

# Solving ordinary differential equations on the Infinity Computer by working with infinitesimals numerically\*

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## Abstract

There exists a huge number of numerical methods that iteratively construct approximations to the solution  $y(x)$  of an ordinary differential equation (ODE)  $y'(x) = f(x, y)$  starting from an initial value  $y_0 = y(x_0)$  and using a finite approximation step  $h$  that influences the accuracy of the obtained approximation. In this paper, a new framework for solving ODEs is presented for a new kind of a computer – the Infinity Computer (it has been patented and its working prototype exists). The new computer is able to work numerically with finite, infinite, and infinitesimal numbers giving so the possibility to use different infinitesimals numerically and, in particular, to take advantage of infinitesimal values of  $h$ . To show the potential of the new framework a number of results is established. It is proved that the Infinity Computer is able to calculate derivatives of the solution  $y(x)$  and to reconstruct its Taylor expansion of a desired order numerically without finding the respective derivatives analytically (or symbolically) by the successive derivation of the ODE as it is usually done when the Taylor method is applied. Methods using approximations of derivatives obtained thanks to infinitesimals are discussed and a technique for an automatic control of rounding errors is introduced. Numerical examples are given.

**Key Words:** Ordinary differential equations, numerical infinitesimals, combining finite and infinitesimal approximation steps, Infinity Computer.

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# 1 Introduction

The number of applications in physics, mechanics, and engineering where it is necessary to solve numerically ordinary differential equations (ODEs) with a given initial value is really enormous. Since many ordinary differential equations cannot be solved analytically, people use numerical algorithms<sup>1</sup> for finding approximate solutions (see [3, 4, 14, 27]). In this paper, we want to approximate the solution  $y(x)$ ,  $x \in [a, b]$ , of the initial value problem (also called the Cauchy problem) for a differential equation

$$y'(x) = f(x, y), \quad y(x_0) = y_0, \quad x_0 = a, \quad (1)$$

where  $a$  and  $b$  are finite numbers and  $y(x_0) = y_0$  is called the initial condition. We suppose that  $f(x, y)$  is given by a computer procedure. Since very often in scientific and technical applications it can happen that the person who wants to solve (1) is not the person who has written the code for  $f(x, y)$ , we suppose that the person solving (1) does not know the structure of  $f(x, y)$ , i.e., it is a black box for him/her.

In the literature, there exist numerous numerical algorithms constructing a sequence  $y_1, y_2, y_3, \dots$  approximating the exact values  $y(x_1), y(x_2), y(x_3), \dots$  that the solution  $y(x)$  assumes at points  $x_1, x_2, x_3, \dots$  (see [4, 13, 17]). The explicit Euler algorithm is the simplest among explicit methods for the numerical integration of ODEs. It uses the first two terms of the Taylor expansion of  $y(x)$  constructing so the linear approximation around the point  $(x_0, y(x_0))$ . The  $(n + 1)$ th step of the Euler algorithm describes how to move from the point  $x_n$  to  $x_{n+1} = x_n + h, n > 0$ , and is executed as follows

$$y_{n+1} = y_n + hf(x_n, y_n). \quad (2)$$

Traditional computers work with finite values of  $h$  introducing so errors at each step of the algorithm. In order to obtain more accurate approximations it is necessary to decrease the step  $h$  increasing so the number of steps of the method (the computations become more expensive). In any case,  $h$  always remains finite and its minimal acceptable value is determined by technical characteristics of each concrete computer the method is implemented on. Obviously, the same effects hold for more sophisticated methods, as well (see [4, 13, 14, 17]). Another approach to solve (1) on a traditional computer is the use of an automatic differentiation software executing pre-processing of (1) (see [12] and references given therein).

In this paper, we introduce a new numerical framework for solving ODEs related to the usage of a new kind of computer – the Infinity Computer (see [31, 33, 37]). It is able to work *numerically* with finite, infinite, and infinitesimal quantities. The Infinity Computer is based on an applied point of view (see [30, 33, 39])

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<sup>1</sup>There exist also symbolic techniques but they are not considered in this paper dedicated to numerical computations.

on infinite and infinitesimal numbers. In order to see the place of the new approach in the historical panorama of ideas dealing with infinite and infinitesimal, see [21, 22, 36, 38, 43]. The new methodology has been successfully applied for studying numerical differentiation and optimization (see [8, 35, 40, 48]), fractals (see [32, 34, 42, 45]), percolation (see [16, 45]), Euclidean and hyperbolic geometry (see [23, 29]), the first Hilbert problem and Turing machines (see [38, 43, 44]), cellular automata (see [9]), infinite series (see [36, 41, 47]), functions and their derivatives that can assume infinite and infinitesimal values (see [36]), etc.

With respect to the initial value problem (1), the possibility to work numerically with infinitesimals allows us to use *numerical infinitesimal values of  $h$* . It is proved that under reasonable conditions the Infinity Computer is able to calculate *exact* values of the derivatives of  $y(x)$  and to reconstruct its Taylor expansion with a desired accuracy by using infinitesimal values of  $h$  without finding the respective derivatives analytically (or symbolically) by the successive derivation of (1) as it is usually done when the Taylor method is applied.

The rest of the paper is organized as follows. Section 2 briefly presents the new computational methodology. Section 3 introduces the main theoretical results and describes how derivatives of  $y(x)$  can be calculated numerically on the Infinity Computer. Section 4 introduces a variety of examples of the usage of infinitesimals for ODEs numerical solving. First, it presents two simple iterative methods. Then, it describes a technique that can be used to obtain approximations of derivatives of the solution  $y(x)$  at the point  $x_{n+1}$  using infinitesimals and the information obtained at the point  $x_n$ . Finally, a technique for an automatic control of rounding errors that can occur during evaluation of  $f(x, y)$  is introduced. Through the paper, theoretical results are illustrated by numerical examples.

## 2 A fast tour to the new computational methodology

Numerous trials have been done during the centuries in order to evolve existing numeral systems<sup>2</sup> in such a way that infinite and infinitesimal numbers could be included in them (see [2, 5, 7, 19, 20, 25, 28, 46]). Particularly, in the early history of the calculus, arguments involving infinitesimals played a pivotal role in the derivation developed by Leibniz and Newton (see [19, 25]). The notion of an infinitesimal, however, lacked a precise mathematical definition and in order to provide a more rigorous foundation for the calculus, infinitesimals were gradually replaced by the d'Alembert-Cauchy concept of a limit.

Since new numeral systems appear very rarely, in each concrete historical period their importance for Mathematics is very often underestimated (especially by

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<sup>2</sup>We are reminded that a *numeral* is a symbol or group of symbols that represents a *number*. The difference between numerals and numbers is the same as the difference between words and the things they refer to. A *number* is a concept that a *numeral* expresses. The same number can be represented by different numerals. For example, the symbols '7', 'seven', and 'VII' are different numerals, but they all represent the same number.

pure mathematicians). In order to illustrate their importance, let us remind the Roman numeral system that does not allow one to express zero and negative numbers. In this system, the expression III-X is an indeterminate form. As a result, before appearing the positional numeral system and inventing zero (by the way, the second event was several hundred years later with respect to the first one) mathematicians were not able to create theorems involving zero and negative numbers and to execute computations with them.

There exist numeral systems that are even weaker than the Roman one. They seriously limit their users in executing computations. Let us recall a study published recently in *Science* (see [11]) that describes a primitive tribe – Pirahã – living in Amazonia. These people use a very simple numeral system for counting: one, two, many. For Pirahã, all quantities larger than two are just ‘many’ and such operations as  $2+2$  and  $2+1$  give the same result, i.e., ‘many’. Using their weak numeral system Pirahã are not able to see, for instance, numbers 3, 4, 5, and 6, to execute arithmetical operations with them, and, in general, to say anything about these numbers because in their language there are neither words nor concepts for that.

In the context of the present paper, it is very important that the weakness of Pirahã’s numeral system leads them to such results as

$$\text{‘many’} + 1 = \text{‘many’}, \quad \text{‘many’} + 2 = \text{‘many’}, \quad (3)$$

which are very familiar to us in the context of views on infinity used in the traditional calculus

$$\infty + 1 = \infty, \quad \infty + 2 = \infty. \quad (4)$$

The arithmetic of Pirahã involving the numeral ‘many’ has also a clear similarity with the arithmetic proposed by Cantor for his Alephs<sup>3</sup>:

$$\aleph_0 + 1 = \aleph_0, \quad \aleph_0 + 2 = \aleph_0, \quad \aleph_1 + 1 = \aleph_1, \quad \aleph_1 + 2 = \aleph_1. \quad (5)$$

Thus, the modern mathematical numeral systems allow us to distinguish a larger quantity of finite numbers with respect to Pirahã but give results that are similar to those of Pirahã when we speak about infinite numbers. This observation leads us to the following idea: *Probably our difficulties in working with infinity is not connected to the nature of infinity itself but is a result of inadequate numeral systems that we use to work with infinity, more precisely, to express infinite numbers.*

Let us compare the usage of numeral systems in Mathematics emphasizing differences that hold when one works, on the one hand, with finite quantities and, on

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<sup>3</sup>This similarity becomes even more pronounced if one considers another Amazonian tribe – Mundurukú (see [26]) – who fail in exact arithmetic with numbers larger than 5 but are able to compare and add large approximate numbers that are far beyond their naming range. Particularly, they use the words ‘some, not many’ and ‘many, really many’ to distinguish two types of large numbers using the rules that are very similar to ones used by Cantor to operate with  $\aleph_0$  and  $\aleph_1$ , respectively.

the other hand, with infinities and infinitesimals. In our every day activities with finite numbers the *same* finite numerals are used for *different* purposes (e.g., the same numeral 4 can be used to express the number of elements of a set and to indicate the position of an element in a finite sequence). When we face the necessity to work with infinities or infinitesimals, the situation changes drastically. In fact, in this case *different* symbols are used to work with infinities and infinitesimals in *different* situations:

- $\infty$  in standard Analysis;
- $\omega$  for working with ordinals;
- $\aleph_0, \aleph_1, \dots$  for dealing with cardinalities;
- non-standard numbers using a generic infinitesimal  $h$  in non-standard Analysis, etc.

In particular, since the mainstream of the traditional Mathematics very often does not pay any attention to the distinction between numbers and numerals (in this occasion it is necessary to recall constructivists who studied this issue), many theories dealing with infinite and infinitesimal quantities have a symbolic (not numerical) character. For instance, many versions of the non-standard Analysis are symbolic, since they have no numeral systems to express their numbers by a finite number of symbols (the finiteness of the number of symbols is necessary for organizing numerical computations). Namely, if we consider a finite  $n$  then it can be taken  $n = 5$ , or  $n = 103$  or any other numeral used to express finite quantities and consisting of a finite number of symbols. In contrast, if we consider a non-standard infinite  $m$  then it is not clear which numerals can be used to assign a concrete value to  $m$ .

Analogously, in non-standard Analysis, if we consider an infinitesimal  $h$  then it is not clear which numerals consisting of a finite number of symbols can be used to assign a value to  $h$  and to write  $h = \dots$ . In fact, very often in non-standard Analysis texts, a *generic* infinitesimal  $h$  is used and it is considered as a symbol, i.e., only symbolic computations can be done with it. Approaches of this kind leave unclear such issues, e.g., whether the infinite  $1/h$  is integer or not or whether  $1/h$  is the number of elements of an infinite set. Another problem is related to comparison of values. When we work with finite quantities then we can compare  $x$  and  $y$  if they assume numerical values, e.g.,  $x = 4$  and  $y = 6$  then, by using rules of the numeral system the symbols 4 and 6 belong to, we can compute that  $y > x$ . If one wishes to consider two infinitesimals  $h_1$  and  $h_2$  then it is not clear how to compare them because numeral systems that can express infinitesimals are not provided by non-standard Analysis techniques.

The approach developed in [30, 33, 39] proposes a numeral system that uses *the same numerals* for several different purposes for dealing with infinities and infinitesimals: in Analysis for working with functions that can assume different infinite, finite, and infinitesimal values (functions can also have derivatives assuming

different infinite or infinitesimal values); for measuring infinite sets; for indicating positions of elements in ordered infinite sequences; in probability theory, etc. It is important to emphasize that the new numeral system avoids situations of the type (3)–(5) providing results ensuring that if  $a$  is a numeral written in this system then for any  $a$  (i.e.,  $a$  can be finite, infinite, or infinitesimal) it follows  $a + 1 > a$ .

The new numeral system works as follows. A new infinite unit of measure expressed by the numeral  $\textcircled{1}$  called *grossone* is introduced as the number of elements of the set,  $\mathbb{N}$ , of natural numbers. Concurrently with the introduction of grossone in the mathematical language all other symbols (like  $\infty$ , Cantor's  $\omega$ ,  $\aleph_0, \aleph_1, \dots$ , etc.) traditionally used to deal with infinities and infinitesimals are excluded from the language because grossone and other numbers constructed with its help not only can be used instead of all of them but can be used with a higher accuracy. Grossone is introduced by describing its properties postulated by the Infinite Unit Axiom (see [33, 39]) added to axioms for real numbers (similarly, in order to pass from the set,  $\mathbb{N}$ , of natural numbers to the set,  $\mathbb{Z}$ , of integers a new element – zero expressed by the numeral 0 – is introduced by describing its properties).

The new numeral  $\textcircled{1}$  allows us to construct different numerals expressing different infinite and infinitesimal numbers and to execute computations with them. As a result, in Analysis, instead of the usual symbol  $\infty$  used in series and integration different infinite and/or infinitesimal numerals can be used (see [36, 41, 47]). Indeterminate forms are not present and, for example, the following relations hold for  $\textcircled{1}$  and  $\textcircled{1}^{-1}$  (that is infinitesimal), as for any other (finite, infinite, or infinitesimal) number expressible in the new numeral system

$$0 \cdot \textcircled{1} = \textcircled{1} \cdot 0 = 0, \quad \textcircled{1} - \textcircled{1} = 0, \quad \frac{\textcircled{1}}{\textcircled{1}} = 1, \quad \textcircled{1}^0 = 1, \quad 1^{\textcircled{1}} = 1, \quad 0^{\textcircled{1}} = 0, \quad (6)$$

$$0 \cdot \textcircled{1}^{-1} = \textcircled{1}^{-1} \cdot 0 = 0, \quad \textcircled{1}^{-1} > 0, \quad \textcircled{1}^{-2} > 0, \quad \textcircled{1}^{-1} - \textcircled{1}^{-1} = 0, \\ \frac{\textcircled{1}^{-1}}{\textcircled{1}^{-1}} = 1, \quad \frac{\textcircled{1}^{-2}}{\textcircled{1}^{-2}} = 1, \quad (\textcircled{1}^{-1})^0 = 1, \quad \textcircled{1} \cdot \textcircled{1}^{-1} = 1, \quad \textcircled{1} \cdot \textcircled{1}^{-2} = \textcircled{1}^{-1}.$$

The new approach gives the possibility to develop a new Analysis (see [36]) where functions assuming not only finite values but also infinite and infinitesimal ones can be studied. For all of them it becomes possible to introduce a new notion of continuity that is closer to our modern physical knowledge. Functions assuming finite and infinite values can be differentiated and integrated.

**Example 1.** The function  $f(x) = x^2$  has the first derivative  $f'(x) = 2x$  and both  $f(x)$  and  $f'(x)$  can be evaluated at infinite and infinitesimal  $x$ . Thus, for infinite  $x = \textcircled{1}$  we obtain infinite values

$$f(\textcircled{1}) = \textcircled{1}^2, \quad f'(\textcircled{1}) = 2\textcircled{1}$$

and for infinitesimal  $x = \textcircled{1}^{-1}$  we have infinitesimal values

$$f(\textcircled{1}^{-1}) = \textcircled{1}^{-2}, \quad f'(\textcircled{1}^{-1}) = 2\textcircled{1}^{-1}.$$

If  $x = 5\mathbb{1} - 10\mathbb{1}^{-1}$  then we have

$$f(\mathbb{1}^{-1}) = (5\mathbb{1} - 10\mathbb{1}^{-1})^2 = 25\mathbb{1}^2 - 100 + 100\mathbb{1}^{-2},$$

$$f'(\mathbb{1}^{-1}) = 10\mathbb{1} - 20\mathbb{1}^{-1}.$$

We can also work with functions defined by formulae including infinite and infinitesimal numbers. For example, the function  $f(x) = \frac{1}{\mathbb{1}}x^2 + \mathbb{1}x$  has a quadratic term infinitesimal and the linear one infinite. It has the first derivative  $f'(x) = \frac{2}{\mathbb{1}}x + \mathbb{1}$ . For infinite  $x = 3\mathbb{1}$  we obtain infinite values

$$f(\mathbb{1}) = 3\mathbb{1}^2 + 9\mathbb{1}, \quad f'(\mathbb{1}) = \mathbb{1} + 6$$

and for infinitesimal  $x = \mathbb{1}^{-1}$  we have

$$f(\mathbb{1}^{-1}) = 1 + \mathbb{1}^{-3}, \quad f'(\mathbb{1}^{-1}) = \mathbb{1} + 2\mathbb{1}^{-2}. \quad \square$$

By using the new numeral system it becomes possible to measure certain infinite sets and to see, e.g., that the sets of even and odd numbers have  $\mathbb{1}/2$  elements each. The set,  $\mathbb{Z}$ , of integers has  $2\mathbb{1} + 1$  elements ( $\mathbb{1}$  positive elements,  $\mathbb{1}$  negative elements, and zero). Within the countable sets and sets having cardinality of the continuum (see [21, 38, 39]) it becomes possible to distinguish infinite sets having different number of elements expressible in the numeral system using grossone and to see that, for instance,

$$\frac{\mathbb{1}}{2} < \mathbb{1} - 1 < \mathbb{1} < \mathbb{1} + 1 < 2\mathbb{1} + 1 < 2\mathbb{1}^2 - 1 < 2\mathbb{1}^2 < 2\mathbb{1}^2 + 1 <$$

$$2\mathbb{1}^2 + 2 < 2\mathbb{1} - 1 < 2\mathbb{1} < 2\mathbb{1} + 1 < 10\mathbb{1} < \mathbb{1}^{\mathbb{1}} - 1 < \mathbb{1}^{\mathbb{1}} < \mathbb{1}^{\mathbb{1}} + 1.$$

The Infinity Computer used in this paper for solving the problem (1) works with numbers having finite, infinite, and infinitesimal parts. To represent them in the computer memory records similar to traditional positional numeral systems can be used (see [33, 37]). To construct a number  $C$  in the new numeral positional system<sup>4</sup> with base  $\mathbb{1}$ , we subdivide  $C$  into groups corresponding to powers of  $\mathbb{1}$ :

$$C = c_{p_m} \mathbb{1}^{p_m} + \dots + c_{p_1} \mathbb{1}^{p_1} + c_{p_0} \mathbb{1}^{p_0} + c_{p_{-1}} \mathbb{1}^{p_{-1}} + \dots + c_{p_{-k}} \mathbb{1}^{p_{-k}}. \quad (7)$$

<sup>4</sup>At the first glance the numerals (7) can remind numbers from the Levi-Civita field (see [20]) that is a very interesting and important precedent of algebraic manipulations with infinities and infinitesimals. However, the two mathematical objects have several crucial differences. They have been introduced for different purposes by using two mathematical languages having different accuracies and on the basis of different methodological foundations. In fact, Levi-Civita does not discuss the distinction between numbers and numerals. His numbers have neither cardinal nor ordinal properties; they are build using a generic infinitesimal and only its rational powers are allowed; he uses symbol  $\infty$  in his construction; there is no any numeral system that would allow one to assign numerical values to these numbers; it is not explained how it would be possible to pass from a generic infinitesimal  $h$  to a concrete one (see also the discussion above on the distinction between numbers and numerals). In no way the said above should be considered as a criticism with respect to results of Levi-Civita. The above discussion has been introduced in this text just to underline that we are in front of two different mathematical tools that should be used in different mathematical contexts.

Then, the record

$$C = c_{p_m} \mathbb{1}^{p_m} \dots c_{p_1} \mathbb{1}^{p_1} c_{p_0} \mathbb{1}^{p_0} c_{p_{-1}} \mathbb{1}^{p_{-1}} \dots c_{p_{-k}} \mathbb{1}^{p_{-k}} \quad (8)$$

represents the number  $C$ , where all numerals  $c_i \neq 0$ , they belong to a traditional numeral system and are called *grossdigits*. They express finite positive or negative numbers and show how many corresponding units  $\mathbb{1}^{p_i}$  should be added or subtracted in order to form the number  $C$ . Note that in order to have a possibility to store  $C$  in the computer memory, values  $k$  and  $m$  should be finite.

Numbers  $p_i$  in (8) are sorted in the decreasing order with  $p_0 = 0$

$$p_m > p_{m-1} > \dots > p_1 > p_0 > p_{-1} > \dots > p_{-(k-1)} > p_{-k}.$$

They are called *grosspowers* and they themselves can be written in the form (8). In the record (8), we write  $\mathbb{1}^{p_i}$  explicitly because in the new numeral positional system the number  $i$  in general is not equal to the grosspower  $p_i$ . This gives the possibility to write down numerals without indicating grossdigits equal to zero.

The term having  $p_0 = 0$  represents the finite part of  $C$  because, due to (6), we have  $c_0 \mathbb{1}^0 = c_0$ . The terms having finite positive grosspowers represent the simplest infinite parts of  $C$ . Analogously, terms having negative finite grosspowers represent the simplest infinitesimal parts of  $C$ . For instance, the number  $\mathbb{1}^{-1} = \frac{1}{\mathbb{1}}$  mentioned above is infinitesimal. Note that all infinitesimals are not equal to zero. Particularly,  $\frac{1}{\mathbb{1}} > 0$  because it is a result of division of two positive numbers.

A number represented by a numeral in the form (8) is called *purely finite* if it has neither infinite not infinitesimal parts. For instance, 2 is purely finite and  $2 + 3\mathbb{1}^{-1}$  is not. All grossdigits  $c_i$  are supposed to be purely finite. Purely finite numbers are used on traditional computers and for obvious reasons have a special importance for applications.

All of the numbers introduced above can be grosspowers, as well, giving thus a possibility to have various combinations of quantities and to construct terms having a more complex structure. However, in this paper we consider only purely finite grosspowers. Let us give an example of multiplication of two infinite numbers  $A$  and  $B$  of this kind (for a comprehensive description see [33, 37]).

**Example 2.** Let us consider numbers  $A$  and  $B$ , where

$$A = 14.3\mathbb{1}^{56.2}5.4\mathbb{1}^0, \quad B = 6.23\mathbb{1}^3 1.5\mathbb{1}^{-4.1}.$$

The number  $A$  has an infinite part and a finite one. The number  $B$  has an infinite part and an infinitesimal one. Their product  $C$  is equal to

$$C = B \cdot A = 89.089\mathbb{1}^{59.2} 21.45\mathbb{1}^{52.1} 33.642\mathbb{1}^3 8.1\mathbb{1}^{-4.1}. \quad \square$$

We conclude this section by emphasizing that there exist different mathematical languages and numeral systems and, if they have different accuracies, it is not possible to use them together. For instance, the usage of ‘many’ from the language of Pirahã in the record  $4 + \text{‘many’}$  has no any sense because for Pirahã it is not



clear what is 4 and for people knowing what is 4 the accuracy of the answer ‘many’ is too low. Analogously, the records of the type  $\mathbb{1} + \omega$ ,  $\mathbb{1} - \aleph_0$ ,  $\mathbb{1}/\infty$ , etc. have no sense because they belong to languages developed for different purposes and having different accuracies.

### 3 Numerical reconstruction of the Taylor expansion of the solution on the Infinity Computer

Let us return to the problem (1). We suppose that a set of elementary functions ( $a^x$ ,  $\sin(x)$ ,  $\cos(x)$ , etc.) is represented at the Infinity Computer by one of the usual ways used in traditional computers (see, e.g. [24]) involving the argument  $x$ , finite constants, and four arithmetical operations. Then the following theorem holds (the word *exact* in it means: with the accuracy of the computer programme implementing  $f(x, y)$  from (1)).

**Theorem 1** *Let us suppose that for the solution  $y(x)$ ,  $x \in [a, b]$ , of (1) there exists the Taylor expansion (unknown for us) and at purely finite points  $s \in [a, b]$ , the function  $y(s)$  and all its derivatives assume purely finite values or are equal to zero. Then the Infinity Computer allows one to reconstruct the Taylor expansion for  $y(x)$  up to the  $k$ -th derivative with exact values of  $y'(x)$ ,  $y''(x)$ ,  $y^{(3)}(x)$ ,  $\dots$ ,  $y^{(k)}(x)$  after  $k$  steps of the Euler method with the step  $h = \mathbb{1}^{-1}$ .*

**Proof.** Let us start to execute on the Infinite Computer steps of the Euler method following the rule (2) and using the infinitesimal step  $h = \mathbb{1}^{-1}$ . Since the problem (1) has been stated using the traditional finite mathematics,  $x_0$  is purely finite. Without loss of generality let us consider the first  $k = 4$  steps of the Euler method (the value  $k = 4$  is sufficient to show the way of reasoning; we shall use the formulae involved in this case later in a numerical illustration). We obtain

$$y_1 = y_0 + \mathbb{1}^{-1}f(x_0, y_0), \quad y_2 = y_1 + \mathbb{1}^{-1}f(x_1, y_1), \quad (9)$$

$$y_3 = y_2 + \mathbb{1}^{-1}f(x_2, y_2), \quad y_4 = y_3 + \mathbb{1}^{-1}f(x_3, y_3). \quad (10)$$

The derivatives of the solution  $y(x)$  can be approximated in different ways and with different orders of accuracy. Let us consider approximations (see, e.g., [10]) executed by forward differences  $\Delta_h^j$ ,  $1 \leq j \leq k$ , with the first order of accuracy and take  $h = \mathbb{1}^{-1}$  as follows

$$\Delta_{\mathbb{1}^{-1}}^k = \sum_{i=0}^k (-1)^i \binom{k}{i} y_{x_0 + (k-i)\mathbb{1}^{-1}}. \quad (11)$$

Then we have

$$y'(x_0) \approx \frac{\Delta_{\mathbb{1}^{-1}}^1}{\mathbb{1}^{-1}} + O(\mathbb{1}^{-1}) = \frac{y_1 - y_0}{\mathbb{1}^{-1}} + O(\mathbb{1}^{-1}), \quad (12)$$

$$y''(x_0) \approx \frac{\Delta_{\mathbb{1}^{-1}}^2}{\mathbb{1}^{-2}} + O(\mathbb{1}^{-1}) = \frac{y_0 - 2y_1 + y_2}{\mathbb{1}^{-2}} + O(\mathbb{1}^{-1}), \quad (13)$$

$$y^{(3)}(x_0) \approx \frac{\Delta_{\mathbb{1}^{-1}}^3}{\mathbb{1}^{-3}} + O(\mathbb{1}^{-1}) = \frac{-y_0 + 3y_1 - 3y_2 + y_3}{\mathbb{1}^{-3}} + O(\mathbb{1}^{-1}), \quad (14)$$

$$y^{(4)}(x_0) \approx \frac{\Delta_{\mathbb{1}^{-1}}^4}{\mathbb{1}^{-4}} + O(\mathbb{1}^{-1}) = \frac{y_0 - 4y_1 + 6y_2 - 4y_3 + y_4}{\mathbb{1}^{-4}} + O(\mathbb{1}^{-1}). \quad (15)$$

Since due to (1) we can evaluate directly  $y'(x_0) = f(x_0, y_0)$ , let us start by considering the formula (13) (the cases with values of  $k > 2$  are studied by a complete analogy). Since  $x_0$  is purely finite, then due to our assumptions  $y''(x_0)$  is also purely finite. This means that  $y''(x_0)$  does not contain infinitesimal parts. Formula (13) states that the error we have when instead of  $y''(x_0)$  use its approximation

$$\widetilde{y''(x_0)} = \frac{\Delta_{\mathbb{1}^{-1}}^2}{\mathbb{1}^{-2}} \quad (16)$$

is of the order  $\mathbb{1}^{-1}$ . The Infinity Computer works in such a way that it collects different orders of  $\mathbb{1}$  in separate groups. Thus,  $\Delta_{\mathbb{1}^{-1}}^2$  will be represented in the format (8)

$$\Delta_{\mathbb{1}^{-1}}^2 = c_0\mathbb{1}^0 + c_{-1}\mathbb{1}^{-1} + c_{-2}\mathbb{1}^{-2} + \dots + c_{-m_2}\mathbb{1}^{-m_2}, \quad (17)$$

where  $m_2$  is a finite integer, its value depends on each concrete  $f(x, y)$  from (1). Note that (17) cannot contain fractional grosspowers because the step  $h = \mathbb{1}^{-1}$  having the integer grosspower  $-1$  has been chosen in (9), (10).

It follows from (13) and the fact that  $y''(x_0)$  is purely finite that  $\widetilde{y''(x_0)}$  contains a purely finite part and can contain infinitesimal parts of the order  $\mathbb{1}^{-1}$  or higher. This means that grossdigits  $c_0 = c_{-1} = 0$ , otherwise after division on  $\mathbb{1}^{-2}$  the estimate  $\widetilde{y''(x_0)}$  would have infinite parts and this is impossible. Thus  $y''(x_0)$  has the following structure

$$\widetilde{y''(x_0)} = c_{-2}\mathbb{1}^0 + c_{-3}\mathbb{1}^{-1} + c_{-4}\mathbb{1}^{-2} + \dots + c_{-m}\mathbb{1}^{-m+2}. \quad (18)$$

It follows from (13) that  $\widetilde{y''(x_0)}$  can contain an error of the order  $\mathbb{1}^{-1}$  or higher. Since all the derivatives of  $y(x)$  are purely finite at  $x_0$  and, in particular,  $y''(x_0)$  is purely finite, the fact that the finite part and infinitesimal parts in (18) are separated gives us that  $c_{-2} = y''(x_0)$ . Thus, in order to have the exact value of  $y''(x_0)$  it is sufficient to calculate  $\Delta_{\mathbb{1}^{-1}}^2$  from (18) and to take its grossdigit  $c_{-2}$  that will be equal to  $y''(x_0)$ .

By a complete analogy the exact values of higher derivatives can be obtained from (12) – (15) and analogous formulae using forward differences (11) to approximate the  $k$ -th derivative  $y^{(k)}(x_0)$ . It suffices just to calculate  $\Delta_{\mathbb{1}^{-1}}^k$  and to take the grossdigit  $c_{-k}$  that will be equal to the exact value of the derivative  $y^{(k)}(x_0)$ .  $\square$

Let us consider an illustrative numerical example. We emphasize that the Infinity Computer solves it numerically, not symbolically, i.e., it is not necessary to translate the procedure implementing  $f(x, y)$  in a symbolic form.

**Example 3.** Let us consider the problem

$$y'(x) = x - y, \quad y(0) = 1, \quad (19)$$

taken from [1]. Its exact solution is

$$y(x) = x - 1 + 2e^{-x}. \quad (20)$$

We start by applying formulae (9) to calculate  $y_1$  and  $y_2$ :

$$\begin{aligned} y_1 &= 1 + \mathbb{1}^{-1} \cdot (0 - 1) = 1 - \mathbb{1}^{-1}, \\ y_2 &= 1 - \mathbb{1}^{-1} + \mathbb{1}^{-1}(\mathbb{1}^{-1} - 1 + \mathbb{1}^{-1}) = 1 - 2\mathbb{1}^{-1} + 2\mathbb{1}^{-2}. \end{aligned}$$

We have now the values  $y_0$ ,  $y_1$ , and  $y_2$ . Thus, we can apply formula (13) and calculate  $\Delta_{\mathbb{1}^{-1}}^2$  as follows

$$\Delta_{\mathbb{1}^{-1}}^2 = y_0 - 2y_1 + y_2 = 1 - 2 + 2\mathbb{1}^{-1} + 1 - 2\mathbb{1}^{-1} + 2\mathbb{1}^{-2} = 2\mathbb{1}^{-2}.$$

Thus,  $c_{-2} = 2$ . Let us now verify the obtained result and calculate the exact derivative  $y''(0)$  using (20). Then we have  $y''(x) = 2e^{-x}$ , and  $y''(0) = 2$ , i.e.,  $c_{-2} = y''(0)$ . Note that in this simple illustrative example  $c_{-m} = 0$ ,  $m > 2$ , where  $m$  is from (13). In general, this is not the case and  $c_{-m} \neq 0$  can occur.

Let us proceed and calculate  $y_3$  following (10). We have

$$y_3 = 1 - 2\mathbb{1}^{-1} + 2\mathbb{1}^{-2} + \mathbb{1}^{-1}(2\mathbb{1}^{-1} - 1 + 2\mathbb{1}^{-1} - 2\mathbb{1}^{-2}) = 1 - 3\mathbb{1}^{-1} + 6\mathbb{1}^{-2} - 2\mathbb{1}^{-3}.$$

It then follows from (14) that

$$\begin{aligned} \Delta_{\mathbb{1}^{-1}}^3 &= -y_0 + 3y_1 - 3y_2 + y_3 = \\ &= -1 + 3(1 - \mathbb{1}^{-1}) - 3(1 - 2\mathbb{1}^{-1} + 2\mathbb{1}^{-2}) + 1 - 3\mathbb{1}^{-1} + 6\mathbb{1}^{-2} - 2\mathbb{1}^{-3} = -2\mathbb{1}^{-3}. \end{aligned}$$

We can see that  $c_{-3} = -2$ . The exact derivative obtained from (20) is  $y^{(3)}(x) = -2e^{-x}$ . As a consequence, we have  $y^{(3)}(0) = -2$ , i.e.,  $c_{-3} = y^{(3)}(0)$ .

To calculate  $y^{(4)}(0)$  we use (10) and have

$$\begin{aligned} y_4 &= 1 - 3\mathbb{1}^{-1} + 6\mathbb{1}^{-2} - 2\mathbb{1}^{-3} + \mathbb{1}^{-1}(3\mathbb{1}^{-1} - 1 + 3\mathbb{1}^{-1} - 6\mathbb{1}^{-2} + 2\mathbb{1}^{-3}) = \\ &= 1 - 4\mathbb{1}^{-1} + 12\mathbb{1}^{-2} - 8\mathbb{1}^{-3} + 2\mathbb{1}^{-4}. \end{aligned}$$

From (15) we obtain

$$\begin{aligned} \Delta_{\mathbb{1}^{-1}}^4 &= y_0 - 4y_1 + 6y_2 - 4y_3 + y_4 = 1 - 4(1 - \mathbb{1}^{-1}) + 6(1 - 2\mathbb{1}^{-1} + 2\mathbb{1}^{-2}) - \\ &= -4(1 - 3\mathbb{1}^{-1} + 6\mathbb{1}^{-2} - 2\mathbb{1}^{-3}) + 1 - 4\mathbb{1}^{-1} + 12\mathbb{1}^{-2} - 8\mathbb{1}^{-3} + 2\mathbb{1}^{-4} = \end{aligned}$$

Table 1: Comparison of methods solving the problem (19) where  $n_f$  is the number of evaluation of  $f(x, y)$  executed by a method to obtain an approximated solution  $y_n$  at the point  $x = 1$

Method	$n_f$	$y_n$	$\varepsilon$
Heun, $h = 0.2$	10	0.741480	-0.005721
Runge–Kutta ( $4^{th}$ order), $h = 0.2$	20	0.735770	-0.0000116
$y(x, 0) = 1 - x + x^2$	2	1	-0.264241118
$y(x, 0) = 1 - x + x^2 - \frac{x^3}{3}$	3	0.666666667	0.069092216
$y(x, 0) = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12}$	4	0.75	-0.014241118
$y(x, 0) = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{60}$	5	0.733333333	0.002425549
$y(x, 0) = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{60} + \frac{x^6}{360}$	6	0.736111111	-0.000352229
$y(x, 0) = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{60} + \frac{x^6}{360} - \frac{x^7}{2520}$	7	0.7357142857	0.000044597
$y(x, 0) = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{60} + \frac{x^6}{360} - \frac{x^7}{2520} + \frac{x^8}{20160}$	8	0.735763889	-0.000005007

$$-4(1 - 3\mathbb{1}^{-1} + 6\mathbb{1}^{-2} - 2\mathbb{1}^{-3}) + 1 - 4\mathbb{1}^{-1} + 12\mathbb{1}^{-2} - 8\mathbb{1}^{-3} + 2\mathbb{1}^{-4} = 2\mathbb{1}^{-4}.$$

Again we obtain that  $c_{-4} = y^{(4)}(0) = 2$ .

Thus, four steps of the explicit Euler method with the infinitesimal step  $h = \mathbb{1}^{-1}$  have been executed on the Infinity Computer. As a result, the first five exact items of the Taylor expansion of  $y(x)$  in the neighborhood of  $x_0 = 0$  can be written:

$$y(x) = x - 1 + 2e^{-x} \approx 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12}. \quad (21)$$

By a complete analogy it is possible to obtain additional terms in the expansion that correspond to higher derivatives of  $y(x)$ .

In Table 1, we present results of experiments executed (see [1]) with the methods of Heun ( $2^d$  order) and Runge–Kutta ( $4^{th}$  order) solving the problem (19). Both methods use  $h = 0.2$ . Then we present results of the new methods that execute first  $k$  infinitesimal steps with  $k$  going from 2 to 8 and then executing one finite step from the point  $x_0 = 0$  to the point  $x = 1$ . The value  $n_f$  is the number of evaluation of  $f(x, y)$  executed by each method. The column  $y_n$  shows the obtained approximation at the point  $x = 1$  and the column  $\varepsilon$  shows the respective error  $\varepsilon = y(1) - y_n$  where  $n = 5$  for the methods of Heun and Runge–Kutta and  $n = 1$  for the new methods.  $\square$

In cases, where it is not possible to evaluate  $f(x, y)$  at the points  $x_n + \mathbb{1}^{-1}$ ,  $x_n + 2\mathbb{1}^{-1}$ ,  $x_n + 3\mathbb{1}^{-1}$ ,  $\dots$  (for instance, when we should solve the problem over an interval  $[a, b]$  and  $x_n = b$ ) the following corollary can be useful.

**Corollary 1** *Under conditions of Theorem 1 the backward differences calculated at the points  $x_n - \mathbb{1}^{-1}$ ,  $x_n - 2\mathbb{1}^{-1}$ ,  $x_n - 3\mathbb{1}^{-1}$ ,  $\dots$ ,  $x_n - k\mathbb{1}^{-1}$  can be used to calculate the derivatives of  $y(x)$  at the point  $x_n$ .*

**Proof.** The backward difference (see e.g., [10]) of the order  $k$  with  $h = \mathbb{1}^{-1}$  is calculated as follows

$$\nabla_{\mathbb{1}^{-1}}^k = \sum_{i=0}^k (-1)^i \binom{k}{i} y_{x_0 - i\mathbb{1}^{-1}}.$$

The rest of the proof is completely analogous to the proof of the theorem and is so omitted.  $\square$

Thus, if the region of interest  $[a, b]$  from (1) belongs to the region of convergence of the Taylor expansion for the solution  $y(x)$  around the point  $x_0$  then it is not necessary to construct iterative procedures involving several steps with finite values of  $h$  and it becomes possible to calculate approximations of the desired order by executing only one finite step.

## 4 Examples of the usage of infinitesimals in the new computational framework

The approach introduced in the previous section gives the possibility to construct a variety of new numerical methods for the Infinity Computer by using both infinitesimal and finite values of  $h$ . The general step  $n$  of a method of this kind for solving (1) can be described as follows:

- (i) take the point  $(x_n, y_n)$ , choose a value  $k_n$ , and execute  $k_n$  steps of the Euler method starting from  $x_n$  by using  $h = \mathbb{1}^{-1}$ ;
- (ii) calculate exact values of  $y'(x)$ ,  $y''(x)$ ,  $y^{(3)}(x), \dots, y^{(k_n)}(x)$  at the point  $(x_n, y_n)$  following the rules described in Theorem 1;
- (iii) construct the truncated Taylor expansion of the order  $k_n$ ;
- (iv) execute a single step from the point  $x_n$  to  $x_{n+1} = x_n + h_i$  using the constructed Taylor expansion and a finite value of  $h_n$  (steps of the kind  $h_n - \mathbb{1}^{-1}$  or  $h_n + \mathbb{1}^{-1}$  can be also used).

The general step described above allows one to construct numerical algorithms for solving (1) by executing several iterations of this kind. Many numerical methods (see [4, 14, 27]) can be used as a basis for such developments. Due to the easy way allowing us to calculate exact higher derivatives at the points  $(x_n, y_n)$ , methods that use higher derivatives are of the main interest. The fact that to increase the accuracy it is necessary just to execute one additional infinitesimal step without performing additional finite steps (i.e., the whole work executed at a lower level of accuracy is used entirely at a higher level of accuracy) is an additional advantage and suggests to construct adaptive methods (for instance, if one wishes to change the finite step from  $h_1$  to  $h_2 > h_1$  or  $h_3 < h_1$  then the same Taylor expansion can be used in all the cases). A study of such methods will be done in a separate paper.

Hereinafter, since the usage of numerical infinitesimals is a new topic, we give a number of examples showing how the new computational framework can be used in the context of numerical solving ODEs.

The rest of this section is organized as follows. In the first subsection, we present two simple iterative methods using low derivatives (a lower order of derivatives is used for expository reasons). In the second subsection, we present a technique that can be used to obtain an additional information with respect to approximations of derivatives of the solution. In the last subsection, we discuss how an automatic control of rounding errors can be executed during the evaluation of  $f(x, y)$  at the points  $(x_n, y_n)$ .

#### 4.1 A simple method and possibilities of its improvements

We start by introducing the *Method 1* that uses only the first and the second derivatives at each iteration to construct the Taylor expansion by applying formulae (9), (12), and (13). Thus, this method at the current point  $x_n$  executes twice the Euler step with  $h = \mathbb{1}^{-1}$  and then makes the step with a finite value of  $h$  by using the obtained Taylor expansion. Therefore, during these three steps (two infinitesimal steps and one finite step) the function  $f(x, y)$  is evaluated twice and only during the infinitesimals steps. Let us use the number step  $n$  to count the executed finite steps and denote by  $y(x, z)$  the Taylor expansion of the solution  $y(x)$  calculated by the method during the infinitesimal steps at the neighborhood of the point  $z = x_{n-1}$ . Then, we can calculate  $y_n$  as  $y_n = y(h, x_{n-1})$  with a finite value of  $h$ .

**Example 4.** We test this method on the problem (19) with the finite step  $h = 0.2$  (see Table 2). By applying the procedure described above with six digits after the dot we have that

$$\begin{aligned} y(x, 0) &= 1 - x + x^2, & y_1 &= y(0.2, 0) = 0.84, & (22) \\ y(x, 0.2) &= 0.84 - 0.64x + 0.82x^2, & y_2 &= y(0.2, 0.2) = 0.7448, \\ y(x, 0.4) &= 0.7448 - 0.3448x + 0.6724x^2, & y_3 &= y(0.2, 0.4) = 0.702736, \\ y(x, 0.6) &= 0.702736 - 0.102736x + 0.551368x^2, & y_4 &= y(0.2, 0.6) = 0.704244, \\ y(x, 0.8) &= 0.704244 + 0.095756x + 0.452122x^2, & y_5 &= y(0.2, 0.8) = 0.741480. \end{aligned}$$

It can be seen from Table 2 that the results obtained by the new method (see column Method 1.0) for the values  $y_n$  coincide (see [1]) with the results obtained by applying the modified Euler's method (called also Heun's method) that evaluates  $f(x, y)$  twice at each iteration as the new method does.

As it can be seen from the formulae above, the Method 1 at each point  $x_n$  provides us not only with the value  $y_n$  but also with the first and the second derivatives of  $y(x)$  at the point  $(x_n, y_n)$ . We shall denote them as  $y'_n(x_n)$  and  $y''_n(x_n)$  where

$$y'_n(x_n) = y'(0, x_n), \quad y''_n(x_n) = y''(0, x_n).$$

Table 2: Two versions of Method 1 constructing the Taylor expansion using the values of the first and second derivatives calculated during two infinitesimal Euler steps at the points  $(x_n, y_n)$

$n$	$x_n$	Method 1.0		Method 1.1		
		$y_n$	$\varepsilon_n$	$y_n^c$	$c_n$	$\varepsilon_n$
0	0.0	1.000000	0.000000	1.000000	0.000000	0.000000
1	0.2	0.840000	-0.002538	0.839200	0.000800	-0.001738
2	0.4	0.744800	-0.004160	0.743344	0.001456	-0.002704
3	0.6	0.702736	-0.005113	0.700742	0.001994	-0.003119
4	0.8	0.704244	-0.005586	0.701808	0.002436	-0.003150
5	1.0	0.741480	-0.005721	0.738682	0.002798	-0.002923

For instance, we have

$$y'(x, 0.4) = -0.3448 + 1.3448x, \quad y_2'(0.4) = -0.3448,$$

$$y''(x, 0.4) = 1.3448, \quad y_2''(0.4) = 1.3448. \quad \square$$

Note that the values of derivatives calculated at the points  $(x_n, y_n)$  are exact in the sense that they provide values of the derivatives for the solution with the initial condition  $(x_n, y_n)$ . They can be used to estimate derivatives of  $y(x)$  at the point  $(x_n, y(x_n))$ .

The possibility to improve the accuracy of a solution going forward and backward has been the main idea of many numerical methods for ODEs (see, e.g., [6, 15]). Results for the Method 1.0 presented in Table 2 can be improved using similar ideas in the new framework using the values of derivatives that the method provides during its functioning. Let us consider the exact solution  $y(x)$  from (20) and its approximation  $y(x, 0)$  from (22) on the interval  $[0, 0.5]$  (we choose this interval that is larger than the step  $h = 0.2$  used in Table 2 in order to be able to show the idea more clearly graphically). In Fig. 1, the graph of  $y(x, 0)$  is represented by the top (black) solid line and the graph of  $y(x)$  is represented by the lower (grey) solid line. Naturally, both of them start at the point  $(0, 1)$  and we then have that

$$y(0.5) = 0.713061319, \quad y(0.5, 0) = 0.75.$$

Thus, the error  $\varepsilon_1$  at the point  $x_1 = 0.5$  is

$$\varepsilon_1 = y(0.5) - y(0.5, 0) = -0.036938681. \quad (23)$$

By using our equation (19) and executing two infinitesimal steps from the point  $x_1 = 0.5$  to the points  $0.5 + \mathbb{1}^{-1}$  and  $0.5 + 2\mathbb{1}^{-1}$  we obtain

$$y(x, 0.5) = 0.75 - 0.25x + 0.625x^2. \quad (24)$$

In the Method 1.0, this formula would be used to go forward from the point  $x_1 = 0.5$  by executing a new finite step. Instead of this, let us use this formula to go

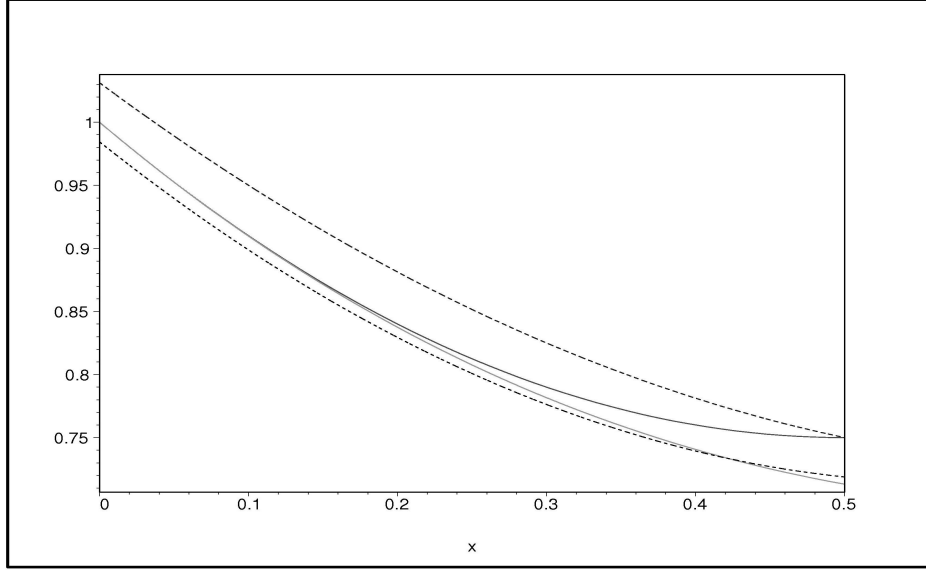


Figure 1: The graph of  $y(x, 0)$  is represented by the top (black) solid line and the graph of  $y(x)$  is represented by the lower (grey) solid line (both  $y(x, 0)$  and  $y(x)$  start at the point  $(0, 1)$ ); the graph of the function  $\bar{y}(x)$  is shown by the top dashed line; the graph of the function  $r_1(x)$  is shown by the lower dashed line.

backward from the point  $x_1 = 0.5$  to the starting point  $x_0 = 0$ . Since in (24)  $x$  is intended to be a removal from the point  $x_1 = 0.5$ , we need to take into account this fact. The graph of the obtained function

$$\bar{y}(x) = 0.75 + 0.25(0.5 - x) + 0.625(0.5 - x)^2 = 1.03125 - 0.875x + 0.625x^2$$

is shown in Fig. 1 by the top dashed line.

It can be seen that  $y(x, 0)$  from (22) does not coincide with the obtained function  $\bar{y}(x)$ . Let us construct a new quadratic approximation  $r_1(x)$  oriented on a better result to be obtained at the point  $x_1 = 0.5$ . The function  $r_1(x)$  is built using coefficients from both  $y(x, 0)$  and  $\bar{y}(x)$  by taking their average with the weights  $\frac{1}{2}$  and  $\frac{1}{2}$  (the general way to mix them is, obviously,  $\tau$  and  $1 - \tau$ ,  $0 < \tau < 1$ ) as follows:

$$\begin{aligned} r_1(x) &= y(0, 0) + \frac{1}{2}(y(0, 0) - \bar{y}(0)) + \frac{1}{2}(y'(0, 0) + \bar{y}'(0))x + \frac{1}{4}(y''(0, 0) + \bar{y}''(0))x^2 = \\ &= 1 + \frac{1}{2}(1 - 1.03125) + \frac{1}{2}(-1 - 0.875)x + \frac{1}{2}(1 + 0.625)x^2 = \\ &= 0.984375 - 0.9375x + 0.8125x^2. \end{aligned} \quad (25)$$

In Fig. 1, the graph of the function  $r_1(x)$  is shown by the lower dashed line. The function  $r_1(x)$  provides us the value  $r_1(0.5) = 0.718750$  and the respective error

$$\bar{\varepsilon}_1 = y(0.5) - r_1(0.5) = -0.005688681. \quad (26)$$



that is better than the error  $\varepsilon_1 = -0.036938681$  from (23) that is obtained by calculating  $y(0.5, 0)$ .

The possibility to calculate corrections to approximations opens the doors to various modifications. For instance, it is possible to execute two additional infinitesimal steps at the point  $x_1 = 0.5$  using the value  $r_1(0.5)$  instead of  $y(0.5, 0)$ . In general, this means that instead of setting  $y_n = y(x_n, x_{n-1})$  as it is done by the Method 1.0 we put  $y_n = r_n(x_n)$ . Obviously, this means that it is necessary to evaluate  $f(x, y)$  two times more with respect to the Method 1.0. Otherwise it is also possible to use the corrected value  $r_n(x_n)$  with the derivatives that have been calculated for  $y(x_n, x_{n-1})$ .

Another possibility would be the use of the functions  $y(x, x_n)$  at each point  $x_n$ , i.e., to put  $y_n = y(x_n, x_{n-1})$ , and to calculate the global correction following the rule

$$c_n = c(x_n) = c(x_{n-1}) + r_n(x_n) - y(x_n, x_{n-1}), \quad (27)$$

starting from the first correction (in our example  $c(x_1) = c(0.5) = 0.031250$ )

$$c(x_1) = r_1(x_1) - y(x_1, x_0).$$

In this way we can approximate the exact solution  $y(x_n)$  by the corrected value

$$y_n^c = y(x_n, x_{n-1}) + c(x_n).$$

In Table 2, results for this algorithm are presented in the column *Method 1.1* where the error  $\varepsilon_n$  is calculated as

$$\varepsilon_n = y(x_n) - y_n^c.$$

Notice that the correction obtained at the final point has been calculated using Corollary 1.

We conclude this subsection by a reminder that Theorem 1 gives us the possibility to easily construct higher-order methods. Two methods described above just show examples of the usage of infinitesimals for building algorithms for solving ODEs.

## 4.2 Approximating derivatives of the solution

In this subsection, we show how approximations of derivatives at the point  $x_n$  can be obtained using the information calculated at the point  $x_{n-1}$ . For this purpose, instead of the usage of a finite step  $h$ , the steps  $h - \mathbb{1}^{-1}$  or  $h + \mathbb{1}^{-1}$  can be used. To introduce this technique we need to recall the following theorem from [40].

**Theorem 2** *Suppose that: (i) for a function  $s(x)$  calculated by a procedure implemented at the Infinity Computer there exists an unknown Taylor expansion in a finite neighborhood  $\delta(z)$  of a purely finite point  $z$ ; (ii)  $s(x), s'(x), s''(x), \dots, s^{(k)}(x)$  assume purely finite values or are equal to zero at purely finite  $x \in \delta(z)$ ; (iii)  $s(x)$*

has been evaluated at a point  $z + \mathbb{1}^{-1} \in \delta(z)$ . Then the Infinity Computer returns the result of this evaluation in the positional numeral system with the infinite radix  $\mathbb{1}$  in the following form

$$s(z + \mathbb{1}^{-1}) = c_0 \mathbb{1}^0 c_{-1} \mathbb{1}^{-1} c_{-2} \mathbb{1}^{-2} \dots c_{-(k-1)} \mathbb{1}^{-(k-1)} c_{-k} \mathbb{1}^{-k}, \quad (28)$$

where

$$s(z) = c_0, \quad s'(z) = c_{-1}, \quad s''(z) = 2!c_{-2}, \quad \dots \quad s^{(k)}(z) = k!c_{-k}. \quad (29)$$

The theorem tells us that if we take a purely finite point  $z$  and evaluate on the Infinity Computer  $s(x)$  at the point  $z + \mathbb{1}^{-1}$  then from the computed  $s(z + \mathbb{1}^{-1})$  we can easily extract  $s(z)$ ,  $s'(z)$ ,  $s''(z)$ , etc. To apply this theorem to our situation we can take as  $s(x)$  the Taylor expansion for  $y(x)$  constructed up to the  $k$ th derivative using infinitesimal steps  $\mathbb{1}^{-1}$ ,  $2\mathbb{1}^{-1}$ ,  $\dots$ ,  $k\mathbb{1}^{-1}$ . Then, if we take as  $z$  a purely finite step  $h$  and evaluate on the Infinity Computer  $s(x)$  at the point  $h + \mathbb{1}^{-1}$  then we obtain  $s(h)$ ,  $s'(h)$ ,  $s''(h)$ , etc.

For instance, let us take  $s(x) = s_2(x) = y(x, 0)$ , where  $y(x, 0)$  is from (22) and  $s_2(x)$  indicates that we use two derivatives in the Taylor expansion. Then, we have

$$s_2(0.2 + \mathbb{1}^{-1}) = 1 - (0.2 + \mathbb{1}^{-1}) + (0.2 + \mathbb{1}^{-1})^2 = 0.84 - 0.6\mathbb{1}^{-1} + \mathbb{1}^{-2}.$$

We have the exact (where the word ‘‘exact’’ again means: with the accuracy of the implementation of  $s(x)$ ) values  $s(0.2) = 0.84$ ,  $s'(0.2) = -0.6$ ,  $s''(0.2) = 1$  for the function  $s(x)$ . These values can be used to approximate the respective values  $y(0.2)$ ,  $y'(0.2)$ ,  $y''(0.2)$  we are interested in. Moreover, we can adaptively obtain an information on the accuracy of our approximations by consecutive improvements. If we calculate now  $y^{(3)}(0)$  from (14) then we can improve our approximation by setting  $s(x) = s_3(x)$  where

$$\begin{aligned} s_3(0.2 + \mathbb{1}^{-1}) &= s_2(0.2 + \mathbb{1}^{-1}) - \frac{1}{3}(0.2 + \mathbb{1}^{-1})^3 = \\ &= s_2(0.2 + \mathbb{1}^{-1}) - 0.002667 - 0.04\mathbb{1}^{-1} - 0.2\mathbb{1}^{-2} - \frac{1}{3}\mathbb{1}^{-3} = \\ &= 0.837333 - 0.64\mathbb{1}^{-1} + 0.8\mathbb{1}^{-2} - \frac{1}{3}\mathbb{1}^{-3}. \end{aligned} \quad (30)$$

Note, that to obtain this information we have calculated only the additional part of  $s_3(0.2)$  taking the rest from the already calculated value  $s_2(0.2)$ .

Analogously, if we calculate now  $y^{(4)}(0)$  from (15) then we can improve our approximation again by setting  $s(x) = s_4(x)$  where

$$\begin{aligned} s_4(0.2 + \mathbb{1}^{-1}) &= s_3(0.2 + \mathbb{1}^{-1}) + \frac{1}{12}(0.2 + \mathbb{1}^{-1})^4 = \\ &= s_3(0.2 + \mathbb{1}^{-1}) + 0.000133 + 0.002667\mathbb{1}^{-1} + 0.02\mathbb{1}^{-2} + 0.066667\mathbb{1}^{-3} + \frac{1}{12}\mathbb{1}^{-4} = \end{aligned}$$

$$0.837466 - 0.637333\mathbb{1}^{-1} + 0.82\mathbb{1}^{-2} - 0.266667\mathbb{1}^{-3} + \frac{1}{12}\mathbb{1}^{-4}.$$

Since we have used the convergent Taylor expansion of the fourth order, the errors in calculating  $y(0.2)$ ,  $y'(0.2)$ , and  $y''(0.2)$  are of the orders 5, 4, and 3, respectively.

### 4.3 An automatic control of rounding errors

In the previous sections, we have supposed that the evaluation of derivatives of  $y(x)$  was done exactly, i.e., the procedure for evaluating  $f(x, y)$  was sufficiently precise and it was possible to neglect rounding errors. The executed numerical examples presented above satisfied this assumption. Let us now study what the Infinity Computer can give us when  $f(x, y)$  is calculated with errors. Hereinafter we suppose again that for the solution  $y(x)$ ,  $x \in [a, b]$ , of (1) there exists the Taylor expansion (unknown for us) and at purely finite points  $s \in [a, b]$ , the function  $y(s)$  and all its derivatives assume purely finite values or are equal to zero. In addition, we assume that the same conditions hold for all approximations of  $y(x)$  the method will deal with.

Let us consider formulae (9) for calculating  $y_1$  and  $y_2$  together with formulae (12) and (13) used for approximating  $y'(x_0)$  and  $y''(x_0)$ . Suppose that  $f(x_0, y_0)$  is calculated with an unknown error  $\epsilon_1$ . Then we have that instead of the derivative  $y'(x_0)$  and the point  $y_1$  we have

$$\tilde{y}'(x_0) = f(x_0, y_0) - \epsilon_1 = y'(x_0) - \epsilon_1 \quad (31)$$

$$\tilde{y}_1 = y_0 + \mathbb{1}^{-1}(f(x_0, y_0) - \epsilon_1) = y_1 - \epsilon_1\mathbb{1}^{-1},$$

Analogously, calculation of the point  $y_2$  will give us  $\tilde{y}_2$  and  $\Delta_{\mathbb{1}^{-1}}^2$  will be calculated with errors, as well, giving us  $\tilde{\Delta}_{\mathbb{1}^{-1}}^2$ . Let us study the structure of this forward difference

$$\tilde{\Delta}_{\mathbb{1}^{-1}}^2 = y_0 - 2\tilde{y}_1 + \tilde{y}_2 = \tilde{y}_2 - \tilde{y}_1 - (\tilde{y}_1 - y_0) = \tilde{y}_2 - \tilde{y}_1 - (y_1 - y_0) + \epsilon_1\mathbb{1}^{-1}. \quad (32)$$

By applying the argumentation analogous to that used in Theorem 1 together with our assumptions on purely finiteness of all derivatives of approximations of  $y(x)$  we have that

$$\tilde{y}_2 - \tilde{y}_1 - (y_1 - y_0) = \tilde{c}_{-2}\mathbb{1}^{-2} + \dots + \tilde{c}_{-m_2}\mathbb{1}^{-m_2}, \quad (33)$$

where the coefficients  $\tilde{c}_{-2}, \dots, \tilde{c}_{-m_2}$  are affected by rounding errors and errors incorporated in  $\tilde{y}_1$  and  $\tilde{y}_2$ . Thus, instead of the exact second derivative that has been obtained in Theorem 1 from the coefficient  $c_{-2}$  of  $\mathbb{1}^{-2}$ , the coefficient  $\tilde{c}_{-2}$  gives us an approximation  $\tilde{y}''(x_0)$  of  $y''(x_0)$ , namely,

$$\tilde{c}_{-2} = \tilde{y}''(x_0) = y''(x_0) - \epsilon_2,$$

where  $\epsilon_2$  is an error we have got during the calculation of  $y''(x_0)$ .

Let us rewrite now (32) in the decreasing orders of the powers of grossone using the representation (33), i.e., as the Infinity Computer does it. We have

$$\tilde{\Delta}_{\mathbb{1}-1}^2 = \epsilon_1 \mathbb{1}^{-1} + \tilde{c}_{-2} \mathbb{1}^{-2} + \dots + \tilde{c}_{-m_2} \mathbb{1}^{-m_2}. \quad (34)$$

This means that by calculating  $\tilde{\Delta}_{\mathbb{1}-1}^2$  we have obtained also the error  $\epsilon_1$  that we have got at the previous infinitesimal step (see (31)). We are able now to reestablish the exact value of the first derivative  $y'(x_0)$  using the approximative value  $\tilde{y}'(x_0)$  calculated in (31) and the grossdigit corresponding to  $\mathbb{1}^{-1}$  by taking it from  $\tilde{\Delta}_{\mathbb{1}-1}^2$  in (34), i.e., we have

$$y'(x_0) = \tilde{y}'(x_0) + \epsilon_1.$$

By a complete analogy we can continue and calculate

$$\tilde{\Delta}_{\mathbb{1}-1}^3 = -\epsilon_1 \mathbb{1}^{-1} + \epsilon_2 \mathbb{1}^{-2} + \tilde{c}_{-3} \mathbb{1}^{-3} + \dots + \tilde{c}_{-m_3} \mathbb{1}^{-m_3}, \quad (35)$$

$$y''(x_0) = \tilde{y}''(x_0) + \epsilon_2.$$

Note that in (35)  $\epsilon_1$  (that can be either positive or negative) appears with the alternated sign following the formulae of forward differences. In fact, in  $\tilde{\Delta}_{\mathbb{1}-1}^3$  we have  $y_1 - y_0$  whereas in  $\tilde{\Delta}_{\mathbb{1}-1}^2$  we have  $-(y_1 - y_0)$ . Analogously, the same alternation happens for higher derivatives.

In general, in order to calculate the  $(k-1)$ th derivative  $y^{(k-1)}(x_0)$  it is necessary to calculate the approximation  $\tilde{y}^{(k-1)}(x_0) = \tilde{c}_{-(k-1)}$  and then to extract the error  $\epsilon_{k-1}$  (that can be negative or positive) from

$$\begin{aligned} \tilde{\Delta}_{\mathbb{1}-1}^k &= (-1)^k \epsilon_1 \mathbb{1}^{-1} + \dots + (-1)^{k-i-1} \epsilon_i \mathbb{1}^{-i} + \dots \\ &- \epsilon_{k-2} \mathbb{1}^{-(k-2)} + \epsilon_{k-1} \mathbb{1}^{-(k-1)} + \tilde{c}_{-k} \mathbb{1}^{-k} + \dots + \tilde{c}_{-m_k} \mathbb{1}^{-m_k}, \\ y^{(k-1)}(x_0) &= \tilde{y}^{(k-1)}(x_0) + \epsilon_{k-1}. \end{aligned} \quad (36)$$

If there exists an index  $j$ ,  $1 \leq j < k$ , such that  $\epsilon_1 = \dots = \epsilon_j = 0$ , then  $y^{(k-1)}(x_0)$  is calculated again by the formula (36) but it follows

$$\begin{aligned} \tilde{\Delta}_{\mathbb{1}-1}^k &= (-1)^{k-j-1} \epsilon_{j+1} \mathbb{1}^{-(j+1)} + \dots \\ &+ \epsilon_{k-1} \mathbb{1}^{-(k-1)} + \tilde{c}_{-k} \mathbb{1}^{-k} + \dots + \tilde{c}_{-m_k} \mathbb{1}^{-m_k}. \end{aligned} \quad (37)$$

Thus, either  $f(x, y)$  is evaluated exactly or rounding errors are present, the Infinity Computer is able to calculate the derivatives of the solution exactly. Let us illustrate the theoretical results presented above by a numerical example.

**Example 5.** Let us consider the following test problem<sup>5</sup> taken from [18]. The ODE

$$y'(x) = -\frac{x-c}{s^2}(y-1) \quad (38)$$

<sup>5</sup>The author thanks Prof. H. P. Langtangen for drawing the author's attention to this nice example.

Table 3: Calculating approximations for derivatives  $y^{(i)}(0)$ ,  $1 \leq i \leq 12$ , for the problem (38) at the point  $(0, y(0))$  by applying the automatic control of rounding errors  $\varepsilon_i$  and providing so the final accuracy  $\delta_i$

$i$	$\tilde{y}^{(i)}(0)$	$\varepsilon_i$	$y^{(i)}(0)$	$\delta_i$
1	$0.182759757 \cdot 10^{-6}$	0.000000000	$0.182759757 \cdot 10^{-6}$	$-0.60449198 \cdot 10^{-28}$
2	$0.207127725 \cdot 10^{-5}$	$0.609199190 \cdot 10^{-7}$	$0.213219716 \cdot 10^{-5}$	$-0.69190731 \cdot 10^{-27}$
3	$0.233932489 \cdot 10^{-4}$	$0.731039028 \cdot 10^{-6}$	$0.241242879 \cdot 10^{-4}$	$-0.78192941 \cdot 10^{-26}$
4	$0.254888941 \cdot 10^{-3}$	$0.901614801 \cdot 10^{-5}$	$0.263905089 \cdot 10^{-3}$	$-0.83928642 \cdot 10^{-25}$
5	$0.266975453 \cdot 10^{-2}$	$0.111117932 \cdot 10^{-3}$	$0.278087246 \cdot 10^{-2}$	$-0.85899500 \cdot 10^{-24}$
6	$0.267228880 \cdot 10^{-1}$	$0.136947978 \cdot 10^{-2}$	$0.280923676 \cdot 10^{-1}$	$-0.82216871 \cdot 10^{-23}$
7	$0.253489245 \cdot 10^0$	$0.168782291 \cdot 10^{-1}$	$0.270367474 \cdot 10^0$	$-0.71132365 \cdot 10^{-22}$
8	$0.224980672 \cdot 10^1$	$0.208016667 \cdot 10^0$	$0.245782339 \cdot 10^1$	$-0.50790463 \cdot 10^{-21}$
9	$0.182784086 \cdot 10^2$	$0.256371293 \cdot 10^1$	$0.208421215 \cdot 10^2$	$-0.19195037 \cdot 10^{-20}$
10	$0.13002719 \cdot 10^3$	$0.315966218 \cdot 10^2$	$0.161623816 \cdot 10^3$	$0.25832403 \cdot 10^{-19}$
11	$0.71638662 \cdot 10^3$	$0.389414314 \cdot 10^3$	$0.110580093 \cdot 10^4$	$0.86669850 \cdot 10^{-18}$
12	$0.13588050 \cdot 10^4$	$0.479935826 \cdot 10^4$	$0.615816329 \cdot 10^4$	$0.16535675 \cdot 10^{-16}$

has the exact solution that is the following Gaussian function

$$u(x) = 1 + e^{-\frac{1}{2}\left(\frac{x-c}{s}\right)^2} \quad (39)$$

centered around  $t = c$  and with characteristic width (standard deviation)  $s$ . The initial condition is taken as the exact value  $y(0) = u(0)$  and the parameters are taken as  $c = 3, s = 0.5$ .

In Table 3, we calculate derivatives  $y^{(i)}(0)$ ,  $1 \leq i \leq 12$ , using (36) with 30 digits in the mantissa, in order to be able to catch the final accuracy  $\delta_k$  presented in the last column. It shows the final error obtained by subtracting from the derivatives calculated using the explicit solution (39) and the derivatives  $y^{(i)}(0)$ ,  $1 \leq i \leq 12$ , i.e.,

$$\delta_i = u^{(i)}(0) - y^{(i)}(0), \quad 1 \leq i \leq 12.$$

Let us make a few remarks regarding Table 3. First, at  $x = 0$  with  $c = 3$  and  $s = 0.5$  it follows from (38) that  $-\frac{x-c}{s^2} = 12$ . In order to illustrate the situation (37), we have calculated  $y_1$  using (instead of the original expression  $12(y_0 - 1)$  from (38) leading to  $\varepsilon_1 \neq 0$ ) the expression  $12y_0 - 12$  that provides  $\varepsilon_1 = 0$  when it is used in  $y_1 = y_0 + \mathbb{1}^{-1}(12y_0 - 12)$ .

Then, it is worthwhile to notice that almost through the whole Table 3 (and in spite of large values of higher derivatives) the relative error has the constant order equal to  $10^{-22}$  (it can be easily seen from Table 3 that  $m - n = 22$  where  $m$  is the exponent of  $y^{(i)}$  and  $n$  is the exponent of  $\delta_i$ ). Notice also that at the last line of the Table the error  $\varepsilon_{12}$  is even larger than the approximation  $\tilde{y}^{(12)}(0)$ .  $\square$

## 5 Conclusion

In this paper, a new framework for solving ODEs has been introduced. The new approach allows us to work numerically not only with usual finite numbers but also

with different infinitesimal and infinite values on a new kind of a computational device called the Infinity Computer (it has been patented and its working prototype exists). The structure of numbers we work on the new computer is more complex and, as a result, we face new computational possibilities. In particular, the presence of different numerical infinitesimals makes it possible to use infinitesimal steps for solving ODEs. The following results have been established in the new framework.

i. It has been shown that (under the assumption that the person solving the ODE does not know the structure of  $f(x, y)$ , i.e., it is a “black box” for him/her) the Infinity Computer is able to calculate numerical values of the derivatives of  $y(x)$  of the desired order without the necessity of an analytical (or symbolical) computation of the respective derivatives by the successive derivation of the ODE as it is usually done when the Taylor method is applied.

ii. If the region of our interest  $[a, b]$  belongs to the region of convergence of the Taylor expansion for the solution  $y(x)$  in the neighborhood of the point  $x_0$ , then it is not necessary to construct iterative procedures involving several steps with finite values of  $h$ . It becomes possible to calculate approximations of the desired order  $k$  by executing  $k$  infinitesimal steps and only one finite step.

iii. Approximations of derivatives of  $y(x)$  at the point  $x_n$  can be obtained using the information calculated at the point  $x_{n-1}$ . For this purpose, instead of the usage of a finite step  $h$ , the steps  $h - \mathbb{1}^{-1}$  or  $h + \mathbb{1}^{-1}$  can be used. Methods going forward and backward and working with approximations of derivatives can be proposed in the new framework.

iv. The last subsection of the manuscript shows that either  $f(x, y)$  is evaluated exactly or rounding errors are present, the Infinity Computer is able to perform, by means of a smart usage of infinitesimals, an automatic control of the accuracy of computation of the derivatives of the solution.

v. Theoretical results have been illustrated by a number of numerical examples.

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