

# Independence of the grossone-based infinity methodology from non-standard analysis and comments upon logical fallacies in some texts asserting the opposite

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## Abstract

This paper considers non-standard analysis and a recently introduced computational methodology based on the notion of  $\textcircled{1}$  (this symbol is called *grossone*). The latter approach was developed with the intention to allow one to work with infinities and infinitesimals numerically in a unique computational framework and in all the situations requiring these notions. Non-standard analysis is a classical purely symbolic technique that works with ultrafilters, external and internal sets, standard and non-standard numbers, etc. In its turn, the  $\textcircled{1}$ -based methodology does not use any of these notions and proposes a more physical treatment of mathematical objects separating the objects from tools used to study them. It both offers a possibility to create new numerical methods using infinities and infinitesimals in floating-point computations and allows one to study certain mathematical objects dealing with infinity more accurately than it is done traditionally. In these notes, we explain that even though both methodologies deal with infinities and infinitesimals, they are independent and represent two different philosophies of Mathematics that are not in a conflict. It is proved that texts [14, 15, 19] asserting that the  $\textcircled{1}$ -based methodology is a part of non-standard analysis unfortunately contain several logical fallacies. Their attempt to show that the  $\textcircled{1}$ -based methodology can be formalized within non-standard analysis is similar to trying to show that constructivism can be reduced to the traditional Mathematics.

## 1 Introduction

The  $\textcircled{1}$ -based infinity theory has been introduced in [32, 34, 39] (see also recent surveys [45, 47]) where this new numeral,  $\textcircled{1}$ , called *grossone*, and the related computational methodology have been described. This new way of looking at infinity is not related either to Cantor's cardinals and ordinals or to non-standard analysis of Robinson or to Levi-Civita field. As is well known, there exists a variety of mathematics: traditional, formalistic, intuitionistic, and other philosophies of mathematics (see,

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e.g., [20, 21, 53]). The  $\textcircled{1}$ -based methodology proposes one more way to look at Mathematics and Computer Science following the example of Physics where tools used to observe objects limit our possibilities of the observation.

This methodological proposal has attracted an appreciable amount of attention both of the pure and applied mathematical communities. A number of papers studying connections of the  $\textcircled{1}$ -based approach to the historical panorama of ideas dealing with infinities and infinitesimals (see [22–24, 26, 49]) have been published. In particular, metamathematical investigations on the new theory and its consistency can be found in [23]. A number of reviews were published in *MIT Technology Review* and in international scientific journals (see [1, 27, 28, 54, 55]), etc. The author received several international prizes and other distinctions<sup>1</sup> for these results and was invited to present them as plenary lectures and tutorials at more than 60 international congresses.

The  $\textcircled{1}$ -based methodology has been successfully applied in several areas of Mathematics and Computer Science: single and multiple criteria optimization and ill-conditioning (see [5, 10–12, 51]), cellular automata (see [7, 8]), Euclidean and hyperbolic geometry (see [25]), percolation (see [18, 56]), fractals (see [4, 33, 35, 42, 46, 56]), infinite series and the Riemann zeta function (see [36, 41, 45, 47, 48, 57]), the first Hilbert problem and supertasks (see [30, 38, 47]), Turing machines and probability (see [29, 47, 49, 50]), numerical differentiation and solution of ordinary differential equations (see [2, 40, 43, 52]), etc.

However, in the paper [14] published in *Foundations of Science*<sup>2</sup> and in the papers [15, 19] published in two journals printed by the Institute where the authors of [15, 19] work, there are numerous attacks on the  $\textcircled{1}$ -based methodology and its author. The paper [15] announces ‘a trivial formalization of the theory of grossone’ using non-standard analysis. The paper [19] does not have any mathematical substance (not a single formula) and consists of insults and accusations starting from the title. The paper [14] in all seriousness attacks a playful note [44] regarding several ways of counting Olympic medals won by different countries and appeals to [15, 19] trying again to show that the  $\textcircled{1}$ -based methodology can be reduced to non-standard analysis.

Before going to technicalities, let us compare goals of the two methodologies. The successfully reached goal of the creator of non-standard analysis, Abraham Robinson, was to reformulate classical analysis and show that ideas of Leibniz can be put in a form satisfying the requirements of rigor of the  $XX^{\text{th}}$  century. In fact, he wrote in paragraph 1.1 of his book [31]: ‘It is shown in this book that Leibniz’s ideas can be fully vindicated and that they lead to a novel and fruitful approach to *classical analysis* and to many other branches of Mathematics’ (italics mine). In fact, classical analysis reformulated using the language of Robinson was then used to address such areas as topology, probability, etc. showing a notable potential of this approach. However, as Davis writes in the introduction to his book ‘Applied Nonstandard Analysis’ (see [9]), ‘Nonstandard analysis is a technique rather than a subject. Aside from theorems that

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<sup>1</sup>Khwarismi International Award, assigned by The Ministry of Science and Technology of Iran, 2017; Honorary Fellowship, the highest distinction of the European Society of Computational Methods in Sciences, Engineering and Technology, 2015; Outstanding Achievement Award from the 2015 World Congress in Computer Science, Computer Engineering, and Applied Computing, USA; Degree of Honorary Doctor from Glushkov Institute of Cybernetics of National Academy of Sciences of Ukraine, 2013; Pythagoras International Prize in Mathematics, Italy, assigned by the city of Crotona (where Pythagoras lived and founded his famous scientific school) and the Calabria Region under the high patronage of the President of the Italian Republic, Ministry of Cultural Assets and Activities and the Ministry of Education, University and Research, 2010; Lagrange Lecture, Turin University, Italy, 2010; etc.

<sup>2</sup>In Section 8, the authors of [14] inform the reader that before appearing in *Foundations of Science* their paper has been 5 times rejected by *The Mathematical Intelligencer*.

tell us that nonstandard notions are equivalent to corresponding standard notions, all the results we obtain can be proved by standard methods.’

The goals of the  $\textcircled{1}$ -based theory are different. The first goal is to bridge the gap between modern Physics and Mathematics at least partially, insofar as it concerns the separation of an object and a tool used to observe it existing in Physics and often not present in Mathematics. The difference between numbers and numerals is emphasized and it is shown that numeral systems limit our capabilities in describing numbers and other mathematical objects<sup>3</sup>. Traditionally, various kinds of infinity are perceived by people as *distinct* mathematical objects. The  $\textcircled{1}$ -based methodology shows that this is not the case. It argues that the objects – infinite numbers – are *the same*, they are just *viewed in different ways* by different theories and with different accuracies. For example, it is shown that in analysis and set theory,  $\infty$  and  $\aleph_0$  are not two different objects, they are two aggregative images of infinite quantities that people see through different ‘lenses’ provided by analysis and set theory.

The second goal of the  $\textcircled{1}$ -based methodology is to describe a numeral system expressing infinities and infinitesimals that can be used in *all* occasions where we need to work with them. This is done in order to have a situation similar to what we have with finite numbers where numerals expressing them can be used in all the occasions we need finite quantities. The viewpoint on infinities and infinitesimals expressed in the  $\textcircled{1}$ -based methodology does not require a knowledge of cardinals, ordinals, ultrafilters, standard and non-standard numbers, internal and external sets, etc. It avoids a number of set theoretical paradoxes related to infinity and, in general, introduces a significant simplification in several fields of mathematics related to infinities and infinitesimals.

Finally, the third goal consists of describing a computational methodology using the introduced numeral system and a computational device called the Infinity Computer (patented in USA and EU, see [37]) working *numerically*<sup>4</sup> with finite, infinite, and infinitesimal numbers in a unique computational framework and in accordance with Euclid’s Common Notion no. 5 ‘The whole is greater than the part’. This gives the possibility to propose numerical algorithms of a new type working *in the same way* with numbers that can have different infinite, finite, and infinitesimal parts. Analogously, traditional computers do not consider as special cases numbers having thousands and numbers hundreds of units. They elaborate all of them in the same way.

Even from this brief introduction it can be seen that the goals and the methodological platforms of the two approaches are very different. In fact, non-standard analysis is a purely symbolic technique that works with ultrafilters, external and internal sets, standard and non-standard numbers, etc. whereas the  $\textcircled{1}$ -based theory does not use any of these notions, is focused on numerical computations, separates mathematical objects from tools used to study them, and takes into account that, as in Physics, the tools used limit our possibilities of the observation and determine the accuracy of the observed

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<sup>3</sup>A *numeral* is a symbol (or a group of symbols) that represents a *number* that is a concept. The same number can be represented by different numerals. For example, symbols ‘4’, ‘four’, ‘IIII’, and ‘IV’ are different numerals, but they all represent the same number.

<sup>4</sup>Recall that numerical computations are performed with floating-point numbers that can be stored in a computer memory. Since the memory is limited, mantissa and exponent of these numbers can assume only certain values and, therefore, the quantity and the form of numerals that can be used to express floating-point numbers are fixed. Due to this fact, approximations are required during computations with them because an arithmetic operation with two floating-point numbers usually produces a result that is not a floating-point number and, as a consequence, this result should be approximated by a floating-point number. In their turn, symbolic computations are the exact algebraic manipulations with mathematical expressions containing *variables that have not any given value*. These manipulations are more computationally expensive than numerical computations and only relatively simple codes can be elaborated in this way.

results. Therefore, the aim preset by the authors of [14], i.e., to prove that the  $\textcircled{1}$ -based methodology is a part of non-standard analysis, is doomed to failure. Such attempt is similar to trying to show that constructivism can be reduced to traditional Mathematics.

Let us now describe the structure of this paper. First, several logical fallacies<sup>5</sup> present in the papers [14, 15, 19] are listed and analyzed in Section 2. The subsequent Section is dedicated to some general considerations comparing the two methodologies under scrutiny. Section 4 shows that weak numeral systems limit our capabilities in measuring infinite sets. Traditional and  $\textcircled{1}$ -based numeral systems are compared. Then, since the authors of [14] criticize not only the  $\textcircled{1}$ -based methodology but also a lexicographic rank described in [44] and propose their own algorithm to count Olympic medals, the two methods are compared briefly in Section 5. Finally, Section 6 concludes the paper.

## 2 Logical fallacies committed in texts [14, 15, 19]

In the following, five main logical fallacies present in [14, 15, 19] are discussed.

**1. Confirmation Bias.** (*We have a proclivity to see and agree with ideas that fit our preconceptions, and to ignore and dismiss information that conflicts with them*<sup>6</sup>). The authors of [14, 15, 19] have an expertise in non-standard analysis and try to defend it from the  $\textcircled{1}$ -based theory saying many times that ‘Sergeyev opposes his system to nonstandard analysis’ (see, e.g., [15], page 1). In so doing they seem not to notice that in his papers, Sergeyev does not attack non-standard analysis. On the contrary, he repeats again and again that his approach does not oppose to any existing methodology, it is just one more view on Mathematics.

The foundational platform of the  $\textcircled{1}$ -based approach consists of three methodological postulates and an axiom (the Infinite Unit Axiom describing properties of grossone) that is added to axioms for real numbers (see Appendix). In order to prove that Sergeyev’s methodology is a part of non-standard analysis it is necessary to show that all the postulates and the axiom can be modeled using non-standard analysis. However, Postulates 1 and 2 are not mentioned at all in [14, 15]. The paper [19] lists the postulates but the only comment related to them is the ironic exclamation ‘The scientific depth of Sergeyev’s postulates transpires’. There is no other explanation showing that the postulates can be (or were) considered by non-standard analysis. The Infinite Unit Axiom is discussed in [15], but only partially. The authors of [15] write explicitly in page 2 that they do not accept its main part (this point will be debated below in detail). The above said suffices to affirm that papers [14, 15, 19] have not proved that Sergeyev’s methodology is a part of non-standard analysis. However, there are other fallacies to add.

**2. Personal Incredulity.** (*Because you found something difficult to understand, or are unaware of how it works, you made out like it is probably not true*<sup>7</sup>). In Section 2 of [14], its authors write: ‘Sergeyev’s attempted definition of  $\textcircled{1}$  as somehow the number of elements of the set  $\mathbb{N}$  contradicts other passages where  $\textcircled{1}$  is included as a member of  $\mathbb{N}$ ’. This personal incredulity is a consequence of the previous fallacy.

<sup>5</sup>A logical fallacy (see [16]) is a flaw in reasoning. Logical fallacies are like tricks or illusions of thought, and they are often very sneakily used by politicians and the media to manipulate people. This and the following footnotes explaining the meaning of fallacies were taken from [16].

<sup>6</sup>You could say that this is the mother of all biases, as it affects so much of our thinking through motivated reasoning.

<sup>7</sup>Complex subjects require some amount of understanding before one is able to make an informed judgement about the subject at hand; this fallacy is usually used in place of that understanding.

The authors identify the set of natural numbers with its representation offered by non-standard analysis. This representation does not consider an entity being the number of elements of  $\mathbb{N}$ . Then, they apply non-standard analysis to the  $\textcircled{1}$ -based theory without noticing that methodological platforms of the two approaches are different thus committing a fallacy. Both approaches observe the same set but do it using different mathematical tools (in particular, different notations) and do not contradict one another. The attempt to apply non-standard analysis tools in the  $\textcircled{1}$ -based framework is a fallacy similar to appealing to proofs by contradiction in the frame of constructivism.

The Infinite Unit Axiom introduces  $\textcircled{1}$  as the number of elements of the set  $\mathbb{N}$  (see Appendix and [47] for a detailed discussion). This is performed by extrapolating from finite to infinite the idea that  $n$  is both the number of elements of the set  $\{1, 2, 3, \dots, n-1, n\}$  and the last element of this set. The fact that non-standard analysis does not offer this vision is one more confirmation that the two theories are independent.

Another difficulty that the authors of [14] confess in Section 4.1 in their appeal to works of Shamseddine (who, by the way, twice delivered plenary lectures at conferences organized by Sergeev) is that they do not understand the difference between numbers and numerals and between symbolic computations and numerical ones. Even though Sergeev in his papers dedicated a lot of space to these topics, the above confession suggests that it is necessary to return to them. These issues will be re-discussed one more time below.

**3. The Fallacy Fallacy.** (*Presuming a claim to be necessarily wrong because a fallacy has been committed*<sup>8</sup>). Rough expository of Sergeev's concepts in discussion sections in [14, 15, 19] are seized upon and used to argue against grossone. At the same time, the paper [14] does not contain a definition of grossone anywhere in it, nor even its properties. It is never explained what exactly is being attacked. A similar situation holds in [15] where the Infinite Unit Axiom is mentioned but methodological postulates are not provided. Vice versa, [19] writes about postulates but does not mention the axiom. The impression is that the authors of [14, 15, 19] avoid to present the  $\textcircled{1}$ -based theory in its complete form in order to make their arguments more convincing.

**4. Appeal to Nature.** (*Making the argument that because something is 'natural' it is therefore valid, justified, inevitable, good, or ideal*<sup>9</sup>). The authors write in [15] that they do not accept that  $\textcircled{1}$  is the number of elements of the set  $\mathbb{N}$  'if for no other reason than the fact that the set  $\mathbb{N}$  of naturals (in the popular sense of this fundamental notion) has no greatest element'. Can 'the popular sense' be the reason to refuse a mathematical definition?

**5. Strawman.** (*Misrepresenting someone's argument to make it easier to attack*<sup>10</sup>). This is the main fallacy. The paper [15] announces 'a trivial formalization of the theory of grossone' using non-standard analysis. However, instead of working with  $\textcircled{1}$  introduced by Sergeev as the number of elements of the set  $\mathbb{N}$ , the authors of [15] introduce a non-standard object, declare that *it is not* the number of elements of the set  $\mathbb{N}$ , call the introduced object *grossone*, use the symbol  $\textcircled{1}$  to indicate it, and then proceed to rip it down. How can this operation be called 'formalization of the theory of grossone'? Certainly, one can define a nonstandard object and study it but

<sup>8</sup>It is entirely possible to make a claim that is false yet argue with logical coherency for that claim, just as it is possible to make a claim that is true and justify it with various fallacies and poor arguments.

<sup>9</sup>Many 'natural' things are also considered good, and this can bias our thinking; but naturalness itself does not make something good or bad.

<sup>10</sup>By exaggerating, misrepresenting, or just completely fabricating someone's argument, it is much easier to present your own position as being reasonable, but this kind of deceitfulness serves to undermine rational debate.

how is this related to what Sergeyev does?

### 3 Comments emphasizing some other differences between the $\mathbb{1}$ -based methodology and non-standard analysis

Let us start this section by discussing the object proposed to be taken as *grossone* in [15] and, in addition to the arguments provided above that are already sufficient to show the independence of the two methodologies, explain why this concrete proposal does not work. The authors of [15] write at page 1:

‘Fix an arbitrary infinitely large natural  $\nu$  and denote its factorial by  $\mathbb{1}$ :

$$\mathbb{1} = \nu!, \text{ where } \nu \in \mathbb{N}, \nu \approx \infty. \quad (1)$$

In order to avoid the confusion present in papers [14, 15, 19], let us use hereinafter the symbol  $\mathbb{1}$  for the original *grossone* introduced by Sergeyev and the symbol  $n(\nu)$  for the object used by the authors of [15], i.e., instead of (1) we write

$$n(\nu) = \nu!, \text{ where } \nu \in \mathbb{N}, \nu \approx \infty. \quad (2)$$

The authors of [15] correctly affirm that  $n(\nu)$  shares with  $\mathbb{1}$  *Infinity* and *Identity* properties of the Infinite Unit Axiom (see Appendix) and  $n(\nu)$  is divisible by all finite numbers as  $\mathbb{1}$ . However, this is not sufficient to prove that  $n(\nu) = \mathbb{1}$ . In fact, there are several problems related to the introduction of  $n(\nu)$  in (2) as a candidate for  $\mathbb{1}$ .

First, since  $\nu$  is *an arbitrary* infinitely large non-standard natural number,  $n(\nu)$  is not the number of elements of  $\mathbb{N}$  (in fact, at page 2 the authors of [15] say this explicitly). However, it can be seen from *Divisibility* part of the Infinite Unit Axiom that, by taking  $k = n = 1$  in (13) (see Appendix) we obtain that the number of elements of  $\mathbb{N}$  is equal to  $\mathbb{1}$ . Since  $n(\nu)$  does not satisfy this condition, it cannot be chosen as  $\mathbb{1}$ .

Second, the expression  $\nu \approx \infty$  is meaningless in the  $\mathbb{1}$ -based framework since the traditional symbols  $\infty, \aleph_0, \omega, \aleph_1$ , etc. are not defined in it and, as a result, cannot be used.

The third difficulty regards the words ‘Fix an arbitrary infinitely large natural  $\nu$ ’. The authors of [15] do not explain how they intend to execute this operation of fixing. In the  $\mathbb{1}$ -based methodology, *fixing a variable* means that a value is assigned to it, as it happens with variables in the finite case. For instance, if we consider a finite  $n$  then we can use, e.g., the numeral 34 and assign this value to it, i.e.,  $n = 34$ . Analogously, for an infinite  $n$ ,  $\mathbb{1}$ -based numerals can be used and we can assign values to  $n$  using these numerals, e.g.,  $n = \mathbb{1} - 1$  or  $n = 3\mathbb{1}^2$ , fixing so  $n$ .

In contrast, if we consider a non-standard infinite  $\nu$  then it is not clear which numerals consisting of a finite number of symbols<sup>11</sup> can be used to assign a concrete value to  $\nu$  since non-standard analysis does not provide numeral systems that can be used for this purpose. In fact, all computations in non-standard analysis theories are executed in a symbolic way (recall footnote 4) using a generic infinite non-standard variable, e.g.,  $\nu$ , and there is no possibility to assign a value to  $\nu$ . Among other things, this means that non-standard analysis is able to describe only those properties of infinite numbers that are shared by all of them, since it does not provide any instrument that would allow one

<sup>11</sup>Notice that the finiteness of the number of symbols in the numeral is necessary for executing practical computations since we should be able to write down and store values we execute operations with.

to individualize an infinite number, to distinguish it from another infinite number, and to compare it with other infinite numbers. For instance, if one considers two infinite non-standard numbers  $\nu$  and  $\xi$ , where  $\nu$  is not expressed in terms of  $\xi$  (and vice versa), then it is not clear how to compare them because of absence in non-standard analysis of numeral systems that can express different values of non-standard infinities. Notice that when we work with finite quantities, then we can compare some  $n$  and  $k$  if they assume numerical values, e.g.,  $k = 25$  and  $n = 78$ . Then, by using rules of the numeral system the symbols 25 and 78 belong to, we can compute that  $n > k$ . The same possibility is provided in the  $\mathbb{1}$ -based framework. For example, if  $k = \mathbb{1} - 1$  and  $n = 3\mathbb{1}^2$  then we can compute

$$n - k = 3\mathbb{1}^2 - (\mathbb{1} - 1) = \mathbb{1}(3\mathbb{1} - 1) + 1 > 0$$

and conclude that  $n > k$ .

It should be noticed here that this kind of difficulties present in non-standard analysis exists also in approaches dealing with Levi-Chivita field, since they work with a *generic* infinitesimal  $\varepsilon$ . Again it is not clear which numerals can be used to assign a value to  $\varepsilon$  and to write  $\varepsilon = \dots$  (for instance, the nice web-based calculator *Inf* mentioned in [19] and offered in [6] operates with a generic symbol  $d$  and there is no possibility to assign a value to  $d$ ). Moreover, approaches of this kind leave unclear such issues as, e.g., whether the infinite  $1/\varepsilon$  is integer or not or whether  $1/\varepsilon$  is the number of elements of a concrete infinite set. The absence of numeral systems allowing one to express quantities in non-standard analysis and Levi-Civita field leads to a symbolic character of both theories.

One more difference between  $n(\nu)$  and  $\mathbb{1}$  can be indicated with respect to infinite sequences. An *infinite sequence*  $\{a_n\}, a_n \in A, n \in \mathbb{N}$ , is a function having as the domain the set of natural numbers,  $\mathbb{N}$ , and as the codomain a set  $A$ . A *subsequence* is obtained from a sequence by deleting some (or possibly none) of its elements. In a sequence  $a_1, a_2, \dots, a_{n-1}, a_n$  the number  $n$  is the number of elements of the sequence. Traditionally, only finite values of  $n$  are considered. Grossone-based numerals give us the possibility to observe infinite numbers and, therefore, to see not only the initial elements of an infinite sequence  $a_1, a_2, \dots$  but also its final part  $\dots, a_{n-1}, a_n$  where  $n$  can assume different infinite values. In other words,  $\mathbb{1}$ -based theory allows one to distinguish infinite sequences of different lengths. For instance, the following two infinite sequences

$$\begin{aligned} \{a_n\} &= \{5, 10, \dots, 5(\mathbb{1} - 1), 5\mathbb{1}\}, \\ \{b_n\} &= \{5, 10, \dots, 5(\frac{2\mathbb{1}}{5} - 1), 5 \cdot \frac{2\mathbb{1}}{5}\}, \end{aligned}$$

have the same general element  $a_n = b_n = 5n$  but they are different because the first sequence has  $\mathbb{1}$  elements and the second sequence has  $\frac{2\mathbb{1}}{5}$  elements.

Notice that since the set of natural numbers,  $\mathbb{N}$ , has  $\mathbb{1}$  elements, any sequence cannot have more than  $\mathbb{1}$  elements (see [47] for a detailed discussion). This fact is very important in several research areas, in particular, in that of Turing machines (see [49]) where it allows one to distinguish infinite tapes of different lengths. The possibility to establish the maximal possible number of elements in a sequence is not provided either by  $n(\nu)$  or by non-standard analysis, in general.

## 4 Numbers, numerals and measuring infinite sets

Another important difference in comparison to non-standard analysis consists of the fact that the  $\textcircled{1}$ -based methodology through its postulates 1 and 2 emphasizes the importance of numeral systems being among our tools used to observe mathematical objects. Due to postulate 1, the number of symbols we can use to write down numbers is finite. Therefore, the choice of symbols and their meaning limit our capabilities of observation of mathematical objects and influence theoretical considerations, as well. The separation of numbers (concepts) and numerals (symbols used to represent numbers) was not discussed in depth traditionally and is not discussed in non-standard analysis at all.

As an example, let us recall the Roman numeral system. It is not able to express zero and negative numbers and such expressions as III – VIII or X – X are indeterminate forms in this numeral system. As a result, before the appearance of positional systems and the invention of zero (the second event was several hundred years later with respect to the first one) mathematicians were not able to create theorems involving zero and negative numbers and to execute computations with them. The appearance of the positional numeral system not only has allowed people to execute new operations but has led to new theoretical results, as well. Thus, numeral systems not only limit us in practical computations, they induce constraints on reasoning in theoretical considerations, as well.

Even a more significant, in the context of infinity, example of limitations induced by numeral systems is provided by the numeral system of Pirahã, a tribe living in Amazonia nowadays and described in *Science* in 2004 (see [13]). These people use an extremely simple numeral system for counting: one, two, many. For Pirahã, all quantities larger than two are just ‘many’ and such operations as 2+2 and 2+1 give the same result, i.e., ‘many’. Using their weak numeral system Pirahã are not able to see, for instance, numbers 3, 4, and 5, to execute arithmetical operations with them, and, in general, to say anything about these numbers because in their language there are neither words nor concepts for that<sup>12</sup>.

The poverty of the numeral system of Pirahã leads also to the following results

$$\text{‘many’} + 1 = \text{‘many’}, \quad \text{‘many’} + 2 = \text{‘many’}, \quad \text{‘many’} + \text{‘many’} = \text{‘many’} \quad (3)$$

that are crucial for changing our outlook on infinity. In fact, by changing in these relations ‘many’ with  $\infty$  we get relations used to work with infinity in the traditional calculus

$$\infty + 1 = \infty, \quad \infty + 2 = \infty, \quad \infty + \infty = \infty. \quad (4)$$

Analogously, if we consider Cantor’s cardinals (where, as usual, numeral  $\aleph_0$  is used for cardinality of denumerable sets and numeral  $\mathfrak{c}$  for cardinality of the continuum) we have similar relations

$$\aleph_0 + 1 = \aleph_0, \quad \aleph_0 + 2 = \aleph_0, \quad \aleph_0 + \aleph_0 = \aleph_0, \quad (5)$$

$$\mathfrak{c} + 1 = \mathfrak{c}, \quad \mathfrak{c} + 2 = \mathfrak{c}, \quad \mathfrak{c} + \mathfrak{c} = \mathfrak{c}. \quad (6)$$

This comparison suggests that our difficulty in working with infinity is not a consequence of the *nature* of infinity but is a result of *weak numeral systems* having too little

<sup>12</sup>It should be noticed that the astonishing numeral system of Pirahã is not an isolated example of this way of counting. In fact, the same counting system, one, two, many, is used by the Warlpiri people, aborigines living in the Northern Territory of Australia (see [3]).



numerals to express the multitude of infinite numbers. As the possibility to have numerals allowing us to express numbers 4, 5, etc. gives the opportunity to execute more precise computations with finite numbers in comparison with (3), the introduction of the variety of ①-based numerals gives the possibility to execute more precise w.r.t. (4)–(6) computations with infinities (and infinitesimals). Notice that this is not a new situation when the introduction of new symbols and respective concepts allowed mathematicians to have a progress in certain directions (it is sufficient to mention  $0$ ,  $\infty$ ,  $i$ ,  $e$ , and  $\pi$ ).

The separation of numbers from their representation is stressed in the ①-based theory whereas traditionally mathematicians do not pay a particular attention to this issue. For instance, very often mathematicians speak directly about real numbers and do not specify which numeral systems are used to represent them. This can lead to ambiguity in some cases.

We illustrate this statement by considering the following simple phrase ‘Let us consider all  $x \in [1, 2]$ ’. For Pirahã, *all numbers* are just 1 and 2. For people who do not know irrational numbers (or do not accept their existence) *all numbers* are fractions  $\frac{p}{q}$  where  $p$  and  $q$  can be expressed in a numeral system they know. If both  $p$  and  $q$  can assume values 1 and 2 (as it happens for Pirahã), *all numbers* in this case are: 1,  $1 + \frac{1}{2}$ , and 2. For persons knowing positional numeral systems *all numbers* are those numbers that can be written in a positional system. Thus, in different historical periods and in different cultures the phrase ‘Let us consider all  $x \in [1, 2]$ ’ has different meanings. As a result, without fixing the numeral system we use to express numbers we cannot fix the numbers we deal with and an ambiguity holds.

In contrast, in Physics, assertions are made with respect to what is visible at a ‘lens’ and not about the object located behind the lens. This is done since observations can be performed using different instruments and without specifying the instruments assertions regarding results have no meaning. For instance, the question: ‘What do you see in this direction?’ is meaningless without indication a tool used for the observation. In fact, by eye the observer will see certain things, by microscope other things, by telescope again other things, etc.

The ①-based theory follows physicists and does not talk, e.g., about the set of real numbers at the interval  $[1, 2)$  but about the set of real numbers expressible in a fixed numeral system chosen to represent real numbers over  $[1, 2)$ . Notice that Cantor’s cardinals do not allow us to distinguish the quantities of real numbers over  $[1, 2)$  written in the binary and in the decimal systems providing the same answer: both sets have the cardinality of continuum. As Table 1 illustrates (see [47] for its detailed explanation), the ①-based methodology allows us to register this difference. In fact, the number of numerals expressing real numbers over  $[1, 2)$  in the binary positional system is equal to  $2^{\textcircled{1}}$  and the number of numerals expressing real numbers over  $[1, 2)$  in the decimal positional system is equal to  $10^{\textcircled{1}}$ , where the number  $10^{\textcircled{1}}$  is infinitely larger than  $2^{\textcircled{1}}$ . Moreover, sets of measure zero are not present in the ①-based framework and the accuracy of measuring infinite sets is equal to one element (i.e., if an infinite set  $A$  has  $k$  elements where  $k$  is expressed using ①-based numerals then exclusion/addition of one element from/to  $A$  gives the resulting set having exactly  $k - 1/k + 1$  elements, exactly as it happens with finite sets). This accuracy is significantly higher than measuring executed by Cantor’s cardinals. For instance, for numerable sets one can see from Table 1 that excluding one number from  $\mathbb{Z}$  can be registered by ①-based numerals. Analogously, it can be seen from the fourth and the third lines from the end in Table 1 that the number of numerals expressed in the binary numeral system and counted over

Table 1: Cardinalities and the number of elements of some infinite sets.

Description of infinite sets	Cantor's cardinalities	Number of elements
the set of natural numbers $\mathbb{N}$	countable, $\aleph_0$	$\textcircled{1}$
$\mathbb{N} \setminus \{3, 5, 10, 23, 114\}$	countable, $\aleph_0$	$\textcircled{1}-5$
the set of even numbers $\mathbb{E}$ (the set of odd numbers $\textcircled{O}$ )	countable, $\aleph_0$	$\frac{\textcircled{1}}{2}$
the set of integers $\mathbb{Z}$	countable, $\aleph_0$	$2\textcircled{1}+1$
$\mathbb{Z} \setminus \{0\}$	countable, $\aleph_0$	$2\textcircled{1}$
squares of natural numbers $\mathbb{G} = \{x : x = n^2, x \in \mathbb{N}, n \in \mathbb{N}\}$	countable, $\aleph_0$	$[\sqrt{\textcircled{1}}]$
pairs of natural numbers $\mathbb{P} = \{(p, q) : p \in \mathbb{N}, q \in \mathbb{N}\}$	countable, $\aleph_0$	$\textcircled{1}^2$
the set of numerals $\mathbb{Q}_1 = \{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}$	countable, $\aleph_0$	$4\textcircled{1}^2 + 2\textcircled{1}$
the set of numerals $\mathbb{Q}_2 = \{0, -\frac{p}{q}, \frac{p}{q} : p \in \mathbb{N}, q \in \mathbb{N}\}$	countable, $\aleph_0$	$2\textcircled{1}^2 + 1$
the power set of the set of natural numbers $\mathbb{N}$	continuum, $\mathfrak{c}$	$2^{\textcircled{1}}$
the power set of the set of even numbers $\mathbb{E}$	continuum, $\mathfrak{c}$	$2^{0.5\textcircled{1}}$
the power set of the set of integers $\mathbb{Z}$	continuum, $\mathfrak{c}$	$2^{2\textcircled{1}+1}$
the power set of the set of numerals $\mathbb{Q}_1$	continuum, $\mathfrak{c}$	$2^{4\textcircled{1}^2 + 2\textcircled{1}}$
the power set of the set of numerals $\mathbb{Q}_2$	continuum, $\mathfrak{c}$	$2^{2\textcircled{1}^2 + 1}$
numbers $x \in [1, 2)$ expressible in the binary numeral system	continuum, $\mathfrak{c}$	$2^{\textcircled{1}}$
numbers $x \in [1, 2]$ expressible in the binary numeral system	continuum, $\mathfrak{c}$	$2^{\textcircled{1}} + 1$
numbers $x \in [1, 2)$ expressible in the decimal numeral system	continuum, $\mathfrak{c}$	$10^{\textcircled{1}}$
numbers $x \in [0, 2)$ expressible in the decimal numeral system	continuum, $\mathfrak{c}$	$2 \cdot 10^{\textcircled{1}}$

$[1, 2)$  and  $[1, 2]$  are different.

To stress again the difference between numbers and numerals, notice that the sets  $\mathbb{Q}_1$  and  $\mathbb{Q}_2$  in Table 1 are sets of different rational numerals and not the sets of different rational numbers. For example, in the numeral system  $\mathbb{Q}_1$  the number 0 can be expressed by  $2\textcircled{1}$  different numerals

$$\frac{0}{-\textcircled{1}}, \frac{0}{-\textcircled{1}+1}, \frac{0}{-\textcircled{1}+2}, \dots, \frac{0}{-2}, \frac{0}{-1}, \frac{0}{1}, \frac{0}{2}, \dots, \frac{0}{\textcircled{1}-2}, \frac{0}{\textcircled{1}-1}, \frac{0}{\textcircled{1}}.$$

Analogously, e.g., such numerals as  $\frac{-1}{-2}$ ,  $\frac{1}{2}$ , and  $\frac{2}{4}$  representing the same number have been calculated as three different numerals.

The provided comparison between the traditional way to measure infinite sets and the one provided by the  $\textcircled{1}$ -based numerals shows that the latter allow us to measure infinite sets more accurately and gives one more argument to the impossibility of formalization of the  $\textcircled{1}$ -based methodology within non-standard analysis.

There are other arguments to expose but what has been said is already sufficient. The recent survey [47] provides more information and shows that  $\textcircled{1}$ -based approach

can be successfully used in a variety of occasions where we need infinite and infinitesimal quantities<sup>13</sup>. On the one hand,  $\textcircled{1}$ -based numerals provide a higher accuracy of results with respect to traditional symbols such as  $\infty, \aleph_0, \omega$ , etc. and, on the other hand, eliminate the necessity to use different symbols in different situations related to infinity. The fact that the same symbols expressing infinities and infinitesimals can be used in all the occasions we need them creates a unique framework with the finite case, since the same numerals expressing finite quantities are used in all the occasions we need them, as well.

In concluding this section, let us express a general doubt regarding the meaning of attempts to reduce the  $\textcircled{1}$ -based methodology to non-standard analysis. As was discussed above, the  $\textcircled{1}$ -based methodology was developed to simplify the treatment of infinite sets and to do away with any distinction between external and internal sets, standard and non-standard numbers, cardinals and ordinals, etc. In fact, it can be easily seen that the representation of infinities and infinitesimals in non-standard analysis using these concepts is more tedious than the treatment offered by the  $\textcircled{1}$ -based methodology (as was mentioned in [22], the  $\textcircled{1}$ -based approach ‘... is simpler than non standard enlargements in its conception, it does not require infinitistic constructions and affords easier and stronger computation power.’).

In their attempt to model  $\textcircled{1}$  using non-standard analysis the authors of [14, 15, 19] not only commit fallacies but also make things significantly more cumbersome with respect to Sergeev’s approach (e.g., in [15] (see page 1 in bottom), its authors try to render  $\textcircled{1}$  using external and internal sets departing radically from Sergeev’s motivation). Traditionally, a complication of a theory is acceptable if it produces more precise results. Unfortunately, results provided in [14, 15, 19] are *less* precise. It is sufficient to mention that in [15] (see the last page), its authors consider a set of numerals involving  $\textcircled{1}$  (where again both concepts, numerals and grossone, do not coincide with their definitions used by Sergeev) and conclude that this set is countable. However, as was already discussed above, *countable* is a very rough result with respect to the  $\textcircled{1}$ -based approach since the introduction of  $\textcircled{1}$  allows one to measure infinite sets with the precision of one element (see Table 1).

Thus, what is the advantage to take a more precise and intuitive theory and to try to reduce it to a less precise one that, in addition, is more cumbersome?

## 5 Once again on Olympic medals

The note [44] entitled *The Olympic Medals Ranks, Lexicographic Ordering, and Numerical Infinities* proposes a funny application of the  $\textcircled{1}$ -based theory related to counting Olympic medals. It is well known that there exist several ways to rank countries with respect to medals won during Olympics. If  $g_A$  is the number of gold medals,  $s_A$  is the number of silver medals, and  $b_A$  is the number of the bronze ones won by

<sup>13</sup>For instance,  $\textcircled{1}$ -based numerals can be used for working with functions and their derivatives that can assume different infinite, finite, and infinitesimal values and can be defined over infinite and infinitesimal domains. The notions of continuity and derivability can be introduced not only for functions assuming finite values but for functions assuming infinite and infinitesimal values, as well. Limits  $\lim_{x \rightarrow a} f(x)$  are substituted by expressions and  $f(x)$  can be evaluated at concrete infinite or infinitesimal  $x$  in the same way as it is done with finite  $x$ . Series are substituted by sums having a concrete infinite number of addends and for different number of addends results (that can assume different infinite, finite or infinitesimal values) are different as it happens for sums with a finite number of summands. There are no divergent integrals, limits of integration can be concrete different infinite, finite or infinitesimal numbers and results can assume different infinite, finite or infinitesimal values. A number of set theoretical paradoxes can be avoided, etc.

country  $A$ , then many ranks have the form

$$R(\alpha, \beta, \gamma) = \alpha g_A + \beta s_A + \gamma b_A, \quad (7)$$

where  $\alpha > 0, \beta > 0$ , and  $\gamma > 0$  are certain weights. However, the unofficial rank used by the Olympic Committee does not allow one to use a numerical counter of the type (7) for ranking since it uses the lexicographic ordering to sort countries. The rule applied for this purpose is the following: one gold medal is more precious than any number of silver medals and one silver medal is more precious than any number of bronze medals. The paper [44] shows how it is possible to quantify these words *more precious* by introducing a counter that for any a priori unknown finite number of medals allows one to compute a numerical rank of a country using the number of gold, silver, and bronze medals in such a way that the higher resulting number puts the country in the higher position in the rank. This can be easily done by applying numerical computations with  $\mathbb{1}$ -based numerals.

More formally, the problem considered in [44] is to introduce as a counter of the type (7) a number  $n(g_A, s_A, b_A)$  that should be calculated in such a way that for countries  $A$  and  $B$  it follows that

$$n(g_A, s_A, b_A) > n(g_B, s_B, b_B), \text{ if } \begin{cases} g_A > g_B, \\ g_A = g_B, s_A > s_B, \\ g_A = g_B, s_A = s_B, b_A > b_B. \end{cases} \quad (8)$$

In addition,  $n(g_A, s_A, b_A)$  should be introduced under condition that the number  $K > \max\{g_A, s_A, b_A\}$  being an upper bound for the number of medals of each type that can be won by each country is unknown.

It is shown in [44] that  $n(g_A, s_A, b_A)$  can be calculated using  $\mathbb{1}$  as follows

$$n(g_A, s_A, b_A) = g_A \mathbb{1}^2 + s_A \mathbb{1}^1 + b_A \mathbb{1}^0. \quad (9)$$

This formula gives us the rank of the type (7) for the country and this rank satisfies condition (8). For instance, let us consider the data

$$g_A = 2, s_A = 0, b_A = 1, \quad g_B = 1, s_B = 11, b_B = 3. \quad (10)$$

Since  $\mathbb{1}$  is larger than any finite number (see the Infinite Unit Axiom in Appendix), it follows from (9) that

$$\begin{aligned} n(g_A, s_A, b_A) &= 2 \cdot \mathbb{1}^2 + 0 \cdot \mathbb{1}^1 + 1 \cdot \mathbb{1}^0 = 2\mathbb{1}^2 + 1 > \\ n(g_B, s_B, b_B) &= 1 \cdot \mathbb{1}^2 + 11 \cdot \mathbb{1}^1 + 3 \cdot \mathbb{1}^0 = 1\mathbb{1}^2 + 11\mathbb{1}^1 + 3 \end{aligned}$$

since

$$2\mathbb{1}^2 + 1 - (1\mathbb{1}^2 + 11\mathbb{1}^1 + 3) = 1\mathbb{1}^2 - 11\mathbb{1}^1 - 2 = \underbrace{\mathbb{1}(\mathbb{1} - 11)}_{\text{positive and infinite}} - 2 > 0.$$

Thus, to the set of other existing ranks discussed in [44] the grossone-based rank (9) has been added<sup>14</sup>. The paper [44] is concluded by the obligatory comment that this way for counting ranks can be applied in all situations that require the lexicographic ordering and not only for three groups of objects but for any finite number of them.

<sup>14</sup>Notice that the paper [44] does not say that the rank (9) is ‘the best one’. It is just one more way to rank countries that can be useful in certain situations.

Let us now return to [14]. In section 7, its authors intend to ‘demonstrate that the approach suggested by Sergeyev is useless’. Unfortunately, in trying to do this, they once again show lack of understanding of the meaning of the words ‘numerical computations’. The authors of [14] write that Sergeyev regards (9) ‘as a “numerical” rank just because it is a “number” in the sense of his grossone theory’. This is not the case. The rank (9) is numerical, since it allows one to work on a computer (the Infinity Computer) where  $\textcircled{1}$ -based numbers are not manipulated as symbols and both  $g_A, s_A, b_A$  and the exponents 2, 1, 0 are expressed by floating-point numbers (see footnote 4 to recall a brief explanation on the essence of numerical computations).

Then the authors of [14] in Section 7 ‘indicate a very simple and honest method’ (are there dishonest methods?) for constructing a rank (7) for lexicographic ordering. It consists of using the binary numeral system to compute the rank as follows

$$r(g_A, s_A, b_A) = 0.\underbrace{11\dots\dots 11}_{{g_A \text{ positions}}}\underbrace{011\dots\dots 11}_{{s_A \text{ positions}}}\underbrace{011\dots\dots 11}_{{b_A \text{ positions}}}. \quad (11)$$

For the data (10) this gives

$$r(2, 0, 1) = 0.11001 > r(1, 11, 3) = 0.101111111111110111.$$

However, some comments can be done upon the very simple and honest method proposed by the authors of [14]:

1. It does not assign the equal weights to all medals of the same class. Indeed, the authors of [14] explicitly state this because in their opinion the first medal of a given class is more significant achievement than the second one, that in its turn, is more important than the third one, etc.<sup>15</sup> Thus, the very simple and honest method does not address the same problem that is solved in [44] (i.e., we face the Strawman fallacy again).
2. The very simple and honest method is not practical since the binary numeral system requires very long sequences of binary digits to represent the rank (11). In fact, even a very modest number of medals, e.g., 20 gold, 20 silver, and 20 bronze, is already non representable in the IEEE 754 double-precision binary floating-point format having mantissa with 52 digits. In fact, the quantity of medals mentioned above requires 62 bits (60 positions for medals that are represented by 1 each and two zeros that separate groups of medals). Notice that the total number of medals in Olympics is usually significantly higher than 60. For instance, during The 2016 Summer Olympics in Rio de Janeiro there were 306 sets of medals. Since countries can have several athletes in each competition, and, therefore, a country can win gold, silver, and bronze in each contest, 920 bits are required to compute the rank proposed by the authors of [14]. This means that a special data structure should be developed to implement the very simple and honest method.
3. We recall that the whole story with the Olympics paper [44] was conceived as a divertissement. However, the rank (9) proposed in [44] is not a joke, since it can be used in practical applications regarding the lexicographic ordering where *the ordered quantities are not necessary integer*. For example, in [5], lexicographic multi-objective linear programming problems have been considered, the

<sup>15</sup>Notice that this point of view implies that the first competition is more important than the second one, etc. violating so the principle of equality of all sportive disciplines.

gross-simplex algorithm using the rank (9) has been proposed and implemented, results of numerical experiments have been provided. The very simple and honest method proposed in [14] cannot be used for this purpose since, due to (11), it is able to work with integer quantities  $g_A$ ,  $s_A$ , and  $b_A$  only.

Thus, the two ranks, (9) and (11), solve different problems, have different areas of applicability, and, therefore, are not in competition.

## 6 A concluding remark

The author of this paper hopes that this text will help people interested in non-standard analysis to see that different points of view on infinity are eligible. The  $\textcircled{1}$ -based approach does not attack non-standard analysis in any way and there is no need to defend it (especially, so inelegantly as it is done in [14, 15, 19]). Non-standard analysis has been proposed in the middle of the previous century, it has its vision of infinity, some scientists practice it and obtain results in certain areas of Mathematics. This is fine, no objection.

In its turn, the  $\textcircled{1}$ -based methodology offers another vision of infinity proposing a physically oriented Philosophy of Mathematics and a number of related numerical algorithms. Some other scientists use it in a variety of applications in pure and applied research areas. In fact, nowadays there exist applications related to numerical differentiation and optimization, ill-conditioning, ODEs, traditional and blinking fractals, cellular automata, Euclidean and hyperbolic geometry, percolation, probability, infinite series and the Riemann zeta function, set theory and the first Hilbert problem, Turing machines, etc. (the interested reader is invited to consult the recent survey [47] and the dedicated web-page [17] for more information).

In conclusion, let us quote a person who knew something about infinity: ‘The essence of Mathematics lies entirely in its freedom.’ Certainly, the reader remembers that these words belong to Georg Cantor.

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## Appendix

The  $\textcircled{1}$ -based methodology is one of the possible views on infinite and infinitesimal quantities and Mathematics, in general. It is added to other existing philosophies of Mathematics such as logicism, formalism, intuitionism, structuralism, etc. (their comprehensive analysis can be found, e.g., in [20, 21, 53]). Three postulates and an axiom that is added to axioms for real numbers form the methodological platform of the proposal. They are given below to make this paper self-contained. A special attention in the  $\textcircled{1}$ -based methodology is paid to the fact that numeral systems that are among our tools used to observe mathematical objects limit our capabilities of the observation. A detailed discussion on this methodological platform can be found in [47].

**Methodological Postulate 1** *We postulate existence of infinite and infinitesimal objects but accept that human beings and machines are able to execute only a finite number of operations.*

**Methodological Postulate 2** We shall not tell *what are* the mathematical objects we deal with; we just shall construct more powerful tools that will allow us to improve our capacities to observe and to describe properties of mathematical objects.

**Methodological Postulate 3** We adopt the principle ‘The part is less than the whole’ to all numbers (finite, infinite, and infinitesimal) and to all sets and processes (finite and infinite).

**The Infinite Unit Axiom.** The infinite unit of measure is introduced as the number of elements of the set,  $\mathbb{N}$ , of natural numbers. It is expressed by the numeral  $\textcircled{1}$  called *grossone* and has the following properties:

*Infinity.* Any finite natural number  $n$  is less than grossone, i.e.,  $n < \textcircled{1}$ .

*Identity.* The following relations link  $\textcircled{1}$  to identity elements 0 and 1

$$0 \cdot \textcircled{1} = \textcircled{1} \cdot 0 = 0, \quad \textcircled{1} - \textcircled{1} = 0, \quad \frac{\textcircled{1}}{\textcircled{1}} = 1, \quad \textcircled{1}^0 = 1, \quad 1^{\textcircled{1}} = 1, \quad 0^{\textcircled{1}} = 0. \quad (12)$$

*Divisibility.* For any finite natural number  $n$  sets  $\mathbb{N}_{k,n}$ ,  $1 \leq k \leq n$ , being the  $n$ th parts of the set,  $\mathbb{N}$ , of natural numbers have the same number of elements indicated by the numeral  $\frac{\textcircled{1}}{n}$  where

$$\mathbb{N}_{k,n} = \{k, k + n, k + 2n, k + 3n, \dots\}, \quad 1 \leq k \leq n, \quad \bigcup_{k=1}^n \mathbb{N}_{k,n} = \mathbb{N}. \quad (13)$$

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