (see **Figure 2**). This system is concerned with information or data bases in computer science: $\varphi \in \{N\}$ is the set in which no information occurs about φ; $\varphi \in \{F\}$ is the set with an information saying that φ fails; $\varphi \in \{T\}$ is the set with an information saying that φ holds; and $\varphi \in \{B\}$ is the set with an information saying that φ both holds and fails.
- *Overclassical Logic*, by Newton C. A. da Costa and Jean-Yves Béziau (1997) (see **Figure 2**). Such a system presents non-classical diagrams with a *relative* complement: the pair <+,-> means that a formula φ is 'absolutely true', *i.e.* is located in the class of true formulas but not in the class of false formulas; <+,+> means that φ is 'relatively true', *i.e.* is located both in the class of true formulas and the class of false formulas; <-,-> means that φ is 'relatively false', *i.e.* is located neither in the class of true formulas nor in the class of false formulas; and <-,+> means that φ is 'absolutely false', *i.e.* is located in the class of false formulas but not in the class of true formulas.

Finally, a case for *indefinitely* many truth-values is some species of fuzzy logics [0,1], where an infinity of truth-values occurs between the extreme cases of falsity {0} and truth {1}; infinitary-valued logics are closely related to probabilitary logics or even to Jerzy Łoś's logic of assertion (in 1948), where each truth-value corresponds to a specific standpoint within a group of speakers.

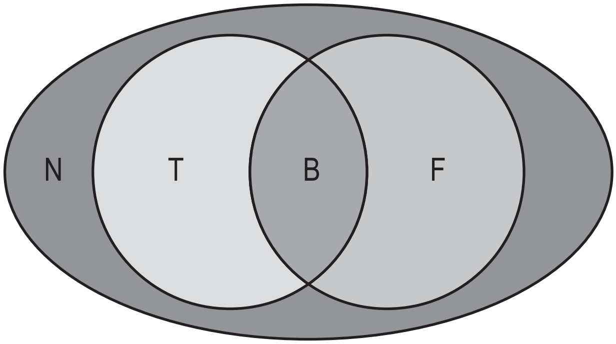
How to define the logical constant of negation, given that the output value of a formula may change according to the cardinality of a semantic set? If, for instance, a given formula φ is neither true nor false in a logical system, it is pretty sure that the output value of the resulting $\sim\varphi$ will be neither false nor true and, thus, won't meet the algebraic definition of classical negation (from 1 to 0, or from 0 to 1). Now that does not mean that the main features of classical negation should be deeply revised in non-classical systems: such is the main thesis to be defended in this paper.

As a minimal and necessary precondition for being a logical constant of negation in any system, it will be argued that negation operates as a *dichotomy*: As in Plato's *Sophist*, each dichotomy splits each set (say, x_i for any i) into two subsets x_i and not-x_i of being and not-being (see **Figure 3**).

According to Buridan's distinction, logical negation discriminates a set of elements $\{\{x_1\},...,\{x_i\}\}$ from the remaining ones $\{\{x_i\},...,\{x_n\}\}$ either positively (*negatio infinitans*), or negatively (*negatio negans*). Thus, to say

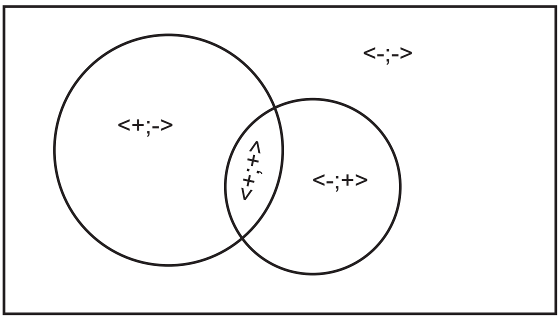
Figure 2

Belnap's Relevance Logic



B	both true and false	{1,0}
T	only true	{1}
F	only false	{0}
N	neither true nor false	{∅}

da Costa and Béziau's Overclassical Logic



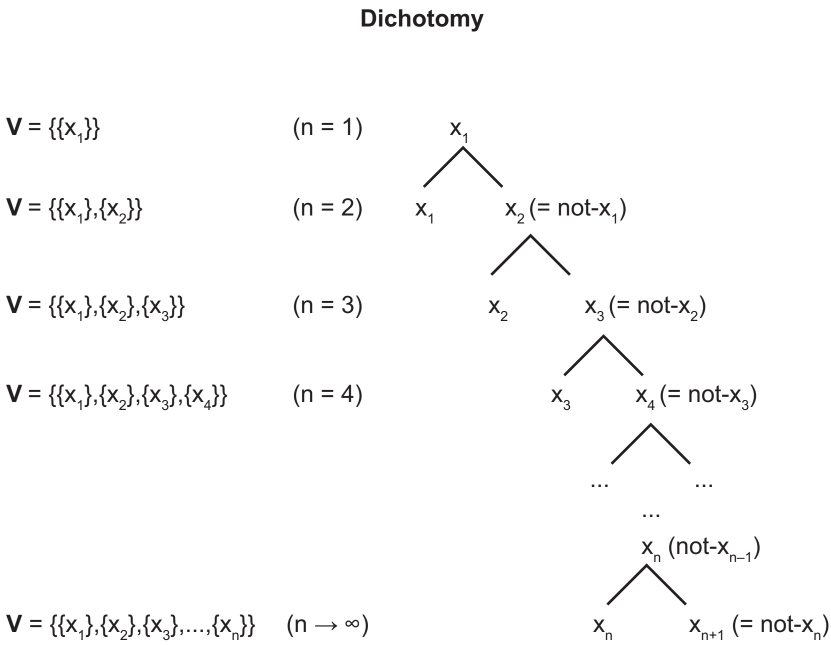
<+;->	absolutely true (only true)	{1}
<+;+>	relatively true (true, false)	{1,0}
<-;+>	absolutely false (only false)	{0}
<-;->	relatively false (not true, not false)	{∅}

that ‘the table is not-red’ positively entails that the table is either blue, or black, or green ... *i.e.* possesses another definite colour. It follows that to be ‘not-true’ amounts to be another value, and not just ‘not to be true’.

Does it entail that the Law of Excluded Middle still holds as an excluded third with n = 2 truth-values in V, as an excluded fourth with n = 3, ..., as an excluded $(x_{i+1})^{th}$ with $n = x_i$? If so, to define negation as a dichotomy seems to result in a *regressio ad infinitum* when applied to excluded middle, as argued by Church (1928) with respect to Burali-Forti’s Paradox.[4] But it is not so: logical negation can be properly defined as a dichotomy whatever V may be.

Against Church’s former objection to non-classical semantic sets with n > 2, Barzin and Errera (1929) insist that negation essentially proceeds as a *negatio negans*, *i.e.* as an indefinite process of exclusion out of a class.[5] The same observation will be applied in the following, thus yielding a general definition of negation as a dichotomy and restoring the above properties of the so-called ‘classical’ negation even within non-classical systems. Isn’t this absurd, given that the classical rules of inference for negation

Figure 3



(contraposition, *reductio ad absurdum*, and the like) normally hold as a whole in *classical* logic only?

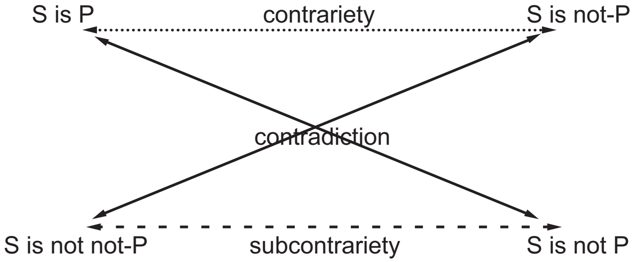
A more careful attention upon 'classical laws' is in order in the following, so as to have a more comprehensive view of the notion of *dichotomy*. Two crucial pairs of notions are concerned here, namely: truth and falsity, on the one hand; affirmation and negation, on the other hand. How are they combined, and do many-valued logics really entail a revision of 'classical' negation? The point is that the classical properties of negation fail in non-classical systems when defined *extensively*, that is, in terms of truth-values. But the present paper wants to show that the process of dichotomy can be introduced within these non-classical systems in order to maintain such 'commonsensical' properties as excluded middle and non-contradiction while defining logical negation as a general constant (*i.e.* for any V). Negation can be distinguished from conjunction, disjunction, conditional and other logical constants as a peculiar process of dichotomy. For this purpose, let us look back on the past and Aristotle's Term Logic in order to support our 'intensional' definition of the concept of negation (beyond its extensional view in terms of output values).

In Aristotle's Logic of Terms, a more fine-grained distinction between basic propositions was made as opposed to modern sentential logic. Such a distinction was called 'the Four' by Aristotle: given a basic predication 'S is P' for every atomic proposition, such an affirmative form can be enriched if we introduce negation in it (see **Figure 4**). We thus obtain: the denial 'S is not P', the contraffirmation 'S is not-P' and, finally, the contradenial 'S is not not-P'. An example of an affirmation is 'Socrates is a philosopher', to be added with 'Socrates is *not* a philosopher', 'Socrates is a *non*-philosopher', and 'Socrates is *not* a *non*-philosopher'. Please note that a clear-cut distinction is to be made between denial and contraffirmation: as noted by Englebretsen (1981), if Socrates doesn't exist the above denial is true whereas the contraffirmation is not. Such a difference disappeared in modern logic, given that negation as a term predicate doesn't make sense therein and only serves as a sentential operator. But this very distinction will turn out to be crucial for the following argument.

Let L_φ be a language of φ-order, φ any formula from L_φ and V_φ a set $\{\{x_1\},...,\{x_{n-1}\},\{x_n\}\}$ with n truth-values. Again, any *valuation* is an assignment of truth-values upon sentences, *i.e.* a mapping from an arbitrary φ of L_φ into a set in V_φ: $\varphi \rightarrow \{x_i\}$. Let $\{1\}$ be the set of true propositions and $\{0\}$ the set of false propositions. Then we can render φ's being true as $\varphi \in \{1\}$, and φ's being false as $\varphi \in \{0\}$. Now two different levels of negation occur in our ordinary speech-acts: a first, linguistic negation '~' is attached to the entire sentence. If φ is read "φ is true", then $\sim\varphi$ means "φ is *not*-true".

<u>Figure 4</u>

'The Four' in Term Oppositions



Affirmation	S is P	("Socrates is a philosopher")
Denial	S is not P	("Socrates is a *not*-philosopher")
Contraffirmation	S is not-P	("Socrates is *not* a philosopher")
Contradenial	S is not not-P	("Socrates is *not* a *not*-philosopher")

Oppositions as pairs of valuations

Contrariety
A and B cannot be both *true*, A and B can be both *false*

Contradiction
If A is *true*, then B is *false*; if A is *false*, then B is *true*

Subcontrariety
A and B cannot be both *false*; A and B can be both *true*

A second, metalinguistic negation '$\notin$' is attached to the truth-values of sentences. If $\varphi\in\{1\}$ is read "It is the case that φ is true" (or, equivalently, "φ is true"), then $\varphi\notin\{1\}$ means "It is *not* the case that φ is true" (or "φ is *not* true"). It follows from the latter that if φ doesn't belong to the set $\{x_i\}$ of truth-values, then it belongs to any other set in V_φ than $\{x_i\}$. As a case of metalinguistic negation, let V be a set of three truth-values $\{\{x_1\},\{x_2\},\{x_3\}\}$;

thus according to Barzin and Errera's view of negation as exclusion out of a class, if n = 3 then $\varphi\in\{x_1\}$ means that $\varphi\notin\{x_2\}$ or $\varphi\notin\{x_3\}$, $\varphi\in\{x_2\}$ means that $\varphi\notin\{x_1\}$ or $\varphi\notin\{x_3\}$, and $\varphi\in\{x_3\}$ means that $\varphi\notin\{x_1\}$ or $\varphi\notin\{x_2\}$. Generally speaking, if $V_\varphi = \{\{x_1\},\{x_2\},...,\{x_a\},...,\{x_{n-1}\},\{x_n\}\}$, then $\varphi\in\{x_n\}$ if and only if (hereafter: iff) $\varphi\in\bigcup_{V\varphi-x^n}$, just as φ is not true iff $\varphi\notin\{1\}$ and φ is not false iff $\varphi\notin\{0\}$.

In the light of these two different levels of negation, the difference between Classical Logic (hereafter: **CL**) and Non-Classical Logics (hereafter: **NCL**) can be rephrased set-theoretically.

Negation in **CL** concerns logics in which $V_\varphi = \{\{1\},\{0\}\}$, *i.e.* a set of n = 2 sets of truth-values, so that $\varphi\in\{1\}$ whenever $\varphi\in\{0\}$ and $\varphi\in\{0\}$ whenever $\varphi\in\{1\}$. In a nutshell, **CL** is the class of bivalent semantic sets with n = 2 elements ({1} and {0}, say). Since $V_\varphi = \{\{1\},\{0\}\}$ in **CL**, it follows that ~φ is true iff φ is not-true, *i.e.* φ is false ($\sim\varphi\in\{1\}$ iff $\varphi\notin\{1\}$, *i.e.* $\varphi\in\{0\}$), and ~φ is false iff φ is not-false, *i.e.* φ is true ($\sim\varphi\in\{0\}$ iff $\varphi\notin\{0\}$, *i.e.* $\varphi\in\{1\}$). According to such a bivalent relation between truth ({1}, say) and falsity ({0}, say), we thus obtain the following 'laws' in **CL**:

- *Law of Bivalence* (hereafter: **LB**): every sentence is either true or false

  In other terms, there are only n = 2 truth-values: for any φ, ($\varphi\in\{1\} \cup \varphi\in\{0\}$), so that $\varphi\in\{1\}$ iff $\varphi\notin\{0\}$ and $\varphi\in\{0\}$ iff $\varphi\notin\{1\}$.

- *Law of Excluded Middle* (hereafter: **LEM**): either an affirmation or its denial is true

  Either (S is P) is true or (S is not P) is true;

  Either (S is not-P) is true or (S is not not-P) is true, so that $\varphi\in\{1\}$ or $\sim\varphi\in\{1\}$ for any φ.

- *Law of (Non-)Contradiction* (hereafter: **LC**): an affirmation and its denial cannot be both true

  (S is P) and (S is not P) are not true;

  (S is not-P) and (S is not not-P) are not true, so that $\varphi\notin\{1\}$ or $\sim\varphi\notin\{1\}$ for any φ.

Negation in **NCL** concerns logics in which V_φ is a set of n > 2 sets of truth-values. **LB** notably fails in **NCL**: for some φ's, $\varphi\in\{1/2\}$ means that $\varphi\notin\{1\}$ and $\varphi\notin\{0\}$. Note that the 'intermediary' set {1/2} needn't be a unit-class: whatever is neither true nor false fills the bill.

The linguistic negation '~' is *truth-functional*, that is: the output value of ~φ is *uniquely* determined by the input value of φ. In **CL**, $V_\varphi = \{\{1\},\{0\}\}$, so that ~φ is true iff φ is false ($\sim\varphi\in\{1\}$ iff $\varphi\in\{0\}$); in **NCL**, if V_φ =

$\{\{1\},\{1/2\},\{0\}\}$ then $\sim\varphi$ is true iff φ is not-true ($\sim\varphi \in \{1\}$ iff $\varphi \notin \{1\}$), and $\sim\varphi$ is indeterminate iff φ is indeterminate ($\sim\varphi \in \{1/2\}$ iff $\varphi \in \{1/2\}$). As in **CL**, the negation of $\{1\}$ and $\{0\}$ still yields $\{0\}$ and $\{1\}$ with so-called *normal* negations in **NCL**;[6] and although the negation of $\{1/2\}$ yields the same output value $\{1/2\}$, such a redundant operation doesn't constitute a counterexample against the general definition of negation as dichotomy or 'otherness'-operator, however; this will be argued later by means of a metalinguistic characterization of negation.

The metalinguistic negation is not *strictly truth-functional* in **NCL**. That is: the value of φ is not *uniquely* determined by the value of φ in a bijective mapping. Thus, φ is not true iff φ is either false or indeterminate ($\varphi \notin \{1\}$ iff $\varphi \in \{0\}$ or $\varphi \in \{1/2\}$); φ is not indeterminate iff φ is either true or false ($\varphi \notin \{1/2\}$ iff $\varphi \in \{1\}$ or $\varphi \in \{0\}$); and φ is false iff φ is either true or indeterminate ($\varphi \notin \{0\}$ iff $\varphi \in \{1\}$ or $\varphi \in \{1/2\}$).

Another point to clarify is about the meaning of '1/2' as an intermediary value between 0 and 1. If φ is *neither* true *nor* false, then it could be claimed that φ doesn't have any value so that $\{1/2\}$ stands for the empty set $\{\varnothing\}$; if φ is *both* true and false, then φ has two values and $\{1/2\}$ stands for the non-empty set $\{1,0\}$. Now any logic with $\{1/2\}$ as a proper set is a 3-valued logic, insofar as $\{\varnothing\}$ and $\{1,0\}$ count as two distinct subsets in V_φ in addition to $\{1\}$ and $\{0\}$. The case of *overdeterminacy* (being both true and false) is not $\{\{1\},\{0\}\}$ but $\{1,0\}$, and the case of *indeterminacy* (being neither true nor false) can also be seen as a three-valued system with the empty set as a proper set in its own part (see **Figure 5**). In the light of such definitions, let us turn to special cases of **NCL** and the purported reasons to revise the properties of classical negation.

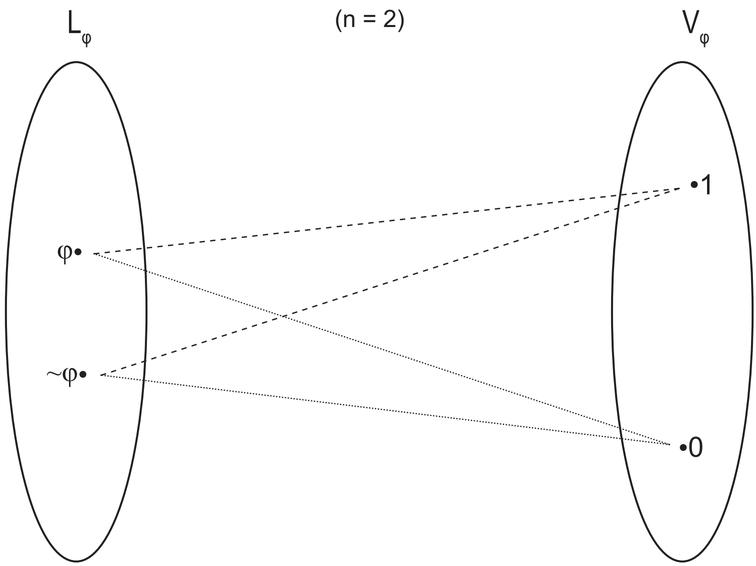
By *gappy* logics, we mean the class of *paracomplete* logics in which some sentences are neither true nor false: φ is neither true nor false whenever φ is not true and φ is not false ($\varphi \in \{1/2\}$ whenever $\varphi \notin \{1\}$ and $\varphi \notin \{0\}$, *i.e.* $\varphi \in \{\varnothing\}$). **LEM** fails in gappy logics: if $\varphi \in \{1/2\}$, then $\varphi \notin \{1\}$ and $\varphi \notin \{0\}$, *i.e.* $\sim\varphi \notin \{1\}$, so that $(\varphi \vee \sim\varphi) \notin \{1\}$ with $\varphi \in \{1/2\}$.

By *glutty* logics, we mean the class of *paraconsistent* logics in which some sentences are both true and false: φ is both true and false whenever φ is true and φ is false ($\varphi \in \{1/2\}$ whenever $\varphi \in \{1\}$ and $\varphi \in \{0\}$, *i.e.* $\varphi \in \{1,0\}$. **LC** fails in glutty logics: if $\varphi \in \{1/2\}$, then $\varphi \notin \{0\}$ and $\varphi \notin \{1\}$, *i.e.* $\sim\varphi \notin \{0\}$, so that $\sim(\varphi \wedge \sim\varphi) \notin \{1\}$ with $\varphi \in \{1/2\}$.

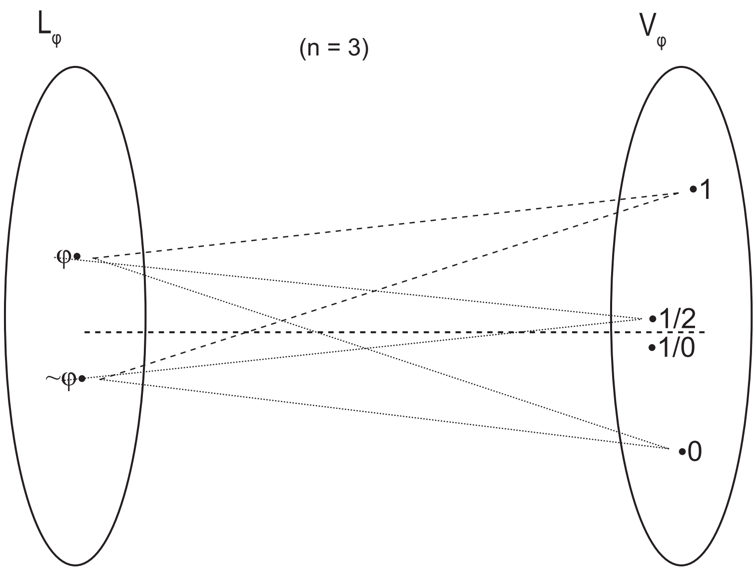
Defining negation as a process of dichotomy generally works only if it is considered from a *metalinguistic* point of view. When negation is about sentences, we note this by $\sim\varphi \in \{x_i\}$; when negation is about truth-values,

Figure 5

1/2 as the ouput value from a total function



1/2 as the output value from a total function



we note this by $\varphi \notin \{x_i\}$. In order to extensionalize this metaproperty, we propose to make use of internalization as follows.

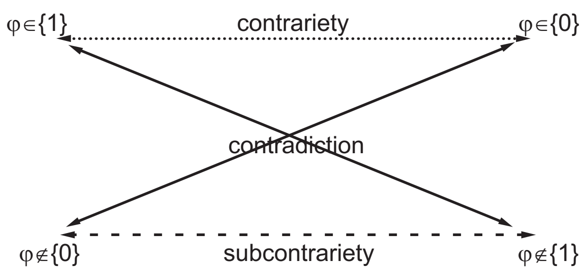
$L_{\varphi+1}$ is said to be an *internalization* of L_φ when some metalinguistic symbols with respect to L_φ are introduced into the object-language of $L_{\varphi+1}$. For instance, '$\notin$' and '$\{x_i\}$' are two metalinguistic symbols, whereas 'φ' and '~' are two linguistic symbols. The truth-values in L_φ are semantic (*i.e.* metalinguistic) predicates in $L_{\varphi+1}$, whereas valuations (upon L_φ in V_φ) are unary operators in $L_{\varphi+1}$.

Turning back to 'the Four' (see **Figure 6**), we can interpret more extensively the general form 'S is P' as the semantic judgment 'φ is true' ($\varphi \in \{1\}$), the contraffirmation 'S is not-P' as 'φ is not-true' ($\varphi \notin \{1\}$, *i.e.* $\varphi \in \{0\}$), the denial 'S is not P' as 'φ is not true' ($\varphi \notin \{1\}$, *i.e.* $\varphi \in \{0\}$ or $\varphi \in \{1/2\}$), and the contradenial 'S is not not-P' as 'φ is not not-true' ($\varphi \notin \{0\}$, *i.e.* $\varphi \in \{1\}$ or $\varphi \in \{1/2\}$). Therefore, the *contrary* relation between $\varphi \in \{1\}$ and $\varphi \in \{0\}$ means that these cannot be both true and both false.

Now if $\{1\}$ means 'to be true', the aforementioned contrariety amounts to say that being true and being false cannot be true together. We thus iterate the notion of truth, but without entailing any antinomy with these judgments. For if I say: the judgment '$\varphi \in \{0\}$' is true, that does not entail $\varphi \in \{1\}$ because a typed distinction is made between the value of the sentence φ and the value of a judgment about it. In order to clarifiy such a distinction between sentences and judgments, let us call for a semantic distinction between plain values and *designated* values.

Figure 6

Semantic Oppositions



Affirmation	$\varphi \in \{1\}$	("It is true that Socrates is a philosopher")
Denial	$\varphi \notin \{1\}$	("It is *not* true that Socrates is wise")
Contraffirmation	$\varphi \in \{0\}$	("It is false that Socrates is a philosopher")
Contradenial	$\varphi \notin \{0\}$	("It is *not* false that Socrates is a philosopher")

By a ‘designated value’ {T} is meant a specific subset of truth-values in V; such a subset may count only one or several elements. For example, if ‘Bydgoszcz is in Poland’ is true in L_φ, *i.e.* $\varphi \in \{1\}$, then "that Bydgoszcz is in Poland is true" is true in $L_{\varphi+1}$, *i.e.* (T{1} ∈ {T}). In Bochvar (1938), a similar use of judgment-operators was made with the so-called *external* operator of assertion Aφ: 'it is true that φ', to be read as 'φ∈{1}'. By this distinction between internal (sentential) and external (judgmental) operators, Bochvar made a typed distinction between affirmations and negations on the one hand, assertions and denials on the other hand. The same rationale will be used in the following in order to internalize the normally metalinguistic notions of truth-values. As a matter of fact, the set {1} of true sentences is taken to be the only case of designated value; but some non-classical logicians supplement the subset of designated values with {1/2} (see **Figure 7**). When a truth-value does not belong to the subset of designated values, it is said to be a *non-designated* value {⊥}.[7] By extension, designated values are used to define the *logical truth* of any formula semantically, namely: for any formulas φ and ψ, ψ is a *logical consequence* of φ iff if ψ∈{T} whenever φ∈{T}. The several properties of logical negation can be thus understood as preserving the designated value from the premises to the conclusion: $\varphi \Rightarrow \psi$.

Here is the core point of the paper: by means of internalization and designated values, and in accordance with Suszko’s thesis, any V_φ with n > 2: $\{\{x_1\},\{x_2\},...\{x_n\}\}$ sets of truth-values can be reduced to a $V_{\varphi+1}$ with n' = 2: $\{\{x_i\},\{x_j,...,x_n\}\}$ subsets of truth-values.

In rephrasing affirmation and denial as '∈' and '∉', the properties of negation can be 'classicized' by both internalizing truth-values while following Suszko's thesis. This will be done in two successive steps, namely: in section **2**, an increasing step from bivalent logics (with n = 2) to many-valued logics (with n > 2); and in section **3**, a decreasing step from many-valued logics to their bivalued (but not bivalent!) counterparts (with n' = 2). Just recall that the step from bivalence to bivaluation does not amount to a back to **CL** at all: it is a transition from so-called ‘algebraic’ values in **NCL** to ‘logical’ values. Such a reduction has been supported by Suszko and will be exemplified by internalizing several non-classical logics.[8]

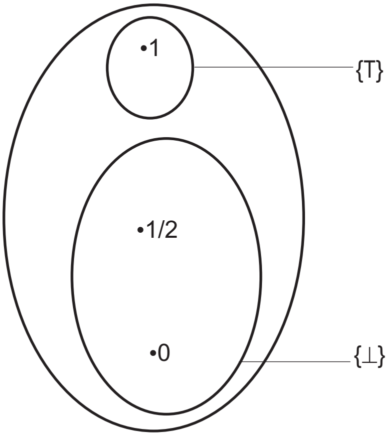
2. From bivalence to many-valuation

From $V_\varphi = \{\{0\},\{1\}\}$ we can derive the following subsets of truth-values: true and false, *i.e.* {1}∩{0} = {1,0}; true and not false, *i.e.* {1}∩~{0} = {1}; not true and false, *i.e.* ~{1}∩{0} = {0}; not true and not false, i.e; ~{1}∩~{0} = {∅}.

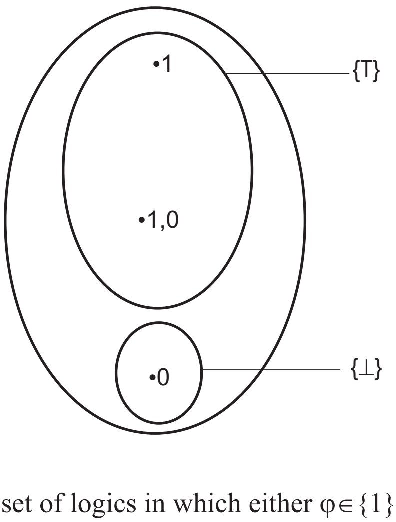
Figure 7

Designated and non-designated values in NCL

In Ł3 (Łukasiewicz's 3-valued logic)



In PLP (Priest's 3-valued logic of paradox)



Given **CL** as the set of logics in which either $\varphi \in \{1\}$ or $\varphi \in \{0\}$ (*tertium non datur*) for every φ, and **NCL** as the set of logics in which $\varphi \notin \{1\}$ and $\varphi \notin \{0\}$, *i.e.* $\varphi \in \{1,0\}$ or $\varphi \in \{\varnothing\}$ for some φ's (with $\{\{1,0\},\{\varnothing\}\} \subseteq \{1/2\}$), two sorts of many-valued logics can be discriminated according to the meaning of {1/2}, namely: gappy logics (including $\mathbf{K}_3$, **DL**), and glutty logics (including **PLP**).

Some general rules of valuation can be afforded for both sorts of **NCL** (whether paracomplete or paraconsistent), provided that the same ordering relation obtains between their truth-values. As it is the case for both $\mathbf{K}_3$ and **PLP**, let us state the following rules of valuation.

Given an ordering relation {1}>{1/2}>{0}, for any two 1-0 truth-values $\{x_1\},\{x_2\}$ we have the following rules for logical constants:

$\sim\{x_1\}\in|\{1\text{-}x_1\}|$

$\{x_1\} \vee \{x_2\}\in \boldsymbol{max}\{x_1,x_2\}$

$\{x_1\} \wedge \{x_2\}\in \boldsymbol{min}\{x_1,x_2\}$

$\{x_1\} \rightarrow \{x_2\}\in \boldsymbol{max}\{1\text{-}x_1,x_2\}$

$\oplus\varphi\in \boldsymbol{min}(\oplus\{x_1,y_1\},\oplus\{x_2,y_2\})$,[9] for any pairs of 1-0 truth-values $\{x_1,x_2\},\{y_1,y_2\}$ and any 1-ary or 2-ary operator $\oplus$.

2.1. Gappy Logics

An example of gappy logic with {1/2} = {∅} is **Heyting's Intuitionistic Logic** (hereafter: **HIL**). In accordance with Brouwer's objections to the dual opposition of truth and falsity and the realist approach of mathematical reasoning, the philosophy of intuitionism roughly consists in reading proofs as mental constructions and refuses to assign any value to a sentence so long as no proof has been constructed for it. Consequently, such classical properties of negation as **LEM** and Double Negation (hereafter: **LDN**) are cancelled by intuitionists because of their wrongly dual treatment of affirmations and negations.

Following Heyting's original notation, we'll make use of '¬' as a symbol for intuitionistic or *strong* negation.[10] Given the truth-conditions for intuitionistic negation, **LB** and **LEM** both fail in **HIL** with $\varphi\in\{1/2\}$. But the core question rather concerns the identity of **LEM** from a logical system to another one: how can **LEM** be said to have the same *meaning* in both classical and intuitionistic logics, assuming that logical negation is *not* the same for classicists and intuitionists? Pending an answer for this question, let us note that the two following properties of classical negation still hold when interpreted intuitionistically:

(H1) $\neg\varphi \Rightarrow (\varphi \rightarrow \psi)$

(H2) $((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \neg\psi)) \Rightarrow \neg\varphi$

However, the following invalid formulas (*) making use of negation are not logical truths in **HIL**:

(H1*) $(\neg\varphi \vee \psi) \Rightarrow (\varphi \rightarrow \psi)$

(H2*) $(\varphi \rightarrow \psi) \Rightarrow \neg(\varphi \wedge \neg\psi)$

(H3*) $(\varphi \vee \psi) \Rightarrow (\neg\varphi \rightarrow \psi)$

(H4*) $(\varphi \vee \psi) \Rightarrow \neg(\neg\varphi \wedge \neg\psi)$

(H5*) $(\varphi \wedge \psi) \Rightarrow \neg(\neg\varphi \vee \neg\psi)$

In order to give a semantic interpretation of these formulas, a language L and a 3-valued matrix V for **HIL** have been supplied by Kleene (1952)'s $\mathbf{K}_3$-system, namely:

$$\mathsf{L}_{K3} = \langle \sim,\wedge,\vee,\rightarrow \rangle \text{ and } \mathsf{V}_{K3} = \langle \{1\},\{1/2\},\{0\} \rangle,$$

in which classical negation '~' can be used as a primitive operator defining intuitionistic negation, that is: $\neg\varphi$ means '$\varphi \in \{0\}$', *i.e.* '$\sim\varphi \in \{1\}$'. The truth-conditions for the logical constants in L_{K3} are truth-functionally defined in matrices (see **Figure 8**).

Another case of gappy logic is **Rogowski's Directional Logic** (hereafter: **DL**), the aim of which was to formalize Hegel's dialectical logic of change. Several kinds of modal operators[11] are introduced into a 4-valued matrix in **DL**, namely:

$$\mathsf{L}_{DL} = \langle N^{\rightarrow},\wedge,\vee,\rightarrow \rangle, \text{ and } \mathsf{V}_{DL} = \langle \{1,1\},\{1,0\},\{0,1\},\{0,0\} \rangle$$

As for most of non-classical systems, the designated value corresponds to the single value $\{\{1,1\}\} \subseteq \{\mathsf{T}\}$, whereas non-designated values are the three remaining ones $\{\{1,0\},\{0,1\},\{0,0\}\} \subseteq \{\bot\}$.

In accordance with the motivation of **DL**, *i.e.* to express changes between 'being' and 'not-being', an interpretation for the four values in **DL** yields $\varphi \in \{1,1\}$ as 'it is true that φ', $\varphi \in \{1,0\}$ as 'it comes to be false that φ', $\varphi \in \{0,0\}$ as 'it is false that φ', and $\varphi \in \{0,1\}$ as 'it comes to be true that φ'.

Given the ordering relation $\{1,1\} > \{0,1\} > \{1,0\} > \{0,0\}$, the general rules of valuation for formulas in **DL** can be defined within matrices (see **Figure 9**) in basic terms of classical negation '~'.

For any pairs of 1-0 values $\{x_1,y_1\},\{x_2,y_2\}$, we have the following valuations for connectives:

$N\{x_1,y_1\} \in \{\sim x_1,\sim y_1\}$;

$N^{+}\{x_1,y_1\} \in \{\sim x_1,\sim y_1\}$ iff $x_1 = y_1$; $N^{+}\{x_1,y_1\} \in \{1,1\}$, otherwise;

$N^{\rightarrow}\{x_1,y_1\} \in \{x_1,\sim y_1\}$ iff $x_1 = y_1$; $N^{\rightarrow}\{x_1,y_1\} \in \{\sim x_1,y_1\}$, otherwise;

$N^{\leftarrow}\{x_1,y_1\} \in \{\sim x_1,y_1\}$ iff $x_1 = y_1$; $N^{\leftarrow}\{x_1,y_1\} \in \{x_1,\sim y_1\}$, otherwise;

Figure 8

Logical matrices for K_3

φ	$\sim\varphi$
{1}	{0}
{1/2}	{1/2}
{0}	{1}

$\varphi \wedge \psi$	{1}	{1/2}	{0}
{1}	{1}	{1/2}	{0}
{1/2}	{1/2}	{1/2}	{0}
{0}	{0}	{0}	{0}

$\varphi \vee \psi$	{1}	{1/2}	{0}
{1}	{1}	{1}	{1}
{1/2}	{1}	{1/2}	{1/2}
{0}	{1}	{1/2}	{0}

$\varphi \rightarrow \psi$	{1}	{1/2}	{0}
{1}	{1}	{1/2}	{0}
{1/2}	{1/2}	{1/2}	{1/2}
{0}	{1}	{1}	{1}

$T\{x_1,y_1\} = N^+N\{x_1,y_1\}$;

$H^{\rightarrow}\{x_1,y_1\} \in \{x_1,y_1\}$ iff $x_1 = y_1$; $H^{\rightarrow}\{x_1,y_1\} \in \{\sim x_1,y_1\}$, otherwise;

$H^{\leftarrow}\{x_1,y_1\} \in \{x_1,y_1\}$ iff $x_1 = y_1$; $H^{\leftarrow}\{x_1,y_1\} \in \{x_1,\sim y_1\}$, otherwise.

Together with the following valuations for the set of classical connectives $\{\vee,\wedge,\rightarrow\}$, namely:

$\{x_1,x_2\} \vee \{y_1,y_2\} \in \boldsymbol{max}(\{x_1,x_2\},\{y_1,y_2\})$;

Figure 9

Logical matrices for DL

φ	Nφ	N^{+}φ	$N^{\rightarrow}$φ	$N^{\leftarrow}$φ	Tφ	$H^{\leftarrow}$φ	$H^{\rightarrow}$φ
{1,1}	{0,0}	{0,0}	{1,0}	{0,1}	{1,1}	{1,1}	{1,1}
{0,1}	{1,0}	{1,1}	{1,1}	{0,0}	{0,0}	{0,0}	{1,1}
{1,0}	{0,1}	{1,1}	{0,0}	{1,1}	{0,0}	{1,1}	{0,0}
{0,0}	{1,1}	{1,1}	{0,1}	{1,0}	{0,0}	{0,0}	{0,0}

φ ∨ ψ	{1,1}	{0,1}	{1,0}	{0,0}
{1,1}	{1,1}	{1,1}	{1,1}	{1,1}
{0,1}	{1,1}	{0,1}	{0,1}	{0,1}
{1,0}	{1,1}	{0,1}	{1,0}	{1,0}
{0,0}	{1,1}	{0,1}	{1,0}	{0,0}

φ ∧ ψ	{1,1}	{0,1}	{1,0}	{0,0}
{1,1}	{1,1}	{0,1}	{1,0}	{0,0}
{0,1}	{0,1}	{0,1}	{1,0}	{0,0}
{1,0}	{1,0}	{1,0}	{1,0}	{0,0}
{0,0}	{0,0}	{0,0}	{0,0}	{0,0}

φ → ψ	{1,1}	{0,1}	{1,0}	{0,0}
{1,1}	{1,1}	{0,1}	{1,0}	{0,0}
{0,1}	{1,1}	{0,1}	{1,0}	{1,0}
{1,0}	{1,1}	{0,1}	{0,1}	{0,1}
{0,0}	{1,1}	{1,1}	{1,1}	{1,1}

$\{x_1,x_2\} \wedge \{y_1,y_2\} \in \boldsymbol{min}(\{x_1,x_2\},\{y_1,y_2\})$;

$\{x_1,x_2\} \rightarrow \{y_1,y_2\} \in \boldsymbol{max}(N\{x_1,x_2\},\{y_1,y_2\})$.

LEM, **LC**, and the Law of Identity (φ → φ)[12] fail in **DL** with φ∈{1,0} and φ∈{0,1}. Furthermore, the same does with all classical properties of negation that fail with the non-static truth-values {1,0} and {0,1}.

Thus, such 'obvious' laws as **LEM** or **LC** are rejected within many-valued systems; just as the former are invalidated in some gappy logics, most of the properties of logical negation are also discarded in glutty logics.

2.2. Glutty logics

In many-valued logics, the main case for gluttiness is the family of *paraconsistent* logics, *i.e.* systems interpreted by logical matrices with $\varphi \in \{1/2\}$: 'φ is both true and false'.

A sample of glutty logic is **Priest's Logic of Paradox** (hereafter: **PLP**).

Since paraconsistent negation is not attached only to a classical value $\{1\}$ or $\{0\}$, let us symbolize it as '–'.

Just as in gappy logics and any many-valued logics, **LB** and **LC** both fail in **PLP**. The main difference between **PLP** and the preceding gappy systems concerns the sets of designated and not-designated values: $\{\top\}$ and $\{\bot\}$ differ in extension given that $\{\top\} = \{\{1/2\},\{1\}\}$ in **PLP**.[13]

Now just as intuitionistic negation didn't have the same meaning as classical negation, how can we say that paraconsistent negation, and the properties of logical negation, do have the same meaning in **CL** and **NCL**?

Pending an answer to this matter of meaning for logical constants, a semantic interpretation for **PLP** was given within logical matrices by Priest (1979)'s 3-valued logic **PLP**. It relies on a language to be interpreted in a 3-valued matrix (see **Figure 10**):

$$\mathsf{L}_{PLP} = \langle -,\wedge,\vee,\rightarrow\rangle \text{ and } \mathsf{V}_{PLP} = \langle\{1\},\{1/2\},\{0\}\rangle$$

As just observed, **LC** does not 'classically' hold ($\varphi \notin \{1\}$ for some φ's) but still holds in **PLP** with $\varphi \in \{1/2\}$: $(1/2 \wedge {\sim}1/2) = (1/2 \wedge 1/2) = 1/2$, therefore **LC**$\notin \{1\}$ with $\varphi \in \{1/2\}$; now $\{1/2\} \in \{\top\}$ in **PLP**, then **LC** 'weakly' holds with $\{1/2\}$.

Valuations in **PLP** turn on the three following truth-values: $\{1\}$ as 'only true', *i.e.* $\{1,1\}$; $\{1/2\}$ as 'both true and false', *i.e.* $\{1,0\}$ or $\{0,1\}$; and $\{0\}$ as 'only false', *i.e.* $\{0,0\}$, within an ordering relation $\{1\}>\{1/2\}>\{0\}$.

The following formulas with logical negation still hold in **PLP**:

(P 1) $(\varphi \rightarrow \psi) \Rightarrow (-\psi \rightarrow -\varphi)$

(P 2) $(-\varphi \wedge -\psi) \Rightarrow -(\varphi \vee \psi)$

(P 3) $(-\varphi \rightarrow -\psi) \Rightarrow (\psi \rightarrow \varphi)$

(P 4) $-(\varphi \vee \psi) \Rightarrow -\varphi$

(P 5) $\varphi \Rightarrow - -\varphi$

<u>**Figure 10**</u>

Logical matrices for PLP

φ	$\sim\varphi$
{1}	{0}
{1/2}	{1/2}
{0}	{1}

$\varphi \wedge \psi$	{1}	{1/2}	{0}
{1}	{1}	{1/2}	{0}
{1/2}	{1/2}	{1/2}	{0}
{0}	{0}	{0}	{0}

$\varphi \vee \psi$	{1}	{1/2}	{0}
{1}	{1}	{1}	{1}
{1/2}	{1}	{1/2}	{1/2}
{0}	{1}	{1/2}	{0}

$\varphi \rightarrow \psi$	{1}	{1/2}	{0}
{1}	{1}	{1/2}	{0}
{1/2}	{1/2}	{1/2}	{1/2}
{0}	{1}	{1}	{1}

(P 6) $- -\varphi \Rightarrow \varphi$

(P 7) $-\varphi \Rightarrow -(\varphi \wedge \psi)$

(P 8) $-(\varphi \rightarrow \psi) \Rightarrow \varphi$

(P 9) $(\varphi \wedge -\psi) \Rightarrow -(\varphi \rightarrow \psi)$

(P10) $-\varphi \Rightarrow (\varphi \rightarrow \psi)$

(P11) $(\varphi \rightarrow -\varphi) \Rightarrow -\varphi$

Some of the logical truths in **CL** fail in **PLP**, namely:

(P1*) $(\varphi \wedge -\varphi) \Rightarrow \psi$

(P2*) $(\varphi \wedge (-\varphi \vee \psi)) \Rightarrow \psi$

(P3*) $((\varphi \rightarrow \psi) \wedge -\psi) \Rightarrow -\varphi$

(P4*) $(\varphi \rightarrow (\psi \wedge -\psi)) \Rightarrow -\varphi$

The invalidity of (P1*) symbolizes non-triviality in **PLP** and means that, from a pair of inconsistent formulas, we cannot derive anything; (P2*) is a rejection of disjunctive syllogism, normally used in order to deduce triviality in an inconsistent system; finally, (P3*) and (P4*) are two variants of the principle of *reductio ad absurdum* (hereafter: **RA**) and mean that any formula entailing an inconsistency is not to be always rejected as such.

In the light of these three non-classical systems and their many-valued semantics, it is established as a commonplace that most of the classical properties of logical negation (especially **LEM** or **LC**) do not hold universally and fail whenever the semantic frame V includes more than the two classical truth-values {0} and {1}.

Does it mean that our alleged commensensical reading of negation is incompatible with the preceding commonplace, or that it should be restricted to some current interpretations of negation? In order to concile the current (classical, bivalent) view of negation and its special (non-classical, many-valued) uses, it will be claimed in the following that:

- logical negation is defined with respect to its arguments, *i.e.* the truth-values it maps onto; by this way, non-classical logics don't appear as a deviation but as an extension of negation in **CL** (from n = 2 to n > 2);
- the commensensical reading of logical negation can be preserved even in **NCL**, by internalizing the truth-values as supplementary unary operators; most of the classical properties of logical negation may be thus restored, depending upon the translation of **LEM**, **LC**, and the like in the internalized systems;
- beyond **CL** and **NCL**, logical negation can be viewed as a general operator of dichotomy, while clearly making a distinction between bivalence and bivaluation (*i.e.* bipartition) in V.

3. From many-valuation to bivaluation

According to *Suszko's thesis*, every many-valued logical matrix can be reduced to a two-valued one, whenever a distinction is made between *algebraic* values ($\{x_1\},...,\{x_n\}$) in L_φ and *logical* values ($\{\top\}$ and $\{\bot\}$) in

$\mathsf{L}_{\varphi+1}$. Two internalized systems may exemplify this thesis, namely: $\mathbf{K}_{3+}$ as an internalization of $\mathbf{K}_3$ for the semantics of **HIL**, and $\mathbf{PLP}_+$ as an internalization of **PLP**.

3.1. Gappy logics

An internalization of $\mathbf{K}_3$ thus yields the Logic of Truth $\mathbf{K}_{3+}$:

$$\mathsf{L}_{3+} = \langle \sim,\wedge,\vee,\rightarrow,\mathrm{T}\rangle \text{ and } \mathsf{V}_{3+} = \langle\{1\},\{1/2\},\{0\}\rangle,$$

in which classical negation '~' is used as a basic term for defining its single intuitionistic counterpart.

Following logical truth in $\mathbf{K}_3$, the only designated value in $\mathbf{K}_{3+}$ is $\{\{1\}\}\subseteq\{\mathsf{T}\}$, so that the not-designated values are $\{\{1/2\},\{0\}\}\subseteq\{\bot\}$.

$\mathbf{K}_{3+}$ is a *semicomplete* logic, extending $\mathbf{K}_3$ with T as a primitive unary operator together with two other interdefinable operators F,I. Tφ helps to internalize $\varphi\in\{1\}$ and stands for 'It's established as true that φ'; a counterpart for Tφ in Modal Logic (hereafter: **ML**) is the notion of necessity, $\Box\varphi$. Fφ internalizes $\varphi\in\{0\}$ and stands for 'It's established as false that φ'; a counterpart for Fφ in **ML** is the notion of impossibility, $\Box\sim\varphi$. Iφ internalizes $\varphi\in\{1/2\}$ and stands for 'Nothing is established about φ'; a counterpart for Iφ in **ML** is the notion of contingency or two-sided possibility, $\nabla\varphi =\mathrm{df} (\sim\Box\varphi \wedge \sim\Box\sim\varphi)$. We thus get the following τ-translations of the valuations in V_{K3} into formulas in L_{K3+}:

$$\tau(\varphi\in\{1\}) = \mathrm{T}\varphi;\ \tau(\varphi\in\{1/2\}) = \mathrm{I}\varphi;\text{ and } \tau(\varphi\in\{0\}) = \mathrm{F}\varphi$$

The language for $\mathbf{K}_{3+}$ helps to translate the formulas from **HIL** and, above all, to compare classical and intuitionistic views of negation. For instance, a τ-translation of the left-sided formulas from **HIL** yields the right-sided formulas from L_{K3+}:

$$\tau(\varphi) = \mathrm{T}\varphi,\ i.e.\ \varphi\in\{1\};\text{ and } \tau(\neg\varphi) = \mathrm{T}\sim\varphi,\ i.e.\ \varphi\in\{0\}$$

According to the logical matrices in $\mathbf{K}_{3+}$ (see **Figure 11**), both values {1} and {0} in **HIL** uniquely correspond to Tφ and T~φ in $\mathbf{K}_{3+}$. Let us note however that, while Fφ (= T$\neg\varphi$) corresponds to a well-formed formula (hereafter: wff) in **HIL**, no wff corresponds to the resulting Iφ (= ~Tφ) in $\mathbf{K}_{3+}$. It is so because of the strictly *strong* meanings of truth and falsity as established values in **HIL**; in other terms, there is no room for contingent

Figure 11

Logical matrices for K_{3+}

Wffs in HIL φ	Tφ Wffs in K_{3+}
{1}	{T}
{1/2}	{⊥}
{0}	{⊥}

φ ¬φ	Fφ
{1} {0}	{⊥}
{1/2} {1/2}	{⊥}
{0} {1}	{T}

-	Iφ
{1}	{⊥}
{1/2}	{T}
{0}	{⊥}

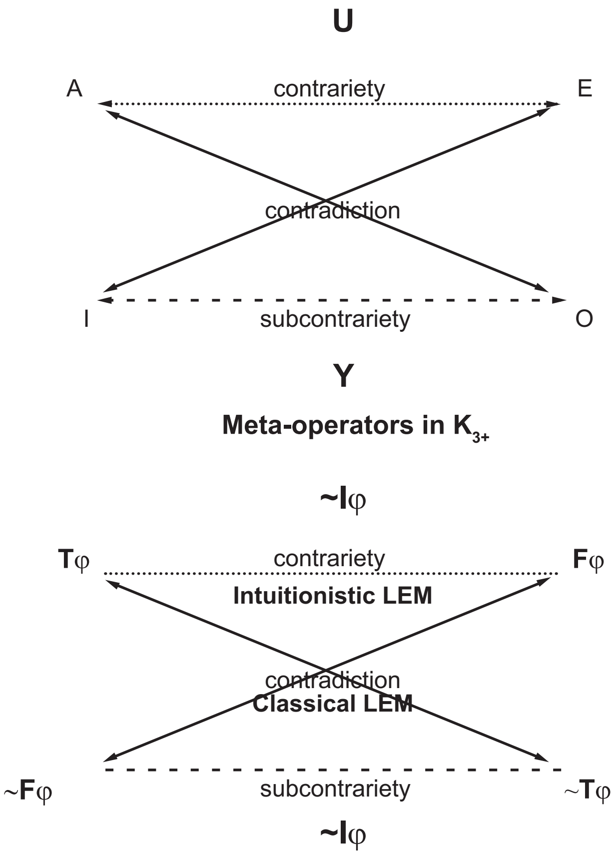
(unestablished) truth in Heyting's view of intuitionistic logic. Now if such a strong reading of truth and falsity is translated in $\mathbf{K}_{3+}$, it seems that the intuitionistic version of **LEM**, namely: (Tφ ∨ Fφ), actually corresponds to the metalinguistic **LB** and thus expresses a relation between *contrary* values, rather than *contradictory* ones.

A comparison between the classical and intuitionistic readings of **LEM** can be made within Blanché's hexagon of oppositions (see **Figure 12**), thus yielding a contrast between the syntactic (*i.e.* according to their logical forms) and semantic formulations (*i.e.* according to their truth-values) of **LEM**: either the latter contains a pair affirmation-denial and thus corresponds to the classical version only; or it is about a pair truth-falsity and thus corresponds to both classical and intuitionistic versions.

By analogy with a famous paper by Slater (1995): "Paraconsistent Logics?", who doubted about their foundations because paraconsistent negation is not a contradictory — but *subcontrary*-forming operator, the same question can

Figure 12

Blanché's Hexagon of Oppositions



be asked about intuitionistic negation given that it is not a contradictory- but *contrary*-forming operator (see **Figure 12**). It is a thing that not every logical negation must be a contradictory-forming operator, as witnessed in the history of logic by several forms of contrary negations within ancient and medieval logics;[14] but it is another thing to say that not-contradictory-forming operators of negation occur in **LEM** as such. Is excluded middle strictly related to classical negation, or does it ultimately express a semantic relation between truth-values, *i.e.* irrespective of its logical form and displayed opposition?

According to Wiredu (1975), the intuitionistic objection against **LEM** does miss the point: a disjunctive relation is stated therein between an affirmation ('is true') and its *contraffirmation* ('is false'), whereas **LEM** is stated as an opposition between and affirmation ('is true') and its *denial* ('is not true'). In a nutshell, the identity of **LEM** seems to vacillate between its syntactic and semantic definition.

Anyway, some iteration laws can be put in $\mathbf{K}_{3+}$ in order to simplify its modal formulas:

$$T\varphi \Leftrightarrow TT\varphi;\ T\odot\varphi \Leftrightarrow \odot\varphi \text{ (for any } \odot \in \{T,F,I\});\ FT\varphi \Leftrightarrow \sim T\varphi;\ F\varphi \Leftrightarrow TF\varphi.$$

The class of theorems in $\mathbf{K}_{3+}$ are **S5**-valid when intuitionistic negation is translated as ~T, so that this class is larger than in Gödel (1933)'s translation of intuitionistic negation as a **S4**-modal system. The reason is that Gödel's translation squares with Heyting's version of '¬' as 'T~' (= 'F'), and not '~T' (the latter does not make sense in **HIL**, again).

It can be established both (see the Appendix) that:

- K_{3+} is a translation of **HIL** whenever '¬' is translated by the strong negation 'T~', in accordance with Heyting's modal interpretation of intuitionistic negation;
- the properties of classical negation can be preserved in $\mathbf{K}_{3+}$ whenever '¬' is translated by the weak, classical negation '~T';
- as for what the 'genuine' translation of **LEM** in $\mathbf{K}_{3+}$ is, namely: $(T\varphi \vee T{\sim}\varphi)$ or $(T\varphi \vee {\sim}T\varphi)$, the question remains open.

Another case of internalization concerns **DL**, with a Logic of Being-the-Case $\mathbf{DL}^+$.

DL is self-internalizing, *i.e.* L_{DL} already contains an operator of assertion T that may be used in order to internalize truth-values and preserve the 'classical' properties of negation. Examples of internalized assertions in $\mathbf{DL}^+$ are 'It is the case that it is true that φ', *i.e.* $T\{1,1\} \in \{T\}$; 'it is the case that it begins to be true that φ', *i.e.* $T\{0,1\} \in \{\bot\}$; 'it is the case that it begins to be false that φ', *i.e.* $T\{1,0\} \in \{\bot\}$; and, finally,'it is the case that it is false that φ', *i.e.* $T\{0,0\} \in \{\bot\}$.

A proof of the validity for **LEM** and **LC** in $\mathbf{DL}^+$ is given in the Appendix below.

3.2. Glutty logics

An internalization of **PLP** yields the Logic of Veridication $\mathbf{PLP}_+$, as suggested by Strössner and Strobach (2007):

$$\mathsf{L}_{PLP_+} = \langle \sim,\wedge,\vee,\rightarrow,V\rangle \text{ and } \mathsf{V}_{PLP_+} = \langle\{1\},\{1/2\},\{0\}\rangle$$

in which classical negation is used as a basic constant defining its possible paraconsistent counterparts.

Following logical truth in **PLP**, there are two designated values in **PLP**$_+$, namely: $\{\{1\},\{1/2\}\}\subseteq\{\mathsf{T}\}$, whereas the single not-designated value is $\{\{0\}\}\subseteq\{\bot\}$.

PLP$_+$ is a semiconsistent logic extending **PLP**, with V as a primitive unary operator together with one other interdefinable operator W. Wφ helps to internalize $\varphi\in\{1\}$ and stands for 'It's only true that φ'; a counterpart of Wφ in **ML** is $\Box\varphi$. W$\sim\varphi$ internalizes $\varphi\in\{0\}$ and stands for 'It's only false that φ'; a counterpart for W$\sim\varphi$ in **ML** is $\Box{\sim}\varphi$; Vφ internalizes either $\varphi\in\{1\}$ or $\varphi\in\{1,0\}$ and stands for 'It's at least true that φ'; a counterpart for Vφ in **ML** is $\Box\varphi$. We thus have the following translations of the valuations in V_{PLP} into the formulas in L_{PLP}:

$$\tau(\varphi\in\{1\}) \supseteq \{W\varphi,V\varphi\};\ \tau(\varphi\in\{1/2\}) = V\varphi;\text{ and } \tau(\varphi\in\{0\}) \supseteq \{W{\sim}\varphi,V{\sim}\varphi\}$$

Note also that Wφ and Vφ are duals, *i.e.* $V\varphi \Leftrightarrow {\sim}W{\sim}\varphi$.

The language for **PPL**$_+$ helps to translate the formulas from **PPL** and, above all, to compare classical and paraconsistent views of negation. As the meaning of paraconsistent negation is not as strong as its intuitionistic counterpart but still differs from the classical one, we thus have the following possible translations from left-sided formulas in **PPL** to right-sided formulas in **PLP**$_+$:

$$\tau(\varphi)= V\varphi \text{ or } W\varphi;\text{ and } \tau(-\varphi)= V{\sim}\varphi \text{ or } W{\sim}\varphi.$$

According to the logical matrices in **PLP**$_+$ (see **Figure 13**), both classical values $\{1\}$ and $\{0\}$ in **PPL** correspond to Wφ or Vφ and W$\sim\varphi$ or V$\sim\varphi$ in **PLP**$_+$; Wφ (*i.e.* ${\sim}V{\sim}\varphi$), W$\sim\varphi$ (*i.e.* ${\sim}V\varphi$) and Vφ (*i.e.* ${\sim}W{\sim}\varphi$) correspond to wffs in **PPL**, whereas the non-classical value $\{\{1\},\{0\}\}$ uniquely corresponds to Vφ. Contrary to the translations from **HIL**-formulas to **K**$_{3+}$-formulas, in which the resulting $\tau(\varphi)$ and $\tau(\neg\varphi)$ are definite, the translations from **PLP**-formulas to **PLP**$_+$-formulas can thus vary according to the adopted translations (syntactic interpretations) of φ and $\sim\varphi$ in **PLP**$_+$.

A comparison between classical and both possible paraconsistent readings of **LC** can be made within Blanché's hexagon of oppositions (see **Figure 14**), thus yielding a contrast between the syntactic (*i.e.* their logical forms) and semantic formulations (*i.e.* their truth-values) of **LC**.

Figure 13

Logical matrices for PLP_+

Wffs in PPL φ	$W\varphi$	$V\varphi$ Wffs in PLP_+
{1}	{T}	{T}
{1/2}	{⊥}	{T}
{0}	{⊥}	{⊥}

φ	$–\varphi = V\sim\varphi$
{1}	{⊥}
{1/2}	{T}
{0}	{T}

φ	$–\varphi = W\sim\varphi$
{1}	{⊥}
{1/2}	{⊥}
{0}	{T}

Some iteration laws can be put in **PLP**$_+$, thus:

$W\varphi \Leftrightarrow Ww\varphi$; $W\odot\varphi \Leftrightarrow W\varphi$ (for any $\odot \in \{W,V,W\sim\}$); and $W\sim\odot\varphi \Leftrightarrow W\sim\varphi$.

Theorems in **PLP**$_+$ are **S4**-valid when paraconsistent negation '–' is read as 'W~' (like Gödel's **S4**, given that F and W~ share the same valuations); whereas they are **S5**-valid when '–' is read as '~W', *i.e.* as 'V~'.

It can be shown (see the Appendix) that:

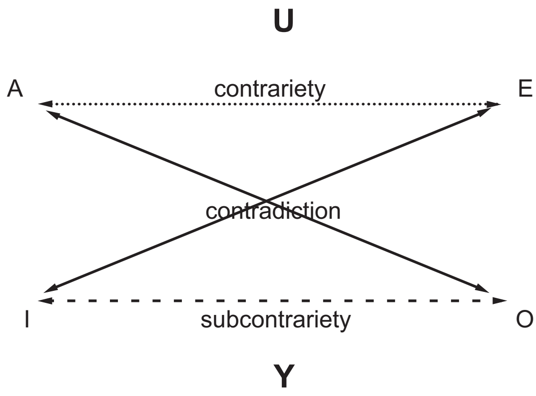
- every logical truth of **PLP**$_+$ is a theorem of **PLP** whenever '–' is read as thre weak negation 'V~';
- every classical property of negation can be restored in **PLP**$_+$ whenever '–' is read as the strong negation 'W~'.

4. Conclusion: Negation as Dichotomy

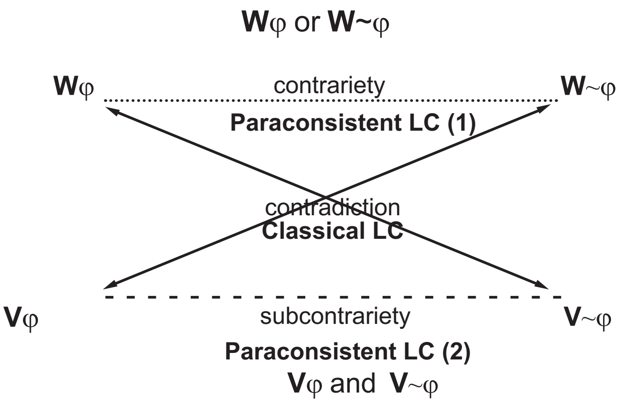
In conclusion, the properties of negation in **CL** may be maintained depending upon the *meaning* of truth and falsity in **NCL**; and assuming that two

Figure 14

Blanché's Hexagon of Oppositions



Meta-Operators in PLP$_+$

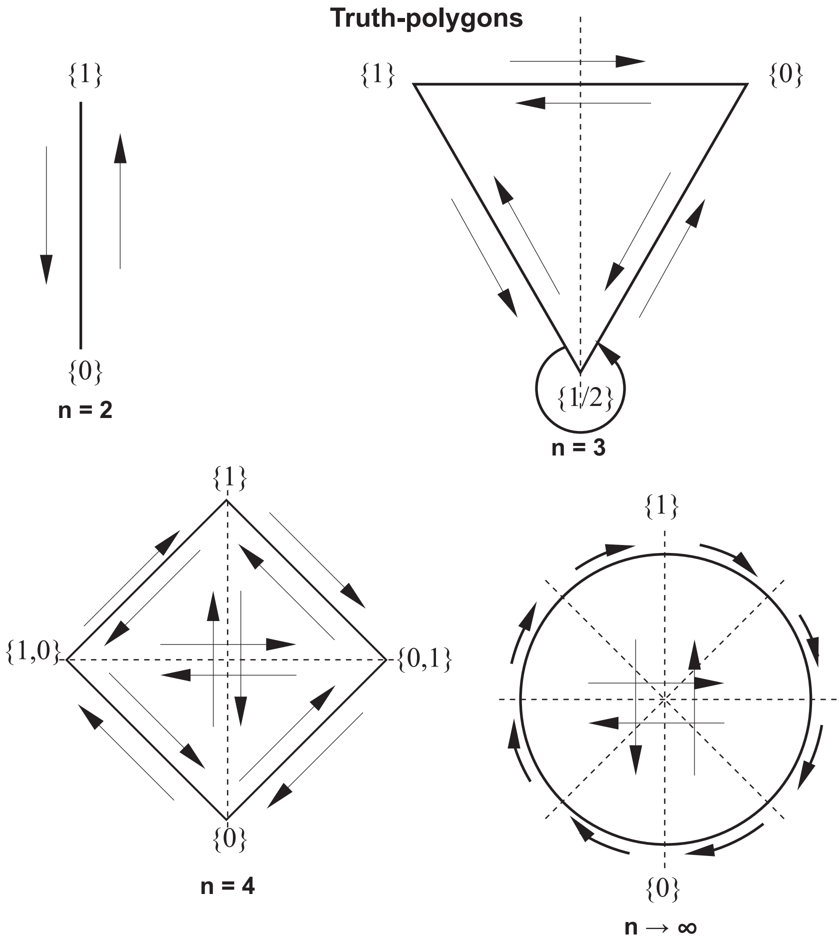


formulas are equivalent iff they have the same meaning, a translation of intuitionistic and paraconsistent negations '¬' and '–' by '~T' and '~W' (*i.e.* 'V~'), respectively, would maintain **LEM**, **LC**, **LDN**, and the like within non-classical (many-valued) systems.

As depicted by a first transition from bivalent to many-valued logics and, then, by a second transition from many-valued to bipartitioned logics, the classical properties of negation in **CL** can be maintained in applying Suszko's thesis and replacing bivalence with bipartition.

Indeed, every property from **RA** to **DM→** may thus be saved when logical consequence and logical truth are defined in $V_{\varphi+1} = \{T,\bot\}$ rather than in $V_\varphi = \{x_1,...x_n\}$; here are the main advantages and shortcomings of defining negation as a dichotomy:

Figure 15



4.1 Dichotomy and n-chotomies

A definition of dichotomy is the following: 'being twofold; a classification into two opposed parts or subclasses', in which it relates to bipartition of classes and don't need to be synonymous with bivalence (it is so only when each subset is a unit-class, *i.e.* only in **CL**).

The conclusive claim of this paper is that, beyond the variety of logical negations from a semantic perspective, negation can be viewed as a metalinguistic process of dichotomy, *i.e.* as a *bipartition* of any two subsets of truth-values. It follows from this definition that:

- **CL** is a peculiar case in which the number of *algebraic* values (metalinguistic predicates: {0}, {1}, ..., $\{x_n\}$) is the same as the number of logical values (designated values);
- 'classical negation' is a minimal negation, in the sense that no lower V_n than n = 2 can be used in logic (if n < 2, then no consequence relation can be stated between formulas);
- every negation consists in dividing a set of n elements into 2 subsets of truth-values, and not always a set of n = 2 elements into 2 subsets (this latter case holds in **CL** only);
- any *n-chotomy* is an increasing set of algebraic values tending to infinitely many, whereas negation as a *dichotomy* differs from these "referential" assignments and only concerns logical (impredicable) values $\{\top,\bot\}$.

4.2 Negations as specific functions

Such an intensional definition of negation as dichotomy makes the mapping *non-truthfunctional*, that is: each particular (*i.e.* cyclic, strong, weak, external, internal, and so on) negation (see **Figure 15** and **Figure 16**) helps to fix one and only one value for $\sim\{x_i\}$ by means of $\{x_i\}$ in V_φ, but the same does not hold in $V_{\varphi+1}$ since different formulas can have the same truth-value.

Several specific functions are related to negation, namely:

- *complementary* negation (classical negation as a contradictory-forming operator)

  $$\varphi \in \{x_i\} \text{ in } V_{\varphi+1}, \text{ iff } \sim\varphi \in \cup V_{X-\{x_i\}} \text{ in } V_\varphi$$

- *polar* negation (intuitionistic negation as a contrary-forming operator)

  $$\varphi \in \{x_1\} \text{ in } V_\varphi = \{x_1,...,x_n\} \text{ iff } \sim\varphi \in \{x_n\}$$

- *symetric* negation

  For any scale of values in

  $$V_\varphi = \{0,...,x_i=1/n,...,1\}, |\{x_i\} + \{\sim x_i\}| = 1 \text{ in } V_\varphi$$

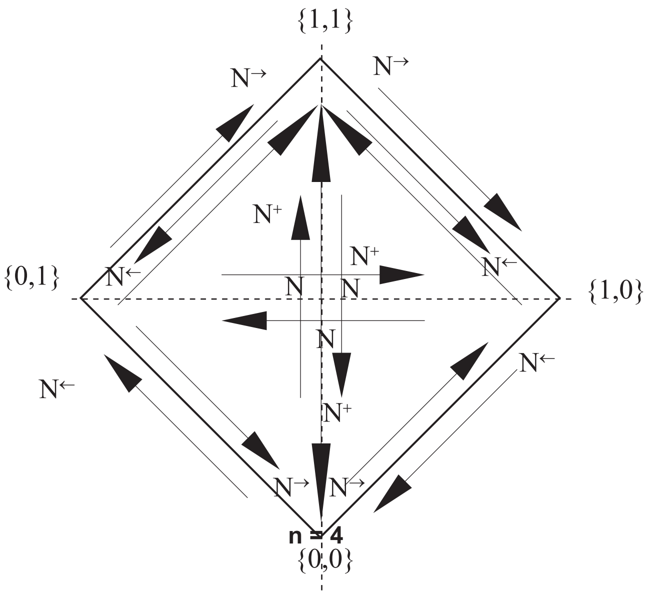
- *cyclic* negation (directional negation as a "backward-or-forward"-forming operator)

  $$\text{If } \varphi \in \{x_i\} \text{ in } V_\varphi, \text{ then } \sim\varphi \in \{x_{i-1}\} \text{ or } \sim\varphi \in \{x_{i+1}\} \text{ in } V_\varphi$$

DN1 and **DN2** can be replaced by cyclic laws of *n-fold* negation: $\sim_n\varphi \Leftrightarrow \varphi$ (for any $V\varphi$)

Figure 16

An example of 4-valued polygon: Directional Logic, with its cyclic (blue) and reflective (red) negations



Values in DL
{1,1} Truth {1,0} Sub-falsity {0,1} Sub-truth {0,0} Falsity

Negations in DL
N⁺φ: strong negation
reflective (↓), 1-fold clockwise (↗), 1-fold counterclockwise (↖)

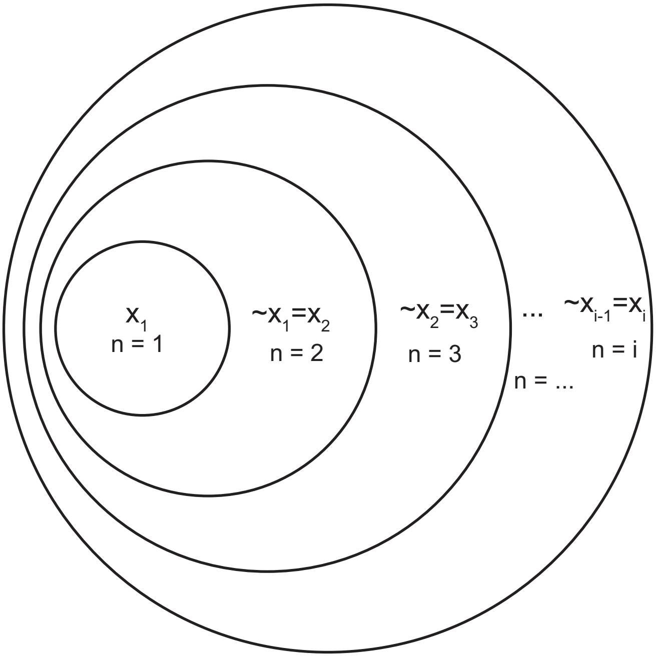
Nφ: weak negation
reflective (↑,↓ / →,←)

N→φ: initiation
n-fold clockwise cyclic (↘,↙,↖,↗,↘,...)

N← φ: finalization
counter-clockwise cyclic (↙,↘,↗,↖,↙,...)

Figure 17

Nagation in concentric spheres



In sum, the commensensical view of logical negation is closely related to the metalinguistic negation '$\notin$' (as applied to sets of truth-values) and not to the linguistic negation '~' (as applied to sets of formulas). Our usual confusion between both negations is due to our usual employment of notions like 'affirmation' and 'negation' from a bivalent point of view.

A distinction between both views of negations helps to restore the 'classical' properties of negation even in **NCL**, whenever a proper translation of many-valuations is given in internalized modal systems while following Suszko's thesis about logical values.

Appendix: Internalizing many-valued logics

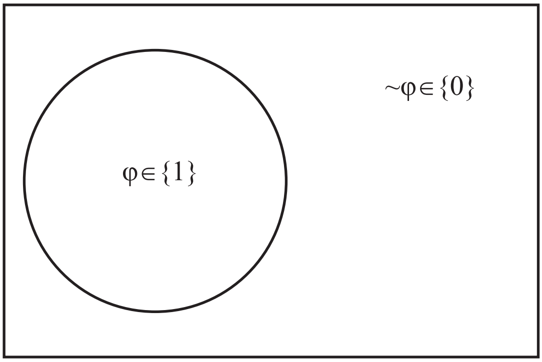
(I) From K_3 to K_{3+}

Theorem 1. Every set of theorems T_{K3} is T_{K3+}.

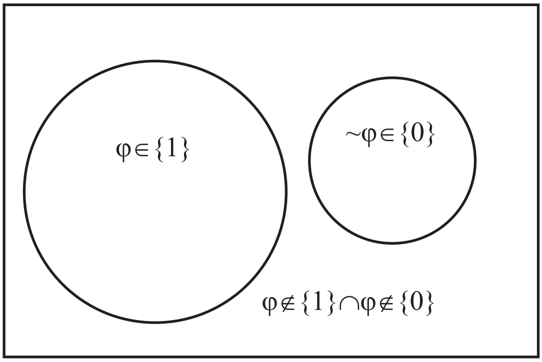
Proof: by induction upon the list of axioms in **HIL** (see below).

Figure 18

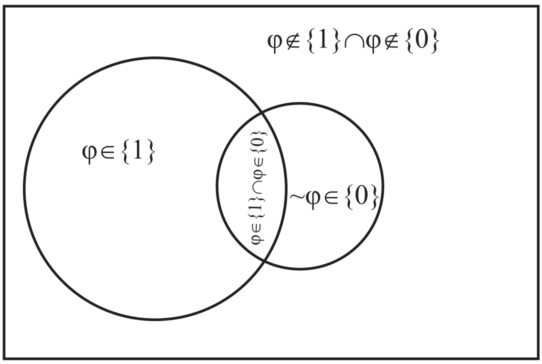
CL: complete, consistent



NCL: paracomplete, consistent



NC: paracomplete, paraconsistent



Theorem 2. $\mathbf{K}_{3+}$ is equivalent with $\mathbf{K}_3$.

Proof: Every logic L_1 is equivalent with another logic L_2 iff (1) $\mathsf{L}_{L1} = \mathsf{L}_{L2}$ and (2) $\mathsf{T}_{L1} = \mathsf{T}_{L2}$.

(1) Let L_1 be L_{HIL} and L_2 be L_{K3+}. $\mathsf{L}_{HIL} = \mathsf{L}_{K3+}$ iff wffs in $\mathbf{K}_3$ are identical with wffs in $\mathbf{K}_{3+}$ or wffs in $\mathbf{K}_3$ are translated into wffs of $\mathbf{K}_{3+}$. L_{K3+} are translations of L_{K3}, hence $\mathsf{L}_{K3} = \mathsf{L}_{K3+}$

(2) $\mathsf{T}_{K3} = \mathsf{T}_{K3+}$, by Theorem 1.

Proof of $\mathsf{T}_{K3} = \mathsf{T}_{K3+}$.
Every translation of TK_3 is TK_{3+}, as can be checked in the following:

(K1) $T\varphi \Rightarrow (T\varphi \wedge T\varphi)$

(K2) $(T\varphi \wedge T\psi) \Rightarrow (T\psi \wedge T\varphi)$

(K3) $(T\varphi \rightarrow T\psi) \Rightarrow ((T\varphi \wedge T\chi) \rightarrow (T\psi \wedge T\chi))$

(K4) $((T\varphi \rightarrow T\psi) \wedge (T\psi \rightarrow T\chi)) \Rightarrow (T\varphi \rightarrow T\chi)$

(K5) $T\psi \Rightarrow (T\varphi \rightarrow T\psi)$

(K6) $T\varphi \Rightarrow ((T\varphi \rightarrow T\psi) \rightarrow T\psi)$

(K7) $T\varphi \Rightarrow (T\varphi \vee T\psi)$

(K8) $(T\varphi \vee T\psi) \Rightarrow (T\psi \vee T\varphi)$

(K9) $((T\varphi \rightarrow T\chi) \wedge (T\psi \rightarrow T\chi)) \Rightarrow ((T\varphi \vee T\psi) \rightarrow T\chi)$

(K10) $F\varphi \Rightarrow (T\varphi \rightarrow T\psi)$

(K11) $((T\varphi \rightarrow T\psi) \Rightarrow (T\varphi \rightarrow F\psi)) \rightarrow F\varphi$

The following **HIL**-logical truths (a)-(e) are also $\mathbf{K}_{3+}$-logical truths:

(a) $(\neg\varphi \vee \psi) \Rightarrow (\varphi \rightarrow \psi)$ $\quad$ $(F\varphi \vee T\psi) \Rightarrow (T\varphi \rightarrow T\psi)$

(b) $(\varphi \rightarrow \psi) \Rightarrow \neg(\varphi \wedge \neg\psi)$ $\quad$ $(T\varphi \rightarrow T\psi) \Rightarrow F(T\varphi \wedge F\psi)$

(c) $(\varphi \vee \psi) \Rightarrow (\neg\varphi \rightarrow \psi)$ $\quad$ $(T\varphi \vee T\psi) \Rightarrow (F\varphi \rightarrow T\psi)$

(d) $(\varphi \vee \psi) \Rightarrow \neg(\neg\varphi \wedge \neg\psi)$ $\quad$ $(T\varphi \vee T\psi) \Rightarrow F(F\varphi \wedge F\psi)$

(e) $(\varphi \wedge \psi) \Rightarrow \neg(\neg\varphi \vee \neg\psi)$ $\quad$ $(T\varphi \vee T\psi) \Rightarrow F(F\varphi \vee F\psi)$

Theorem 3: Any converse of the preceding formulas is not valid in $\mathbf{K}_{3+}$.

$\mathbf{K}_{3+}$ is equivalent with $\mathbf{K}_3$, by Theorem 2. Hence if (a)*-(b)* are not logical truths in **HIL**, the same does in $\mathbf{K}_{3+}$.

Proof: by induction upon the converses of (a)-(e).

(a)* $(\varphi \rightarrow \psi) \Rightarrow (\neg\varphi \vee \psi)$	$(T\varphi \rightarrow T\psi) \Rightarrow (F\varphi \vee T\psi)$
(b)* $\neg(\varphi \wedge \neg\psi) \Rightarrow (\varphi \rightarrow \psi)$	$F(T\varphi \wedge F\psi) \Rightarrow (T\varphi \rightarrow T\psi)$
(c)* $(\neg\varphi \rightarrow \psi) \Rightarrow (\varphi \vee \psi)$	$(F\varphi \rightarrow T\psi) \Rightarrow (T\varphi \vee T\psi)$
(d)* $\neg(\neg\varphi \wedge \neg\psi) \Rightarrow (\varphi \vee \psi)$	$F(F\varphi \wedge F\psi) \Rightarrow (T\varphi \vee T\psi)$
(e)* $\neg(\neg\varphi \vee \neg\psi) \Rightarrow (\varphi \wedge \psi)$	$F(F\varphi \vee F\psi) \Rightarrow (T\varphi \wedge T\psi)$

(a)*-(e)* are of the form $(A \Rightarrow B)$, so that any of these is not a logical truth iff $A \in \{\top\}$ and $B \notin \{\top\}$, *i.e.* $B \in \{\bot\}$ for some assignment(s) of their components φ and ψ. Thus we have:

(a)* $(T\varphi \rightarrow T\psi) \in \{\top\}$ and $(F\varphi \vee T\psi) \in \{\bot\}$ with $\varphi \in \{1/2\}$ and $\psi \in \{1/2\}$ or $\psi \in \{0\}$

(b)* $(F(T\varphi \wedge F\psi)) \in \{\top\}$ and $(T\varphi \rightarrow T\psi) \in \{\bot\}$ with $\varphi \in \{1\}$ and $\psi \in \{1/2\}$

(c)* $(F\varphi \rightarrow T\psi) \in \{\top\}$ and $(T\varphi \vee T\psi) \in \{\bot\}$ with $\varphi \in \{1/2\}$ and $\psi \in \{1/2\}$ or $\psi \in \{0\}$

(d)* $F(F\varphi \wedge F\psi) \in \{\top\}$ and $(T\varphi \vee T\psi) \in \{\bot\}$ with $\varphi \in \{0\}$ and $\psi \in \{1/2\}$ or $\varphi \in \{1/2\}$ and $\psi \in \{0\}$

(e)* $(F(F\varphi \vee F\psi) \in \{\top\}$ and $(T\varphi \wedge T\psi) \in \{\bot\}$ with $\varphi \in \{1\}$ and $\psi \in \{1/2\}$ or $\varphi \in \{1/2\}$ and $\psi \in \{1\}$

Theorem 4: Any converse (a)*-(e)* is valid in $\mathbf{K}_{3+}$ when F is translated as ~T.

Proof: for any member of (a)*-(e)*, either $A \in \{\top\}$ and then $B \in \{\top\}$; or $B \in \{\bot\}$ and then $A \in \{\bot\}$. Therefore $A \Rightarrow B$ for any value of A,B in V_{K3+}.

(II) From DL to DL$^+$

Theorem 5: LEM and **LC** are theorems in DL$^+$.

Proof: τ(**LEM**) = $T(\odot\varphi) \vee NT(\odot\varphi)$ in **DL**$^+$ and τ(**LC**) = $N(T(\odot\varphi) \wedge NT(\odot\varphi))$ in **DL**$^+$. Any formula φ in **DL** is a theorem iff $\varphi \in \{1,1\}$; now

(**LEM**)$\in\{1,1\}$ and (**LC**)$\in\{1,1\}$ for any interpretation of φ in **DL**. Therefore, **LEM**$\in\{T\}$ in **DL**$^+$ and **LC**$\in\{T\}$ in **DL**$^+$.

Theorem 6: any property of negation is classicized in **DL**$^+$.

Proof: by induction on the properties of classical negation, with the translation $\odot\varphi =_{df} T\odot\varphi$ for any modal formula $\odot\varphi$ in **DL**.

(III) From PLP to PLP$_+$

Theorem 7. Every T_{PLP} is T_{PLP_+}.

Proof: by induction upon the list of axioms in **HIL** (see below).

Theorem 8. **PLP$_+$** is equivalent with **PLP**.

Proof: Every logic L_1 is equivalent with another logic L_2 iff (1) $\mathsf{L}_{L1} = \mathsf{L}_{L2}$ and (2) $\mathsf{T}_{L1} = \mathsf{T}_{L2}$.

(1) Let L_1 be L_{PLP} and L_2 be L_{PLP+}. $\mathsf{L}_{PLP+} = \mathsf{L}_{PLP+}$ iff wffs in **PLP** are identical with wffs in **PLP$_+$** or wffs in **PLP** are translated into wffs of **PLP$_+$**. L_{PLP+} are translations of L_{PLP}, hence $\mathsf{L}_{PLP} = \mathsf{L}_{PLP+}$

(2) $\mathsf{T}_{PLP} = \mathsf{T}_{PLP+}$, by Theorem 1.

Proof of $\mathsf{T}_{PLP} = \mathsf{T}_{PLP+}$.
Every translation of T_{PLP} is T_{PLP+}.

We assume that, for some plausible translations τ_1 in **PLP$_+$** of a formula $(A \Rightarrow B)$ in **PLP**, $\tau(A \Rightarrow B)\in\{\mathsf{T}\}$. It can then be verified that any of (P1)-(P11) are logical truths in **PLP$_+$**, especially the following ones:

(P1) $(\varphi \rightarrow \psi) \Rightarrow (-\psi \rightarrow -\varphi)$

(P2) $(-\varphi \wedge -\psi) \Rightarrow -(\varphi \vee \psi)$

(P3) $(-\varphi \rightarrow -\psi) \Rightarrow (\psi \rightarrow \varphi)$

(P4) $-(\varphi \vee \psi) \Rightarrow -\varphi$

(P5) $\varphi \Rightarrow --\varphi$

(P6) $--\varphi \Rightarrow \varphi$

(P7) $-\varphi \Rightarrow -(\varphi \wedge \psi)$

(P8) $-(\varphi \rightarrow \psi) \Rightarrow \varphi$

(P9) $(\varphi \wedge -\psi) \Rightarrow -(\varphi \rightarrow \psi)$

(P10) $-\varphi \Rightarrow (\varphi \rightarrow \psi)$

(P11) $(\varphi \rightarrow -\varphi) \Rightarrow -\varphi$

Theorem 9: (f)*-(i)* are not logical truths in $\mathbf{PLP}_{+}$ for some τ_1-translation.

$\mathbf{PLP}_{+}$ is equivalent with **PLP** by Theorem 2. Hence if (f)*-(i)* are not logical truths in **PLP**, the same does in $\mathbf{K}_{3+}$ under some τ_1-translation.
Proof: by induction upon the translated formulas of **PLP**.

(f)* $(\varphi \wedge -\varphi) \Rightarrow \psi$	$(V\varphi \wedge V{\sim}\varphi) \Rightarrow V\psi$
(g)* $(\varphi \wedge (-\varphi \vee \psi)) \Rightarrow \varphi$	$(V\varphi \wedge (V{\sim}\varphi \vee V\psi)) \Rightarrow V\varphi$
(h)* $((\varphi \rightarrow \psi) \wedge -\psi) \Rightarrow -\varphi$	$((V\varphi \rightarrow V\psi) \wedge V{\sim}\psi) \Rightarrow V{\sim}\varphi$
(i)* $(\varphi \rightarrow (\psi \wedge -\psi)) \Rightarrow -\varphi$	$(V\varphi \rightarrow (V\psi \wedge V{\sim}\psi)) \Rightarrow V{\sim}\varphi$

(f)*-(i)* are of the form $(A \Rightarrow B)$, so that any of these is not a logical truth iff $A \in \{\top\}$ and $B \notin \{\top\}$, *i.e.* $B \in \{\bot\}$ for some assignment(s) of their components φ and ψ. Thus we have:

(f)* $(V\varphi \rightarrow V\psi) \in \{\top\}$ and $(V\psi) \in \{\bot\}$ with $\varphi \in \{1/2\}$ and $\psi \in \{1/2\}$ or $\psi \in \{0\}$

(g)* $(V\varphi \wedge (V{\sim}\varphi \vee V\psi)) \in \{\top\}$ and $(V\psi) \in \{\bot\}$ with $\varphi \in \{1/2\}$ and $\psi \in \{0\}$

(h)* $((V\varphi \rightarrow V\psi) \wedge V{\sim}\psi) \in \{\top\}$ and $V{\sim}\varphi \in \{\bot\}$ with $\varphi \in \{0\}$ and $\psi \in \{1/2\}$

(i)* $(V\varphi \rightarrow (V\psi \wedge V{\sim}\psi)) \in \{\top\}$ and $V{\sim}\varphi \in \{\bot\}$ with $\varphi \in \{0\}$ and $\psi \in \{1/2\}$

Theorem 10: Any converse (f)*-(i)* is valid in $\mathbf{PLP}_{+}$ when $V{\sim}$ is substituted by $W{\sim}$.

Proof: for any member of (f)*-(i)*, either $A \in \{\top\}$ and then $B \in \{\top\}$; or $B \in \{\bot\}$ and then $A \in \{\bot\}$.
Therefore $A \Rightarrow B$ for any value of A,B in $\mathsf{V}_{\mathrm{PLP}_+}$.

Notes

* I am especially grateful to Konrad Turzynski for his very helpful comments on Rogowski's Directional Logic, as well as his detailed observations during private correspondences; see also his: "The temporal functors in the directional logic of Rogowski — some results." *Bulletin of the Section of Logic.* Vol. 19 (1990), pp. 80-82.

1 "The original motivation [of many-valued investigations] was an abstractly mathematical one: the pursuit of analogy and generalization. From such a perspective, many-valued logic is logic by analogy only; indeed, it is an uninterpreted theory, or abstract algebra." (Quine 1973, p. 124)

2 Łukasiewicz is the chief perpetrator of a magnificent conceptual deceit lasting out in mathematical logic to the present day." (Suszko 1977, p. 377)

3 "Among all these many-valued systems, just two are entitled to claim to some philosophical involvement: the three-valued, and the infinitely-valued one. For if any other values than 0 and 1 are read as 'the possible,' we can reasonably distinguish only two cases: either we assume that the possible does not include degrees, so that we get the three-valued system; or we assume the contrary, so that it is natural to recognize, as in the calculus of probability, that there is infinitely many degrees of the possible, what leads to the system with infinitely many values." (Łukasiewicz 1930, p. 72)

4 If excluded middle is rephrased as the view that no third value stands besides 1 and 0 in a bivalent system with $n = 2$, no fourth value stands besides 1, 0 and 1/2 in a trivalent system with $n = 3$ etc., Church claims that such a recurrent sequence cannot avoid a paradox with transfinite ordinal numbers of truth-values: $n = \omega$, $n = \omega+1$, ... As a conclusion, Church says, "this paradox, in fact, compels us to regard as *illegitimate* the consideration of this sequence as a whole." (Church 1928, p. 78)

5 "... after characterizing a number of groups by some *positive* properties, we'll reject the whole residue into a last group which won't have any special properties, except that its components won't have any of the properties that would have let them introduced into one of the first divisions. The same does for logical classification. Classical logic acknowledges a *true*, which it characterizes by a positive property, e.g. correspondence with an external reality; then the *non-true* or *false*, that is, all that doesn't have such a feature. In other places, especially in mathematics, it characterizes the false by a positive property, *i.e. contradiction*; so that the true is the non-false according to it. Now if we define the true and the false each by some positive quality, there might be a residue into our classification. Such a residue will be the tiers, which could be defined as being the neither-true-nor-false." (Barzin and Errera 1929, pp. 9-10)

6 A unary operator of negation is said to be "normal" iff if $\sim\{1/2\} = \{1/2\}$. Post's *cyclic* negation is not a normal negation, given that the negation of 1/2 yields 1 in it.

7 The pairs of elements $\{1\}$-$\{0\}$ and subsets $\{\top\}$-$\{\bot\}$ are one and the same in **CL**: obviously, the bivalent frame of classical logics entails that whatever is true is designated, and whatever is false is not-designated. Therefore, the distinction between designated and not-designated values is relevant only in **NCL** (with $n > 2$).

8 Tsuji sketches Suszko's thesis of reduction for logical matrices as follows: "In short, according to [Suszko], many-valued logics are in essence two-valued logics with many-valued *referential* (semantic) correlates; these semantic correlates are not to be confused with *truth* and *falsity*, which are after all the only possible logical values in such cases" (Tsuji 1998, p. 302)

9 Suszko meant by "true" and "false" that what is symbolized here by $\{\top\}$ (the designated value) and $\{\bot\}$ (the non-designated).

Let us give three examples of valuation with pairs of truth-values, assuming throughout that {1,1} = {1}, {1,0} = {0,1} = {1/2}, and {0,0} = {0}:

a) Let ⊕ be '~', and φ with $\{x_1,y_1\} = \{1/2\}$.

Then $\sim\varphi \in \boldsymbol{min}\{\sim x_1, \sim y_1\}$, *i.e.* $\sim\varphi \in \boldsymbol{min}(\{0,1\},\{1,0\})$, hence $\sim\varphi \in \{1/2\}$.

b) Let ⊕ be '∧', φ with $\{x_1,y_1\} = \{1/2\}$ and ψ with $\{x_2,y_2\} = \{1\}$, *i.e.* {1,1}

Then $(\varphi \wedge \psi) \in \boldsymbol{min}(\{1\wedge 1, 0\wedge 1\},\{0\wedge 1, 1\wedge 1\})$, *i.e.* $(\varphi \wedge \psi) \in \boldsymbol{min}(\{1,0\},\{0,1\})$, hence $(\varphi \wedge \psi) \in \{1/2\}$.

c) Let ⊕ be '→', φ with $\{x_1,y_1\} = \{1/2\}$, and ψ with $\{x_2,y_2\} = \{1/2\}$

Then $(\varphi \rightarrow \psi) \in \boldsymbol{min}(\{1\rightarrow 1, 0\rightarrow 0\},\{1\rightarrow 0, 0\rightarrow 1\},\{0\rightarrow 1, 1\rightarrow 0\},\{0\rightarrow 1, 0\rightarrow 1\})$; hence

$(\varphi \rightarrow \psi) \in \boldsymbol{min}(\{1,1\},\{0,1\}\},\{1,0\},\{1,1\})$, *i.e.* $(\varphi \wedge \psi) \in \{1/2\}$.

Note that the above truth-conditions don't obtain in Łukasiewicz's three-valued system, where $\{1/2 \rightarrow 1/2\} = \{1\}$ and not {1/2}.

10 See von Wright (1959)'s strong negation, the truth-conditions of which are more stringent that the classical or 'weak' negation. In a nutshell, this strong negation amounts to the modal notion of *impossibility*: "By saying that ~p means 'the proposition p is not true', it is meant 'it is impossible for p to be true'." (Heyting 1932, p. 122) Hence the intuitionistic definition of negation as absurdity: $\neg\varphi =_{df} (\varphi \rightarrow \bot)$.

11 These are "modal" in the sense that they proceed as *unary* operators upon sentences, but they could be said not to be "modal" insofar as that they proceed truth-functionally within a *functionally complete* matrix. The informal readings of modal operators in **DL** are the following: negations include, besides the primitive negation $N^{\rightarrow}$ as initiation, $N^{\leftarrow}$ as finalization, N as weak negation (it is not the case that), and N^{+} as strong negation; the other modal operators are $H^{\rightarrow}$ as protention, $H^{\leftarrow}$ as retention, and T as assertion.

12 The failure of self-identity doesn't matter for our present purposes concerning negation, but it is a special feature of **DL**: according with Hegel's dialectic, objects don't have any static properties and, hence, no identity as such.

13 A consequence of this extended definition of logical truth is the distinction between contradiction, inconsistency, and triviality in **PLP**: $(\varphi \wedge -\varphi)$ and $-(\varphi \wedge -\varphi)$ are logically true in **PLP**, but that the former contains inconsistent theorems φ and –φ does not entail that anything is true in **PLP** (according to the classical law *ex contradictio sequitur quodlibet*: $(\varphi \wedge -\varphi) \Rightarrow \psi$, for any ψ). Triviality thus fails with $\psi \in \{0\}$.

14 A case for non-classical negations is given by Dutilh-Novaes (2003), for instance. The writer rightly insists that the problem with paraconsistent logics is not so much about non-classical negation in general than about contradiction: there *are* some non-classical negations in logic, but how to accept a negation N with $(\varphi \wedge N\varphi) \in \{1\}$?

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