A comprehensive review of diamonds and their incarnations in the Langlands program

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Abstract A comprehensive review of diamonds, in the sense of Scholze, is presented. The diamond formulations of the Fargues-Fontaine curve and Bun_G are stated. Principal results centered on the diamond formalism in the global Langlands correspondence and the geometrization of the local Langlands correspondence are given. We conclude with a discussion of future geometrizations, and conjecture a diamond reformulation of quantum computational complexity towards a diamond $ER = EPR$.

Keywords diamond; v-stack; perfectoid space; geometrization of local Langlands

1 Introduction

A diamond [17] is a functorial-geometric construction with an incredibly rich structure and equally elegant formalism, whose arithmetic-geometric foundations live in rigid analytic geometry. The sophisticated definition of the diamond itself is commendable, let alone its principal utility in the Langlands Program, which is often, informally, claimed to be a "Grand Unified Theory" of mathematics. The notion of a *diamond* makes its first appearance in the *Berkeley Lectures on p-adic Geometry* [21]. A six operations formalism is then constructed in the *Étale cohomology of diamonds* [17] (See Appendix). Diamonds are like bridges between objects that admit universal constructions.

The foundation of diamonds abides in nonarchimedean geometry; namely in adic spaces. An adic space is a particular geometric object that resembles a scheme. The adic form of "locally ringed space" is constructed as a "topologically ringed topological space equipped with valuations" [21]. Specifically, let A be a Huber ring,

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and A^+ a ring of integral elements. The pair (A, A^+) is called a "Huber pair" ([21] Definition 2.2.12). We have the following definition of the adic spectrum.

Definition 1. *([21] Definition 2.3.2) The adic spectrum $Spa(A, A^+)$ is the set of equivalence classes of continuous valuations $| * |$ on A such that $|A^+| \leq 1$ for a sheafy Huber pair (A_i, A_i^+).*

An adic space is formally defined as follows.

Definition 2. *([21] Definition 3.2.1) We define a category (V) as follows. The objects are triples $(X, \mathcal{O}_X, (| * (x)|)_{x \in X})$, where X is a topological space, $\mathcal{O}_x$ is a sheaf of topological rings, and for each $x \in X$, $| * (x)|$ is an equivalence class of continuous valuations on $\mathcal{O}_{X,x}$. (Note that this data determines $\mathcal{O}_X^+$). The morphisms are maps of toplogically ringed topological spaces $f : X \to Y$ (so that the map $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ is continuous for each open $V \subset Y$) that make the following diagram commute up to equivalence for all $x \in X$:*

$$\begin{array}{ccc} \mathcal{O}_{Y,f(x)} & \longrightarrow & \mathcal{O}_{X,x} \\ \downarrow & & \downarrow \\ \Gamma_{f(x)} \cup \{0\} & \longrightarrow & \Gamma_x \cup \{0\} \end{array}$$

Fig. 1 Commutative diagram for adic space construction.

An adic space is an object $(X, \mathcal{O}_X, (| * (x)|)_{x \in X})$ of (V) that admits a covering by spaces U_i such that the triple $(U_i, \mathcal{O}_{X|U_i}, (|*(x)|)_{x \in U_i})$ is isomorphic to $Spa(A_i, A_i^+)$ for a sheafy Huber pair (A_i, A_i^+).

For sheafy (A, A^+), the topological space $X = Spa(A, A^+)$ together with its structure sheaf and continuous valuations is an affinoid adic space, which we continue to write as $Spa(A, A^+)$.

The "intuitive definition of diamonds [21]" centers around certain adic spaces called perfectoid spaces [14], and the *tilting functor*, which is a "functor from perfectoid spaces to perfectoid spaces of characteristic p [21]." The notion of *perfectoid space* is first introduced in Scholze's manuscript *Perfectoid Spaces* [14].

Definition 3. *([21] Definition 6.1.1) A complete Tate ring R is perfectoid if R is uniform and there exists a pseudo-uniformizer $\bar{\omega} \in R$ such that $\bar{\omega}^p | p$ holds in R°, and such that the pth power Frobenius map*

$$\Phi : R^\circ / \bar{\omega} \to R^\circ / \bar{\omega}^p$$

is an isomorphism.

Let R^+ denote a perfectoid ring of integral elements [17]. A perfectoid space is formally defined as follows.

Definition 4. *([21] Definition 7.1.2) A perfectoid space is an adic space covered by affinoid adic spaces* $Spa(R, R^+)$ *with R perfectoid.*

A few examples of Perfectoid spaces are:

- The perfectoid shimura variety used in studying "torsion in the cohomology of compact unitary Shimura varieties [18]";
- Any completion of an "arithmetically profinite extension," in the sense of Fontaine and Wintenberger [14]; and
- The Lubin Tate tower at infinite level [20].

We briefly recall the tilting operation.

Definition 5. *([21] Definition 6.2.1) Let R be a perfectoid Tate ring. The tilt of R is*

$$R^b = \varprojlim_{x \to x^p} R, \text{ given the inverse limit topology.}$$

If R° is the subring of power bounded elements, we have the following lemma:

Lemma 1. *([21] Lemma 6.2.5). The set of rings of integral elements* $R^+ \subset R^\circ$ *is in bijection with the set of rings of integral elements* $R^{b+} \subset R^{b\circ}$*, via* $R^{b+} = \varprojlim_{x \to x^p} R^+$*. Also,* $R^{b+}/\bar{\omega}^b = R^+/\bar{\omega}$*.*

The following theorem highlights the utility of the tilting operation.

Theorem 1. *([21] Theorem 6.2.7) Let R be a perfectoid ring with tilt* R^b*. Then there is an equivalence of categories between perfectoid R-algebras and perfectoid* R^b*-algebras, via* $S \to S^b$*.*

The tilting functor has an inverse functor called the *untilt*, which is a certain sheaf.

Lemma 2. *([21] Lemma 6.2.8). Let* $(R^\#, R^{\#+})$ *be an untilt of* (R, R^+)*, i.e. a perfectoid Tate ring* $R^\#$ *together with an isomorphism* $R^{\#b} \to R$ *such that* $R^{\#+}$ *and* R^+ *are identified under Lemma 6.2.5.*

1. There is a canonical surjective ring homomorphism

$$\theta : W(R^+) \to R^{\#+}$$

$$\sum_{n \geq 0} [r_n] p^n \to \sum_{n \geq 0} r_n^\# p^n.$$

2. The kernel of θ *is generated by a nonzero divisor* ξ *of the form* $\xi = p + [\bar{\omega}]\alpha$ *where* $\bar{\omega} \in R^+$ *is a pseudo-uniformizer, and* $\alpha \in W(R^+)$*.*

As explained in ([21] *Section 8.1 Diamonds: Motivation*), the construction of a diamond mirrors the construction of an algebraic space, classically formed by taking a quotient of a scheme by an étale equivalence relation. To wit, the motivation of diamonds is the construction of a functor

- $\{$analytic adic spaces over $Z_p\} \to \{$diamonds$\}$

- $\{X\} \to \{X^{\diamond}\}$

that “forgets the structure morphism to Z_p [21]. ”

The tilting functor $X \to X^b$ exhibits this property, when X is a perfectoid space [21]. This idea can be generalized. Notice that for any analytic adic space X/Z_p,

X is pro-étale locally perfectoid:

$$X = \mathrm{Coeg}(\tilde{X} \times_X \tilde{X} \rightrightarrows \tilde{X}),$$

where $\tilde{X} \to X$ is a pro-étale perfectoid cover;...the equivalence relation $R = \tilde{R} \times_X \tilde{X}$ is also perfectoid (as it is pro-étale over $\tilde{X}$), at least after passing to a uniform completion [21].

Let R^b be the *tilt* of R and $\tilde{X}^b$ be the *tilt* of $\tilde{X}$. Then, the desired functor above should take X to $\mathrm{Coeg}(R^b \rightrightarrows \tilde{X}^b)$. However, two questions immediately arise:

Question 1. *What category does this object live in [21]?*

Question 2. *There is also the question of whether this construction depends on the choices made. Whatever this object is, it is pro-étale under a perfectoid space in characteristic p, namely $\tilde{X}^b$ [21].*

The following example provides the needed clarification.

Example 1. *([17] Example 8.1.1) If $X = Spa(Q_p)$, then a pro-étale perfectoid cover of X is $\tilde{X} = Spa(Q_p^{cycl})$. Thus $R = \tilde{X} \times_X \tilde{X}$ is essentially $\tilde{X} \times Z_p^x$. This is a perfectoid space, and so $X^{\diamond}$ should be the coequalizer of $\tilde{X}^b \times Z_p^x \rightrightarrows \tilde{X}$, which comes out to be the quotient $Spa((Q_p^{cycl})^b)/Z_p^x$, whatever this means.*

It turns out that the quotient $Spa(Q_p^{cycl})^b/Z_p^x$ lives in a “category of sheaves on the site of perfectoid spaces with pro-étale covers [21].” Once formalized, this quotient will be our principal example of a diamond, discussed in Section 2.1.

A diamond takes the form of a Grothendieck functor of points [9], that *maps in* perfectoid spaces. In particular, a diamond is an “algebraic space for the pro-étale topology in Perf” ([5] Definition 1.12), where Perf denotes the category of perfectoid spaces of characteristic p, which is the full subcategory of the κ-small category of perfectoid spaces. Specifically, diamonds are particular “pro-étale sheaves on Perf [17].”

Diamonds are so named because their properties reflect certain properties of mineralogical diamonds. In particular, their geometric points are, informally, a mathematical representation of mineralogical impurities. To wit, let C be an algebraically closed affinoid field and $\mathcal{D}$ a diamond.

A geometric point $Spa(C) \to \mathcal{D}$ is something like an impurity within a gem which produces a color. This impurity cannot be seen directly, but produces many reflections of this color on the surface of the diamond. Likewise, the geometric point cannot be seen directly, but when we pull it back through a quasi-pro-étale cover $X \to \mathcal{D}$, the result is profinitely many copies of $Spa(C)$. Often one can produce multiple such covers $X \to \mathcal{D}$, which result in multiple descriptions of the geometric points of $\mathcal{D}$ ([21] Figure 9.1).

A diamond is formally defined as follows:

Definition 6. *([21] Definition 1.3) Let Perfd be the category of perfectoid spaces. Let Perf be the subcategory of perfectoid spaces of characteristic p. Let Y be a pro-étale sheaf on Perf. Then Y is a diamond if Y can be written as the quotient X/R with X a perfectoid space of characteristic p and R a pro-étale equivalence relation $R \subset X \times X$.*

The meaning of the pro-étale equivalence relation is developed in the following proposition.

Proposition 1. *([21] Proposition 11.3) Let X be a perfectoid space of characteristic p, and let R be a perfectoid space with two pro-étale maps $s, t : R \to X$ such that the induced map $R \to X \times X$ is an injection making R an equivalence relation on X. Then $D = X/R$ is a diamond and the natural map $R \to X \times X$ is an isomorphism.*

Diamonds appear in the global Langlands correspondence for function fields and in Scholze and Fargues' exceptional geometrization of the local Langlands correspondence [7]. The motivation of diamonds in the geometrization of the local Langlands correspondence is referenced in the following remark [7]:

> The local Langlands conjecture, including its expected functorial behavior with respect to passage to inner forms and Levi subgroups, then still predicts that for any irreducible sheaf $\mathcal{F}$ - necessarily given by an irreducible representation π_b of $G_b(E)$ for some $b \in B(E, G)$ - one can associate an L-parameter $\phi_{\mathcal{F}} : W_E \to \check{G}(C)$.
>
> To go further, we need to bring geometry into the picture. Indeed, it will be via geometry that (sheaves on the groupoid of) G-torsors on $Spec\check{E}/\phi^{\mathbb{Z}}$ will be related to the fundamental group W_E of $Spec\check{E}/\phi^{\mathbb{Z}}$. The key idea is to study a moduli stack of G-torsors on $Spec\check{E}/\phi^{\mathbb{Z}}$. [7]

Indeed the geometry needed is precisely that of the *diamond formalism*.

Diamonds have many *incarnations*. Principal examples of diamonds are discussed in Section 2.2. Two of the incarnations are:

Example 2. $SpdQ_p = Spa(Q_p^{cycl})^b/\underline{Z_p^\times}$*; "a sheaf for the pro-étale topology on Perf" ([21]Definition 9.4.1).*

Example 3. $\mathcal{Y}_{S,E}^\diamond = S \times (Spa\mathcal{O}_E)^\diamond$*: the diamond relative Fargues-Fontaine curve in the geometrization of the local Langlands correspondence [5].*

In [21], it is explained that the underlying topological space of a diamond "can be quite pathological. It may not even be T_0." Consider the following example:

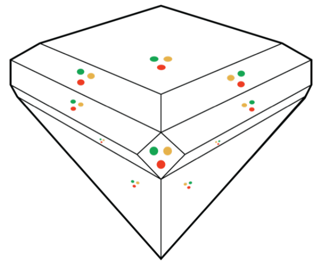

Fig. 2 Representation of Figure 9.1 [21]. Diamond $\mathcal{D}$ with geometric point Spa $C \to \mathcal{D}$ pulled back through a quasi-pro-étale cover $X \to \mathcal{D}$, the result of which is "profinitely many copies of Spa C[21]." [Image @ShannaDobson.]

Example 4. *([21])*

The quotient of the constant perfectoid space Z_p over a perfectoid field by the equivalence relation "congruence modulo Z" produces a diamond with underlying topological space Z_p/Z.

A special class of diamonds, *qcqs diamonds*, are T_0. (See [21] Proposition 10.3.4).

A diamond is further categorized as a *v-sheaf*, which is "a sheaf for the v-topology on Perfd, [21]" the category of perfectoid spaces. The following Proposition generalizes the diamond formalism so that "all diamonds are v-sheaves [21]," mirroring Gabber's result that "algebraic spaces are fpqc sheaves[21]."

Proposition 2. *(Proposition 11.9) Let Y be a diamond. Then Y is a v-sheaf.*

Two powerful descent results follow:

Proposition 3. *([17] Proposition 9.3) The functor which assigns to a totally disconnected affinoid perfectoid X the category {Y/X affinoid perfectoid} is a stack for the v-topology.*

Lemma 3. *([17] Lemma 17.1.8) The fibered category sending any $X \in$ Perfd to the category of locally finite free $\mathcal{O}_X$-modules is a stack on the v-site on Perfd.*

v-sheaves are further classified as "small" and "spatial", which we discuss in detail below. Using the v-topology, one can show that "certain pro-étale sheaves on Perf are diamonds without finding an explicit pro-étale cover [21]." In particular, it is explained that "the more general formalism of v-sheaves makes it possible to consider not only analytic adic spaces as diamonds, but also certain non-analytic objects as v-sheaves [21]." The following is an example of a v-sheaf that is not a diamond.

Let X be any pre-adic space over $\mathbb{Z}_l$. Consider the presheaf $X^\diamond$ on Perf whose S-valued points for $S \in Perf$ are given by the untilts $S^\#$ of S together with a map $S^\# \to X$. If $X = Spa\mathbb{F}_p$, this is the trivial functor, sending any S to a point [21].

This is encapsulated in the following lemma:

Lemma 4. *([21] Lemma 18.1.1) For any pre-adic space X over Z_p, the presheaf $X^\diamond$ is a v-sheaf.*

As our aim is a comprehensive and complete review of diamonds, we exclude proofs of the constructions and key results, and direct the reader to the original articles for further detail. As the diamond construction is extremely technical, our survey consists of a review of fundamental definitions, followed by a collection of principal results. The elegance of the diamond formalism is reflected in its many incarnations.

It is hoped that our comprehensive review will illuminate the importance of the diamond as a very powerful construction used in arithmetic geometry, the future forms of which are promising. While our focus is to illuminate the idea that to *geometrize* the local Langlands correspondence is to use the diamond formalism to construct the geometric Langlands correspondence over the Fargues-Fontaine curve, it is hoped that our survey, which takes the form of a small compendium, can inspire others to develop further incarnations of diamonds.

2 Diamonds

We commence our review with a comprehensive overview of diamonds and v-sheaves. First, we discuss the pro-étale topology and the v-topology, which is a "big topology" on Perfd that is subcanonical and "finer than the pro-étale topology [17]." We then summarize principal results.

2.1 Pro-étale topology and v-topology

Definition 7. *([17] Definition 8.1) Let Perfd be the category of κ-small perfectoid spaces.*

- *The big pro-étale site is the Grothendieck topology on Perfd for which a collection $\{f_i : Y_i \to X\}_{i\in I}$ of morphisms is a covering if all f_i are pro-étale, and for each quasicompact open subset $U \subset X$, there exists a finite subset $J \subset I$ and quasicompact open subsets $V_i \subset Y_i$ for $i \in J$, such that $U = \bigcup_{i\in J} f_i(V_i)$.*
- *Let X be a perfectoid space. The small pro-étale site of X is the Grothendieck topology on the category of perfectoid spaces $f : Y \to X$ pro-étale over X, with covers the same as in the big pro-étale site.*
- *The v-site is the Grothendieck topology on Perfd for which a collection $\{f_i : Y_i \to X\}_{i\in I}$ of morphisms is a covering if for each quasicompact open subset $U \subset X$, there exists a finite subset $J \subset I$ and quasicompact open subsets $V_i \subset Y_i$ for $i \in J$, such that $U = \bigcup_{i\in J} f_i(V_i)$.*

The v-topology is subcanonical.

Theorem 2. *([17] Theorem 1.2) The v-topology on Perf is subcanonical, and for any affinoid perfectoid space* $X = Spa(R, R^+)$, $H^0_v(X, O^+_X) = R^+$ *and for* $i > 0, H^i_v(X, O_X) = 0$ *and* $H^i_v(X, O^+_X)$ *is almost zero.*

There is a promising result that "the structure sheaf is a sheaf for the v-topology on Perfd [21]."

Theorem 3. *([17] Theorem 8.7, Proposition 8.8) The functors* $X \to H^0(X, O_X)$ *and* (X, O_X) *and* $X \to H^0(X, O^+_X)$ *are sheaves on the v-site. Moreover if X is affinoid then* $H^i_v(X, O_X) = 0$ *for* $i > 0$*, and* $H^i_v(X, O^+_X)$ *is almost zero for* $i > 0$.

The Corollary immediately follows:

Corollary 1. *([21] Corollary 17.1.5) Representable presheaves are sheaves on the v-site.*

We now review affinoid pro-étale morphisms.

Definition 8. *([21] Definition 8.2.1) A morphism* $f : Spa(B, B^+) \to Spa(A, A^+)$ *of affinoid perfectoid spaces is affinoid pro-étale if*

- $(B, B^+) = \hat{\varinjlim}(A_i, A^+_i)$ *is a completed filtered colimit of pairs* (A_i, A^+_i) *with* A_i *perfectoid, such that*
- $Spa(A_i, A^+_i) \to Spa(A, A^+)$ *is étale.*

A morphism $f : X \to Y$ *of perfectoid spaces is pro-étale if it is locally on the source and target affinoid pro-étale.*

Nice properties of pro-étale morphisms include the following.

Proposition 4. *([21] Proposition 8.2.5)*

1. *Compositions of pro-étale maps are pro-étale.*
2. *Pullbacks of pro-étale morphisms are pro-étale.*

Example 5. *([21] Example 8.2.2) If X is any perfectoid space and S is a profinite set, we can define a new perfectoid space* $X \times \underline{S}$ *as the inverse limit of* $X \times S_i$*, where* $S = \varprojlim S_i$ *is the inverse limit of finite sets* S_i *. Then* $X \times \underline{S} \to X$ *is pro-étale. This construction extends to the case that S is locally profinite.*

2.2 Examples of Diamonds

2.2.1 $SpdQ_p$

We first discuss $SpdQ_P$, the principal example of a diamond.

Example 6. $SpdQ_p$ *is "a sheaf for the pro-étale topology on Perf [21]." It is defined twice:*

Definition 9. *([21] Section 8.4 The Example of $SpdQ_p$).*

$$SpdQ_p = Spa(Q_p^{cycl})^b/\underline{Z_p^\times} = Spa(F_p((t^{\frac{1}{p^\infty}})))/Z_p^\times,$$

where $Z_p^\times$ acts on $F_p((t^{\frac{1}{p^\infty}}))$ via $\gamma(t) = (1+t)^\gamma - 1$ for all $\gamma \in Z_p^\times$.

This notation is explained as follows:

If X is a perfectoid space and G is a profinite group, then one can define what it means for G to act continuously on X; cf [Sch18]. Equivalently, this is an action of the pro-étale sheaf of groups $\underline{G}$ on X, where $\underline{G}(T)$ is the set of continuous maps from $|T|$ to G, for any $T \in Perf$. In particular, given a continuous action of G on X, one can define $R = X \times \underline{G}$ (which agrees with the previous definition of $X \times \underline{S}$ for a profinite set S), which comes with two maps $R \rightrightarrows X$, given by the projection to the first component, and the action map. If the induced map $R \to X \times X$ is an injection, then X/R is a diamond that we also denote by $X/\underline{G}$ [21].

$SpdQ_p$ is further described by the following Proposition.

Proposition 5. *([21] Proposition 8.4.1). If $X = Spa(R, R^+)$ is an affinoid perfectoid space of characteristic p, then $(SpdQ_p)(X)$ is the set of isomorphism classes of data of the following shape.*

- *A $\underline{Z_p^x}$-torsor $R \to \tilde{R}$; that is, $\tilde{R}$= $(\hat{\varprojlim} R_n)$, where R_n/R is finite étale with Galois group $(Z/p^nZ)^\times$.*
- *A topologically nilpotent element unit $t \in \tilde{R}$ such that for all $\gamma \in Z_p^\times$, $\gamma(t) = (1+t)^\gamma - 1$.*

Definition 10. *([21]Definition 9.4.1) $SpdQ_p = Spa(Q_p^{cycl})^b/\underline{Z_p^\times}$. That is, $SpdQ_p$ is the coequalizer of*

$$\underline{Z_p^\times} \times Spa(Q_p^{cycl})^\flat \rightrightarrows Spa(Q_p^{cycl})^\flat,$$

where one map is the projection and the other is the action.

Per the lemma and corollary below, this construction is well-defined.

Lemma 5. *([21] Lemma 9.4.2). Let $g : \underline{Z_p^\times} \times SpdQ_p^{cycl} \to SpdQ_p^{cycl} \times SpdQ_p^{cycl}$ be the product of the projection onto the second factor and the group action. Then g is an injection.*

Corollary 2. *([21] Corollary 9.4.3). The map $SpaQ_p^{cycl} \to SpdQ_p$ is a $\underline{Z_p^\times}$-torsor and the description of $SpdQ_p$ in Proposition 8.4.1 holds true.*

The following theorem and lemma result.

Theorem 4. *([21] Theorem 9.4.4) The following categories are equivalent:*

- *Perfectoid spaces over Q_p.*
- *Perfectoid spaces X of characteristic p equipped with a "structure morphism" $X \to SpdQ_p$.*

Let Untilt be the presheaf on Perf which assigns to X the set of pairs $(X^\#, \iota)$, where $X^\#$ is a perfectoid space (of whatever characteristic), and $\iota : X^{\#b} \cong X$ is an isomorphism. Then

Lemma 6. *([21] Lemma 9.4.5, [17] Lemma 15.1 (i)) Untilt is a sheaf on Perf.*

2.2.2 Other Key Examples

We now summarize other examples of diamonds.

Example 7. *([21] Definition 15.5). Let Y be an analytic adic space over Z_p. The diamond associated to Y is the v-sheaf defined by*

- $Y^\diamond : X \to \{((X^\#, \iota), f : X^\# \to Y)\}/\simeq$,
- *where $X^\#$ is a perfectoid space with an isomorphism $\iota : (X^\#)^b \simeq X$.*

Example 8. *([21] Theorem 10.1.5) Let X be an analytic pre-adic space over $SpaZ_p$. The presheaf $X^\diamond$ is a diamond.*

Example 9. *For $X = Spa(R, R^+)$, $Spd(R, R^+) = Spa(R, R^+)^\diamond$ [21].*

Example 10. $SpdQ_p \times_\diamond SpdQ_p = \mathcal{Y}^\diamond_{(0,\infty)}/\underline{G_{Q_p}}$*: the diamond self product [21];*

Example 11. *"Every connected component of a spatial v-sheaf $\mathcal{F}$ is a geometric point $Spd(C, C^+)$, for C an algebraically closed nonarchimedean field and $C^+ \subset C$ an open and bounded valuation subring [17]";*

Example 12. $SpdQ_p = Spa(Q_p^{cycl})/\underline{Z_p^x}$ *for* $\underline{Z_p^x}$ *the profinite group* $Gal(Q_p^{cycl}/Q_p)$*[21];*

Example 13. *"All diamonds are v-sheaves" in the v-topology [17];*

Example 14. *Diamond functor ([9],[**?**] Proposition 6.11) For an analytic adic space X/Z_p, the diamond functor $X^\diamond : S \in Perf \to \{S^\#/Z_p$ untilts of S plus map $S^\# \to X\}$ defines a locally spatial diamond.*

Example 15. *The Fargues-Fontaine Curve X_{FF} is a regular noetherian scheme of Krull dimension 1 which is locally the spectrum of a principal ideal domain. The set of closed points of X_{FF} is identified with the set of characteristic 0 untilts of C^b modulo Frobenius. For C an algebraically closed perfectoid field of characteristic $p > 0$ and ϕ the Frobenius automorphism of C we have*

- $X^\diamond_{FF} \cong (SpdC \times SpdQ_p)/(\phi \times id)$ *[21].*

Example 16. *The "mirror curve:" $Div^1 = (Spa\check{E})^\diamond/\phi^{\mathbb{Z}}$ [7].*

Example 17. $\mathcal{Y}^\diamond_{S,E} = S \times (Spa\mathcal{O}_E)^\diamond$*: the diamond relative Fargues-Fontaine curve in the geometrization of the local Langlands correspondence [5];*

Example 18. *([21] Proposition 8.3.7). Let $\mathcal{D}$ and $\mathcal{D}'$ be diamonds. Then the product sheaf $D \times_\diamond D'$ is also a diamond. [21].*

Example 19. $SpdQ_p = Spd(Q_p^{cycl})/\underline{Z_p^x}$ *where* $\underline{Z_p^x}$ *is the profinite group* $Gal(Q_p^{cycl}/Q_p)$ *[21].*

Example 20. $SpdQ_p \times_\diamond SpdQ_p$ *[21].*

Example 21. *The moduli space $Sht_{\mathcal{G},b,\{\mu_i\}}$: "the moduli space of mixed-characteristic local G-shtukas is a locally spatial diamond [21]" fibered over the m-fold product*

- $SpaQ_p \times SpaQ_p \times \cdots_m \times SpaQ_p$;

Example 22. *$\pi_1((SpdQ_p)^n/p.Fr.) \simeq G^m_{Q_p}$: the diamond version of Drinfeld's lemma for the $n = 2$ case for global Langlands for function fields [21];*

Example 23. *All Banach-Colmez spaces are diamonds [21].*

Example 24. *Any closed subset of a diamond is a diamond [21].*

Example 25. *([17] Proposition 11.5). A pro-étale sheaf Y on Perf is a diamond if and only if there is a surjective quasi-pro-étale map [1] $X \to Y$ from a perfectoid space X.*

2.2.3 v-sheaves

Recall that "all diamonds are v-sheaves [21]."

Proposition 6. *([17] Proposition 11.9) Let Y be a diamond. Then Y is a sheaf for the v-topology.*

v-sheaves are classified as *small* and *spatial*.

2.3 Small v-Sheaves

Definition 11. *([21] Definition 17.2.1) A v-sheaf $\mathcal{F}$ on Perf is small if there is a surjective map of v-sheaves $X \to \mathcal{F}$ from the sheaf represented by a perfectoid space X.*

Any small v-sheaf admits geometric structure.

Proposition 7. *([21] Proposition 17.2.2 (Sch17, Proposition 12.3). Let $\mathcal{F}$ be a small v-sheaf, and let $X \to \mathcal{F}$ be a surjective map of v-sheaves from a diamond X (e.g., a perfectoid space). Then $R = X \times_{\mathcal{F}} X$ is a diamond, and $\mathcal{F} = X/R$ as v-sheaves.*

Remark 1. *([21] Remark).*

> *So to access v-sheaves takes two steps. First, we analyze diamonds as quotients of perfectoid spaces by representable equivalence relations. Second, then we analyze small v-sheaves as quotients of perfectoid spaces by diamond equivalence relations [21].*

We define the underlying topological space of a small v-sheaf.

Definition 12. *([21] Definition 17.2.3). Let $\mathcal{F}$ be a small v-sheaf, and let $X \to \mathcal{F}$ be a surjective map of v-sheaves from a diamond X, with $R = X \times_{\mathcal{F}} X$. Then the underlying topological space of $\mathcal{F}$ is $|\mathcal{F}| = |X|/|R|$. This is well-defined and functorial by Proposition 12.7.*

---

[1] ([17] Definition 9.2.2). Consider the site Perf of perfectoid spaces of characteristic p with the pro-étale topology. A map $f : \mathcal{F} \to \mathcal{G}$ of sheaves on Perf is quasi-pro-étale if it is locally separated and for all strictly totally disconnected perfectoid spaces Y with a map $Y \to \mathcal{G}$ (i.e., an element of $\mathcal{G}(Y)$), the pullback $\mathcal{F} \times_{\mathcal{G}} Y$ is representable by a perfectoid space X and $X \to Y$ is pro-étale.

There exists a restricted class of diamonds, called spatial v-sheaves, with $|\mathcal{F}|$ well-behaved. We discuss these spatial diamonds and summarize their main properties.

2.3.1 Spatial v-sheaf

Diamonds for which $|\mathcal{F}|$ is well-behaved are defined as follows.

Definition 13. *([21] Definition 17.3.1). A v-sheaf $\mathcal{F}$ is spatial if*

- *1. $\mathcal{F}$ is qcqs (in particular, small), and*
- *2. $|\mathcal{F}|$ admits a neighborhood basis consisting of $|\mathcal{G}|$, where $\mathcal{G} \subset \mathcal{F}$ is quasicompact open.*

We say $\mathcal{F}$ is locally spatial if it admits a covering by spatial open subsheaves [21].

Remark 2. *([21] Remark 17.3.2).*

- *For algebraic spaces, (1) implies (2); however (1) does not imply (2) in the context of small v-sheaves, or even diamonds. See Remark 17.3.6 below.*
- *If $\mathcal{F}$ is quasicompact, then so is $|\mathcal{F}|$. Indeed, any open cover of $|\mathcal{F}|$ pulls back to a cover of $\mathcal{F}$. However, the converse need not hold true, but it does when $\mathcal{F}$ is locally spatial; cf. [Sch17, Proposition 12.14 (iii)].*
- *If $\mathcal{F}$ is quasiseparated, then so is any subsheaf of $\mathcal{F}$. Thus if $\mathcal{F}$ is spatial, then so is any quasicompact open subsheaf.*

Remark 3. *[21]*

We demand maps to be representable in locally spatial diamonds so that $Rf_!$ (from the six operations in étale cohomology of diamonds) commutes with all direct sums.

We summarize key examples and results of spatial diamonds.

Example 26. *([21] Example 17.3.3). Let K be a perfectoid field in characteristic p, and let $\mathcal{F} = SpaK/Frob^{Z}$, so that $|\mathcal{F}|$ is one point. Then $\mathcal{F}$ is not quasiseparated. Indeed if $X = Y = SpaK$ (which are quasicompact), then $X \times_{\mathcal{F}} Y$ is a disjoint union of Z copies of $SpaK$. So $X \times_{\mathcal{F}} Y$ is not quasicompact. While $\mathcal{F}$ is not spatial, $\mathcal{F} \times SpaF_p((t^{\frac{1}{p^\infty}})) = (D^*_K/Frob^{Z})^\diamond$ is spatial.*

Proposition 8. *([21] Proposition 17.3.4) Let $\mathcal{F}$ be a spatial v-sheaf. Then $|\mathcal{F}|$ is a spectral space, and for any perfectoid space X with a map $X \to \mathcal{F}$, the map $|X| \to |\mathcal{F}|$ is a spectral map.*

We use the following proposition to check if a small v-sheaf is spatial.

Proposition 9. *([21] Proposition 17.3.5). Let X be a spectral space, and $R \subset X \times X$ a spectral equivalence relation such that each $R \to X$ is open and spectral. Then X/R is a spectral space, and $X \to X/R$ is spectral.*

Remark 4. *([21] Remark 17.3.6). It is important to note that counterexamples to Proposition 17.3.5 exist for the case that $R \to X$ is generalizing but not open. For an example, take X and R are profinite sets. Then one can produce any compact Hausdorff space as X/R. For T any compact Hausdorff space, we can find a surjection $X \to T$ from a profinite set X (e.g., the Stone-Cech compactification of T considered as a discrete set). Then, $R \subset X \times X$ is closed and therefore profinite. If we repeat this construction in the world of diamonds, taking $SpaK \times \underline{X}/SpaK \times \underline{R}$, the result is a qcqs diamond $\mathcal{D}$ with $|\mathcal{D}| = T$.*

The corollary is immediate.

Corollary 3. *([21] Corollary 17.3.7). Let $\mathcal{F}$ be a small v-sheaf. Assume there exists a presentation $R \rightrightarrows X \to F$, for R and X spatial v-sheaves (e.g., qcqs perfectoid spaces), and each $R \to X$ is open. Then $\mathcal{F}$ is spatial.*

Proposition 10. *([21] Proposition 17.3.8). If X is a qcqs analytic adic space over $SpaZ_p$, then $X^\diamond$ is spatial.*

We arrive at a culminating theorem. As we stated in the introduction, using the v-topology, one can show that "certain pro-étale sheaves on Perf are diamonds without finding an explicit pro-étale cover [21]." This means that a spatial v-sheaf is actually a diamond "as soon as its points are sufficiently nice [21]."

Theorem 5. *([21] Theorem 17.3.9 [17] Theorem 12.18). Let $\mathcal{F}$ be a spatial v-sheaf. Assume that for all $x \in |\mathcal{F}|$, there is a quasi-pro-étale map $X_x \to \mathcal{F}$ from a perfectoid space X_x such that x lies in the image of $|X_x| \to |\mathcal{F}|$. Then $\mathcal{F}$ is a diamond.*

Additionally, there exists the following characterization and important example of a spatial diamond.

Definition 14. *([17] Definition 1.4) A diamond Y is spatial if it is quasicompact and quasiseparated (qcqs), and $|Y|$ admits a basis for the topology given by $|U|$, where $U \subset Y$ ranges over quasicompact open subdiamonds. More generally, Y is locally spatial if it admits an open cover by spatial diamonds.*

Example 27. *([17] Remark)*

> *Any perfectoid space X defines a locally spatial diamond. This diamond is spatial precisely when X is qcqs. We see this as follows. If X is a (locally) spatial diamond, then $|X|$ is a (locally) spectral topological space, and X is quasicompact (resp. quasiseparated) as a v-sheaf precisely when $|X|$ is quasicompact (resp. quasiseparated) as a topological space. Now, if X is a locally spatial diamond, we define $X_{ét}$ as consisting of (locally separated) étale maps $\mathcal{D} \to X$ from diamonds $\mathcal{D}$ (automatically locally spatial).*

Lemma 7. *([17] Lemma 11.27) Let Y be a spatial diamond. Assume that every connected component of Y is representable by an affinoid perfectoid space. Then Y is representable by an affinoid perfectoid space.*

We have two *permanence* properties.

Corollary 4. *([17] Corollary 11.28) Let Y be a locally spatial diamond, and $Y' \to Y$ a quasi-pro-étale map of pro-étale sheaves. Then Y' is a locally spatial diamond.*

Corollary 5. *([17] Corollary 11.29) A fibre product of (locally) spatial diamonds is (locally) spatial.*

We have two further characterizations of spatial diamonds.

Proposition 11. *([17] Proposition 11.26) Let Y be a spatial diamond. Assume that any surjective étale map $\tilde{Y} \to Y$ that can be written as a composite of quasicompact open immersions and finite étale maps splits. Then Y is a strictly totally disconnected perfectoid space.*

Proposition 12. *([17] Proposition 13.6). Let $f : Y' \to Y$ be a separated map of v-stacks. Then f is quasi-pro-étale if and only if it is representable in locally spatial diamonds and for all complete algebraically closed fields C with a map $Spa(C, O_C) \to Y$, the pullback $Y' \times_Y Spa(C, O_C) \to Spa(C, O_C)$ is pro-étale.*

Spatial morphisms are now discussed.

Definition 15. *([17] Definition 13.1) A map $f : Y' \to Y$ of v-stacks is representable in diamonds if for all diamonds X with a map $X \to Y$, the fibre product $Y' \times_Y X$ is a diamond. This notion is well-behaved.*

Proposition 13. *([17] Proposition 13.2) Let $f : Y' \to Y$ and $\tilde{Y} \to Y$ be maps of v-stacks, with pullback $\tilde{f} : \tilde{Y}' = Y' \times_Y \tilde{Y} \to \tilde{Y}$.*

- *If Y is a diamond, then f is representable in diamonds if and only if Y' is a diamond.*
- *If f is representable in diamonds, then $\tilde{f}$ is representable in diamonds.*
- *If $\tilde{Y} \to Y$ is surjective as a map of pro-étale stacks and $\tilde{f}$ is representable in diamonds, then f is representable in diamonds.*

The definition of locally spatial morphisms follows.

Definition 16. *([17] Definition 13.3) A map $f : Y' \to Y$ of v-stacks is representable in (locally) spatial diamonds if for all (locally) spatial diamonds X with a map $X \to Y$, the fibre product $Y' \times_Y X$ is a (locally) spatial diamond.*

We compare the pro-étale and v-topology.

Definition 17. *([17] Definition 14.1). Assume that Y is a diamond.*

- *The quasi-pro-étale site $Y_{qproét}$ is the site whose objects are (locally separated) quasi-pro-étale maps $Y' \to Y$, with coverings given by families of jointly surjective maps.*
- *The v-site Y_v is the site whose objects are all maps $Y' \to Y$ from small v-sheaves Y', with coverings given by families of jointly surjective maps.*

We conclude with a characterization of the topoi $Y_{\text{ét}}$.

Proposition 14. *([17] Proposition 14.2) The topoi $Y_{ét}$ respectively, $Y_{qproét}$ respectively Y_v for a locally spatial diamond respectively diamond respectively small v-stack Y are algebraic. If Y is 0-truncated (i.e., if Y is a small v-sheaf), then an object is quasicompact (respectively quasiseparated) if and only if it quasicompact (respectively quasiseparated) as a small v-stack on Perf.*

3 Diamonds in Global Langlands Correspondence

We now review the use of diamonds in the global Langlands Correspondence, and a diamond reformulation of Drinfeld's lemma [21]. We recall the global Langlands correspondence over number fields.

Conjecture 1. *([16] Conjecture I.1. Global Langlands (Clozel-Fontaine-Mazur) Conjecture)). Let F be a number field, p some rational prime, and fix an isomorphism* $\mathbb{C} \simeq \bar{\mathbb{Q}_p}$. *Then for any* $n \geq 1$ *there is a unique bijection between the set of L-algebraic cuspidal automorphic representations of* $GL_n(\mathbb{A}_F)$, *and the set of (isomorphism classes of) irreducible continuous representations* $Gal(\bar{F}/F) \to GL_n(\bar{\mathbb{Q}_p})$ *which are almost everywhere unramified, and de Rham at places dividing p, such that the bijection matches Satake parameters with eigenvalues of Frobenius elements* [2].

Drinfeld studies the moduli spaces of "X-shtukas" to obtain the $n = 2$ contribution to the global Langlands Correspondence over function fields, where X/F_p is a smooth projective curve and K is a function field ([Dri80] [21]). To wit, Scholze studies the moduli space of "mixed-characteristic local G-shtukas," $Sht_{(\mathcal{G},b,\{\mu_i\})}$, which Scholze identifies as a locally spatial diamond.

To formally define $Sht_{(\mathcal{G},b,\{\mu_i\})}$, we first recall the definition of the adic space $S \dot{\times} SpaZ_p$.

Proposition 15. *([21] Proposition 11.2.1) If* $S = Spa(R, R^+)$ *is an affinoid perfectoid space of characteristic p, we can define an analytic adic space* $S \dot{\times} SpaZ_p$ *such that there is a natural isomorphism*

- $(S \dot{\times} SpaZ_p)^{\diamond} = S \times SpdZ_p$.

Proposition 16. *([21] Proposition 11.3.1) Let* $S \in Perf$. *The following sets are naturally identified:*

- *Sections of* $S \dot{\times} SpaZ_p \to S$.
- *Morphisms* $S \to SpdZ_p$, *and*
- *Untilts* $S^{\#}$ *of S.*

Moreover, given these data, there is a natural map

- $S^{\#} \hookrightarrow S \dot{\times} SpaZ_p$ *of adic spaces over* Z_p

that is the inclusion of a closed Cartier divisor.

A shtuka over a perfectoid space takes the following form [21]:

> For an object $S \in Perf$, a shtuka over S should be a vector bundle over an adic space $S \times SpaZ_p$ together with a Frobenius structure, where the product is a fiber product...It's associated diamond is the product of sheaves on $Perfd$.

---

[2] Recall, $\mathbb{A}_F = \prod_v' F_v$ denotes the adeles of F, which is the restricted product of the completions F_v at all finite or infinite places of F.

The definition of a "mixed-characteristic shtuka of rank n over S with legs $x_1, \cdots, x_m$ [21]" follows.

Definition 18. *([21] Definition 11.4.1) Let S be a perfectoid space of characteristic p. Let* $x_1, \cdots, x_m : S \to SpdZ_p$ *be a collection of morphisms. For* $i = 1, \cdots, m$ *let*

- $\Gamma_{x_i} : S_i^{\#} \to S \dot{\times} SpaZ_p$

be the corresponding closed Cartier divisor. A mixed-characteristic shtuka of rank n over S with legs $x_1, \cdots, x_m$ *is a rank n vector bundle* $\mathcal{E}$ *over* $S \dot{\times} SpaZ_p$ *together with an isomorphism*

- $\phi_{\mathcal{E}} : (Frob_S^* \mathcal{E})|_{S \dot{\times} SpaZ_p \setminus \bigcup_{i=1}^m \Gamma_{x_i}} \to \mathcal{E}|_{S \dot{\times} SpaZ_p \setminus \bigcup_{i=1}^m \Gamma_{x_i}}$

that is meromorphic along $\bigcup_{i=1}^m \Gamma_{x_i}$.

This formalism is extended to the *moduli space of mixed-characteristic local G-shtukas*.

Definition 19. *([21] Definition 23.1.1.) Let G be a reductive group over* Q_p*. G does not live over* Z_p *in the mixed characteristic setting. So we choose a smooth group scheme* $\mathcal{G}$ *over* Z_p *with generic fiber G and connected special fiber. Now let* $S = Spa(R, R^+)$ *be an affinoid perfectoid space of characteristic p, with pseudouniformizer* $\hat{\omega}$*. Take a discrete algebraically closed field, and* $L = W(k)[1/p]$*. Let*

$$(\mathcal{G}, b, \{\mu_i\})$$

be a triple consisting of a smooth group scheme $\mathcal{G}$ *with reductive generic fiber G and connected special fiber, an element* $b \in G(L)$*, and a collection* $\mu_1, \cdots, \mu_m$ *of conjugacy classes of cocharacters* $G_m \to G_{\bar{Q_p}}$*. For* $i = 1, \cdots, m$*, let* E_i/Q_p *be the field of definition of* μ_i *, and let* $\hat{E_i} = E_i \cdot L$*. The moduli space*

$$Sht_{\mathcal{G}, \lfloor, \{\mu \rangle} \to Spd\hat{E}_1 \times_{Spdk} \cdots \times_{Spdk} Spd\hat{E}_m$$

of shtukas associated with $(\mathcal{G}, b, \{\mu_i\})$ *is the presheaf on* $Perf_k$ *sending* $S = Spa(R, R^+)$ *to the set of quadruples* $(\mathcal{P}, \{S_i^{\#}\}, \phi_{\mathcal{P}}, \iota_r\}$ *where:*

- $\mathcal{P}$ *is a G-torsor on* $S \dot{\times} SpaZ_p$*,*
- $S_i^{\#}$ *is an untilt of S to* $\hat{E}_i$ *for* $i = 1, \cdots, m$*,*
- $\phi^{\mathcal{P}}$ *is an isomorphism* $\phi_{\mathcal{P}} : (Frob_S^* \mathcal{P})|_{(S \times_{F_q} X) \setminus \bigcup_{i=1}^m \Gamma_{x_i}} \tilde{\to} \mathcal{P}|_{(S \times_{F_q} X) \setminus \bigcup_{i=1}^m \Gamma_{x_i}}$ *and finally*
- ι_r *is an isomorphism* $\iota_r : \mathcal{P}|_{\mathcal{Y}_{[r,\infty)}(S)} \tilde{\to} G \times \mathcal{Y}_{[r,\infty)}(S)$ *for large enough r, under which* $\phi^{\mathcal{P}}$ *gets identified with* $b \times Frob_S$*.*

The culminating result follows:

Theorem 6. *([21] Theorem 23.1.4.) The moduli space* $(\mathcal{G}, b, \{\mu_i\})$ *is a locally spatial diamond.*

A further powerful result is that Drinfeld's lemma has a diamond reformulation, the consequence of replacing with the diamond SpdQ_p *all* the connected schemes X_i. The following is Drinfeld's lemma for diamonds:

Theorem 7. *([21] Theorem 16.3.1)* $(\pi_1((SpdQ_p)^n/p.Fr.) \simeq G^m_{Q_p}$.

4 Diamonds in the Geometrization of the Local Langlands Correspondence

We now review diamonds in the *Geometrization of the local Langlands correspondence* [7][**?**]. In [7], the authors construct the foundational tools for a geometrization of the local Langlands correspondence. Specifically, the authors prove many applications and

> define a category of ℓ-adic sheaves on the stack Bun_G of G-bundles on the Fargues-Fontaine curve, prove a geometric Satake equivalence over the Fargues-Fontaine curve, and study the stack of L-parameters.

Recall the local Langlands correspondence. Let G be a split reductive group and let E be a non-archimedean local field, such as $F_q((t))$ or a finite extension of Q_p.

Conjecture 2. *([**?**] Local Langlands correspondence (Conjecture 1.6)). Consider representations over* $L = \boldsymbol{C}$*. There exists a natural map*

- $Irrep(G, E))/\sim \longrightarrow Hom(W_E, \hat{G}(\boldsymbol{C}))/G(\hat{\boldsymbol{C}})$

where $\hat{G}$ *is the Langlands dual group,* W_E *is the Weil group of* E *which is surjective with finite fibers (called L-packets) defined as the pre-image of* $Z \subset \hat{Z}$ *under the surjection* $Gal(\bar{E}/E) \rightarrow \bar{Z}$ *corresponding to the maximal unramified extension of* E.

Recall the geometric Langlands correspondence.

Conjecture 3. *([12] Geometric Langlands Correspondence (Geometric Langlands Duality)). For* G *a reductive group,* ${}^LG$ *the Langlands dual group, and* $\sum$ *an algebraic curve, there is an equivalence of derived categories of* D*-modules on the moduli stack of* G*-principal bundles on* σ *and quasi-coherent sheaves on the* ${}^LG$*-moduli stack of local systems on* σ*:*

- $\mathcal{O}Mod(Loc_{{}^LG}(\sum) \xrightarrow{\simeq} \mathcal{D}Mod(Bun_G(\sum))$.

4.1 History and Motivation

We first highlight aspects of the motivation and historical account for diamonds and their use in the geometrization of the local Langlands correspondence, as given in ([7] Section I.11. *The origin of the ideas* and [7] Section I.2. *The big picture*). Then we summarize key results of diamonds in the Fargues-Fontaine curve and Bun_G.

A key question is:

Question 3. *What does it mean to geometrize the local Langlands correspondence?*

After all, the authors reveal that the idea was a "completely unexpected conceptual leap [7]." Simply stated, though the construction is *anything but simple*, to *geometrize* the local Langlands correspondence is

> to view the local Langlands correspondence as a geometric Langlands correspondence on the Fargues–Fontaine curve [7].

The notion of *geometrizing* a Langlands correspondence is not novel. In [3], Fargues proved a version of local Langlands that is produced "in the cohomology of basic Rapoport-Zink spaces for GL_n (and $U(3)$ and general miniscule cocharacters)" [7]. Additionally, in [4] Fargues proved that the Lubin-Tate Tower and the Drinfeld Tower exhibit a duality isomorphism, which was a version of *geometrizing* the Jacquet-Langlands correspondence ([4] Préambule Theorem 2). Moreover, in [15], Scholze constructed a new proof of the local Langlands correspondence for the case of GL_n, the results of which illuminated "the idea that there ought to exist certain sheaves on the moduli stack of p-divisible groups, giving a certain geometrization of the local Langlands correspondence, then formulated as a certain character sheaf property [7]." [3] Then, Fargues pursued the construction of "p-adic Hodge theory without Galois actions, i.e. for fields like $\mathbb{C}_p$," while Scholze constructed perfectoid spaces to provide results on the Weight-monodromy conjecture. From the realization that the Lubin-Tate tower at infinite level is a perfect space, came the idea that the duality isomorphism was an isomorphism between perfectoid spaces [21]. During this time, the importance of the Fargues-Fontaine Curve took center stage, and the duality isomorphism was realized as

> two dual descriptions of the space of miniscule modifications $O_X^n \to O_X(\frac{1}{n})$ on the Fargues-Fontaine curve, depending on which bundle is fixed and which bundle is the modification. This was the first clear connection between local Langlands (as encoded in the cohomology of the Lubin-Tate and Drinfeld space) and the theory of vector bundles on the Fargues-Fontaine curve.

Eventually, there was a realization that the Hodge-Tate period map "gives a substitute for the map from the moduli space of elliptic curves to the moduli space of p-divisible groups [7]." In [5], Fargues conjectured to geometrize the local Langlands

[3] The authors state that

> Scholze was always uneasy with the very bad geometric properties of the stack of p-divisible groups [7].

correspondence over the Fargues-Fontaine curve, with a diamond reformulation of Bun_G. Namely,

Proposition 17. *([5] Proposition 2.2). Bun_G is a stack on $PerfF_q$.*

Conjecture 4. *([5] Hope 2.3). Bun_G is a "smooth diamond stack."*

The notion of diamonds readily developed by

> Increasingly, taking the perspective of studying all geometric objects by mapping only perfectoid spaces in, including the possibility of getting several copies of $Spec\mathbb{Q}_p$, and of defining general moduli spacess of p-adic shtukas. [4]

Additionally, there was a concern that the Geometric Langlands Program was not able to include the "subtle arithmetic properties of supercuspidal representations of p-adic groups [7]." The ideas to formulate Fargues' conjecture congealed upon the following realizations, which gave a "compelling geometric origin" of the internal structure of L-packets. In [6], Fargues proved the classification of G-torsors on the Fargues-Fontaine curve. Then, Fargues aimed to construct

> for any (discrete) L-parameter ϕ an associated Hecke eigensheaf A_ϕ on Bun_G with eigenvalue ϕ. This should define a functor $\phi \to A_\phi$, and thus carry an action of the centralizer group $S_\phi \subset \check{G}$ of ϕ, and the corresponding S_ϕ-isotypic decomposition of A_ϕ should realize the internal structures of the L-packets. Moreover, the Hecke eigensheaf property should imply the Kottwitz conjecture on the cohomology of local Shimura varieties [7].

Fargues' conjecture, while having issues which needed to be addressed [5], was promising because if Fargues' conjecture could be formulated, then Lafforgue's ideas [**Laf18**] could be applied to obtain the "automorphic-to-Galois" direction and define (semisimple) L-parameters (as Genestier-Lafforgue [**GL17**] did in equal characteristic"), which was not unlike an arduous task [6].

Notably, developing a "good definition of the category of geometric objects relevant to this picture, i.e. diamonds [7] " is the crucial first step in solidly formulating Fargues' conjecture. In order to prove that "the relevant affine Grassmannians have this property" v-sheaves and the property of v-descent were developed. Next, a six functor formalism was developed for the étale cohomology of diamonds [17] (See Appendix), where a

> central technique of [17] is pro-étale descent, and more generally v-descent. In fact, virtually all theorems of [17] are proved using such descent techniques, essentially reducing them to profinite collections of geometric points. It came as a surprise to

[4] The authors state that "the earliest published incarnation of this idea is [**Wei17**].

[5] Namely, it assumed that the moduli stack Bun_G could be handled with the canonical techniques of algebraic geometry, and it supposed that there existed a geometric Satake equivalence.

[6] The authors poignantly state that

> Since then, it has been a long and very painful process [7].

Scholze that this process of disassembling smooth spaces into profinite sets has any power in proving geometric results, and this realization gave a big impetus to the development of condensed mathematics (which in turn fueled back into the present project).

4.2 Fargues-Fontaine Curve

The Fargues-Fontaine curve is defined and main results are summarized. We follow the conventions of [7]. The set up is the following:

Set up. Let E be a nonarchimedean local field (i.e. $E = F_q((t))$ or a finite extension of Q_p) with residue field $\mathbb{F}_q$ of characteristic p. Let $O_E \subset E$ be the ring of integers and π be a uniformizing element in E. Let S be a perfectoid space over $\mathbb{F}_q$. The idea of the curve is conveyed as follows:

Remark 5. *For any perfectoid space S over $\mathbb{F}_q$, we introduce a curve $\mathcal{Y}_S$ to be thought of as the product $S \times_{Spa\mathbb{F}_q} SpaO_E$, together with an open subset $Y_S \subset \mathcal{Y}_S$ given by the locus where $\pi \neq 0$. This carries a Frobenius ϕ induced from the Frobenius on S, and X_S is the quotient $Y_S/\phi^{\mathbb{Z}}$ [7].*

Additionally, let C be a complete algebraically closed nonarchimedean field $C|\mathbb{F}q$. Let $S = SpaC$. Then $\mathcal{Y}_C$ denotes the curve on S. $\mathcal{Y}_C$ is an adic space.

Theorem 8. *([7] Theorem II.0.1 (Proposition II.1.11, Corollary II.1.12, Definition/Proposition 11.1.22). The adic space $\mathcal{Y}_C$ is locally the adic spectrum $Spa(B, B^+)$, where B is a principal ideal domain; the classical points of $Spa(B, B^+) \subset \mathcal{Y}_C$ are in bijection with the maximal ideals of B. For each classical point $x \in \mathcal{Y}_C$, the residue field of x is an untilt $C^\#$ of C over O_E, and this induces a bijection of the classical points of $\mathcal{Y}_C$ with untilts $C^\#$ of C over O_E. A similar result holds true for $Y_C \subset \mathcal{Y}_C$, and the quotient $X_C = Y_C/\phi^{\mathbb{Z}}$.*

Definition 20. *Allowing general $S \in Perf_{\bar{\mathbb{F}}_Q}$, we define the moduli space of degree 1 Cartier divisors as $Div^1 = Spd\check{E}/\phi^{\mathbb{Z}}$. Given a map $S \to Div^1$, one can define an associated closed Cartier divisor $D_S \subset X_S$; locally this is given by an untilt $D_S = S\#$ $\subset X_S$ of S over E, and this embeds Div^1 into the space of closed Cartier divisors on X_S [7].*

The Banach-Colmez space is defined as a locally spatial diamond.

Definition 21. *([7] Definition I.3.5) Let $\mathcal{E}$ be a vector bundle on X_S. The Banach-Colmez space $\mathcal{BC}(\mathcal{E})$ associated with $\mathcal{E}$ is the locally spatial diamond over S whose T-valued points, for $T \in Perf_S$, are given by*

$$\mathcal{BC}(\mathcal{E})(T) = H^0(X_T, \mathcal{E}|X_T).$$

Similarly, if $\mathcal{E}$ is everywhere of only negative Harder-Narasimhan slopes, the negative Banach-Colmez space $\mathcal{BC}(\mathcal{E}[1])$ is the locally spatial diamond over S whose T-valued points are

$$\mathcal{BC}(\mathcal{E}[1])(T) = H^1(X_T, \mathcal{E}|X_T).$$

Remark 6. *Implicit here is that this functor actually defines a locally spatial diamond [7].*

The Fargues-Fontaine curve is constructed in three steps:

1. First, one constructs a curve $\mathcal{Y}_S$, an adic space over O_E which carries a Frobenius action ϕ.
2. Pass to the locus $Y_S = \mathcal{Y}_S \setminus \{\pi = 0\}$, i.e. the base change to E, the action of ϕ is free and totally discontinuous, so that one can
3. Pass to the quotient $X_S = Y_S/\phi^{\mathbb{Z}}$, which will be the Fargues-Fontaine curve [7].

We quickly discuss $\mathcal{Y}_S$ in the affinoid case.

Remark 7. *If $S = Spa(R, R^+)$ is an affinoid perfectoid space over $\mathbb{F}_q$ and $\bar{\omega} \in R^+$ is a pseudouniformizer (i.e. a topologically nilpotent unit of R), we let*

$$\mathcal{Y}_S = SpaW_{O_E}(R^+) \setminus V([\bar{\omega}]) \text{ [7].}$$

Proposition 18. *([7] Proposition II.1.1) The above defines an analytic adic space $\mathcal{Y}_S$ over O_E. Letting E_∞ be the completion of $E(\pi^{\frac{1}{p^\infty}})$, the base change*

$$\mathcal{Y}_S \times_{SpaO_E} SpaO_{E_\infty}$$

is a perfectoid space, with tilt given by

$$S \times_{\mathbb{F}_q} Spa\mathbb{F}_q[[t^{\frac{1}{p^\infty}}]] = D_{S,perf}$$

a perfectoid open unit disc over S.

Proposition 19. *([7] Proposition II.1.2) For any perfectoid space T over $\mathbb{F}_q$, giving an untilt $T^\#$ of T together with a map $T^\# \to \mathcal{Y}_S$ of analytic adic spaces is equivalent to giving an untilt $T^\#$ together with a map $T^\# \to SpaO_E$, and a map $T \to S$. In other words, there is a natural isomorphism*

$$\mathcal{Y}_S^\diamond \cong SpdO_E \times S.$$

Remark 8. *In particular, there is a natural map*

$$|\mathcal{Y}_S| \cong |\mathcal{Y}_S^\diamond| \cong |(Spa_{O_E})^\diamond \times S| \to |S| \text{ [7].}$$

Remark 9. *([7] Proposition II.1.3) ensures that we may glue $\mathcal{Y}_S$ for general S, i.e. for any perfectoid space S there is an analytic adic space $\mathcal{Y}_S$ equipped with an isomorphism*

$$\mathcal{Y}_S^\diamond \cong SpdO_E \times S$$

(and in particular a map $|\mathcal{Y}_S| \to |S|$) such that for $U = Spa(R, R^+) \subset S$ an affinoid subset, the corresponding pullback of $\mathcal{Y}_S$ is given by $\mathcal{Y}_U$.

Example 28. *([7] Example II.1.6) Assume that $E = \mathbb{F}_q((t))$ is of equal characteristic. Then $\mathcal{Y}_C = \mathbb{D}_C$ is an open unit disc over C, with coordinate t. In particular, inside $|\mathcal{Y}_C|$, we have the subset of classical points $|\mathcal{Y}_C|^{cl} \subset |\mathcal{Y}_C|$, which can be identified as*

$|\mathcal{Y}_C|^{cl} = \{x \in C||x| < 1\}$. *Note that these classical points are in bijection with maps $O_E \to C$ (over $\mathbb{F}_q$), i.e. with "untilts of C over O_E.*

There is a similar result when E has mixed characteristic.

Proposition 20. *([7] Proposition II.1.7/Definition) Any untilt $C^\#$ of C over O_E defines a closed Cartier divisor $SpaC^\# \hookrightarrow \mathcal{Y}_S$, and in particular a closed point of $|\mathcal{Y}_C|$. This induces an injection from the set of such untilts to $|\mathcal{Y}_C|$. The set of classical points $|\mathcal{Y}_C|^{cl} \subset |\mathcal{Y}_C|$ is defined to be the set of such points.*

We are now ready to state our culminating definition of the Fargues-Fontaine curve.

Definition 22. *([7] Definition II.1.15) For any perfectoid space S over $\mathbb{F}_q$, the relative Fargues-Fontaine curve is*

$$X_S = Y_S/\phi^{\mathbb{Z}}$$

where

$$Y_S = \mathcal{Y}_S \times_{SpaO_E} SpaE = \mathcal{Y}_S \setminus V(\pi),$$

which for affinoid $S = Spa(R, R^+)$ with pseudouniformizer $\bar{\omega}$ is given by

$$Y_S = SpaW_{O_E}(R^+) \setminus V(\pi[\bar{\omega}]).$$

For the case when $S = SpaC$ a geometric point, there exists a bijection between the classical points of X_S and the untilts $S^\#$ of S "together with a map $S \to SpaE$, modulo the action Frobenius [7]." Recall that for Z an adic space over $W(\bar{\mathbb{F}_q})$ [7] there exists a functor

$$Z^\diamond\colon \text{Perf} \to \text{Sets}\colon S \longmapsto \{S^\#, f : S^\# \to Z\}$$

sending a perfectoid space S over $\bar{\mathbb{F}_q}$ to pairs $S^\#$ of an untilt of S, and a map $S^\# \to Z$. If Z is an analytic adic space, then $Z^\diamond$ is a diamond, that is a quotient of a perfectoid space by a pro-étale equivalence relation. Then the classical points of X_S are in bijection with the S-valued points of the diamond

$$(Spa\check{E})^\diamond/\phi^{\mathbb{Z}}.$$

This can be generalized.

For any $S \in Perf$, maps $S \to (Spa\check{E})^\diamond \phi^{\mathbb{Z}}$ are in bijection with degree 1 Cartier divisors $D_S \subset X_S$, so we define

[7] If R is a perfect $\mathbb{F}_q$-algebra, then $W(R)$ denotes the "p-typical Witt vectors" and $W_{O_E}(R)$ denotes the "unramified Witt vectors [7]."

$$Div^1 = (Spa\check{E})^\diamond/\phi^{\mathbb{Z}} \text{ [7].}$$

The diamond formulation of the relative Fargues-Fontaine curve follows:

Proposition 21. *([7] Proposition II.1.17) There is a natural isomorphism*

$$Y_S^\diamond \cong S \times Spd(E),$$

descending to an isomorphism

$$X_S^\diamond \cong (S \times Spd(E))/\phi^{\mathbb{Z}} \times id.$$

This is formalized in the following:

Proposition 22. *([7] Proposition II.1.18) The following objects are naturally in bijection.*

1. *Sections of $Y_S^\diamond \to S$;*
2. *Maps $S \to Spd(E)$;*
3. *Untilts $S^\#$ over E of S.*

It is further stated that

> *...Any map $S \to Spd(E)/\phi^{\mathbb{Z}}$ defines a closed Cartier divisor $D \subset X_S$; this gives an injection of $Spd(E)/\phi^{\mathbb{Z}}$ into the space of closed Cartier divisors on X_S [7].*

Definition 23. *([7] Definition II.1.19) A closed Cartier divisor of degree 1 on X_S is a closed Cartier divisor $D \subset X_S$ that arises from a map $S \to Spd(E)/\phi^{\mathbb{Z}}$. Equivalently, it arises locally on S from an untilt $S^\#$ over E of S.*

Remark 10. *In particular we see that the moduli space Div^1 of degree 1 closed Cartier divisors is given by*

$$Div^1 = Spd(E)/\phi^{\mathbb{Z}} \text{ [7].}$$

We conclude our review of the Fargues-Fontaine curve with the following:

Proposition 23. *([7] Proposition II.1.21) The map $Div^1 \to *$ is proper, representable in spatial diamonds, and cohomologically smooth.*

Remark 11. *In particular, the map*

$$|X_S| = |Div^1 \times S| \to |S|$$

is open and closed. We can thus picture X_S as being "a proper and smooth family over S [7]."

In summary, the Fargues-Fontaine curve has the following key incarnations, the culmination of which is the diamond formulation ([7] 1.2. *The big picture*):

1. For any complete algebraically closed nonarchimedean field $C|\bar{\mathbb{F}}_q$, the curve $X_C = X_{C,E}$ is a strongly noetherian adic space over E, locally the adic spectrum of a principal ideal domain. One can also construct a schematic version X_C^{alg}, with the same classical points and the same category of vector bundles. The classical points are in bijection with untilts $C^\#|E$ of C, up to Frobenius.

2. More generally, for any perfectoid space $S \in$ Perf, the "family of curves" X_S, again an adic space over E, but no longer strongly noetherian. If S is affinoid, there is a schematic version X_S^{alg}, with the same category of vector bundles.
3. The "mirror curve" $Div^1 = (Spa\check{E})^\diamond/\phi^{\mathbb{Z}}$, which is only a diamond. For any $S \in$ Perf, this parametrizes "degree 1 Cartier divisors on X_S [7]."

4.3 Bun_G

We now review the diamond formulation of Bun_G and summarize main results. Our notation and convention follows ([7] Chapter III. Bun_G).

Set up. Let E be a nonarchimedean local field (i.e. $E = F_q((t))$ or a finite extension of Q_p). Let G be a reductive group over E. Let k be a complete algebraically closed field over $\mathbb{F}_q$. The Galois group $Gal(\bar{k}/k)$ is denoted $G(k)$. Let S be a perfectoid space over $Spdk$. Denote by $Perf_k$ the category of perfectoid spaces over $Spdk$.

Definition 24. *([7] Definition III.0.1) Let Bun_G be the prestack taking a perfectoid space $S \in Perf_k$ to the groupoid of G-bundles on X_S.*

The following theorem summarizes the main results of ([7] Chapter III. Bun_G).

Theorem 9. *([7] Theorem III.0.2 (Proposition III.1.3; Theorem III.2.2; Theorem III.2.3 and Theorem III.2.7; Theorem III.4.5; Proposition III.5.3). The prestack Bun_G satisfies the following properties.*

1. *The prestack Bun_G is a small v-stack.*
2. *The points $|Bun_G|$ are naturally in bijection with Kottwitz' set $B(G)$ of G-isocrystals.*
3. *The map*

$$\nu : |Bun_G| \to B(G) \to (X_*(T)^+_{\mathbb{Q}})^\Gamma$$

is semicontinuous, and

$$\kappa : |Bun_G| \to B(G) \to \pi_1(G_{\bar{E}})_\Gamma$$

is locally constant. Equivalently, the map $|Bun_G| \to B(G)$ is continuous when $B(G)$ is equipped with the order topology.

4. *The semistable locus Bun_G^{ss} is open, and given by*

$$Bun_G^{ss} \cong \bigsqcup_{b \in B(G)_{basic}} [*/\underline{G_b(E)}]$$ [8]

5. *For any $b \in B(G)$, the corresponding subfunctor*

$$i^b : Bun_G^b = Bun_G \times_{|Bun_G|} \{b\} \subset Bun_G$$

[8] [7] (Theorem I.2.1) $G_b(E)$ is a locally profinite group.

is locally closed, and isomorphic to $[*/\tilde{G_b}]$ where $\tilde{G_b}$ is a v-sheaf of groups such that $\tilde{G_b} \to *$ is representable in locally spatial diamonds with $\pi_0\tilde{G_b} = G_b(E)$. The connected component $\tilde{G}_b^\circ \subset \tilde{G}_b$ of the identity is cohomologically smooth of dimension $< 2\rho, \nu_b >$.

It is stated that "there is a good notion of G-torsors in p-adic geometry [9]."

Proposition 24. *([7] Definition/Proposition III.1.1 ([21] Proposition 19.5.1) Let X be a sousperfectoid space over E. [10] The following categories are naturally equivalent.*

1. *The category of adic spaces $T \to X$ with a G-action such that étale locally on X, there is a G-equivariant isomorphism $T \cong G \times X$.*
2. *The category of étale sheaves Q on X equipped with an action of G such that étale locally, $Q \cong G$.*
3. *The category of exact ⊗-functors*

$$Rep_E G \to Bun(X)$$

to the category of vector bundles on X.
A G-bundle on X is an exact ⊗-functor

$$Rep_E G \to Bun(X);$$

by the preceding, it can equivalently be considered in a geometric or cohomological manner.

As G-torsors up to isomorphism are classified by $H^1_{ét}(X, G)$, we have by ([7] Proposition II.2.1), the following generality:

Definition 26. *([7] Definition III.1.2) Let Bun_G be the v-stack taking a perfectoid space $S \in Perf_k$ to the groupoid of G-bundles on X_S.*

4.4 Geometry of Diamonds

Chapter IV of [7] is devoted entirely to the *Geometry of Diamonds*, where results from the étale cohomology of schemes are reformulated in terms of diamonds. We summarize the principal results.

Example 29. *([7] Example IV.1.7) Any locally spatial diamond is an Artin v-stack [7] [11].*

[9] See Appendix for a review of G-torsors.

[10] [21] "X is locally of the form $Spa(R, R^+)$ with R sousperfectoid, cf. Definition 6.3.1.

Definition 25. *([7] Definition 6.3.1) Let R be a complete Tate-$\mathbf{Z}_p$-algebra. Then R is sousperfectoid if there exists a perfectoid Tate ring $\tilde{R}$ with an injection $R \hookrightarrow \tilde{R}$ that splits as topological R-modules.*

[11] ([7] Definition IV.1.1). An Artin v-stack is a small v-stack X such that the diagonal $\Delta_X : X \to X \times X$ is representable in locally spatial diamonds, and there is some surjective map $f : U \to X$ from a locally spatial diamond U such that f is separated and cohomologicaly smooth.

Proposition 25. *([7] Proposition IV.1.8).*

- *Any fibre product of Artin v-stacks is an Artin v-stack.*
- *Let $S \to *$ be a pro-étale surjective, representable in locally spatial diamonds, separated and cohomologically smooth morphism of v-sheaves. The v-stack X is an Artin v-stack if and only if $X \times S$ is an Artin v-stack.*
- *If X is an Artin v-stack and $f : Y \to X$ is representable in locally spatial diamonds, then Y is an Artin v-stack.*

Example 30. *([7] Example IV.1.9).*

- *(i) According to point (ii) of Proposition IV.1.8, the v-stack X is an Artin v-stack if and only if $X \times SpdE$, resp. $X \times Spa(\mathbb{F}_q((t^{\frac{1}{p^\infty}})))$, is an Artin v-stack. To check that X is an Artin v-stack we can thus replace the base point $*$ by $SpdE$, resp. $Spa(\mathbb{F}_q((t^{\frac{1}{p^\infty}})))$.*
- *(ii) Any small v-sheaf X such that $X \to *$ is representable in locally spatial diamonds is an Artin v-stack; e.g. $X = *$.*
- *(iii) Using point (iii) of Proposition I.V.1.8 and [[17], Proposition 11.20 [17]], we deduce that any locally closed substack of an Artin v-stack is an Artin v-stack.*
- *(iv) Let G be a locally profinite group that admits a closed embedding into $GL_n(E)$ for some n. Then the classifying stack $[*/\underline{G}]$ is an Artin v-stack. For this it suffices to see that $[SpdE/\underline{G}] = Spd\ E \times [*/\underline{G}]$ is an Artin v-stack. Now let $H = GL^{\diamond}_{n,E}$; then there is a closed immersion $\underline{G} \times SpdE \hookrightarrow H$. The map $H \to SpdE$ is representable in locally spatial diamonds, separated, and cohomologically smooth; hence so is $H/\underline{G} \to [SpdE/\underline{G}]$ (by [[17], Proposition 13.4 (iv), Proposition 23.15 [17]]), and $H/\underline{G}$ is a locally spatial diamond (itself cohomologically smooth over $SpdE$ by [[17], Proposition 24.2 [17]] since this becomes cohomologically smooth over the separated étale cover $H/\underline{K} \to H/\underline{G}$ for some compact open pro-p subgroup K of G). It is clear that the diagonal of $[*/\underline{G}]$ is representable in locally spatial diamonds.*

Theorem 10. *([7] Theorem IV. 1.19). The stack Bun_G is a cohomologically smooth Artin v-stack of ℓ-dimension 0. The Beauville-Laszlo map defines a separated cohomologically cover $\bigsqcup_{\tilde{\mu} \in X_*(T)^+/\Gamma} [\underline{G(E)} \backslash Gr_{G,\tilde{\mu}}] \to Bun_G$.*

Definition 27. *([7] Definition IV.1.15). Let $f : Y \to X$ be a cohomologically smooth map of Artin v-stacks. The functor $Rf^! : D_{ét}(X, \Lambda) \to D_{ét}(Y, \Lambda)$ is given by $Rf^! = Rf^! \otimes^{\mathbb{L}}_{\Lambda} f^*$.*

Definition 28. *([7] Definition IV.3.1) Let $f : Y \to X$ be a map of v-stacks. Then f is formally smooth if for any affinoid perfectoid space S of characteristic p with a Zariski closed subspace $S_0 \subset S$, and any commutative diagram*

there is some étale map $S' \to X$ containing S_0 in its image and a map $g : S' \to Y$ fitting in a commutative diagram

Corollary 6. *([7] Corollary IV.3.4). If $f : Y \to X$ is a smooth morphism of analytic adic spaces over Z_p, then $f^\diamond : Y^\diamond \to X^\diamond$ is formally smooth.*

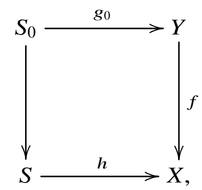

Fig. 3 Commutative diagram for formal smoothness ([7] Definition IV.3.1).

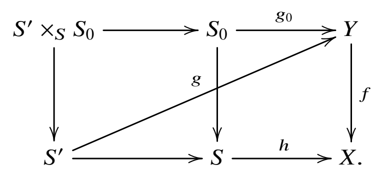

Fig. 4 Commutative diagram for formal smoothness ([7] Definition IV.3.1).

Example 31. *([7] IV.3.2 Examples and basic properties (iv))*

> *For morphisms of locally spatial diamonds, formal smoothness is étale local on the source and the target [7].*

Proposition 26. *([7] Proposition IV.3.5). If $f : Y \to X$ is a formally smooth and surjective map of v-stacks, then f is surjective as a map of étale stacks. Equivalently, in case X is a perfectoid space, the map f splits over an étale cover of X.*

Example 32. *The stack $Bun_G \to *$ is formally smooth [7].*

Theorem 11. *([7] Theorem IV.4.2). Let S be a perfectoid space and let $Z \to X_S$ be a smooth map of sous-perfectoid spaces[12] such that Z admits a Zariski closed immersion into an open subset of (the adic space) $\mathbb{P}^n_{X_S}$ for some $n \geq 0$. Then M_Z is a locally spatial diamond, the map $M_Z \to S$ is compactifiable, and $M_Z^{sm} \to S$ is cohomologically smooth[13]. Moreover, for a geometric point $x : SpaC \to M_Z^{sm}$ given by a map $SpaC \to S$ and a section $s : X_C \to Z$, the map $M_Z^{sm} \to S$ is at x of ℓ-dimension equal to the degree of $s^* T_{Z/X_S}$.*

We note that Theorem IV.4.2 is regarded as

> the most profound in the theory of diamonds so far: While we cannot control much of the geometry of these diamonds, in particular we have no way to relate them to (perfectoid) balls in any reasonable way, we can still prove relative Poincare' duality for them [7].

---

[12] ([7] Definition). Recall that an adic space X is sous-perfectoid if it is analytic and admits an open cover by $U = Spa(R, R^+)$ where each R is a sous-perfectoid Tate algebra, meaning that there is some perfectoid R-algebra $\tilde{R}$ such that $R \to \tilde{R}$ is a split injection in the category of topological R-modules.

[13] "Let S be a perfectoid space and let $Z \to X_S$ be a smooth map of sous-perfectoid adic spaces [7]." M_Z is the v-sheaf of "sections of $Z \to X_S$, sending any perfectoid space $S' \to S$ to the set of maps $X_{S'} \to Z$ lifting $X_{S'} \to X_S$ [7]." (Definition IV.4.1) Let $M_z^{sm} \subset M_Z$ be the open subfunctor of all sections $s : X_{S'} \to Z$ such that $s^* T_{Z/X_S}$ has everywhere positive Harder-Narasimhan slopes.

Lemma 8. *([7]Lemma IV.3.28). Let S be a perfectoid space over* $\mathbb{F}_q$ *and let* $\mathcal{E}$ *be a vector bundle on* X_S *that is everywhere of nonnegative Harder-Narasimhan slopes. There is a perfectoid space* $T \to S$ *that is locally Zariski closed in a perfectoid ball* $\tilde{\mathbb{B}}^n_S$ *over S and that admits a surjective map*

$$T \to (\mathcal{BC}(\mathcal{E}) \setminus \{0\})/\underline{E^\times}$$

over S that is separated, representable in locally spatial diamonds, cohomologically smooth, and formally smooth.

Theorem 12. *([7] Theorem IV.6.5) Braden's Theorem on hyperbolic localization reformalized in diamonds.*

Example 33. *([7]Section IV.7). Drinfeld's Lemma.*

Example 34. *([7])* $D_{\blacksquare}(X)$*. The six functor formalism in [17] is extended to the larger class of solid pro-étale sheaves.*

Lemma 9. *([7] Lemma VII.1.5). Any torsion constructible étale sheaf on the spatial diamond X is represented by a spatial diamond.*

Example 35. *([7] Section VII.2). Four functors are constructed on solid sheaves.*

Definition 29. *([7] Definition VII.6.1). Let X be an Artin v-stack. The full subcategory* $D_{lis}(X, \Lambda) \subset D_{\blacksquare}(X, \Lambda)$ *is the smallest triangulated subcategory stable under all direct sums that contains* $f_\sharp\Lambda$ *for all maps* $f : Y \to X$ *that are separated, representable in locally spatial diamonds, and* ℓ*-cohomologically smooth.*

5 Future Geometrizations

Diamonds have recast the local Langlands Program in the framework of nonarchimedean geometry. The significant results on *representability in locally spatial diamonds* highlight the utility of diamonds for universal constructions. Moreover, the presence of diamonds in the Langlands Program exemplifies how they are like geometric bridges between objects that admit universal properties. From these observations, three questions immediately arise:

Question 4. *What else could and should be geometrized?*

Question 5. *In addition to the Jacquet-Langlands correspondence and the local Langlands correspondence, could the global Langlands correspondence and/or Langlands Functoriality be geometrized over a suitable adic space?*

Question 6. *Can the diamond formalism be extended across disciplines?*

5.1 Diamond *ER = EPR*

Regarding Question 6, it would be interesting to explore the application of diamonds across other p-adic disciplines; namely, in p-adic quantum mechanics. We are particularly interested in a diamond reformulation of Susskind's excellent $ER = EPR$ conjecture [22], which resituates quantum entanglement *(EPR)* in terms of geometry *(ER)* and quantum computational complexity. Leonard Susskind explains the curious nature of entanglement, which mirrors the mineralogical properties of a diamond, where the impurities are never directly visible, but only indirectly so via the colored reflections:

> The peculiar thing about entanglement is that knowing the quantum state in this form is everything that can possibly be known about those two qbits. There is no more to be known. A quantum state like this is the fullest possible description of the two q-bits, and yet it says nothing whatever about what either one of them is doing. It is equally likely that the first qbit is up as that it is down. So, it is a state of being which is absolutely the maximal you can know about the system, and yet you know nothing about the individual constituents [23].

The possible connection between *diamonds - geometrization of local langlands -* $ER = EPR$, is important for two significant reasons:

1. Inserting diamonds into particular principles of quantum mechanics could lead to a functorial-geometric version of computational complexity and/or entanglement; i.e. making entanglement "more geometric."
2. As $ER = EPR$ is often called a conjectural GUT for physics, formulating diamonds in $ER = EPR$ could lay the foundations for a possible link between the two GUTs.

We state our conjecture as follows:

Conjecture 5. ***Diamond ER = EPR***. *Quantum computational complexity is locally representable in spatial diamonds.*

6 Conclusion

We have presented a current and comprehensive review of diamonds, and presented their extensive results in the Langlands Program. Studying diamonds' many mathematical guises all at once, similar to studying the profinite $Gal(\bar{Q}/Q)$ [15], may give us new representations of diamonds. In particular, future work could consist of exploring the universal constructions that diamonds admit. A key question is whether diamonds could be used across disciplines due to their universal properties. Perhaps one could construct a notion of a diamond holography, or a diamond version of the extraordinary $ER = EPR$ conjecture [22], with a diamond-equivalent notion of computational complexity. In any case, diamonds are an extraordinarily powerful and versatile construction, the future use of which is bright and promising.

7 Statements and Declarations

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Appendix

7.1 Six Functor Formalism of Diamonds

In [17], Scholze constructs a six functor formalism on the étale cohomology of diamonds. We summarize key points of the construction.

The set up is the following: A prime p is fixed. X is an analytic adic space "on which p is topologically nilpotent [17]." $X_{\acute{e}t}$ is an étale site.

For any ring Λ such that $n\Lambda = 0$ for some n prime to p, we get a (left-completed) derived category $\mathcal{D}_{\acute{e}t}(X, \Lambda)$ of étale sheaves of Λ-modules on $X_{\acute{e}t}$.

Definition 30. *([17] Definition 1.7) Let X be a small v-stack, and consider the site X_v of all perfectoid spaces over X, with the v-topology. Define the full subcategory*

$$\mathcal{D}_{\acute{e}t}(X, \Lambda) \subset \mathcal{D}(X_v, \Lambda)$$

as consisting of all $A \in \mathcal{D}(X_v, \Lambda)$ such that for all (equivalently, one surjective) map $f : Y \to X$ from a locally spatial diamond Y , f^*A lies in $\hat{\mathcal{D}}(Y_{\acute{e}t}, \Lambda)$.

Now we can state that we have the following operations.

1. A (derived) tensor product

$$\text{-}\bigotimes_{\Lambda}^{\mathbb{L}}\text{-} : D_{\acute{e}t}(X, \Lambda) \times D_{\acute{e}t}(X, \Lambda) \to D_{\acute{e}t}(X, \Lambda).$$

   This is compatible with the inclusions into $D(X_v, \Lambda)$ and the usual derived tensor product on X_v.

2. An internal Hom

$$R\mathcal{H}om_{\Lambda}(-,-) : D_{\acute{e}t}(X, \Lambda)^{op} \times D_{\acute{e}t}(X, \Lambda) \to D_{\acute{e}t}(X, \Lambda).$$

   characterized by the adjunction

$$R\ \mathrm{Hom}_{D_{\acute{e}t}}(X, \Lambda)(A, R\mathcal{H}om_{\Lambda}(B, C)) = R\ \mathrm{Hom}_{D_{\acute{e}t}(X,\Lambda)}(A \otimes_{\Lambda}^{\mathbb{L}} B, C)$$

   for all $A, B, C, \in D_{\acute{e}t}(X, \Lambda)$. In particular, for $A = \Lambda$,

$$R\Gamma(X, R\mathcal{H}om_{\Lambda}(B, C)) = R\ \mathrm{Hom}_{D_{\acute{e}t}(X,\Lambda)}(B, C).$$

   In general, the formation of $R\mathcal{H}om_{\Lambda}$ does not commute with the inclusion $D_{\acute{e}t}(X, \Lambda) \subset D(X_v, \Lambda)$.

3. For any map $f : Y \to X$ of small v-stacks, a pullback functor

$$f^* : D_{\acute{e}t}(X, \Lambda) \to D_{\acute{e}t}(Y, \Lambda).$$

   This is compatible with the inclusions into $D(X_v, \Lambda)$ resp. $D(Y_e v, \Lambda)$, and the pullback functor $D(X_v, \Lambda) \to D(Y_v, \Lambda)$.

4. For any map $f : Y \to X$ of small v-stacks, a pushforward functor

$$\mathcal{R}f_* : D_{\acute{e}t}(Y, \Lambda) \to D_{\acute{e}t}(X, \Lambda)$$

which is right adjoint to f^*. In general, formation of Rf_* does not commute with the inclusions into $D(X_v, \Lambda)$ resp. $D(Y_v, \Lambda)$, but this holds true if f is qcqs and one starts with an object of D^+.

5. For any map $f : Y \to X$ of small v-stacks that is compactifiable (cf. Definition 22.2), representable in locally spatial diamonds (cf. Definition 13.3), and with (locally) dim.trg $f < \infty$ (cf. Definition 21.7), a functor

$$\mathcal{R}f_! : D_{\acute{e}t}(Y, \Lambda) \to D_{\acute{e}t}(X, \Lambda).$$

6. For any map $f : Y \to X$ of small v-stacks that is compactifiable, representable in locally spatial diamonds, and with (locally) dim.trg $f < \infty$, a functor

$$\mathcal{R}f^! : D_{\acute{e}t}(X, \Lambda) \to D_{\acute{e}t}(Y, \Lambda)$$

that is right adjoint to $Rf_!$.

The above constructions are upgraded to "functors of ∞-categories [17]."

Lemma 10. *([17] Lemma 17.1) There is a (natural) presentable stable ∞-category $D_{\acute{e}t}(Y, \Lambda)$ whose homotopy category is $D_{\acute{e}t}(Y, \Lambda)$. More precisely, the ∞-derived category $D(Y_v, \Lambda)$ of Λ-modules on Y_v is a presentable stable ∞-category, and $D_{\acute{e}t}(Y, \Lambda)$ is a full presentable stable ∞-subcategory closed under all colimits.*

Proof [17]

First, $D(Y_v, \Lambda)$ is a presentable stable ∞-category, as this is true for any ringed topos. Next, we check that the full ∞-subcategory $D_{\text{ét}}(Y, \Lambda)$, with objects those of $D_{\text{ét}}(Y, \Lambda)$, is closed under all colimits in $D(Y_v, \Lambda)$. This is clear for cones, so we are reduced to filtered colimits. Those commute with canonical truncations, and filtered colimits of étale sheaves are still étale sheaves, as desired.

By [Lur09, Proposition 5.5.3.12][11], it is enough to prove the claim if Y is a disjoint union of strictly totally disconnected perfectoid spaces. In that case, $D_{\text{ét}}(Y, \Lambda) = D(Y_{\text{ét}}, \Lambda)$ (as the functor of stable ∞-categories $D(Y_{\text{ét}}, \Lambda) \to D(Y_v, \Lambda)$ is fully faithful (as it is on homotopy categories), and has the same objects as $D_{\text{ét}}(Y, \Lambda)$, which is a presentable ∞-category. □

7.2 $\underline{G}$-torsor

In order to understand the important diamond examples $SpdQ_p \times SpdQ_p$ and $\mathcal{Y}^{\diamond}_{(S,E)}$, we follow the excellent exposition in [21] and review their notions of a G-torsor and a $\underline{G}$-torsor.

Definition 31. *([21] Definition Section 9.3)*

If G is a finite group, we have the notion of G-torsor on any topos. This is a map $f : \mathcal{F}' \to \mathcal{F}$ with an action $G \times \mathcal{F}' \to \mathcal{F}'$ over $\mathcal{F}$ such that locally on $\mathcal{F}$, one has a G-equivariant isomorphism $\mathcal{F}' \simeq \mathcal{F} \times G$.

Definition 32. *([21] Discussion Section 9.3)*

There is another notion if G is a group object in a topos, for example in the category of pro-étale sheaves on Perf [21]. For any topological space T, we can introduce a sheaf $\underline{T}$ on Perf, by $\underline{T}(X) = C^0(|X|, T)$. As proétale covers induce quotient mappings by Proposition 4.3.3, we see that $\underline{T}$ is a pro-étale sheaf. If T is a profinite set, this agrees with the definition of $\underline{T}$ given earlier [in Example 8.2.2]. If now G is a topological group, then $\underline{G}$ is a sheaf of groups. If $G = \lim_{\leftarrow i} G_i$ is a profinite group, then in fact $\underline{G} = \lim_{\leftarrow i} \underline{G_i}$.

Note that $\underline{G}$ is not representable, even if G is finite. The problem is that Perf lacks a final object X (in other words, a base). If it had one, then for finite G, the sheaf $\underline{G}$ would be representable by G copies of X. And indeed, $\underline{G}$ becomes representable once we supply the base. If X is a perfectoid space and G a profinite group, then $X \times \underline{G}$ is representable by a perfectoid space, namely

$$X \times \underline{G} = \lim_{\leftarrow} X \times G/H,$$

where $X \times G/H$ is just a finite disjoint union of copies of X [21].

Remarkably, if G is rather a profinite set, the same conclusions hold. A $\underline{G}$-torsor is defined as follows.

Definition 33. *([21] Discussion Section 9.3)*

A $\underline{G}$-torsor is a morphism $f : \mathcal{F}' \to \mathcal{F}$ with an action $\underline{G} \times \mathcal{F}' \to \mathcal{F}'$ such that locally on $\mathcal{F}$ we have a $\underline{G}$-equivariant isomorphism $\mathcal{F}' \simeq \mathcal{F} \times G$.

There exists the following general result for torsors under locally profinite groups G.

Lemma 11. *([17] Lemma 10.13) Let $f : \mathcal{F}' \to \mathcal{F}$ be a $\underline{G}$-torsor, with G profinite. Then for any affinoid $X = Spa(B, B^+)$ and any morphism $X \to \mathcal{F}$, the pullback $\mathcal{F}' \times_{\mathcal{F}} X$ is representable by a perfectoid affinoid $X' = Spa(A, A^+)$. Furthermore, A is the completion of $\lim_{\to H} A_H$, where for each open normal subgroup $H \subset G$, A_H/B is a finite étale G/H-torsor in the algebraic sense.*

7.3 $\mathcal{G}$-torsors

Our brief review of $\mathcal{G}$-torsors follows ([17] (Appendix to Lecture 19)).

The discussion is motivated by the following problem:

There is the problem that in general, if X is an adic space over Z_p, it is not clear whether $\mathcal{G} \times X$ is also an adic space. For this reason, we restrict to one class of spaces where this happens, at least when $\mathcal{G}$ is smooth.

The set up is the following. Let $\mathcal{G}$ be smooth, and let X be a sousperfectoid analytic adic space over Z_p.

$\mathcal{G}$ denotes the adic space $\mathrm{Spa}(R, R^+)$ if $\mathcal{G} = \mathrm{Spec} R$ and $R^+ \subset R$ is the integral closure of Z_p. For every adic space S over Z_p, one has $\mathcal{G}(S) = \mathcal{G}(O_S(S))$.

A $\mathcal{G}$-torsor takes either a geometric, cohomological, or Tannakian form.

Definition 34. *([17] Discussion Appendix to Lecture 19)*

- *(Geometric) A geometric $\mathcal{G}$-torsor is an adic space $\mathcal{P} \to X$ over X with an action of $\mathcal{G}$ over X such that étale locally on X, there is a $\mathcal{G}$-equivariant isomorphism $\mathcal{P} \simeq \mathcal{G} \times X$.*
- *(Cohomological) A cohomological $\mathcal{G}$-torsor is an étale sheaf Q on X with an action of $\mathcal{G}$ such that étale locally on X, there is a $\mathcal{G}$-equivariant isomorphism $Q \simeq \mathcal{G}$.*
- *(Tannakian) A Tannakian $\mathcal{G}$-torsor is an exact $\otimes$-functor $P : Rep\ \mathcal{G} \to Bun(X)$, where $Bun(X)$ is the category of vector bundles on X.*

Theorem 13. *([17] Theorem 19.5.2) The categories of geometric, cohomological, and Tannakian $\mathcal{G}$-torsors on X are canonically equivalent.*

We conclude with the following descent result.

Proposition 27. *([17] Proposition 19.5.3) Let $S \in Perf$ be a perfectoid space of characteristic p and let $U \subset S \dot{\times} SpaZ_p$ be an open subset. The functor on $Perf_S$ sending any $S' \to S$ to the groupoid of $\mathcal{G}$-torsors on*

$$U \times_{S \dot{\times} SpaZ_p} S' \dot{\times} SpaZ_p$$

is a v-stack.

7.3.1 Étale Locus

Following the exposition in [21] we highlight a few results concerning vector bundles on the relative Fargues-Fontaine curve $\mathcal{X}_{FF,S}$,

which are also studied via ϕ-modules on subspaces of $\mathcal{Y}_{(0,\infty)}(S)$ [21].

Two foundational theorems about vector bundles on $\mathcal{X}_{FF,S}$ are proved by Kedlaya-Liu:

The first concerns the semicontinuity of the Newton polygon; the second the open locus where the Newton polygon is 0.

The set up is the following: Let S be a perfectoid space of characteristic p, and let $\mathcal{E}$ be a vector bundle on $\mathcal{X}_{FF,S}$. Passing to an open and closed cover of S, we assume that the rank of $\mathcal{E}$ is constant. For any $s \in S$, we choose a geometric point

- $\bar{s} = \mathrm{Spa}(C, C^+) \to S$

whose closed point maps to s, and pullback $\mathcal{E}$ to a vector bundle $\mathcal{E}_{\bar{s}}$ on $\mathcal{X}_{FF,C}$ [21].

Theorem 14. *([21] Theorem 22.2.1 [KL15]) The map $\nu_{\mathcal{E}}$ is upper semicontinuous.*

To construct examples of the open locus where the Newton polygon is 0, we take a pro-étale $\underline{Q_p}$-local system $\mathbb{L}$ on S and look at

- $\mathcal{E} = \mathbb{L} \otimes_{\underline{Q_p}} \mathcal{O}_{\mathcal{X}_{FF,S}}$.

Using the pro-étale or even v-descent of vector bundles on $\mathcal{Y}_{(0,\infty)}(S)$, and thus on $\mathcal{X}_{FF,S}$ ([21] Proposition 19.5.3), we have a vector bundle on $\mathcal{X}_{FF,S}$, upon descending to the case where $\mathbb{L}$ is trivial [21].

We now have the second theorem of Kedlaya-Liu.

Theorem 15. *([21] Theorem 22.3.1 [KL15) This construction defines an equivalence between the category of pro-étale Q_p-local systems on S and the category of vector bundles $\mathcal{E}$ on $X_{FF,S}$ which are trivial at every geometric point of S (which is to say, the function $\nu_{\mathcal{E}}$ is identically 0).*

For the link to isocrystals and further discussion, see [21] Corollary 22.3.3, Definition 22.4.1, and Remark 22.4.2.

References

1. Bhatt, B. and Scholze,P., *The pro-étale topology for schemes*, Astérisque No. 369 (2015), 99–201.
2. Cais, B., Davis, C., and Lubin, J., *A Characterization of Strictly APF Extensions*, arXiv:1403.6693 [math.NT].
3. Fargues, L., *Cohomologie des espaces de modules de groupes p-divisibles et correspondances de Langlands locales, Variétés de Shimura, espaces de Rapoport-Zink et correspondances de Langlands locales*, no. 291, Astérisque, 2004, pp. 1–199.
4. Fargues, L., *L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld et applications cohomologiques, L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld*, Progr. Math., vol. 262, Birkhäuser, Basel, 2008, pp. 1–325.
5. Fargues, L., *Geometrization of the Local Langlands Correspondence: an Overview*, arXiv:1602.00999 [math.NT] (2016).
6. Fargues, L., *G-torseurs en th ´eorie de Hodge p-adique*, , to appear, Comp. Math., 2018.
7. Fargues, L. and Scholze, P., *Geometrization of the Local Langlands Correspondence*, arXiv:2102.13459 [math.RT] (2021).
8. Fontaine, JM. and Wintenberger,JP, *Extensions algébrique et corps des normes des extensions APF des corps locaux*, C. R. Acad. Sci. Paris Sér. A–B 288(8) (1979), A441–A444.
9. Grothendieck, A. Introduction to functorial algebraic geometry, part 1: affine algebraic geometry, summer school in Buffalo, 1973, lecture notes by Federico Gaeta..
10. Huber, R. *A generalization of formal schemes and rigid analytic varieties*, Math. Z. 217 (1994), no 4, 513-551.
11. Lurie, J., *Higher Topos Theory*, Annals of Mathematics Studies, Vol. 170, Princeton University Press, 2009.
12. ncatlab authors. Geometric Langlands Correspondence. https://ncatlab.org/nlab/show/geometric+Langlands+correspondence (accessed January 29, 2023).
13. Rognes, J.*Chromatic Redshift*, arXiv:1403.4838 [math.AT], 2014.
14. Scholze, P., *Perfectoid Spaces*,Publ. Math. Inst. Hautes Etudes Sci. ´ 116 (2012), 245–313. MR 3090258.
15. Scholze, P., *p-adic Hodge theory for rigid-analytic varieties*,Forum of Mathematics, Pi 1 (2013), no. e1.
16. Scholze, P., *On torsion in the cohomology of locally symmetric varieties*, Annals of Mathematics (2) 182 (2015), no. 3, 945–1066.
17. Scholze, P., *Étale Cohomology of Diamonds*. arXiv:1709.07343 [math.AG], 2017.
18. Caraiani, A. and Scholze, P., *On the generic part of the cohomology of compact unitary Shimura varieties*, Annals of Mathematics (2) 186 (2017), no. 3, 649–766.
19. Scholze, P., *On the p-adic cohomology of the Lubin-Tate tower*, Ann. Sci. Ec. Norm. Super. (4) 51 (2018), no. 4, 811-863.

20. Scholze, P., *P-adic Geometry*, arXiv:1712.03708 [math.AG].
21. Scholze, P. and Weinstein, J., *Berkeley Lectures on P-adic Geometry*, Princeton University Press, Annals of Mathematics Studies Number 207.
22. L. Susskind, ER=EPR, GHZ, and the Consistency of Quantum Measurements, arXiv:1412.8483 [hep-th], 2014..
23. L. Susskind, ER=EPR, or What's Behind the Horizons of Entangled Black Holes?, Stanford Institute for Theoretical Physics,https://www.youtube.com/watch?v=OBPpRqxY8Uw.
24. Weinstein, J., *Gal*($\bar{Q}_p/Q_p$)) *as a geometric fundamental group*, Int. Math. Res. Not. IMRN (2017), no. 10, 2964–2997.