A General Schema for Bilateral Proof Rules

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Abstract

Bilateral proof systems, which provide rules for both affirming and denying sentences, have been prominent in the development of proof-theoretic semantics for classical logic in recent years. However, such systems provide a substantial amount of freedom in the formulation of the rules, and, as a result, a number of different sets of rules have been put forward as definitive of the meanings of the classical connectives. In this paper, I argue that a single general schema for bilateral proof rules has a reasonable claim to inferentially articulating the core meaning of all of the classical connectives. I propose this schema in the context of a bilateral sequent calculus in which each connective is given exactly two rules: a rule for affirmation and a rule for denial. Positive and negative rules for all of the classical connectives are given by a single rule schema, harmony between these positive and negative rules is established at the schematic level by a pair of elimination theorems, and the truth-conditions for all of the classical connectives are read off at once from the schema itself.

Key Words: Bilateralism; Classical Logic; Proof-Theoretic Semantics; Inferentialism; Classical Sequent Calculus

0 Introduction

Logical bilateralism of the sort proposed by Smiley (1996) and Rumfitt (2000) provides rules for both *affirming* and *denying* sentences, prefixing each sentence with a positive or negative force-marker to indicate its affirmation or denial.[1] Formalisms of this sort have been used to provide harmonious natural deduction systems for classical logic and, as such, have been prominent in the development of proof-theoretic semantics

[1]It's worth noting from the outset that this sort of bilateralism contrasts with the sort of bilateralism proposed by Restall (2005) and Ripley (2013) in which multiple conclusion systems are *interpreted* bilaterally. See Simonelli (M.S.) for an account unifying these two approaches.

for classical logic. However, such systems provide a substantial amount of freedom in the formulation of the rules, and, as a result, a number of different sets of rules have been put forward as definitive of the meanings of the classical connectives.

In this paper, I argue that the core meaning of all of the classical connectives is capture by a single general schema for bilateral proof rules. I propose this schema in the context of a new kind of bilateral proof system, a bilateral sequent calculus in which each connective is given exactly two rules: a positive rule saying when one is to affirm a sentence with that connective and a negative rule saying when one is to deny a sentence with that connective. Positive and negative rules for all of the binary connectives (not just conjunction, disjunction and the conditional, but also the Sheffer Stroke, Peirce's Arrow, and the dual of the conditional) are given by a single rule schema. The harmony between positive and negative rules of this form is established at the schematic level by a pair of elimination proofs: a generalized bilateral analogue to Cut Elimination along with a proof that the axiom schema can be restricted to atomic formulas. Finally, the truth-conditions for all of the connectives are be read off at once from the schema itself. Beyond the intrinsic interest for bilateralism in proof-theoretic semantics, these results are of general interest as they illustrate a new method for formulating systems and doing one's metatheory at a higher level of generality, simplifying both the presentation of systems and the proofs of their key properties, in a way that is both technically convenient and conceptually illuminating.[2]

[2] A full comparison of related approaches is not undertaken here, as doing so would distract from the main philosophical aims of this paper, but, for the most similar approaches, compare the generalized approach to Cut Elimination put forward by Baaz, Fermüller, and Zach (1993) and Zach (2021), and the "unifying notation" for tableau systems, developed and deployed by Smullyan (1963, 1968 20-22) and Fitting (1983).

1 The Promise of Bilateralism

Michael Dummett (1991) famously argues that, if we want to think of the meanings of the logical connectives in terms of the rules governing their use in proofs, we should be intuitionists rather than classicalists, since it is intuitionistic natural deduction rather than classical natural deduction that displays that proof-theoretic virtue of *harmony*, with the introduction and elimination rules fitting together as they ought, with each set of rules being neither too strong nor too weak relative to the other.[3] In what is now a classic response to Dummett, Ian Rumfitt (2000), drawing on prior work from Timothy Smiley (1996), shows that if one has a natural deduction system that contains not just rules for *affirming* sentences but rules for *denying* sentences as well, then it is easy to arrive at a harmonious natural deduction system for classical logic. The system Rumfitt proposes is thus *bilateral* in taking affirmation and denial as basic. In such a system, a well-formed formula must be prefaced with a positive or negative force-marker, expressing either affirmation or denial. Thus, the affirmation of a sentence φ can be written as $+\langle\varphi\rangle$, and the denial of φ can be written as $-\langle\varphi\rangle$. Unlike a negation operator, force-markers are neither embeddable nor iterable; there must always be exactly one force-marker and it must always be prefixed to a whole sentence. So, for instance, although both $+\langle p \wedge \neg q\rangle$ and $-\langle \neg p\rangle$ are well-formed, neither $+\langle p \wedge -\langle q\rangle\rangle$ nor $-\langle -\langle p\rangle\rangle$ are well-formed.[4]

There are two key innovations of Smiley/Rumfitt-style bilateral natural deduction systems. The first key innovation is the rules for negation. In Rumfitt's natural deduction system, which contains positive and negative introduction and elimination rules for each connective, they are the

[3] I leave the notion of harmony informal in this preliminary presentation, as there are many differing conceptions of what, exactly, it amounts to. For discussion, see Steinberger (2011b).

[4] I find that the notation I use, where the unsigned sentences to which signs are attached are always enclosed in angle brackets makes this feature of the system more clear.

following:[5]

$$\frac{-\langle \varphi \rangle}{+\langle \neg\varphi \rangle}\ +_{\neg_I} \qquad\qquad \frac{+(\neg\varphi)}{-\langle \varphi \rangle}\ +_{\neg_E}$$

$$\frac{+\langle \varphi \rangle}{-\langle \neg\varphi \rangle}\ -_{\neg_I} \qquad\qquad \frac{-\langle \neg\varphi \rangle}{+\langle \varphi \rangle}\ -_{\neg_E}$$

These rules jointly codify that denying a sentence has the same logical significance as affirming its negation, and affirming a sentence has the same logical significance as denying its negation. They are obviously harmonious, and they clearly define classical negation, as both double negation introduction and elimination are immediately derived through two applications of the I-rules or E-rules, respectively.

The second key innovation of bilateral systems is the introduction of structural rules of a distinctively bilateral sort known as *coordination principles*, so-called because they "coordinate" the opposite stances of affirmation and denial. In his original presentation of a bilateral system for classical logic, Smiley (1996) provides just one coordination principle, which Rumfitt (2000, 804) dubs "Smileian Reductio." Where A and B are signed formulas, and starring a formula yields the oppositely signed formula, the rule can be put as follows:

[5]Following secondary literature (e.g. Kürbis 2021), I will refer to system $\mathcal{B}$, originally put forward by Rumfitt (2000), as "Rumfitt's system," even though Rumfitt ultimately expresses a preference for Smiley's "more compact" system $\mathcal{J}$, which contains only positive rules for conjunction, negative rules for disjunction, and elimination rules (both positive and negative) for the conditional. While Rumfitt expresses a preference for this system on account of its compactness, as Kürbis (2016) points out, his own proof-theoretic orientation gives him decisive reason to favor his own "somewhat unwieldy" system. Most fundamentally, it seems clear that, when giving the rules for a given connective in a bilateral system, one ought to provide rules for both affirming and denying sentences with that connective. That seems to follow directly from the core idea of bilateralism of treating affirmation and denial as conceptually on a par in defining the meanings of the logical connectives. The system I'll eventually propose is just as compact as Smiley's $\mathcal{J}$ (in fact, even more so) and yet, in assigning a rule for affirmation and a rule for denial to each connective, it remains true to the core idea of bilateralism.

$$\dfrac{\dfrac{\begin{array}{c}\overline{A}^{\;u}\\ \vdots\\ B\end{array} \quad B^*}{A^*}\ \mathrm{SR}^u}{}$$

In recent years, most bilateralst have followed Rumfitt in splitting Smileian Reductio into two rules, which I'll call *Conclusion of Incoherence* (CI) and *Reductio* (R):.[6]

$$\dfrac{A \quad A^*}{\perp}\ \mathrm{CI} \qquad\qquad \dfrac{\begin{array}{c}\overline{A}^{\;u}\\ \vdots\\ \perp\end{array}}{A^*}\ \mathrm{R}^u$$

Whichever way one opts to carve up the coordination principles, just the negation rules, coordination principles, along with the standard rules for conjunction, taken as the positive rules, itself constitutes a sound, complete, and harmonious system of classical propositional logic.[7] However, since Rumfitt's motivation is to provide a proof-theoretic account of the meanings of the logical connectives, the main theoretical fruit of the system is meant to be the rules he provides for the whole set of classical connectives.

Beyond the modification of the rules for negation, which crucially make use of the bilateral set-up, Rumfitt sticks to the standard rules from Gentzen (1935) as much as possible in his proposal of a bilateral proof system for classical logic. The positive rules are just those of Gentzen's

[6]Rumfitt calls the first principle "Non-contradiction," whereas whereas Incurvati and Schlöder (2017) and del Valle-Inclan and Schlöder (2023) call the first principle "Rejection." It seems to me that "Conclusion of Incoherence" is a better name, as "Non-contradiction" is ambiguous between the principle involving negation and "Rejection" expresses the negative stance, and the positive stance of acceptance is treated as conceptually on a par with the negative stance of rejection in the context of this rule. Incurvati, Schlöder, and del Valle-Inclan call the second principle "Smileian Reductio." This is also a bit confusing, as Smileian Reductio, as originally formulated by Smiley and Rumfitt, is the principle above which combines these two principles.

[7]It's straightforward (though a bit tedious) to derive the conjunction/negation translations of the standard Hilbert axioms, along with modus ponens from these rules.

natural deduction system with positive signs added to all of the formulas. Exploiting the duality of conjunction and disjunction, the negative disjunction rules are of exactly the same form as the standard (positive) conjunction rules, replacing the positive signs left implicit in Gentzen's unilateral calculus with negative ones:

$$\frac{-\langle\varphi\rangle \quad -\langle\psi\rangle}{-\langle\varphi \vee \psi\rangle}\ -_{\vee_I} \qquad\qquad \frac{-\langle\varphi \vee \psi\rangle}{-\langle\varphi\rangle}\ -_{\vee_{EL}} \qquad\qquad \frac{-\langle\varphi \vee \psi\rangle}{-\langle\psi\rangle}\ -_{\vee_{ER}}$$

Likewise, the negative conjunction rules are of exactly the same form as Gentzen's (positive) disjunction rules:[8]

$$\frac{-\langle\varphi\rangle}{-\langle\varphi \wedge \psi\rangle}\ -_{\wedge_{IL}} \qquad\qquad \frac{-\langle\psi\rangle}{-\langle\varphi \wedge \psi\rangle}\ -_{\wedge_{IR}} \qquad\qquad \frac{-\langle\varphi \wedge \psi\rangle \quad \begin{matrix}\overline{-\langle\varphi\rangle}^{\,u} \\ \vdots \\ \overline{A}\end{matrix} \quad \begin{matrix}\overline{-\langle\psi\rangle}^{\,v} \\ \vdots \\ \overline{A}\end{matrix}}{A}\ -_{\wedge_E}^{u,v}$$

The one set of connective rules that Rumfitt cannot simply inherit from Gentzen are the negative conditional rules, since Gentzen himself does not provide rules for the dual of the conditional. Here, Rumfitt provides the following negative conditional rules:

$$\frac{+\langle\varphi\rangle \quad -\langle\psi\rangle}{-\langle\varphi \to \psi\rangle}\ -_{\to_I} \qquad\qquad \frac{-\langle\varphi \to \psi\rangle}{+\langle\varphi\rangle}\ -_{\to_{EL}} \qquad\qquad \frac{-\langle\varphi \to \psi\rangle}{-\langle\psi\rangle}\ -_{\to_{ER}}$$

Supplementing the (positive) rules for the binary connectives of NK with these negative rules yields a natural deduction system for classical logic that conforms to Dummett's demand of harmony. Thus, bilateralism promises to provide a way for the classical logician to adopt a proof-theoretic semantics for their preferred set of connectives. However, the promise of bilateralism cannot be taken to be truly fulfilled insofar as

[8]Note that, in keeping with the terminology used above, in the elimination rule here, "A" is an arbitrary *signed* formula.

there remains the question of *which* bilateral rules should be taken as definitive of the meanings of the connectives. Let me explain.

The particular set of rules proposed by Rumfitt is widely considered to be the definitive set of rules for bilateral classical logic.[9] However, as several authors have recently noted, there is reason to think that, once one goes bilateral, restricting oneself to the rule forms proposed by Gentzen is somewhat arbitrary.[10] To see this, look again at the negative conditional rules above and see that they are notably different than all of the other binary connective rules in Rumfitt's system. In particular, they're the only binary connective rules in Rumfitt's system that are *bilaterally mixed* in the sense of involving both positively and negatively signed formulas. Once one sees that such rules are possible in a bilateral system, it's natural to wonder why there's not more of them, and once one wonders this, one quickly sees that there is a number of possible harmonious bilateral proof systems with different rules for the connectives. For instance, rather than having positive disjunction rules that correspond to Gentzen's rules, del Valle-Inclan and Schlöder (2023) have recently proposed the following set of rules:

$$\dfrac{\begin{array}{c}\overline{-\langle\varphi\rangle}^{\,u}\\ \vdots\\ \overline{+\langle\psi\rangle}\end{array}}{+\langle\varphi \vee \psi\rangle}\;{+}_{\vee_I}{}^{u} \qquad\qquad \dfrac{+\langle\varphi \vee \psi\rangle \quad -\langle\varphi\rangle}{+\langle\psi\rangle}\;{+}_{\vee_E} \qquad \dfrac{+\langle\varphi \vee \psi\rangle \quad -\langle\psi\rangle}{+\langle\varphi\rangle}\;{+}_{\vee_E}$$

Alternatively, consider the following disjunction introduction rule which can be harmoniously paired with the same elimination rules:[11]

[9]By "widely" here, I mean to be speaking primarily of commentators other than the few who are serious bilateralists themselves. See, for instance, Hjortland (2014), Restall (2020), and Kürbis (2022).

[10]See, for instance, Kürbis (2016), del Valle-Inclan and Julian Schlöder (2023), and Simonelli (forthcoming).

[11]Murzi (2020, 399) proposes a disjunction rule of this form in a unilateral context, where, rather than φ and ψ being negatively signed, they are negated. However, as he points out, in a unilateral context, such a rule suffers from lack of separability from the negation rules. There is no such issue in a bilateral context.

$$\dfrac{\begin{array}{cc}\overline{-\langle\varphi\rangle}^{\,u} & \overline{-\langle\psi\rangle}^{\,v}\\ \vdots & \vdots\end{array}}{\dfrac{\perp}{+\langle\varphi \vee \psi\rangle}\;+_{\vee_I}{}^{u,v}}$$

As these examples show, if one adopts a bilateralist approach to proof-theoretic semantics, one has a great amount of theoretical freedom when it comes to determining the rules that one takes to be definitive of the meanings of the logical connectives. With theoretical freedom, however, comes theoretical responsibility—in particular, the responsibility to justify one's theoretical choices. And it's not at all clear what the justificatory criteria that should be applied to settle the matter actually are.

Whatever the justificatory criteria for counting a set of rules as definitive of a set of meanings are, the most fundamental one is generally given the label "harmony." However, the significance of this label, owed to Dummett, already varies greatly from author to author already in a unilateral context, and this variation is multiplied in a bilateral context in which there is the further constraint of "bilateral harmony" between the positive and negative rules. Here, there is even less clarity than in the more familiar context, with various authors proposing different criteria for bilateral harmony that the specific rules that they propose meet.[12] My own view, which I've argued at more length elsewhere, is that any reasonable criterion of bilateral harmony is going to underdetermine which set of rules is to be preferred in the context of proof-theoretically defining the classical connectives.[13] That is, all of the rules I've just specified are going to be harmonious under any reasonable criterion of harmony. Likewise, all of these rules meet other important proof-theoretic criteria such as seperability. Thus, though bilateralism enables one to provide a proof system for classical logic whose rules meet standard proof-theoretic

[12] In addition to del Valle-Inclan and Schlöder (2023), see Francez (2014, 2018) and Kürbis (2022).

[13] See Simonelli (forthcoming).

constraints, we are left with the question of *which* proof system should be counted as specifying the meaning of the classical connectives.

Beyond the fact that bilateralism offers more possibilities for harmonious rules for the classical connectives, perhaps an even more pressing issue regarding the choice of rules for the bilateralist concerns the fact that, in a bilateral system of the sort just specified, *half of the rules are redundant*. Given the coordination principles and structural rules constitutive of a classical consequence relation, for any set of operational rules for some connective provided in a harmonious bilateral natural deduction system, the negative rules can be derived from its positive rules, the positive rules can be derived from the negative rules, the introduction rules can derived from the elimination rules, and the elimination rules can be derived from the introduction rules. This fact leads actually Rumfitt himself to ultimately opt against his system containing all of these rules and favor of the "more compact" system proposed by Smiley (1996). Smiley's system, however, contains just positive rules for conjunction, just negative rules for disjunction, and just elimination rules (both positive and negative) for the conditional. The choice of taking these particular sets of rules as primitive, while technically viable, is completely conceptually unmotivated. Particularly when it comes to the rules for conjunction and disjunction, Smiley's choice to take just the positive conjunction rules and just the negative disjunction rules as primitive is fundamentally out of line with the core bilateralist idea of treating affirmation and denial as conceptually on a par in defining the meanings of the logical connectives. Thus, while a bilateral system with half the rules as Rumfitt's is possible and, at least by the lights of Smiley and Rumiftt, motivated, no existing bilateral system actually motivates a particular way of selecting just half of the rules of a bilateral system such as Rumfitt's.[14]

[14]I follow Smiley and Rumfitt here of thinking that the bilateralist should appeal to coordination principles in order to propose a more compact system. However, it is worth noting that this is not the only way of responding to the fact that half of the rules are bilateral system are redundant. Another way to go, suggested by del Valle-Inclan

So, the bilateralist faces a basic question regarding the choice of rules along these two different dimensions. Until the question about choice of rules is answered, the promise of bilateralism in the context of proof-theoretic semantics has not been truly fulfilled. In what follows, I'll attempt to answer this question. I'll do so, however, by proposing a system quite different than those that have been proposed thus far.

2 A New Kind of Bilateral System

Bilateralism has principally been proposed as a way of formulating natural deduction systems, and it is in this context that I have raised the question of the choice of rules. To answer this question, however, I want to introduce a new kind of bilateral system: a bilateral *sequent calculus* more in the spirit of Gentzen's LK than NK. There has been little to no development of bilateral sequent calculi along these lines.[15] It's not surprising that this is so. While the classical sequent calculus is technically quite nice, from a proof-theoretic perspective, its sequents feature *multiple conclusions*, and this is thought by many to disqualify it from being appealed to in the context of proof-theoretic semantics.[16] Natural deduction systems, in virtue of their "naturalness" (their close correspondence

and Schlöder (2023), is to restrict coordination principles to atoms. If coordination principles are so restricted, then all rules are needed. Given that the possibility of restricting coordination principles to atoms is del Valle-Inclan and Schlöder's bilateral harmony requirement, there is a tension in their approach between harmony and a compact system. With the new criterion of bilateral harmony I propose here, there is no such tension.

[15]One exception is Ayhan's (2021) development of a bilateral intuitionistic sequent calculus, following Wansing (2016). While bilateral natural deduction systems for classical logic are sometimes formulated in sequent notation (e.g. Hjortland 2014, Francez 2014), to my knowledge, no classical bilateral sequent calculus has been proposed.

[16]See, for instance, Dummett (1991, 187), Garson (2001), Rumfitt (2008), and Steinberger (2011). It's perhaps worth noting that there are single conclusion sequent calculi for classical logic. However, these systems involve simply adding rules, such as the Pierce Rule (Curry 1963, 193, Gordeev 1987) or double negation elimination, to the intuitionistic sequent calculus, and so are not properly classical in the sense that a multiple conclusion sequent calculus is.

with ordinary reasoning practices), are widely thought to be preferable to sequent calculi for the purpose of proof-theoretic semantics. Thus, in the context of proof-theoretic semantics, bilateralism is generally proposed as a way of providing a proof-theoretically virtuous natural deduction system for classical logic. As I'll show, however, this is not all that bilateralism has to offer. Indeed, bilateralism has perhaps even more to offer in a sequent calculus setting, and it is in this setting, I take it, that the question I've just posed to the bilateralist can be most directly answered. I'll now propose a bilateral sequent calculus that I'll argue constitutes an answer to this question.

2.1 The Sequent Calculus

The calculus features single-conclusion sequents of the form $\Gamma \vdash A$, where Γ is a set of signed formulas and A is a single signed formula.[17] Speaking of affirmations and denials as different "stances" one might take towards propositions, $\Gamma \vdash A$ can be read as saying that *taking* all of the stances in Γ, be they affirmations or denials, *commits one* to taking the stance A, be it an affirmation or denial.[18] The sole axiom schema is that of Containment (Contexted Reflexivity):

$$\dfrac{}{\Gamma, A \vdash A}\ \text{Containment}$$

This says that, no matter what stances Γ one takes, taking stance A always commits one to taking stance A. In addition to encoding *commitment* with sequents of the form $\Gamma \vdash A$, this sequent calculus will encode *incoherence* with sequents of the form $\Gamma \vdash$. The coordination principles relate commitment to incoherence at the structural level. Transposing the standard

[17]Note, it is just for simplicity's sake that I am here taking what occurs on the left of a sequent to be a *set* of signed formulas rather than a sequence or a multi-set.

[18]The notion of commitment here is understood along the lines developed by Brandom (1994, 157-166). See Incurvati and Schlöder (2023) for a development of this notion in the context of bilateral systems.

coordination principles of Conclusion of Incoherence and Reductio into this sequent context, we can formulate them as follows:

$$\frac{\Gamma \vdash A \quad \Delta \vdash A^*}{\Gamma, \Delta \vdash} \text{ CI} \qquad\qquad \frac{\Gamma, A \vdash}{\Gamma \vdash A^*} \text{ R}$$

Understood in this context, Conclusion of Incoherence says that if a set of stances Γ commits one to taking the stance A, and a set of stances Δ commits one to taking the opposite stance A^*, then we can conclude that the set of stances Γ, Δ is incoherent. Reductio says that if Γ along with stance A is incoherent, then Γ commits one to the opposite stance, A^*. While these are the two coordination principles generally proposed in bilateral natural deduction systems, in a sequent setting, it also makes good sense to add the inverse of Reductio:

$$\frac{\Gamma \vdash A}{\Gamma, A^* \vdash}$$

This says that if Γ commits one to stance A, then Γ along with A^*, the opposite stance of A, is incoherent. Reductio and its inverse together amount to the coordination rule that Smiley (1996) calls "Reversal" (RV), with the following additional condition:

$$\frac{\Gamma, A \vdash B}{\Gamma, B^* \vdash A^*} \text{ RV}$$

Where {A} or {B} can be null.

Applications of Reversal where $\{A\}$ or $\{B\}$ are null is not permitted in Smiley's formulation of the rule, but it makes sense in the context of a bilateral sequent calculus which, like a standard unilateral sequent calculus, permits sequents with empty right-hand sides that encode incoherence.[19] So,

[19] In a standard unilateral sequent calculus, a sequent with a set of sentences on the left side of the turnstile and an empty right side encodes the incoherence of that set of sentences. Thus, for instance, the sequent $p, q \vdash$ can be read as saying that the set $\{p, q\}$ is incoherent or that p and q are incompatible, as codified by the fact that the negation rules let you move from $p, q \vdash$ to $p \vdash \neg q$ and $q \vdash \neg p$. In this bilateral system, we get the same behavior at the structural level by allowing instances of Reversal, where $\{A\}$ and $\{B\}$ are null.

the three structural rules of this system, which I'll call BK_+ (for reasons which will be clear shortly), are Containment, Conclusion of Incoherence, and Reversal. It is easy to show, as I do in the Appendix, that, given these bilateral structural rules, the more familiar structural rules of Weakening and Cut are admissible.

The significance of Reversal is worth discussing in a bit more detail, since it will be central to the functioning of this system. We've already considered the conceptual significance of the cases in which $\{A\}$ or $\{B\}$ is null. Another case worth considering is where A and B are of opposite signs, that is, instances of Reversal of one of the following two forms:

$$\frac{\Gamma, +\langle\varphi\rangle \vdash -\langle\psi\rangle}{\Gamma, +\langle\psi\rangle \vdash -\langle\varphi\rangle}\ \text{RV} \qquad\qquad \frac{\Gamma, -\langle\varphi\rangle \vdash +\langle\psi\rangle}{\Gamma, -\langle\psi\rangle \vdash +\langle\varphi\rangle}\ \text{RV}$$

To appreciate these instances, suppose first that Γ is null. In the first sort of case, Reversal tells us that if, affirming φ commits one to denying ψ, then affirming ψ commits one to denying φ. Now, if the affirmation of one sentence commits one to denying the other, then these sentences can be said to be *incompatible*. So, in this case, Reversal tells us that the relation of incompatibility between sentences is *symmetric*. Alternately, in the second sort of case, Reversal tells us that if denying φ commits one to affirming ψ, then denying ψ commits one to affirming φ. So, in addition to encoding the symmetry of incompatibility or (to put in Aristotelian vocabulary) *contraeity*, Reversal also encodes the symmetry of *subcontraeity*.[20] If we consider cases in which Γ is not null, then, we can think of two sentences as being contraries or subcontraries *relative to the set* Γ, and Reversal tells us that this notion of context-relative contraeity or subcontriety is also symmetric.

With the conceptual significance of Reversal explicated, we can note that its technical significance is that it enables us to put forward a system with rules for introducing positively and negatively signed formulas on

[20]See Smiley (1996, 8) and Rumfitt (1997, 224-225) for how these traditional logical relations are encoded in a bilateral framework.

just one side of the turnstile, since one can get a signed formula with a given connective on the other side of the turnstile by getting its opposite on the side for which rules are given and using Reversal.[21] Though technically, we could provide either solely left or solely right rules, there is a strong tradition in proof-theoretic semantics, going back to Gentzen (1935) himself, according to which the rules for introducing formulas on the right side of the turnstile are treated as conceptually basic in defining the meanings of the logical connectives. Gentzen influentially stated, in the context of natural deduction, that "The introductions represent, as it were, the 'definitions' of the symbols concerned" (80). In a bilateral context, the introduction rules for a connective tell us when one is to affirm a sentence containing that main connective and when one is to deny such a sentence. Adhering to this dictum of Gentzen's, we will take the specification of these conditions to define the meanings of the connectives.[22] Unlike a natural deduction system, which has both introduction and elimination rules, and unlike a standard sequent calculus, which has both right and left rules, the proposed sequent calculus will be such that *all* of the operational rule are right introduction rules. Accordingly, we can adhere to this claim of Gentzen's and maintain that all of the rules of this proof system are definitive of the meanings of the connectives.

Let us now turn to the connective rules. The negation rules of the system are just the introduction rules proposed by Rumfitt:

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[21]A similar use of Reversal is invoked by Smiley (1996) in providing a pared-down natural deduction system with half of the rules of Rumfitt. However, as noted above, Smiley's system leaves out negative rules for conjunction, positive rules for disjunction, and introduction rules for the conditional. Thus, it seems less that we get a bilateral system with half the rules, and more that we get half a bilateral system. The system proposed here contains half the rules of standard bilateral systems, but in a way that is conceptually well-motivated.

[22]The basic philosophical thought underlying this approach to bilateralist proof-theoretic semantics is nicely summed up by Kremer (2004), articulating a key thought of Wittgenstein's: "One understands a proposition by knowing how to use it—when to assert it and when to deny it," (63).

$$\frac{\Gamma \vdash -\langle\varphi\rangle}{\Gamma \vdash +\langle\neg\varphi\rangle}\ +_{\neg} \qquad\qquad \frac{\Gamma \vdash +\langle\varphi\rangle}{\Gamma \vdash -\langle\neg\varphi\rangle}\ -_{\neg}$$

These say that if Γ commits one to denying φ, then Γ commits one to affirming $\neg\varphi$, and, if Γ commits one to affirming φ, then Γ commits one to denying $\neg\varphi$. I will not discuss these rules further in what follows, as they are not novel to this system and it easy to confirm that they pass all of the harmony tests articulated in what follows. My concern will be with the rules for the binary connectives.

The key feature of this system is that, rather than introducing rules for the different binary connectives one by one, the rules for all of the connectives are introduced at once by way of a general rule *schema*. To do this, I deploy a notation that schematizes over signs, using variables such as $\boldsymbol{a}$ and $\boldsymbol{b}$ to indicate signs that may be either + or − along with a function * that maps + to − and − to +. So, for any signed formula of the form $\boldsymbol{a}\langle\varphi\rangle$, where $\boldsymbol{a} \in \{+, -\}$, if $\boldsymbol{a} = +$ then $\boldsymbol{a}^* = -$, and if $\boldsymbol{a} = -$ then $\boldsymbol{a}^* = +$ (and so $\boldsymbol{a}^{**} = \boldsymbol{a}$).[23] With this notation defined, we add to the above negation rules the following general schema for the binary connective rules:

$$\frac{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle \quad \Gamma \vdash \boldsymbol{b}\langle\psi\rangle}{\Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}_{\circ} \qquad\qquad \frac{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash}{\Gamma \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}^*{}_{\circ}$$

It is important to be clear that this is not *two rule schemas* but a *single two-rule schema*, providing the general form of a pair of rules for opposite stances towards a sentence containing a binary connective $\circ$.[24] The $\boldsymbol{c}_{\circ}$ rule says that if Γ commits one to taking stance $\boldsymbol{a}$ to φ and Γ also commits one to taking stance $\boldsymbol{b}$ to ψ, then Γ commits one taking stance $\boldsymbol{c}$ to $\varphi \circ \psi$. The $\boldsymbol{c}^*{}_{\circ}$ rule says that if Γ along with taking stance $\boldsymbol{a}$ to φ and taking stance $\boldsymbol{b}$ to ψ constitutes and incoherent set of stances, then Γ commits one to taking $\boldsymbol{c}^*$, the opposite of stance $\boldsymbol{c}$, to $\varphi \circ \psi$. Even at this very high level of

[23]So that the star is not ambiguous, we might now say that, where A is shorthand for a formula of the form $\boldsymbol{a}\langle\varphi\rangle$, A^* is shorthand for $\boldsymbol{a}^*\langle\varphi\rangle$.

[24]I thank an anonymous referee for urging me to emphasize this point.

abstraction, we can make sense, conceptually, of how rules of these two forms fit together harmoniously as rules for opposite stances towards a sentence. If the condition for the application of the $\boldsymbol{c}_\circ$ rule holds, then, given the stances one has taken, one is *committed* to taking both stance $\boldsymbol{a}$ to φ and stance $\boldsymbol{b}$ to ψ, whereas, if the condition for the application of the $\boldsymbol{c}^*{}_\circ$ rule holds, then, given the stances one has taken, it is *incoherent* to do just that. So, intuitively, these rules fit together perfectly as rules for opposite stances towards a sentence. Technically, however, we can prove that they fit together harmoniously by proving two important elimination theorems at this schematic level of generality.

2.2 Two Generalized Elimination Theorems

The most important elimination theorem is a generalized bilateral analogue to Cut Elimination that shows the Conclusion of Incoherence rule is eliminable; anything that one can prove with the CI rule can be proven without it. To appreciate the technical significance of this proof of the eliminability of CI when it comes to establishing bilateral harmony, consider an alternative binary connective schema, one that defines four distinct "tonkish" connectives:

$$\frac{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash}{\Gamma \vdash \boldsymbol{c}\langle\varphi \bullet \psi\rangle}\ \boldsymbol{c}_\bullet \qquad\qquad \frac{\Gamma, \boldsymbol{a}^*\langle\varphi\rangle, \boldsymbol{b}^*\langle\psi\rangle \vdash}{\Gamma \vdash \boldsymbol{c}^*\langle\varphi \bullet \psi\rangle}\ \boldsymbol{c}^*{}_\bullet$$

Rules for Prior's (1960) *tonk* can be identified in this system with the specific instance of this schema where $\boldsymbol{a} = -$, $\boldsymbol{b} = -$, and $\boldsymbol{c} = +$ (or, equally, where $\boldsymbol{a} = +$, $\boldsymbol{b} = +$, and $\boldsymbol{c} = -$).[25] However, any instance of this general schema has the same basic problem. This schema says that one

[25] As will be clear shortly, that instance gives us the affirmation condition of disjunction along with the denial condition of conjunction, and this corresponds to the following multiple conclusion (unilateral) sequent rules:

$$\frac{\Gamma \vdash \varphi, \psi, \Delta}{\Gamma \vdash \varphi\ tonk\ \psi, \Delta}\ R_{tonk} \qquad\qquad \frac{\Gamma, \varphi, \psi \vdash \Delta}{\Gamma, \varphi\ tonk\ \psi \vdash \Delta}\ L_{tonk}$$

is committed to taking stance $\boldsymbol{c}$ towards $\varphi \bullet \psi$ just in case taking stance $\boldsymbol{a}$ towards φ along with taking stance $\boldsymbol{b}$ towards ψ is incoherent, and one is committed to taking stance $\boldsymbol{c}^*$ towards $\varphi \bullet \psi$ just in case taking stance $\boldsymbol{a}^*$ towards φ along with taking stance $\boldsymbol{b}^*$ towards ψ is incoherent. The problem is that, given that taking stance $\boldsymbol{a}^*$ towards φ will commit one to taking stance $\boldsymbol{c}$ towards $\varphi \bullet \psi$ and taking stance $\boldsymbol{b}$ towards ψ will commit one to taking stance $\boldsymbol{c}^*$ towards $\varphi \bullet \psi$, these rules let us conclude that taking stance $\boldsymbol{a}^*$ towards φ along with taking stance $\boldsymbol{b}$ to ψ is incoherent, and thus that taking stance $\boldsymbol{a}^*$ towards φ commits one to taking stance $\boldsymbol{b}^*$ to ψ. We can show this with the following proof, considering the case where φ and ψ are atomic:

$$\dfrac{\dfrac{\dfrac{\dfrac{\boldsymbol{a}^*\langle p\rangle, \boldsymbol{b}\langle q\rangle \vdash \boldsymbol{a}^*\langle p\rangle}{\boldsymbol{a}^*\langle p\rangle, \boldsymbol{a}\langle p\rangle, \boldsymbol{b}\langle q\rangle \vdash}\,\text{RV}}{\boldsymbol{a}^*\langle p\rangle \vdash \boldsymbol{c}\langle p \bullet q\rangle}\,\boldsymbol{c}_\bullet \qquad \dfrac{\dfrac{\boldsymbol{a}^*\langle p\rangle, \boldsymbol{b}^*\langle q\rangle \vdash \boldsymbol{b}^*\langle q\rangle}{\boldsymbol{b}\langle q\rangle, \boldsymbol{a}^*\langle p\rangle, \boldsymbol{b}^*\langle q\rangle \vdash}\,\text{RV}}{\boldsymbol{b}\langle q\rangle \vdash \boldsymbol{c}^*\langle p \bullet q\rangle}\,\boldsymbol{c}^*_\bullet}{\boldsymbol{a}^*\langle p\rangle, \boldsymbol{b}\langle q\rangle \vdash}\,\text{CI}}{\boldsymbol{a}^*\langle p\rangle \vdash \boldsymbol{b}^*\langle q\rangle}$$

Depending on the signs assigned to $\boldsymbol{a}$ and $\boldsymbol{b}$, this proof lets us conclude either that p and q are contraries, subcontraries, or, as in the case of *tonk*, that one implies the other. The problem, of course, is that p and q are arbitrary, so any tonkish connective of this form lets us conclude such a thing about *any* sentences. The crucial thing to note about this proof is that the use of CI is *ineliminable*; there is no way to reach this conclusion without the use of CI. The schematic proof of that CI is eliminable for any of our $\circ$ rules establishes that no connective that is introduced in accordance with our schema will produce such tonkish behavior. Concretely, introducing any connective into a language with rules that are an instance of our schema will constitute a *conservative extension* of that language, where no new sequents containing only old vocabulary come to be derivable as a result of the addition of the new vocabulary.

The full CI-elimination proof is provided in the Appendix. To appreciate further its conceptual significance, however, it's worth looking at one crucial step. Consider first that it's clear, given the intuitive description of these rules above, that if a position Γ commits one to one stance towards $\varphi \circ \psi$ and a position Δ commits one to the opposite stance towards $\varphi \circ \psi$, then Γ, Δ is already incoherent in virtue of committing one to opposite stances to the simpler formulas in $\varphi \circ \psi$. We might show this by the following transformation:[26]

$$\dfrac{\dfrac{\dfrac{\vdots n}{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle} \quad \dfrac{\vdots m}{\Gamma \vdash \boldsymbol{b}\langle\psi\rangle}}{\Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\,\boldsymbol{c}_{\circ} \quad \dfrac{\dfrac{\vdots k}{\Delta, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash}}{\Delta \vdash \boldsymbol{c}^{*}\langle\varphi \circ \psi\rangle}\,\boldsymbol{c}^{*}{}_{\circ}}{\Gamma, \Delta \vdash}\,\text{CI}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\vdots n}{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle} \quad \dfrac{\dfrac{\dfrac{\vdots m}{\Gamma \vdash \boldsymbol{b}\langle\psi\rangle} \quad \dfrac{\dfrac{\vdots k}{\Delta, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash}}{\Delta, \boldsymbol{a}\langle\varphi\rangle \vdash \boldsymbol{b}^{*}\langle\psi\rangle}\,\text{RV}}{\Gamma, \Delta, \boldsymbol{a}\langle\varphi\rangle \vdash}\,\text{CI}}{\Gamma, \Delta \vdash \boldsymbol{a}^{*}\langle\varphi\rangle}\,\text{RV}}{\Gamma, \Delta \vdash}\,\text{CI}$$

This is the crucial case of the CI-elimination proof where a proof in which CI has been applied to premises expressing commitments to opposite stances towards $\varphi \circ \psi$ is transformed to one in which it has been applied to premises expressing commitments to simpler signed formulas. The fact that such a reduction in the complexity of formulas to which CI is applied is possible means that the introduction rules for $\circ$ do not enable one to introduce opposite stances towards $\varphi \circ \psi$ in any case in which one's stances do not already commit one to opposite stances towards simpler sentences. In general, proving the eliminability of CI can be understood as showing that connective rules of this form do not generate incoherence when it is not already there. In any case in which we apply CI after

[26]Note, once again, that I am treating what goes on the left side of the turnstile as a *set* of signed formulas, and so there is no need to apply Contraction in the final step of the transformed proof.

applying the positive and/or negative rules yielded by our connective schema, concluding the incoherence of a position, we could have just as well applied CI prior to applying these rules (or without applying them at all), and so that position must have already been incoherent prior to the application of the connective rules.

In addition to the problematic schema for tonkish connectives, it's worth considering another problematic connective schema, of which the rules for the connective that Francez (2015, 81) calls *tunk* are an instance:[27]

$$\frac{\Gamma \vdash \boldsymbol{a}\langle \varphi \rangle \quad \Gamma \vdash \boldsymbol{b}\langle \psi \rangle}{\Gamma \vdash \boldsymbol{c}\langle \varphi \blacksquare \psi \rangle}\ \boldsymbol{c}_{\blacksquare} \qquad\qquad \frac{\Gamma \vdash \boldsymbol{a}^*\langle \varphi \rangle \quad \Gamma \vdash \boldsymbol{b}^*\langle \psi \rangle}{\Gamma \vdash \boldsymbol{c}^*\langle \varphi \blacksquare \psi \rangle}\ \boldsymbol{c}^*_{\blacksquare}$$

This rule schema for tunkish connectives has the *opposite* problem as the previous rule schema for tonkish connectives. Whereas, $\bullet$ lets us conclude of a coherent set of stances, that it commits one to opposite stances towards $\varphi \bullet \psi$ and, as such, is incoherent, in the case of $\blacksquare$, there's no reason why we have to say that taking opposite stances towards $\varphi \blacksquare \psi$ is itself incoherent or, equivalently, that taking some stance towards $\varphi \blacksquare \psi$ commits one to taking that very stance towards $\varphi \blacksquare \psi$. To see this, consider the following non-proof of $\boldsymbol{c}\langle p \blacksquare q \rangle \vdash \boldsymbol{c}\langle p \blacksquare q \rangle$:

$$\dfrac{\dfrac{\dfrac{\boldsymbol{a}^*\langle p \rangle \vdash \boldsymbol{a}^*\langle p \rangle \quad \boldsymbol{a}^*\langle p \rangle \vdash \boldsymbol{b}^*\langle q \rangle}{\boldsymbol{a}^*\langle p \rangle \vdash \boldsymbol{c}^*\langle p \blacksquare q \rangle}\ \boldsymbol{c}^*_{\blacksquare}}{\boldsymbol{c}\langle p \blacksquare q \rangle \vdash \boldsymbol{a}\langle p \rangle}\ \text{RV} \qquad \dfrac{\dfrac{\boldsymbol{b}^*\langle q \rangle \vdash \boldsymbol{a}^*\langle p \rangle \quad \boldsymbol{b}^*\langle q \rangle \vdash \boldsymbol{b}^*\langle q \rangle}{\boldsymbol{b}^*\langle q \rangle \vdash \boldsymbol{c}^*\langle p \blacksquare q \rangle}\ \boldsymbol{c}^*_{\blacksquare}}{\boldsymbol{c}\langle p \blacksquare q \rangle \vdash \boldsymbol{b}\langle q \rangle}\ \text{RV}}{\boldsymbol{c}\langle p \blacksquare q \rangle \vdash \boldsymbol{c}\langle p \blacksquare q \rangle}\ \boldsymbol{c}_{\blacksquare}$$

Here, we're required to use sequents that *aren't* axioms in order to conclude something that *is* an axiom. Since one can coherently maintain that

[27] Whereas *tonk* has the affirmation condition of disjunction (corresponding to its right rule in a unilateral system) and the denial condition of conjunction (corresponding to its left rule), *tunk* has the affirmation condition of conjunction and the denial condition of disjunction. The multiple conclusion sequent rules for *tunk* are the following:

$$\frac{\Gamma \vdash \varphi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi\ \mathit{tunk}\ \psi, \Delta} \qquad\qquad \frac{\Gamma, \varphi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi\ \mathit{tunk}\ \psi \vdash \Delta}$$

taking stance $\boldsymbol{a}^*$ to p *doesn't* commit one to taking stance $\boldsymbol{b}^*$ to q or vice versa, one is under no compulsion to say that taking stance $\boldsymbol{c}$ towards $p \blacksquare q$ commits one to taking stance $\boldsymbol{c}$ towards $p \blacksquare q$. The crucial technical point of note here is that the axiom schema of Containment cannot be restricted to atoms. The proof provided in the Appendix that the axiom schema of Containment can be restricted to atoms for our $\circ$ rules suffices to establish that these rules will never leave one in such a predicament. The key step in the proof is given by the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash \boldsymbol{a}\langle\varphi\rangle \quad \Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash \boldsymbol{b}\langle\psi\rangle}{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}_\circ}{\Gamma, \boldsymbol{c}^*\langle\varphi \circ \psi\rangle, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash}\ \mathrm{RV}}{\Gamma, \boldsymbol{c}^*\langle\varphi \circ \psi\rangle \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}^*_\circ}{\Gamma, \boldsymbol{c}\langle\varphi \circ \psi\rangle \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\ \mathrm{RV}$$

Thus, not only will no instance of our rule schema define a connective that exhibits *tonkish* behavior, but no such connective will exhibit *tunkish* behavior either.[28]

2.3 Harmonious Rules for All Classical Connectives

These schematic elimination proofs establish that the conditions for taking opposite stances towards any connective introduced in accordance with our schema fit together harmoniously. Thus, any instance of our rule schema can be counted as defining a legitimate logical connective. Accordingly, there's no reason not to define all such connectives. That's what I'll do. The proposed sequent system includes negation, as defined by the standard introduction rules, along with all and only the binary

[28]For an influential discussion of the significance of the proof of the eliminability of non-atomic Reflexivity, see Hacking (1979). This proof is very closely related to the proof of *uniqueness*, discussed by Belnap (1962) as a criterion for proof-theoretic definitions. For a comparison of these two approaches to this aspect of harmony, see Naibo and Petrolo (2015).

connectives defined by any assignment of signs to $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$. In particular, I take the following connectives to be defined by the rules given by the following assignments of signs:

$$\wedge\colon \boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = + \qquad \vee\colon \boldsymbol{a} = -, \boldsymbol{b} = -, \boldsymbol{c} = -$$
$$|\colon \boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = - \qquad \downarrow\colon \boldsymbol{a} = -, \boldsymbol{b} = -, \boldsymbol{c} = +$$
$$\rightarrow\colon \boldsymbol{a} = +, \boldsymbol{b} = -, \boldsymbol{c} = - \qquad \succ\!\!-\colon \boldsymbol{a} = -, \boldsymbol{b} = +, \boldsymbol{c} = +$$
$$-\!\!\prec\colon \boldsymbol{a} = +, \boldsymbol{b} = -, \boldsymbol{c} = + \qquad \leftarrow\colon \boldsymbol{a} = -, \boldsymbol{b} = +, \boldsymbol{c} = -$$

So, in addition to providing rules for the standard classical connectives—conjunction, disjunction, and the material conditional—the schema also provides rules for some not-so-standard ones—the Sheffer Stroke, Peirce's Arrow, and the dual of the conditional, for which I use the symbol $\succ\!\!-$, drawing this notation from Wansing (2016) and Ayhan (2021).[29] As you can see here, when the rules for a given connective have been provided in accordance with this schema, the rules for its dual can be obtained simply by taking the opposite of all the signs.

I call the system without the eliminable structural rules, constituted by the axiom schema of atomic Containment, the structural rule of Reversal, the negation rules, and all of the binary connective rules directly given by the schema—every possible assignment of positive/negative signs to $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ in the schema above—"BK." Though, technically, BK contains all of the binary connectives definable by the rule schema, one can take whatever fragment of BK consisting of whatever rules one likes. As I show in the Appendix, any such fragment is sound and complete with respect to the classical semantics of the connectives that fragment contains. If

[29]Though this notation is consistent with Wansing and Ayhan, it's worth pointing out that, working in an intuitionistic context, Wansing and Ayhan actually take the dual of $\rightarrow$ to be $-\!\!\prec$ rather than $\succ\!\!-$ (See Wansing 2016, 427-428 for a justification). I will not comment on this choice of theirs other than to note that, in an intuitionistic context, the notion of duality is more complicated. However, at least in a classical context, the dual C^d of an n-ary connective C can be defined, following Post (1941, 30-31), such that $C(\varphi_1, \varphi_2 \ldots \varphi_n) \equiv \neg C^d(\neg\varphi_1, \neg\varphi_2 \ldots \neg\varphi_n)$. It should be clear, given this definition of duality, why reversing all of the signs in the rules for a given connective yields the rules for its dual, and thus, the dual of classical $\rightarrow$ is $\succ\!\!-$, as defined by these rules.

one wants all of classical logic with as few connectives as possible, one can take the fragment of BK containing just | or ↓, as defined by either of the following pairs of rules:

$$\frac{\Gamma, +\langle\varphi\rangle, +\langle\psi\rangle \vdash}{\Gamma \vdash +\langle\varphi \mid \psi\rangle}\ +_{\mid} \qquad \frac{\Gamma \vdash +\langle\varphi\rangle \quad \Gamma \vdash +\langle\psi\rangle}{\Gamma \vdash -\langle\varphi \mid \psi\rangle}\ -_{\mid}$$

$$\frac{\Gamma \vdash -\langle\varphi\rangle \quad \Gamma \vdash -\langle\psi\rangle}{\Gamma \vdash +\langle\varphi \downarrow \psi\rangle}\ +_{\downarrow} \qquad \frac{\Gamma, -\langle\varphi\rangle, -\langle\psi\rangle \vdash}{\Gamma \vdash -\langle\varphi \downarrow \psi\rangle}\ -_{\downarrow}$$

These rules for the Sheffer Stroke and Peirce's Arrow are equivalent to those proposed by Riser (1967) and Zach (2016). From the perspective taken here, these connectives are not second-class citizens, and, importantly, their meaning is not to be understood in terms of the meanings of "not," "and," or "or," but, rather, directly in terms of these bilateral proof rules. Of more direct relevance to contemporary debates about bilateralism in proof-theoretic semantics is the following fragment of BK:

$$\frac{\Gamma \vdash -\langle\varphi\rangle}{\Gamma \vdash +\langle\neg\varphi\rangle}\ +_{\neg} \qquad \frac{\Gamma \vdash +\langle\varphi\rangle}{\Gamma \vdash -\langle\neg\varphi\rangle}\ -_{\neg}$$

$$\frac{\Gamma \vdash +\langle\varphi\rangle \quad \Gamma \vdash +\langle\psi\rangle}{\Gamma \vdash +\langle\varphi \wedge \psi\rangle}\ +_{\wedge} \qquad \frac{\Gamma, +\langle\varphi\rangle, +\langle\psi\rangle \vdash}{\Gamma \vdash -\langle\varphi \wedge \psi\rangle}\ -_{\wedge}$$

$$\frac{\Gamma, -\langle\varphi\rangle, -\langle\psi\rangle \vdash}{\Gamma \vdash +\langle\varphi \vee \psi\rangle}\ +_{\vee} \qquad \frac{\Gamma \vdash -\langle\varphi\rangle \quad \Gamma \vdash -\langle\psi\rangle}{\Gamma \vdash -\langle\varphi \vee \psi\rangle}\ -_{\vee}$$

$$\frac{\Gamma, +\langle\varphi\rangle, -\langle\psi\rangle \vdash}{\Gamma \vdash +\langle\varphi \to \psi\rangle}\ +_{\to} \qquad \frac{\Gamma \vdash +\langle\varphi\rangle \quad \Gamma \vdash -\langle\psi\rangle}{\Gamma \vdash -\langle\varphi \to \psi\rangle}\ -_{\to}$$

As I show in the Appendix, this fragment of BK is equivalent to Ketonen's (1944) multiple conclusion classical sequent calculus in that every proof in this system corresponds to a unique proof in Ketonen's system and every proof in Ketonen's system corresponds to an equivalence class of proofs in this system under Reversal. Hence "BK" stands for "Bilateral Ketonen."

Ketonen's sequent calculus has several nice formal properties, all of which are preserved in this system.[30] However, despite all of the wonderful proof-theoretic properties of Ketonen's sequent calculus, it is appealed to relatively rarely in the context of proof-theoretic semantics. The reason is that it is essentially a *multiple conclusion* system, and such systems are widely discounted in the context of proof-theoretic semantics because it is difficult to understand multiple conclusion sequents as expressing *arguments* or *inferences*.[31] Accordingly, since proof-theoretic semantics is widely understood in the context of providing an *inferentialist* theory of meaning for the logical connectives, multiple conclusion sequent calculi are often taken to be ruled out as meaning-conferring proof systems.[32] The system proposed here enables one to retain all of the technical benefits of Ketonen's multiple conclusion sequent calculus while maintaining that its rules count as specifications of the meanings of the classical connectives in terms of the inferential rules governing their use. This is because, though the fragment of the BK containing just the rules for the familiar connectives is equivalent to Ketonen's sequent calculus, this bilateral system uses only *single conclusion* sequents, and thus provides an immediately intuitive explication of the sense of the connectives in terms

[30]Following Indrzejczak (2021), I call the Ketonen system to which this system is equivalent "K." K has the same binary connective rules as the system Troelstra and Schwichtenberg (2000) call "G3cp," discussed in Negri and von Plato (2008), but with the standard negation rules of LK. See Indrzejczak (2021, 1-61) for an extended discussion of this sequent calculus and its properties, several of which I discuss below.

[31]Once again, see Dummett (1991, 187), Garson (2001), Rumfitt (2008) and Steinberger (2011) for relevant criticisms of multiple conclusion sequent calculi along these lines.

[32]Kremer (1988) considers multiple conclusion sequent calculi as potentially serving to inferentially specify the meanings of the connectives, but he does not address the issues raised by multiple conclusion. One attempt to deal with this issue is the *bilateral interpretation* of multiple conclusion sequents, owed to Restall (2005) and notably pursued by Ripley (2013, 2017) according to which a sequent of the form $\Gamma \vdash \Delta$ expresses that affirming everything in Γ and denying everything in Δ is incoherent or "out of bounds." As I show in the Appendix (Section 5.3), multiple conclusion unsigned sequents of the form $\Gamma \vdash \Delta$ can be translated into signed sequents of the form $\Gamma \vdash$, which figure in this system, in a way that captures Restall and Ripley's bilateralism. See Simonelli (M.S.) for a fuller development of this idea.

of the conditions under which one is to affirm or deny a sentence with that connective. It is thus a genuine candidate for defining the meanings of the classical connectives in inferential terms. Let me now argue that it does just that.

3 In Defense of Definitiveness

I have provided a single schema that yields the rules for all of the classical connectives. I take it that I've done enough to show that However, it's worth being clear that the schema I've provided is not the only possible one. Instead of the schema we've articulated here, one could alternatively propose the following schema:

$$\frac{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle \quad \Gamma \vdash \boldsymbol{b}\langle\psi\rangle}{\Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\, \boldsymbol{c}_{\circ} \qquad \frac{\Gamma \vdash \boldsymbol{a}^{*}\langle\varphi\rangle}{\Gamma \vdash \boldsymbol{c}^{*}\langle\varphi \circ \psi\rangle}\, \boldsymbol{c}^{*}{}_{\circ_L} \qquad \frac{\Gamma \vdash \boldsymbol{b}^{*}\langle\psi\rangle}{\Gamma \vdash \boldsymbol{c}^{*}\langle\varphi \circ \psi\rangle}\, \boldsymbol{c}^{*}{}_{\circ_R}$$

Whereas our rules correspond to those of Ketonen's classical sequent calculus, rules of this form correspond more closely to the rules of Gentzen's classical sequent calculus LK (with the exception of his conditional rules). Indeed, if one just considers the rules for conjunction and disjunction yielded by this schema this system is equivalent to the fragment of LK with just conjunction and disjunction.[33] Accordingly I'll call the system constituted by Containment, Reversal, the negation rules, and all instances of this alternative binary connective schema "BG" for "Bilateral Gentzen." The first thing to point out is that, insofar all of the axioms are instances of Containment, these two logics define just the same bilateral consequence relation.[34] To see this, note first that the rules of BG are admissible (though not derivable) in BK, since, though BK does not contain

[33]The proof of this claim is directly analogous to the proof of the equivalence with Ketonen's sequent calculus provided in the Appendix.

[34]It's worth noting that, if one adds additional axioms involving non-logical vocabulary, the two consequence relations can come apart. For instance, Hlobil (2018), Kaplan (2017), Brandom (2018), and Hlobil and Brandom (forthcoming) consider adding to Ketonen's sequent calculus non-logical axioms encoding defeasible material inferential relations (for instance, the inference from "bird" to "flies") for which the structural rule

Weakening, Weakening is admissible in BK, and, with Weakening, one can derive the BG rules from the BK rules as follows:

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash \boldsymbol{a}^*\langle\varphi\rangle}{\Gamma, \boldsymbol{b}\langle\psi\rangle \vdash \boldsymbol{a}^*\langle\varphi\rangle}\text{ Weak.}}{\Gamma, \boldsymbol{b}\langle\psi\rangle, \boldsymbol{a}\langle\varphi\rangle \vdash}\text{ RV}}{\Gamma \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}^*{}_\circ \qquad \dfrac{\dfrac{\dfrac{\Gamma \vdash \boldsymbol{b}^*\langle\psi\rangle}{\Gamma, \boldsymbol{a}\langle\varphi\rangle \vdash \boldsymbol{b}^*\langle\psi\rangle}\text{ Weak.}}{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash}\text{ RV}}{\Gamma \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}^*{}_\circ$$

Alternatively, given Contraction (which is built into our treatment of the left side of a sequent as a set), we can derive the rules of BK from those of BG as follows:[35]

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash}{\Gamma, \boldsymbol{a}\langle\varphi\rangle \vdash \boldsymbol{b}^*\langle\psi\rangle}\text{ RV}}{\Gamma, \boldsymbol{a}\langle\varphi\rangle \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}^*{}_{\circ_R}}{\Gamma, \boldsymbol{c}\langle\varphi \circ \psi\rangle \vdash \boldsymbol{a}^*\langle\varphi\rangle}\text{ RV}}{\Gamma, \boldsymbol{c}\langle\varphi \circ \psi\rangle \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}^*{}_{\circ_L}}{\Gamma, \boldsymbol{c}\langle\varphi \circ \psi\rangle, \boldsymbol{c}\langle\varphi \circ \psi\rangle \vdash}\text{ RV}}{\Gamma, \boldsymbol{c}\langle\varphi \circ \psi\rangle \vdash}\text{ Cont.}}{\Gamma \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\text{ RV}$$

So, the question is not about which set of rules yield classical logic—they both do. The question, rather, is about which set of rules is to be taken to be definitive of the meanings of the classical connectives in the context of proof-theoretic semantics. Insofar as this is the question, there are several reasons to prefer the schema I have set out here over this alternative schema. Let me focus on just a few, which I take to be the most fundamental ones.

of Weakening fails. In such a context, the consequence relations of BK and BG actually come apart. I take the fact that BK can be used in such an application to be another important feature that distinguishes it over BG; however, I limit my attention to classical logic for the purposes of this paper. For a development a bilateral system of this sort in which non-logical axioms are integrated, see Simonelli (MS).

[35] I show the step of applying Contraction for clarity, but, officially, it is not actual step in this proof.

The first problem with BG will be familiar to anyone who's worked with Gentzen's LK: proofs of simple theorems often just don't correspond at all to an intuitive justification of those theorems. Consider, for instance, the following proof of the Law of Excluded Middle:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{+\langle p\rangle \vdash +\langle p\rangle}{+\langle p\rangle \vdash +\langle p \vee \neg p\rangle}\ {+_{\vee_L}}}{-\langle p \vee \neg p\rangle \vdash -\langle p\rangle}\ \mathrm{RV}}{-\langle p \vee \neg p\rangle \vdash +\langle \neg p\rangle}\ {+_{\neg}}}{-\langle p \vee \neg p\rangle \vdash +\langle p \vee \neg p\rangle}\ {+_{\vee_L}}}{-\langle p \vee \neg p\rangle, -\langle p \vee \neg p\rangle \vdash}\ \mathrm{RV}}{-\langle p \vee \neg p\rangle \vdash}\ \mathrm{Cnt}}{\vdash +\langle p \vee \neg p\rangle}\ \mathrm{RV}$$

This proof, which involves the required technical trick of deriving the same formula twice and then using Contraction simply does not correspond to any intuitive justification of this theorem. By contrast, as I explain below, the proof of the same theorem in BK is perfectly intuitive. A further problem concerns the fact that, in BG, proving a sequent whose conclusion is the affirmation or denial of some complex sentence φ does not require using rules for all of the connectives in φ. Consider, for instance, the proof in BG of $\vdash +\langle (p \vee \neg p) \vee (q \rightarrow r)\rangle$, which is identical to the above proof with the following additional step:

$$\dfrac{\vdash +\langle p \vee \neg p\rangle}{\vdash +\langle (p \vee \neg p) \vee (q \rightarrow r)\rangle}\ {+_{\vee_L}}$$

Now, plausibly, it's a condition on rationally affirming some sentence that one understands the sentence that one affirms, and, plausibly, understanding the meaning of a disjunction requires understanding the meaning of both disjuncts. However, given that this proof doesn't require the use of the conditional rules, one could affirm this sentence without having knowledge of these rules, and thus, without understanding the meaning of the right disjunct. In BK, by contrast, the proof of any classically valid sequent of the form $\Gamma \vdash \pm\langle\varphi\rangle$ is such at least one of the rules for each logical connective contained in φ is used.[36] Moreover, where one rule is used for some logically complex subformula in φ in the proof of a sequent of the form $\Gamma \vdash +\langle\varphi\rangle$, the other rule is used for that same subformula in the proof of a sequent of the form $\Gamma \vdash -\langle\varphi\rangle$.[37]

I take the above two reasons to be among the most significant reasons *against* the use of BG in the context of proof-theoretic semantics. Let me now turn to what I take to be the most significant reason *for* the use of BK in this context. In BK, there are exactly two rules for each binary connective: a rule that says when one is to affirm a sentence with that main connective and a rule that says when one is to deny a sentence with that main connective. Following the dictum of Gentzen's quoted above, I take it that these rules alone suffice to serve as "definitions" of the connectives. Though one might take the rules of BG to serve as "definitions" in a similar fashion, the positive and negative introduction rules of BK have a crucial feature that elevates their status above the rules of BG in the context of proof-theoretic semantics: they are all *invertible* in that whenever a sequent involving a sentence featuring some connective is derivable, the premise sequents involving the connected sentences are also derivable. Thus, the premise sequents specify *exactly* the conditions under which one is committed to affirming or denying a compound sentence. This licenses us to say that one is committed to affirming $\varphi \circ \psi$ *just in case* condition X holds and one is committed denying $\varphi \circ \psi$ *just in case* condition Y holds, where conditions X and Y are in perfect opposition to one another. If such a specification of the meanings of the classical connectives is possible, it is clearly desirable. The question, then, is whether rules of this form really do define the meanings of the classical connectives with which we are familiar. I'll now argue that they do.

[36]To see this, note that a proof is constructed root-first, and each logically complex sub-formula in φ must be decomposed in order for the proof to be complete. This ensures that some rule for each connective is used.

[37]Proof is straightforward by induction on the complexity of the subformulas in φ.

Consider first the rules for conjunction and disjunction. Though the positive conjunction rule and the negative disjunction rule are the familiar introduction rules from Rumfitt's natural deduction system, the negative conjunction rule and the positive disjunction rule are new to this system. However, they should be easily recognizable as formally codifying the meanings of conjunction and disjunction along with the other member of their pair. The positive conjunction rule says that one is committed to affirming a conjunction just in case one is committed to affirming both of the conjuncts, and the negative conjunction rule says that one is committed to denying a conjunction just in case it is incoherent to do just that. The positive disjunction rule says that one is committed to affirming a disjunction just in case it is incoherent to deny both of the disjuncts, and the negative disjunction rule says that one is committed to denying a disjunction just in case one is committed to doing just that. In this way, these rules fit together perfectly as rules for opposite stances towards a conjunctive or disjunctive proposition. To illustrate how the negative conjunction and positive disjunction rules actually work in the context of this system, consider the proofs of the Principle of Non-Contradiction and the Law of Excluded Middle respectively:

$$\dfrac{\dfrac{\dfrac{+\langle p\rangle \vdash +\langle p\rangle}{+\langle p\rangle \vdash -\langle \neg p\rangle}\,-_{\neg}}{+\langle p\rangle, +\langle \neg p\rangle \vdash}\,+_{\mathrm{RV}}}{\vdash -\langle p \wedge \neg p\rangle} \qquad\qquad \dfrac{\dfrac{\dfrac{-\langle p\rangle \vdash -\langle p\rangle}{-\langle p\rangle \vdash +\langle \neg p\rangle}\,+_{\neg}}{-\langle p\rangle, -\langle \neg p\rangle \vdash}\,+_{\vee}}{\vdash +\langle p \vee \neg p\rangle}$$

The proof on the left reads as follows. Affirming p commits one to affirming p. So affirming p commits one to denying $\neg p$. Accordingly, affirming p and affirming $\neg p$ is incoherent. Thus, one is committed to denying $p \wedge \neg p$. The proof on the right reads as follows. Denying p commits one to denying p. So, denying p commits one to affirming $\neg p$. Accordingly, denying p and denying $\neg p$ is incoherent. Thus, one is committed to affirming $p \vee \neg p$. Unlike the proofs in BG, these seem to me to be perfectly intuitive justifications for LEM and PNC.[38]

Now consider the conditional rules. These rules together very clearly capture the fact that it is the *material* conditional that they define. As with the other rules, the positive and negative rules fit together perfectly as rules for opposite stances towards the material conditional. One is committed to affirming a material conditional just in case one can't coherently affirm its antecedent and deny its consequent, and one is committed to denying a conditional just in case one is committed to doing just that. This is, in fact, how Frege himself (1879/1997) first defines the material conditional in the *Begriffsschrift*.[39] Still, I take it that the rules for the conditional will get the most resistance here, so let me say a few more words in their defense. The first thing to note is that, if one feels attachment to the standard introduction rule by conditional proof, it's worth noting that, given Reversal, one can freely move between $\Gamma, +\langle\varphi\rangle, -\langle\psi\rangle \vdash$ and $\Gamma, +\langle\varphi\rangle \vdash +\langle\psi\rangle$, and so, in that sense, this system does contain the standard conditional introduction rule. However, this is true no less of $\Gamma, -\langle\psi\rangle \vdash -\langle\varphi\rangle$. All three of these sequents can be understood in terms of their relations to the others. Whereas $\Gamma, +\langle\varphi\rangle, -\langle\psi\rangle \vdash$ says that, relative to a context Γ, affirming φ and denying ψ is *incoherent*, $\Gamma, +\langle\varphi\rangle \vdash +\langle\psi\rangle$ and $\Gamma, -\langle\psi\rangle \vdash -\langle\varphi\rangle$ jointly say that, relative to Γ, affirming φ and denying ψ are *incompatible* stances in that, if one takes one of these stances, one is committed to taking the opposite of the other. In general, $\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash$ says that taking all of the stances in Γ, along with taking stance $\boldsymbol{a}$ to φ, and taking stance $\boldsymbol{b}$ to ψ is *incoherent*, whereas $\Gamma, \boldsymbol{a}\langle\varphi\rangle \vdash \boldsymbol{b}^*\langle\psi\rangle$ and $\Gamma, \boldsymbol{b}\langle\psi\rangle \vdash \boldsymbol{a}^*\langle\varphi\rangle$ can be understood as jointly saying that, relative to all of

[38]As an exercise for the reader, construct the proofs of LEM and PNC in BG, and compare.

[39]To introduce the conditional, Frege tells us that, if φ and ψ are judgeable contents, then there are four possibilities regarding the different stances we might take on φ and ψ: (1) φ is affirmed and ψ is affirmed, (2) φ is denied and ψ is affirmed, (3), φ is affirmed and ψ is denied, and (4) φ is denied and ψ is denied. He then says that the judgment of $\varphi \rightarrow \psi$ "denotes the judgment that the third of these possibilities does not obtain," whereas, if $\varphi \rightarrow \psi$ "is denied, then this is to say that the third of of these possibilities does obtain, i.e. that ψ is denied and φ affirmed," (56).

the stances in Γ, taking stance $\boldsymbol{a}$ to φ is *incompatible* with taking stance $\boldsymbol{b}$ to ψ in the sense that taking one of these stances commits you to taking the opposite of the other. Though all of these sequents are immediately inter-provable, equivalent sequents are not necessarily conceptually on a par in the context of proof-theoretic semantics, and one crucial advantage of taking $\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash$ to be conceptually prior to $\Gamma, \boldsymbol{a}\langle\varphi\rangle \vdash \boldsymbol{b}^*\langle\psi\rangle$ in specifying the condition under which one is to take stance $\boldsymbol{c}^*$ to $\varphi \circ \psi$ is that it makes it very clear that there need not be any inferential connection between taking stance $\boldsymbol{a}$ to φ and taking stance $\boldsymbol{b}$ to ψ in order for one to be committed to $\boldsymbol{c}^*\langle\varphi \circ \psi\rangle$.[40] To see the upshot of this choice as it bears on the rules for the conditional, consider the derivation of two of the key "paradoxes" of material implication: $+\langle p\rangle \vdash +\langle q \to p\rangle$ and $-\langle q\rangle \vdash +\langle q \to p\rangle$ in BK:

$$\dfrac{\dfrac{+\langle p\rangle, +\langle q\rangle \vdash +\langle p\rangle}{+\langle p\rangle, +\langle q\rangle, -\langle p\rangle \vdash}\,\text{RV}}{+\langle p\rangle \vdash +\langle q \to p\rangle}\,+_{\to} \qquad\qquad \dfrac{\dfrac{-\langle q\rangle, -\langle p\rangle \vdash -\langle q\rangle}{-\langle q\rangle, +\langle q\rangle, -\langle p\rangle \vdash}\,\text{RV}}{-\langle q\rangle \vdash +\langle q \to p\rangle}\,+_{\to}$$

This makes clear that the condition for affirming $p \to q$ does not require there be any inferential connection between affirming p and affirming q; it simply requires that, given the stances one has taken, affirming p and denying q is incoherent. Now, clearly, given an affirmation of p or a denial of q, it's incoherent to affirm q and deny p, since one would be denying or affirming the very thing one affirms or denies. And thus, an affirmation of p commits one to affirming $q \to p$, as does a denial of q. Once again, we have a perfectly intuitive justification of some key classical validities (and, in this case, ones usually thought to be quite intuitive) through the

[40] I'd like to thank an anonymous reviewer for raising this point in response to an earlier formulation of the $\boldsymbol{c}^*$ rule in which the top sequent was $\Gamma, \boldsymbol{a}\langle\varphi\rangle \vdash \boldsymbol{b}^*\langle\psi\rangle$. Rules of this form are what del Valle-Inclan and Schloder propose in a sequent context. See Simonelli (forthcoming) for an explicit comparison of these different rule forms in a natural deduction context. While, conceptually, I think there is some reason to prefer the formulation endorsed here, technically, it's hard to see how there can be anything decisive to be said in favor of

distinctive rules of BK.[41]

I have provided intuitive justifications for the BK rules for conjunction, disjunction, and the conditional. It should be clear that similar intuitive justifications can be given for the rules of the other less standard connectives defined by the schema.[42] Intuitively, then, these rules can be taken to be definitive of the meanings of the classical connectives. I'll now show that, technically, these rules can be taken to be definitive of the meanings of the classical connectives in that they precisely determine the truth-conditions of the connectives. To do this, let us introduce just a bit of semantic machinery.[43] Let a valuation v be any function $\mathcal{L} \to \{1, 0\}$, that is, any function mapping each sentence of our logical language to a truth-value, 1 or 0. Now, to formally codify the fact that an affirmation of some sentence is correct just in case the sentence affirmed is true, whereas the denial of some sentence is correct just in case the sentence denied is

[41]Now, it may be that these rules so clearly manifest the meaning of the material conditional as to reveal, as Anderson and Belnap (1975, 4) claim, "that material 'implication' is *not* an implication connective" at all. Whether that is so is a question on which I'll remain neutral. I just claim that these rules can be taken to be definitive of the meaning of the horseshoe of classical logic, regardless of whether that logical connective really deserves to be called a "conditional" or "implication" connective.

[42]One might wonder, at this point, about the rules for the biconditional $\leftrightarrow$ or the exclusive disjunction $\veebar$. It is clear how harmonious invertible rules for these connectives can be given, given the rules we already have:

$$\frac{\Gamma, -\langle\varphi\rangle, -\langle\psi\rangle \vdash \quad \Gamma, +\langle\varphi\rangle, +\langle\psi\rangle \vdash}{\Gamma \vdash +\langle\varphi \veebar \psi\rangle}\ +\veebar \qquad \frac{\Gamma, +\langle\varphi\rangle, -\langle\psi\rangle \vdash \quad \Gamma, +\langle\psi\rangle, -\langle\varphi\rangle \vdash}{\Gamma \vdash -\langle\varphi \veebar \psi\rangle}\ -\veebar$$

$$\frac{\Gamma, -\langle\varphi\rangle, +\langle\psi\rangle \vdash \quad \Gamma, -\langle\psi\rangle, +\langle\varphi\rangle \vdash}{\Gamma \vdash +\langle\varphi \leftrightarrow \psi\rangle}\ +_{\leftrightarrow} \qquad \frac{\Gamma, +\langle\varphi\rangle, +\langle\psi\rangle \vdash \quad \Gamma, -\langle\varphi\rangle, -\langle\psi\rangle \vdash}{\Gamma \vdash -\langle\varphi \leftrightarrow \psi\rangle}\ -_{\leftrightarrow}$$

Thus, the condition under which one is to affirm an exclusive disjunction is just the condition under which one is to affirm a disjunction and deny a conjunction, and the condition under which one is to deny an exclusive disjunction is just the condition under which one is to affirm the conditional in both directions. Dually for the biconditional. These rules give precise proof-theoretic sense to the claim that, unlike the connectives defined by way of the schema, exclusive disjunction and the biconditional are not *basic* propositional connectives.

[43]My approach to introducing the requisite machinery for the discussion of this issue closely follows that of Hjortland (2014).

false, let us define:

> **Correctness Function:** The *correctness function* [] is a function from $\{+, -\}$ to $\{1, 0\}$ mapping $+$ to 1 and $-$ to 0.

The expression $[\boldsymbol{a}]$ can be read as "the value that would make stance $\boldsymbol{a}$ correct." This lets us define:

> **Correctness:** Taking some stance $\boldsymbol{a}$ towards some sentence φ, $\boldsymbol{a}\langle\varphi\rangle$, is *correct*, relative to some valuation v, just in case $v(\varphi) = [\boldsymbol{a}]$.

Now, we'll say that a sequent $\Gamma \vdash A$ is *confirmed* by a valuation v just in case, if all of the stances in Γ are correct, relative to v, then A is correct, relative to v. A sequent of the form $\Gamma \vdash$ is confirmed by a valuation v just in case some of the stances in Γ are incorrect, relative to v. A sequent is refuted by some valuation just in case v does not confirm that sequent. We can now define the notion of validity, relative to a set of valuations V:

> **V-Validity:** A sequent rule is *V-valid* just in case there is no $v \in V$ that confirms all of the premise sequents and refutes the conclusion sequent.

We can now consider which constraints our schematic sequent rules put on the set of admissible valuations, assuming no valuation in that set confirms all of the premises and refutes the conclusion of either of our rules. As we will now see, they constrain the valuations to just those of classical propositional logic.

Once again, the rules are the following:

$$\frac{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle \quad \Gamma \vdash \boldsymbol{b}\langle\psi\rangle}{\Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}_\circ \qquad\qquad \frac{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash}{\Gamma \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}^*{}_\circ$$

Consider first the constraint put on valuations by the $\boldsymbol{c}^*{}_\circ$ rule. Suppose some valuation v refutes the conclusion sequent. So, every stance in Γ is correct, relative to v, and $\boldsymbol{c}^*\langle\varphi \circ \psi\rangle$ is incorrect, relative to v, which is to say, $v(\varphi \circ \psi) = [\boldsymbol{c}]$. Then, $\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash$ must be refuted by v, and so all of the stances in $\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle$ must be correct, relative to v. Thus, whenever $v(\varphi \circ \psi) = [\boldsymbol{c}]$, both $v(\varphi) = [\boldsymbol{a}]$ and $v(\psi) = [\boldsymbol{b}]$. Now consider the constraint put on the set of valuations by the $\boldsymbol{c}_\circ$ rule. Suppose every stance in Γ is correct and $\boldsymbol{c}\langle\varphi \circ \psi\rangle$ is incorrect, which is to say, $v(\varphi \circ \psi) = [\boldsymbol{c}^*]$. Then it can't be the case that both $\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle$ is confirmed and $\Gamma \vdash \boldsymbol{b}\langle\psi\rangle$ is confirmed. Given that all of the stances in Γ are correct, this means that either $\boldsymbol{a}\langle\varphi\rangle$ is incorrect or $\boldsymbol{b}\langle\psi\rangle$ is incorrect. Thus, whenever $v(\varphi \circ \psi) = [\boldsymbol{c}^*]$, either $v(\varphi) = [\boldsymbol{a}^*]$ or $v(\psi) = [\boldsymbol{b}^*]$. These constraints can be put as follows:

$$v(\varphi \circ \psi) = \begin{cases} [\boldsymbol{c}], & \text{if } v(\varphi) = [\boldsymbol{a}] \text{ and } v(\psi) = [\boldsymbol{b}] \\ [\boldsymbol{c}^*], & \text{if } v(\varphi) = [\boldsymbol{a}^*] \text{ or } v(\psi) = [\boldsymbol{b}^*] \end{cases}$$

And this *just is* a schematic specification of the classical truth-conditions for all of the connectives we've defined. For instance, we have the following instance for conjunction where $\boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = +$:

$$v(\varphi \wedge \psi) = \begin{cases} 1, & \text{if } v(\varphi) = 1 \text{ and } v(\psi) = 1 \\ 0, & \text{if } v(\varphi) = 0 \text{ or } v(\psi) = 0 \end{cases}$$

Likewise, for all of the other classical connectives. So, given the form of our rules, the set of admissible valuations must be just those of classical propositional logic, no matter which connectives we introduce in accordance with the rule schema. Thus, if we take these rules to be definitive of the meanings of the classical connectives, we can maintain that these meanings, codifying the inferential use of the classical connectives, determine their truth-conditions. This is truly a fulfillment of the promise of bilateralism in proof-theoretic semantics.

4 Conclusion

I have put forward a new kind of bilateral system in which the rules for all of the classical connectives are yielded by a single schema, and I have argued that these rules have a reasonable claim to being uniquely definitive of the meanings of the classical connectives. In doing so, I have responded to the concern I've raised for bilateralism in the context of proof-theoretic semantics regarding the choice of rules.[44] However, beyond providing this response to this potential reason *against* bilateralism, given the conceptual and technical upsides of the sort of bilateral schematization I've illustrated here, I take it that the system I have put forth provides a new reason *for* the bilateralist set-up. Far from the bilateralist being committed to unwieldy systems with an unmanageable number of rules, this schematized approach to bilateralism enables one to put forward entire proof systems with a single schema and do one's meta-theory at that level of generality in a way that is both technically convenient and conceptually illuminating. While I have focused on a sequent calculus formulation here, as I think it gets most to the heart of the matter, elsewhere (Simonelli, forthcoming) I have adopted the same approach to formulating bilateral natural deduction systems. Where there is perhaps the most work to be done in further developing this generalized approach to bilateralism is extending it beyond classical logic, to various non-classical systems, and perhaps beyond bilateralism, to mutlilateral systems. I hope the approach I've developed here in application to classical logic, bilateralism's original context, will prove useful in extending bilateralism to many other contexts.[45]

[44] Of course, there are other worries one might raise for bilateralism which I have not addressed here. See, for instance, Dickie (2011), Restall (2020), Kürbis (2023).

[45] Many thanks to Kevin Davey, Bob Brandom, Ulf Hlobil, Pedro del Valle-Inclan, Michael Kremer, Rea Golan, Shuhei Shimamura, Julian Schlöder, and two anonymous referees of this journal for comments.

5 Appendix

In this appendix, I lay out the formal system officially and provide proofs of the results stated in the body of the paper.

5.1 Basic Set-Up

I define two sequent calculi. The first is what I'll call "BK$_+$," which contains the unrestricted Containment axiom schema and the rule of Conclusion of Incoherence:

BK$_+$

$$\frac{}{\Gamma, A \vdash A}\ \text{CO} \qquad \frac{\Gamma, A \vdash B}{\Gamma, B^* \vdash A^*}\ \text{RV} \qquad \frac{\Gamma \vdash A \quad \Delta \vdash A^*}{\Gamma, \Delta \vdash}\ \text{CI}$$

Where {A} or {B} can be null.

$$\frac{\Gamma \vdash -\langle\varphi\rangle}{\Gamma \vdash +\langle\neg\varphi\rangle}\ +_\neg \qquad \frac{\Gamma \vdash +\langle\varphi\rangle}{\Gamma \vdash -\langle\neg\varphi\rangle}\ -_\neg$$

$$\frac{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle \quad \Gamma \vdash \boldsymbol{b}\langle\psi\rangle}{\Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}_\circ \qquad \frac{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash}{\Gamma \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}^*{}_\circ$$

where

$\wedge$: $\boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = +$ $\qquad$ $\vee$: $\boldsymbol{a} = -, \boldsymbol{b} = -, \boldsymbol{c} = -$

$|$: $\boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = -$ $\qquad$ $\downarrow$: $\boldsymbol{a} = -, \boldsymbol{b} = -, \boldsymbol{c} = +$

$\rightarrow$: $\boldsymbol{a} = +, \boldsymbol{b} = -, \boldsymbol{c} = -$ $\qquad$ $\succ\!\!-$: $\boldsymbol{a} = -, \boldsymbol{b} = +, \boldsymbol{c} = +$

$-\!\!<$: $\boldsymbol{a} = +, \boldsymbol{b} = -, \boldsymbol{c} = +$ $\qquad$ $\leftarrow$: $\boldsymbol{a} = -, \boldsymbol{b} = +, \boldsymbol{c} = -$

For the purposes of the present paper, I treat the left side of the sequents as *sets* of signed formulas. Left sides could alternatively be treated as multi-sets or sequences, and it's worth noting that, if one does opt for such a treatment, Contraction is eliminable in this system, like Ketonen's, but I do not deal with this complication for the purposes of the present paper.

Proposition 1.1: The structural rules of Cut and Weakening:

$$\dfrac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\ \text{Cut} \qquad\qquad \dfrac{\Gamma \vdash A}{\Gamma, B \vdash A}\ \text{Weakening}$$

are admissible in BK_+.

Proof: One can derive Cut as follows:

$$\dfrac{\dfrac{\Gamma \vdash A \quad \dfrac{\Delta, A \vdash B}{\Delta, B^* \vdash A^*}\ \text{RV}}{\Gamma, \Delta, B^* \vdash}\ \text{CI}}{\Gamma, \Delta \vdash B}\ \text{RV}$$

and, given Cut, one can derive $\Gamma, B \vdash A$ from $\Gamma \vdash A$ along with an instance of Containment as follows:

$$\dfrac{\Gamma \vdash A \quad \Gamma, B, A \vdash A}{\Gamma, B \vdash A}\ \text{Cut}$$

□

Proposition 1.2 The connective rules of BK_+ are invertible.

Proof: Given any formula of the form $\Gamma \vdash \boldsymbol{c}\langle \varphi \circ \psi \rangle$ or $\Gamma \vdash \boldsymbol{c}^*\langle \varphi \circ \psi \rangle$, we can derive its premises from instances of Containment as follows:

$$\dfrac{\dfrac{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash \boldsymbol{a}\langle\varphi\rangle \quad \Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash \boldsymbol{b}\langle\psi\rangle}{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}_\circ \qquad \Gamma \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash}\ \text{CI}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash \boldsymbol{a}\langle\varphi\rangle}{\Gamma, \boldsymbol{a}^*\langle\varphi\rangle, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash}\ \text{RV}}{\Gamma, \boldsymbol{a}^*\langle\varphi\rangle \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}^*{}_\circ \qquad \Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}{\Gamma, \boldsymbol{a}^*\langle\varphi\rangle \vdash}\ \text{CI}}{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle}\ \text{RV}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle, \vdash \boldsymbol{b}\langle\psi\rangle}{\Gamma, \boldsymbol{b}^*\langle\psi\rangle, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash}\ \text{RV}}{\Gamma, \boldsymbol{b}^*\langle\psi\rangle \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}^*{}_\circ \qquad \Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}{\Gamma, \boldsymbol{b}^*\langle\psi\rangle \vdash}\ \text{CI}}{\Gamma \vdash \boldsymbol{b}\langle\psi\rangle}\ \text{RV}$$

□

The second sequent calculus, which I just call **BK** is the same as BK_+, but without the CI rule and with CO restricted to the cases where Γ contains only atomic signed formulas and A is a signed atomic formula. The first set of results show that CI and non-atomic CO are eliminable in BK_+, and thus, admissible in BK. I leave out the cases involving negation, as the negation rules are not novel to this system and it's easy to show that the main results hold for them.

5.2 Elimination Proofs and Consequences

Proposition 2.1: Conclusion of Incoherence is eliminable in BK_+.

Proof: Proceeds analogously to standard Cut-Elimination, but at this higher level of generality.[46] I'll refer to the schematic formula "A," in the above CI schema, which gets eliminated through an application of CI, the "CI formula" (analogously to the "Cut formula"). We induct primarily on CI formula weight with a secondary induction on CI height, where:

> **Formula Weight:** The weight of a sentence φ, $w(\varphi)$, is defined inductively where $w(p) = 1$ and $w(\varphi \circ \psi) = w(\varphi) + w(\psi) + 1$. The weight of a signed formula is simply the weight of the sentence that is signed.
>
> **CI Height:** The height of an application of CI is the sum of the heights of the proofs of the premises.

Importantly, uses of RV are not taken to contribute to proof height, since it does not modify the complexity of the formulas and applications of RV can always be immediately undone through another application of RV. In this way, RV is treated analogously to Exchange in standard treatments of sequent calculi in which sequents are treated as involving sequences of formulas rather than sets or multi-sets. It is possible to simplify the

[46]See, for instance, Negri and von Plato (2008) and Indrzejczak (2021) for analogous presentations of Cut-Elimination for Ketonen-style rules.

proofs so that one doesn't need to deal with applications of RV by working the equivalent solely left-sided system laid out below, yet, since it is an important philosophical thesis of this paper that doing one's metatheory at this higher level of generality actually makes good conceptual sense in the bilateralist terms I've laid out, I have done the CI-Elimination proof directly in the main system proposed here.

For the proof, we show six ways in which a proof involving CI can be transformed to one with either lesser CI height or CI formula weight. The outline of the proof in which these transformations figure is as follows:

Primary Induction: On CI formula weight:

1. **Base Case:** CI on atoms is eliminable, proven by:
   (a) **Secondary Induction:** On CI height:
      i. **Base Case:** CI on atoms of height 0 is eliminable, proven by Case Zero.
      ii. **Inductive Step:** If CI on atoms of height n is eliminable, CI on atoms of height $n + 1$ is eliminable, proven by Cases One to Four.
2. **Inductive Step:** If CI on formulas of weight n is eliminable, then CI on formulas of weight $n + 1$ is eliminable, proven by Cases One through Six.

Elaborating this a bit, Case Zero shows that, when both premises of CI are axioms, so too is the Conclusion. Cases One through Four show that CI height can be reduced in any case where the CI formula is not principal in both premises (a formula is said to be principal in a premise of a rule if the last rule applied was to derive that formula). Since the CI formula will never be principal in the case where it is atomic (since it won't be derived at all), this suffices to establish the inductive step of the secondary induction. For the primary inductive step, if the CI formula is not principal in both premises, then some series of transformations of type One through Four, will transform it into a proof in which the CI formula is principal in both premises, and then a transformation of type Five or Six will reduce the weight of the CI formula.

Case Zero: CI of height 0, where both premises are axioms or follow from an axiom via a single application of RV (since applications of RV don't affect proof height, such sequents are counted as axioms for our purposes here). If the left premise is an axiom, then either $A \in \Gamma$ or is some formula B such that $B \in \Gamma$ and $B^* \in \Gamma$. If the right premise is an axiom, then either $A^* \in \Delta$ or is some formula B such that $B \in \Delta$ and $B^* \in \Delta$. So, if both premises are axioms, then either $A \in \Gamma, \Delta$ and $A^* \in \Gamma, \Delta$ or there's some formula B such that $B \in \Gamma, \Delta$ and $B^* \in \Gamma, \Delta$. Either way, $\Gamma, \Delta \vdash$ is an axiom.

Case One: CI formula is not principal in the left premise, where $\Gamma = \Gamma', \boldsymbol{c}\langle\varphi \circ \psi\rangle$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\, n}{\Gamma', A^*, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash}}{\Gamma', A^* \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\,\boldsymbol{c}^*_\circ}{\Gamma', \boldsymbol{c}\langle\varphi \circ \psi\rangle \vdash A}\,\mathrm{RV} \qquad \dfrac{\vdots\, m}{\Delta \vdash A^*}}{\Gamma', \Delta, \boldsymbol{c}\langle\varphi \circ \psi\rangle \vdash}\,\mathrm{CI}$$

We have a CI of height $n + 1 + m$. We can push applications of CI up the proof tree as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots\, n}{\Gamma', A^*, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash}}{\Gamma', \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\varphi\rangle \vdash A}\,\mathrm{RV} \qquad \dfrac{\vdots\, m}{\Delta \vdash A^*}}{\Gamma', \Delta, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash}\,\mathrm{CI}}{\Gamma', \Delta \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\,\boldsymbol{c}^*_\circ}{\Gamma', \Delta, \boldsymbol{c}\langle\varphi \circ \psi\rangle \vdash}\,\mathrm{RV}$$

to get a CI of lesser height $n + m$.

Case Two: CI formula is not principal in the left premise, where $\Gamma = \Gamma', \boldsymbol{c}^*\langle\varphi \circ \psi\rangle$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots n}{\Gamma', A^* \vdash \boldsymbol{a}\langle\varphi\rangle} \quad \dfrac{\vdots m}{\Gamma', A^* \vdash \boldsymbol{b}\langle\varphi\rangle}}{\Gamma', A^* \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\,\boldsymbol{c}_\circ}{\Gamma', \boldsymbol{c}^*\langle\varphi \circ \psi\rangle \vdash A}\,\mathrm{RV} \qquad \dfrac{\vdots k}{\Delta \vdash A^*}}{\Gamma', \Delta, \boldsymbol{c}^*\langle\varphi \circ \psi\rangle \vdash}\,\mathrm{CI}$$

We have a CI of height $max(n, m) + 1 + k$. We can push applications of CI up the proof tree as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots n}{\Gamma', A^* \vdash \boldsymbol{a}\langle\varphi\rangle}}{\Gamma', \boldsymbol{a}^*\langle\varphi\rangle \vdash A}\,\mathrm{RV} \quad \dfrac{\vdots k}{\Delta \vdash A^*}}{\Gamma', \Delta, \boldsymbol{a}^*\langle\varphi\rangle \vdash}\,\mathrm{CI}}{\Gamma', \Delta \vdash \boldsymbol{a}\langle\varphi\rangle}\,\mathrm{RV} \quad \dfrac{\dfrac{\dfrac{\dfrac{\vdots m}{\Gamma', A^* \vdash \boldsymbol{b}\langle\varphi\rangle}}{\Gamma', \boldsymbol{b}^*\langle\varphi\rangle \vdash A}\,\mathrm{RV} \quad \dfrac{\vdots k}{\Delta \vdash A^*}}{\Gamma', \Delta, \boldsymbol{b}^*\langle\psi\rangle \vdash}\,\mathrm{CI}}{\Gamma', \Delta \vdash \boldsymbol{b}\langle\psi\rangle}\,\mathrm{RV}}{\Gamma', \Delta \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\,\boldsymbol{c}_\circ}{\Gamma', \Delta, \boldsymbol{c}^*\langle\varphi \circ \psi\rangle \vdash}\,\mathrm{RV}$$

to get two CIs of lesser heights $n + k$ and $m + k$

Case Three: CI formula is not principal in the right premise, where $\Delta = \Delta', \boldsymbol{c}\langle\varphi \circ \psi\rangle$. Exactly analogous to Case One.

Case Four: CI formula is not principal in the right premise, where $\Delta = \Delta', \boldsymbol{c}^*\langle\varphi \circ \psi\rangle$. Exactly analogous to Case Two.

Case Five: CI formula is principal in both premises, where $A = \boldsymbol{c}\langle\varphi \circ \psi\rangle$

$$\dfrac{\dfrac{\dfrac{\vdots n}{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle} \quad \dfrac{\vdots m}{\Gamma \vdash \boldsymbol{b}\langle\psi\rangle}}{\Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\,\boldsymbol{c}_\circ \qquad \dfrac{\dfrac{\vdots k}{\Delta, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash}}{\Delta \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\,\boldsymbol{c}^*{}_\circ}{\Gamma, \Delta \vdash}\,\mathrm{CI}$$

We have a CI of height $max(n, m) + 1 + k + 1$. We transform the proof tree as follows:[47]

[47]Note, for simplicity's sake, we are treating contexts here as *sets* rather than multi-sets, and so there is no appeal to Contraction in the last step of this transformation.

$$
\begin{prooftree}
\AxiomC{$\vdots_{n}$}
\UnaryInfC{$\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle$}
\AxiomC{$\vdots_{m}$}
\UnaryInfC{$\Gamma \vdash \boldsymbol{b}\langle\psi\rangle$}
\AxiomC{$\vdots_{k}$}
\UnaryInfC{$\Delta, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash$}
\doubleLine
\RightLabel{RV}
\UnaryInfC{$\Delta, \boldsymbol{a}\langle\varphi\rangle \vdash \boldsymbol{b}^{*}\langle\psi\rangle$}
\RightLabel{CI}
\BinaryInfC{$\Gamma, \Delta, \boldsymbol{a}\langle\varphi\rangle \vdash$}
\doubleLine
\RightLabel{RV}
\UnaryInfC{$\Gamma, \Delta \vdash \boldsymbol{a}^{*}\langle\varphi\rangle$}
\RightLabel{CI}
\BinaryInfC{$\Gamma, \Delta \vdash$}
\end{prooftree}
$$

Here, we have CIs of heights $m + k$ and $n + max(m, k)$, and the latter is not necessarily lesser than the original CI height, but the weight of the CI formula has decreased in both cases.

Case Six: CI formula is principal in both premises, where $A = \boldsymbol{c}^{*}\langle\varphi \circ \psi\rangle$. Exactly analogous to Case Five. □

Proposition 2.2: BK_{+} is consistent in the sense that, if $\vdash A$ is derivable, then $\vdash A^{*}$ is not derivable.

Proof: Note first that we cannot derive the empty sequent. Since all of the axioms are of the form $\Gamma, A \vdash A$ (where A is importantly not null), the only way to derive the empty sequent would be through applying the only simplifying rule, CI. Since CI is eliminable, the empty sequent cannot be derived. Suppose now $\vdash A$ and $\vdash A^{*}$. Then, by CI, we could derive the empty sequent. Since the empty sequent is not derivable, it follows that if $\vdash A$ then $\vdash A^{*}$ is not derivable. □

Proposition 2.3: Adding any of the connectives of BK_{+} to a language L_0 to yield a language L constitutes a conservative extension of L_0 in the sense that, where $\Gamma \cup \{A\}$ contains only formulas of L_0, if $\Gamma \nvdash_{L_0} A$, then $\Gamma \nvdash_{L} A$.

Proof: Follows directly from the fact that the only simplifying rule is CI, and this rule is eliminable. □

Proposition 2.4: The axiom schema of Containment can be limited to atoms.

Proof: The proof involves two inductions. For the first induction, we show that any sequent of the $\Gamma, A \vdash A$, where A is atomic, is derivable by induction on complexity of the most complex formulas in Γ. The base case is immediate as an instance of atomic Containment. For the inductive step, we suppose the most complex formulas in Γ are of weight $n+1$ and show that a sequent of the form $\Gamma', \boldsymbol{c}\langle\varphi \circ \psi\rangle, A, \vdash A$ or $\Gamma', \boldsymbol{c}^*\langle\varphi \circ \psi\rangle, A, \vdash A$ can be derived from some number of sequents of the form $\Gamma'', A \vdash A$ in which the most complex formulas in Γ'' are of weight n:

$$\dfrac{\dfrac{\dfrac{\Gamma', \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle, A \vdash A}{\Gamma', A^*, A, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle, \vdash}\,\mathrm{RV}}{\Gamma', A^*, A \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\,\boldsymbol{c}^*{}_\circ}{\Gamma', \boldsymbol{c}\langle\varphi \circ \psi\rangle, A \vdash A}\,\mathrm{RV} \qquad \dfrac{\dfrac{\dfrac{\Gamma', \boldsymbol{a}^*\langle\varphi\rangle, A \vdash A}{\Gamma', A, A^* \vdash \boldsymbol{a}\langle\varphi\rangle}\,\mathrm{RV} \quad \dfrac{\Gamma', \boldsymbol{b}^*\langle\psi\rangle, A \vdash A}{\Gamma', A, A^* \vdash \boldsymbol{b}\langle\psi\rangle}\,\mathrm{RV}}{\Gamma', A, A^* \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\,\boldsymbol{c}_\circ}{\Gamma', \boldsymbol{c}^*\langle\varphi \circ \psi\rangle, A \vdash A}\,\mathrm{RV}$$

For the second induction, we show that any sequent of the form $\Gamma, A \vdash A$ is derivable by induction on the complexity of A. The base case is already established. For the inductive step, we suppose that A is complexity $n+1$ and show we can derive $\Gamma, A \vdash A$ from some number of sequents of the form $\Gamma', B \vdash B$ where B is complexity n. Whether A is of the form $\boldsymbol{c}\langle\varphi \circ \psi\rangle$ or $\boldsymbol{c}^*\langle\varphi \circ \psi\rangle$, the following derivation establishes this:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash \boldsymbol{a}\langle\varphi\rangle \quad \Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash \boldsymbol{b}\langle\psi\rangle}{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\,\boldsymbol{c}_\circ}{\Gamma, \boldsymbol{c}^*\langle\varphi \circ \psi\rangle, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash}\,\mathrm{RV}}{\Gamma, \boldsymbol{c}^*\langle\varphi \circ \psi\rangle \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\,\boldsymbol{c}^*{}_\circ}{\Gamma, \boldsymbol{c}\langle\varphi \circ \psi\rangle \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\,\mathrm{RV}$$

□

Proposition 2.5: CI, non-atomic CO, Cut, and Weakening are all admissible in BK.

Proof: Since BK is just BK_+ without CI and non-atomic CO, and we've just shown that these rules are eliminable in BK_+, they are admissible in BK. Since Cut and Weakening are admissible given these rules (Proposition 1.1), they too are admissible in BK. □

Proposition 2.6: The connective rules of BK are invertible.

Proof: Proposition 1.2 establishes that the connective rules are invertible, given non-atomic CO and CI. Since these structural rules are admissible in BK, the connective rules of BK are invertible. □

5.3 Equivalence with K, Soundness and Completeness

Consider the following solely left-sided version of BK, which I call BK_{ls}:

BK_{ls}:

$$\frac{}{\Gamma, A, A^{*} \vdash}\ \text{Ax}$$

Where Γ and $\{A\}$ contain only signed atoms.

$$\frac{\Gamma, -\langle\varphi\rangle \vdash}{\Gamma, +\langle\neg\varphi\rangle \vdash}\ +_{\neg} \qquad \frac{\Gamma, +\langle\varphi\rangle \vdash}{\Gamma, -\langle\neg\varphi\rangle \vdash}\ -_{\neg}$$

$$\frac{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash}{\Gamma, \boldsymbol{c}\langle\varphi \circ \psi\rangle \vdash}\ \boldsymbol{c}_{\circ} \qquad \frac{\Gamma, \boldsymbol{a}^{*}\langle\varphi\rangle \vdash \quad \Gamma, \boldsymbol{b}^{*}\langle\psi\rangle \vdash}{\Gamma, \boldsymbol{c}^{*}\langle\varphi \circ \psi\rangle \vdash}\ \boldsymbol{c}^{*}{}_{\circ}$$

Proposition 3.1: BK_{ls} is equivalent to BK in that any BK proof corresponds to a unique BK_{ls} proof and any BK_{ls} proof corresponds to an equivalence class of BK proofs under Reversal.

Proof: Straightforward by induction on proof height (where, once again, we do not take applications of Reversal to contribute to proof height). For the base case, any instance of the axiom schema of BK_{ls} of the form $\Gamma, A, A^{*} \vdash$, is obtained by Reversal from a BK axiom of the form $\Gamma, A \vdash A$. For the inductive step, we suppose that we've shown the correspondence of proofs up to height n, and we show that proofs correspond at height $n+1$ by showing that, for any application of a rule of one system, a Reversed form of the conclusion sequent can be obtained, via a rule in other system, from a Reversed form of the premise sequent(s). □

Proposition 3.2: The fragment of BK_{ls} consisting in the rules for negation, conjunction, disjunction, and the conditional is a notational variant of Ketonen's (1944) multiple conclusion sequent calculus, K:

K:

$$\frac{}{\Gamma, \varphi \vdash \varphi, \Delta}$$

Where Γ, Δ, and $\{\varphi\}$ contain only atoms.

$$\frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta}\ \mathrm{L}_{\neg} \qquad \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}\ \mathrm{R}_{\neg}$$

$$\frac{\Gamma, \varphi, \psi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta}\ \mathrm{L}_{\wedge} \qquad \frac{\Gamma \vdash \varphi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \wedge \psi, \Delta}\ \mathrm{R}_{\wedge}$$

$$\frac{\Gamma, \varphi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \vee \psi \vdash \Delta}\ \mathrm{L}_{\vee} \qquad \frac{\Gamma \vdash \varphi, \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}\ \mathrm{R}_{\vee}$$

$$\frac{\Gamma \vdash \varphi, \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \to \psi \vdash \Delta}\ \mathrm{L}_{\to} \qquad \frac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma \vdash \varphi \to \psi, \Delta}\ \mathrm{R}_{\to}$$

Proof: We can provide a one-to-one translation schema to show that the two systems are simply notational variants. To translate a K sequent of the form $\Gamma \vdash \Delta$ to a $\mathrm{BK}_{\mathrm{ls}}$ sequent of the form $\Gamma' \vdash$ let $\Gamma' = \{+\langle\varphi\rangle \mid \varphi \in \Gamma\} \cup \{-\langle\varphi\rangle \mid \varphi \in \Delta\}$. Conversely, to translate a $\mathrm{BK}_{\mathrm{ls}}$ sequent of the form $\Gamma' \vdash$ to a K sequent of the form $\Gamma \vdash \Delta$ let $\Gamma = \{\varphi \mid +\langle\varphi\rangle \in \Gamma'\}$ and $\Delta = \{\varphi \mid -\langle\varphi\rangle \in \Gamma'\}$.

Remark: As an interesting side-note, this notation precisely captures Restall's (2005) bilateral reading of multiple conclusion sequents according to which a sequent of the form $\Gamma \vdash \Delta$ expresses that the position consisting in affirming everything in Γ and denying everything in Δ is incoherent.

Let us define the basic semantic notions needed to state soundness and completeness:

> **Correctness Function:** The *correctness function* [] is a function from $\{+, -\}$ to $\{1, 0\}$ mapping + to 1 and − to 0.
>
> **Correctness:** Taking some stance $\boldsymbol{a}$ towards some sentence φ is *correct*, relative to some valuation v, just in case $[\boldsymbol{a}] = v(\varphi)$.
>
> **Classical Valuations:** Let a *classical* valuation v be any function from $\mathcal{L} \to \{1, 0\}$ such that
>
> 1. $\forall p \in \mathcal{A}$, $v(p) = 1$ or $v(p) = 0$

2. $v(\neg\varphi) = \begin{cases} 1, & \text{if } v(\varphi) = 0 \\ 0, & \text{if } v(\varphi) = 1 \end{cases}$

3. $v(\varphi \circ \psi) = \begin{cases} [\boldsymbol{c}], & \text{if } v(\varphi) = [\boldsymbol{a}] \text{ and } v(\psi) = [\boldsymbol{b}] \\ [\boldsymbol{c}^*], & \text{if } v(\varphi) = [\boldsymbol{a}^*] \text{ or } v(\psi) = [\boldsymbol{b}^*] \end{cases}$

Classical Unsatisfiability: A set of signed formulas Γ is classically unsatisfiable, $\Gamma \vDash$, just in case there is no classical valuation v such that all of the stances in Γ are correct.

Classical Validity: An inference $\Gamma : A$ is classically valid, $\Gamma \vDash A$, just in case there is no classical valuation v such that all of the stances in Γ are correct and A is incorrect.

Proposition 3.3: $\mathrm{BK}_{\mathrm{ls}}$ proves $\Gamma \vdash$ just in case $\Gamma \vDash$

Proof: Equivalence with K, which is known to be sound and complete, suffices to establish this for the standard connectives. For all the non-standard connectives, a direct proof of soudness and completeness is easily obtained by schematizing a proof of soundness and completeness for K. □

Proposition 3.4: BK proves $\Gamma \vdash A$ just in case $\Gamma \vDash A$

Proof: Given the equivalence of BK and $\mathrm{BK}_{\mathrm{ls}}$ under Reversal, it suffices just to point out that $\Gamma : \varphi$ is classically valid just in case Γ, φ^* is classically unsatisfiable. □

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