III. Plural Reference and Set Theory

> Most mathematicians do not perceive the problem which is posed by the abstractness of set theory. They prefer to take an aloof attitude and pretend not to be interested in philosophical (as opposed to purely mathematical) questions. In practice this means that they limit themselves to deducing theorems from axioms which were proposed by some authorities . . . the writings of contemporary set theorists and logicians do not offer very much which could help us in solving these problems.
>
> Mostowski, 1966, 140f.

This essay has three aims, only one of which is furthered in detail. The first, and basic one, is to criticise the conventional interpretation of axiomatic set theories as alternatives in a programme of formalising the 'naive' concept of set, collection or class. The polemic which needs to be directed at the various conceptions of set used in defence of this view has already been convincingly accomplished by Max Black and Erik Stenius, so I need not carry that through here.[1] I shall be more concerned with developing a positive account of what I take the naive conception to amount to. The principal idea, which is Black's, is that sets are to plural terms as individuals are to singular terms. In the previous essay I called such entities manifolds. They entered in the context of the philosophy of number, as bearers of number-properties, whereas in this essay I shall consider them for their own sake and in greater detail. Cantor himself was led to abstract set theory through consideration of number, in particular transfinite numbers. It was he who first showed clearly what it means for one infinite collection to have more members than another infinite collection, and showed that there could be collections with different transfinite cardinality.

The positive theory of manifolds will be treated in § 4. §§ 2–3 prepare the way for this. In § 1 I shall suggest that the basic idea of a manifold, or class as many, has a nobler and longer history than Black and Stenius might suggest, and that echoes of this conception still inform some systems of axiomatic set theory.

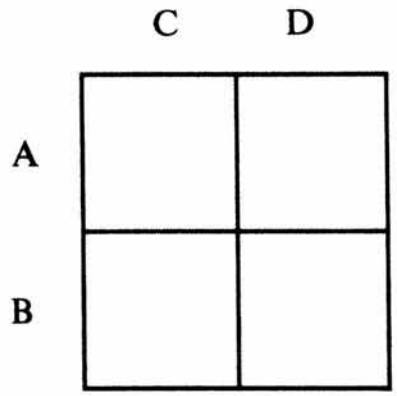
The second aim is reinterpretative. If axiomatic set theory is not a theory of manifolds, then what is it a theory of? The key notion here, that of an individual which is a *representative* of a manifold, is also suggested by Stenius, but again the idea goes back further. This aim will not be pursued in any great detail, though outlines of a theory embodying such a reinterpretation are sketched in § 6.

The third aim arises out of the other two. Because of the power of most systems of axiomatic set theory, sufficient power in most cases to serve as a foundation for finite and transfinite arithmetic and almost all of the rest of mathematics, sets have been massively over-used by logicians and philosophers in ontological investigations, and made to do service for such diverse entities as numbers, properties, relations, orderings, functions, propositions, facts, theories, worlds, persons, material bodies, higher-order objects, and so on. If, as I believe, a theory of manifolds serves to outline the ontology of nothing but manifolds (whatever they are manifolds of), then much of the set-based ontology of modern philosophy represents theft rather than honest toil, and the work for the most part remains to be done. The third aim, which is accomplished if the first two are, is not to *do* this work but to clear the decks for it. The substantive work left to be done is formal ontology, of which manifold theory comprises a small but not insignificant part.

§ 1 Classes as Many and as One: Historical Remarks

Introductory textbooks on set theory usually contain on page 1 a sentence like this: 'A set is a collection of things regarded as a single object', with a warning not to take 'collection' to imply any kind of physical bringing-together of the things in question. Such a conception raises in extreme form the ancient problem of the one and the many. Something which is a collection, i. e. many, is also one. It is completely specified by its members but is distinct from them, even when it has only one. The intelligibility of this kind of stipulation has in recent years been questioned, above all by Black and Stenius. This essay is in large part a development of the line of thought opened up in particular by Black, who first formulated with clarity the view that sets are to plural terms as individuals are to singular terms. The one-many problem cannot be avoided by taking a set to be the whole comprised of its members, their mereological sum. In the first place, such a sum does not in every case exist, or at least

it is not clear that it must.[2] In the second, even when such a whole does exist, it will not usually satisfy the fundamental principle of sets, the principle of extensionality: that sets are the same if, *and only if*, they have the same members. For two different collections may comprise the same whole when summed: this divided square



is the sum of the top half and the bottom half as well as the sum of the left half and the right half. We must accordingly distinguish the sum $A+B$ from the set $\{A,B\}$, for $A+B = C+D$ but $\{A,B\} \neq \{C,D\}$.[3] Sums are wholes, and thus also individuals, whereas manifolds with more than one member are not individuals but pluralities.[4] The whole-part relation $<$ is a relation between individuals, and must therefore be distinguished from the membership relation $\in$.[5, 6] A mereological approach to classes has always held attractions for those of an anti-Platonist turn of mind. Goodman indeed *defined* Platonism, somewhat idiosyncratically, as the acceptance of sets. I would suggest that nominalist scruples about sets as abstract entities, 'high-brow' sets, might be to some extent assuaged by the use of manifolds, which are 'low-brow' sets, no more abstract than their members.

In the face of the successful advances of axiomatic set theory since Zermelo's first axiomatization of 1908, logicians have for the most part simply put aside or ignored the problem of one and many.[7] If we look back, however, to the origins of set theory, when intuitions were perhaps fresher and less apt to be moulded by a tradition, we find a much greater awareness of the issue. In particular I wish to show how the problem made itself felt to three great set theorists: Cantor, Russell and, more recently, Bernays.

§ 1.1 Cantor

Cantor explicitly regarded a set *(Menge)* as a comprehension into a whole *(Zusammenfassung zu einem Ganzen)* of a plurality *(Vielheit)* of different objects. After the appearance of Burali-Forti's paradox, and in view of his own proof that since a set has more subsets than members, the impossibility of there being a universal set (sometimes called Cantor's paradox), he came to realize that it cannot be the case that to all pluralities *(Vielheiten)* there should belong a set *(Menge)*. Those pluralities which can be comprehended into wholes he called *consistent*, those which cannot he called *inconsistent*.[8] That Cantor can accept contradictions with such remarkable equanimity is due not to his being a working mathematician with better things to do[9] but to his having on hand the distinction between sets and pluralities. He even went so far as to outline principles for deciding when pluralities can be comprehended and when they cannot, foreshadowing later developments.[10] It is unfortunately not clear what the nature of this 'comprehension' is, but the important point for our purposes is that Cantor apprehended a distinction between sets and other individuals on the one hand and pluralities on the other, turning it to good use when the paradoxes were discovered.

§ 1.2 Russell

A distinction analogous to that between *Mengen* and *Vielheiten* is to be found, independently of Cantor, in Russell's early work of genius, *The Principles of Mathematics*. The importance of this work for our purposes lies in the circumstance that Russell, in the first flush of his enthusiasm for realism, was more sensitive to fine distinctions than he was to be later, after the success of the theory of descriptions in depopulating much of his universe spurred him to further reductions. In § 70 of this work, Russell distinguishes between a *class as one* and a *class as many*. He regards this as an 'ultimate distinction'.[11] What is especially interesting and important is that Russell, like Cantor, does not introduce the distinction for the express purpose of providing a way out of the antinomies, although, like Cantor, he does thereafter avail himself of the distinction for this purpose.[12] The immediate need for the distinction arises rather in connection with the argument put forward by Peano and Frege for distinguishing singleton sets from their members. The argument

goes thus: suppose we invariably identify x with $\{x\}$. In the case where x is itself a class with more than one member, since $\{x\}$ has just one member, and $x = \{x\}$, it follows that x both has just one member and more than one, a contradiction. Russell takes this argument to establish rather that we should not be tempted to identify classes as one with classes as many: 'the many are only many and are not also one.'[13] For $\{x\}$ can only have one member which is itself a class if 'x' denotes a class as one, while x can only have many members if 'x' denotes a class as many. The Peano-Frege argument turns on an ambiguity and so founders: there can be, from Russell's point of view, no case where a class as many is a *member* of another class, since only *individuals* (Russell's *terms*) can be members. The difference between individuals (including classes as one) and classes (as many) is one of *type*.[14]

The distinction blocks Russell's paradox in that the non-self-membered classes comprise only a class as many: there is no corresponding class as one.[15] This is essentially the same as Cantor's approach.

Russell even anticipates, though somewhat unclearly, Black's view on the crucial role of plural reference:

> In such a proposition as 'A and B are two' there is no logical subject: the assertion is not about A, nor about B, nor about the whole composed of both, but strictly and only about A and B. Thus it would seem that assertions are not necessarily *about* single subjects, but may be about many subjects.[16]

Russell adverts to the use of 'and' to form what he calls 'numerical conjunctions' or 'addition' of individuals: '*A and B* is what is denoted by the concept of a class of which *A* and *B* are the only members.'[17] Russell sways between denying that plural propositions can have genuine logical subjects and allowing that they do.[18] He is also vague to the point of unintelligibility about the status of classes as many:

> In a class as many, the component terms, though they have some kind of unity have less than is required for a whole. They have, in fact, just so much unity as is required to make them many, and not enough to prevent them from being many.[19]

Russell admits that he cannot find any individual like Frege's *Wertverlauf* (a word Russell felicitously translates as 'range') which is distinct from his own class as one. But whereas Frege's range is designed to obey the principle of extensionality, Russell's classes as one are mereological sums, and so do not.[20]

Nevertheless, without a single object to *represent an extension*, Mathematics crumbles . . . But it is exceedingly difficult to discover any such object, and the contradiction proves conclusively that, even if there be such an object sometimes, there are propositional functions for which the extension is not one term.[21]

Russell's exasperation is clear. He is for the most part happy to regard the extension of a concept under which more than one thing falls as a class as many, but feels, in part under Frege's influence, the need for individuals to do the work of extensions. Why should mathematics crumble without these? Russell offers one brief example, and another reason is not hard to find. Firstly, consider a simple combinatorial problem: How many ways can m things be selected from n things, without regard to order, where $m < n$? The answer, $n!/m!(n-m)!$, is usually taken as the cardinality of the set of subsets of cardinality m of a set of cardinality n. This requires that we treat sets as members of other sets, i.e. use classes as one. But, on Russell's mereological view of classes as one, should any of the m things be a part of one of the others, the wrong answer would result. So we appear here to need something like Frege's range, which obeys the principle of extensionality while still being an individual. Secondly, Russell, like Frege, wants to give the logicist account of numbers as classes of equinumerous classes, but again if only classes as one can be members of other classes, the only number which could be thus defined is the number one, and that still remains a class as many. Russell again badly needs Frege's ranges: a number can then be taken as the range of the concept *equinumerous with M*, for suitable choice of concept (or range) *M* (I am ignoring Frege's difficulties about referring to concepts). But Russell's paradox has blocked for ever the unconditional guarantee of such handy individuals. Rather than admit the bankruptcy of logicism, Russell prefers to look to the complications of type theory, which he outlines in the second Appendix to *Principles*. From here on, the distinction between classes as one and as many ceases to play a role, and the whole idea of a class is eventually dropped in favour of a reduction to propositional functions.[22]

§ 1.3 Bernays

Between the wars Zermelo's initial axiomatisation of set theory was modified and improved by various writers. Skolem made more precise

Zermelo's vague notion of a definite property, and Fraenkel proposed the Axiom Scheme of Replacement in place of Zermelo's Axiom Scheme of Separation, to allow unrestrictedly for transfinite ordinals. With Miriamoff's suggestion that all sets should be founded, so that for no set s_0 would there be an infinite descending sequence $\ldots s_{k+1} \in s_k \in \ldots \in s_2 \in s_1 \in s_0$, the shape of what is now always called ZF set theory was complete.[23] In 1925 von Neumann reinjected Cantorian ideas into set theory with a distinction between *sets* and *classes*.[24] This allowed axiom schemata to be replaced by axioms, and set theory was for the first time finitely axiomatized.

In a series of papers from 1937 to 1954, Paul Bernays developed von Neumann's treatment along somewhat similar lines. Bernay's treatment is usually taken as a mere variant of the approach of von Neumann, and the similar approach of Gödel 1940: the three are run together under the title NBG set theory But there is a difference between the treatment of Bernays and those of von Neumann and Gödel which is quite crucial from our point of view. Whereas von Neumann and Gödel both regarded sets as classes, namely those classes which can be elements of other classes, even though Gödel, for example, used different faces for set and class variables, Bernays keeps sets and classes distinct from one another, allowing smaller and more tractable classes to *correspond* to sets. In his development of this theory he uses its finite axiomatization property to interpret it in a two-sorted first-order predicate calculus, with sets and classes comprising the different sorts, and two different primitive membership relations.[25] This is usually regarded as an unnecessary nuisance, since it complicates the symbolism and the treatment of mathematics, and the expedient of identifying sets with their corresponding classes is usually employed. But the thinking behind Bernays' treatment is clearly motivated by philosophical rather than mathematical considerations, as the following passage shows:

> The two kinds of individual [sc. sets and classes], as well known, can in principle be reduced to only one kind, so that we come back to a one-sorted system . . . However it might be asked if we have here really to go as far in the formal analogy with the usual axiomatics. Let us regard the question with respect to the connection between set theory and extensional logic. As well known, it was the idea of Frege to identify sets with extensions (Wertverläufe) of predicates and to treat these extensions on the same level as individuals. That this idea cannot be maintained was shown by Russell's paradox.
> Now one way to escape the difficulty is to distinguish different kinds of individuals and thus to abandon Frege's second assumption; that is the method of type

theory. But another way is to give up Frege's first assumption, that is to distinguish classes as extensions from sets as individuals.[26]

Bernays' axiomatic theory of sets and classes consists in showing how to attain full freedom of set construction according to the intuitive principles laid down by Cantor, with sufficient power to derive classical mathematics, while avoiding the paradoxes. It thus constitutes a fulfilment of the idea, sketched, but never followed through, by Russell in Appendix A of *Principles*, of allowing unrestrictedly classes as extensions of propositional functions, while employing certain individuals as *Ersatz* extensions, Frege's ranges, in order to develop classical mathematics.[27]

This is not to suggest that Bernays regarded classes as manifolds in our sense, that is, as 'many's' of individuals. Rather, he regarded them as individuals, though apparently as less substantial individuals than sets: useful fictions, perhaps.[28] However, he does speak of sets as *representing* classes. It would not therefore do excessive violence to at least the letter of his views if we were to regard classes, the extensions of predicates, as manifolds in our sense, and sets as individuals which are taken for mathematical purposes to represent the more tractable classes. Such an idea will be pursued further in § 6 below.

§ 2 Linguistic Phenomenology of Plural Reference

Plural reference was already introduced in the previous essay. Plural terms are expressions apt for referring to more than one thing at once. They contrast not with general, but with singular terms. A singular term is an expression apt for referring to, denoting or designating an individual. As the name suggests, it is (in Indo-European languages at least) usually inflected or otherwise modified for number, and when the subject of a clause, the main verb of the clause will usually agree with it in number. General terms, such as 'man', 'hooded crow', 'horse with a wooden leg' etc. are unfortunately so called, in that both general and singular terms might be assumed to be subsumed in a single category of terms. But I believe Frege was right in considering such general words and phrases (which I shall henceforth call *common noun phrases* (CNPs), where Frege called them 'concept words') as being inherently predicative rather than referential, although I do not consider CNPs to

be *simply* predicates, but rather to occupy a position intermediate in various respects between predicates and terms, constituting in fact a basic category of expression distinct from terms.[29]

Singular terms should be contrasted rather with plural terms, which are also referential rather than predicative. Whenever we use a term, the syntax of English and many other languages compels us to treat the term as either singular or plural, and modify it accordingly. This can on occasion be a nuisance in ordinary discourse, and would be a considerable drawback in formulating an artificial language for logical purposes. The problem of how to deal formally with modification for grammatical number will be considered in the next section.

As outlined in the previous essay, plural terms fall into the same sub-categories as singular terms, namely proper names, descriptions, demonstrative phrases and pronouns, as well as having sub-categories not available, for obvious reasons, to singular terms, namely term lists. We have already seen how Bolzano, Russell and others drew attention to the possibility of forming term lists by using the word 'and' any number of times, flanked by that number plus one terms. The usual method of writing out a name for a finite set, as '$\{a,b,c\}$' etc., constitutes, for those not under the impression that this expression denotes a new abstract unit, another feasible way of forming plural terms. Plural terms, like singular terms, may be different in sense and yet still designate the same things, while plural demonstratives, pronouns etc. are indexical in exactly the same way as their singular counterparts. Just as a singular term ('that man', 'the owner of 34 High Street'), may be used to refer to different individuals on different occasions of its use, so a plural term ('those men', 'John's children', etc.) may on different occasions of its use refer to different manifolds of things.

A plural term like 'the people in this room' is to be sharply distinguished from the (plural) CNP 'people in this room'. Whether singular or plural, CNPs are not terms. This difference is both syntactic and semantic. Semantically, CNPs do not of themselves make definite reference to things. Apparent exceptions, like 'People in this room have been smoking', can be set aside. In this case, although the CNP occurs alone as subject of the sentence, it is not a referential use, but quantificatory. The sentence means something like '*Some* people in this room have been smoking'. It is doubtful whether there is an exact logic for the quantificatory uses of CNPs in subject position. Sometimes, as in the above case, the meaning is existential, at others, as in 'Men are mortal', it

is universal, at others, as in 'People went home at midnight' it is probably majoritive, meaning something like 'Most people . . .', and in yet other cases ('Tigers have four legs', 'Gentlemen prefer blondes') the meaning is one of vague typicality, perhaps requiring some new kind of typicality-operator. Syntactically the difference varies according to language. In English, terms, unlike CNPs, may not be preceded by articles, demonstrative pronouns or quantifier phrases. In other languages the conventions differ: e. g. in Italian proper names require the definite article. In some languages, such as the Slavonic ones where articles are lacking, the difference is certainly less marked, and it might be preferable to regard the term/CNP distinction as somewhat parochial, expecially in view of the long tradition of grouping proper and common nouns together in the one category of *name*.[30] Nevertheless, while the syntactic distinction may vary in strength according to language, the semantic distinction, between a nominal expression which is, and one which is not, marked for *definiteness*, whether this marking is morphological, syntactic, contextual or whatever, is one which cannot be ignored. As it happens, we shall not employ anything like common nouns in the formal treatment of § 4, but this is essentially a move away from ordinary language to the predicate/variable language of orthodox logic, where there is no CNP category.[31]

Mention must be made of collective nouns, like 'class', 'group', 'set', 'collection', 'aggregate', 'herd', 'flock', 'bunch' and the like. If *c* is a collective noun and *d* is some other CNP then '*c* of *d*s' is a CNP in the singular, yet we rightly regard such phrases as 'this flock of sheep' as referring to many individuals, though not one at a time. In the terminology of the previous essay, the expression may designate each of many sheep without *subdesignating* any of them, i. e. without containing a subterm designating any one. But, unlike a plural expression like 'these sheep' the expression 'this flock of sheep' is syntactically singular, and the question naturally arises whether we have here a singular term or an *ostensibly singular plural term*.[32] Much of the appeal of the trinitarian concept of sets, whatever there is to be said against it, derives from the familiarity of cases where we use a grammatically singular expression to somehow characterise a plurality of individuals. The very words 'set', 'class' etc. are themselves collective nouns used for just this purpose. Do collective noun phrases refer to new, higher-order individuals, constituted by but distinct from their members, or do they simply refer to manifolds of individuals? I believe that, if we consider carefully, we

shall see that they do neither, although they share in part the behaviour of singular terms and in part the behaviour of plural terms referring to a manifold. To facilitate the discussion, I shall annex the word 'group' to describe what such terms refer to, or rather to describe, somewhat weakening my claim, what many or most of them seem to refer to. This answer is important, since on acceptance of it rests my suggestion that set theory (manifold theory) is a poor tool for ontological research (since most groups are not manifolds).[33]

Two facts about groups have to be noticed: we shall then be clearer as to what a group is. Firstly, when we use a collective noun, we never, or hardly ever, use it without an accompanying CNP, linked to it (in English), by 'of'. We have classes of degree, sets of cutlery, clumps of trees, herds of cattle, collections of stamps and so on. In other words, groups are always groups *of* individuals, often of a specified sort. Secondly, to take up a point noticed by Stenius,[34] what makes certain individuals belong to a group is almost always more than their being several of the kind comprising the group. Not just any plurality of trees constitutes a clump, and not just any plurality of postage stamps constitutes a philatelic collection, and so on. The members of the group are linked, tied, connected or associated in some way. To borrow the terminology of Husserl from the first essay, between the members of the group there subsist various *foundation relations.* Such relations may take many forms. It may be that all the individuals in the group have a common relation to one thing, as for example when all the grapes in a bunch are connected, directly or indirectly, to one stem, or all the bees in a swarm are following the one Queen. It may be, alternatively, that the ties are simply relations holding between or among the members of the group, as for instance all the trees in a clump are relatively close to one another and further from other trees, or all the stars in a galaxy are relatively strongly attracted to one another gravitationally, as well as being closer to one another than to stars in other galaxies.

These facts distinguish groups in general from mere manifolds. For it is characteristic of a manifold that its members may be anything whatever. They need have no instrinsic ties or foundation relations: the only tie they need have is the purely extrinsic one of all being designated by one and the same term. Since we may form terms arbitrarily by listing, it is not surprising that the most bizarre bedfellows may be together in a manifold. Most of the manifolds we take any interest in are, mercifully, not of this kind. But the most important feature distinguishing most

groups from manifolds is this: the identity of a manifold is purely parasitic upon the several identities of its members: it obeys the principle of extensionality: manifolds are the same iff they have the same members. Groups on the other hand obey neither the 'if' nor the 'only if' part of this condition. A group may have different members at different times, and still be the same group. If a single tree is felled in a clump, the clump is diminished, but not destroyed. Likewise, if a new tree grows up in the clump, it is the same clump, but now augmented. Similar remarks may be made about other groups. Just as individuals, at least, those individuals which we call substances, may gain or lose *parts* to some extent without loss of identity, so groups may gain or lose *members* without loss of identity. I still attend concerts by the same orchestra I heard ten years ago, although the personnel has changed appreciably over that time. It is in this respect that groups are analogous to individuals, at least to individual substances, meriting the term 'higher-order objects' for groups. On the other hand, groups differ from individuals in being multiply constituted: a group may not *be* a manifold, but at any one time its members constitute a manifold. It is for this reason that the members of a group may be referred to using a plural term: we may refer to the trees in a clump as 'these trees', for example. It may be that the line between groups and individual substances is not a sharp one: a herd of cattle is certainly a group, and a multicelled organism like a man is certainly an individual, but certain colonies of insects resemble single organisms in various ways such as specialisation of role and balance of functions, while there is genuine dispute as to whether sponges are colonies of single-celled organisms or multicelled organisms of a different kind from most.[35]

Because a group is not constituted solely by its members, but is the group it is in part because of the foundation relations among them, one and the same manifold of individuals may constitute, either successively or simultaneously, more than one group. To revert to the example of orchestras: in the days of the Empire, three of the orchestras of Vienna had the same personnel: when they played in the Court Chapel they were the Orchestra of the Court Chapel, when they played in the pit at the opera they were the Court Opera Orchestra, and when they played symphony concerts in the Musikverein they were the Vienna Philharmonic. Similarly two committees may have exactly the same members, yet not be one committee. In cases where two groups have the same members, we shall say they *coincide*. Because different groups have different persis-

tence conditions, two groups may first coincide and then not, or vice versa.

It would be as wrong to regard groups as mere successions of manifolds as it would be to regard individual substances as mere successions of 'genuine' individuals.[36] Just as we may regard individuals which can neither gain nor lose parts without ceasing to exist as a limiting case of individual substances, which can gain or lose parts, so we may regard manifolds as limiting cases of groups: those whose identity is exhausted by that of their members. In such circumstances the 'foundation relation' is the purely formal one of being just these several individuals and no others, although when we *refer* to a manifold using a plural term, this adds the weak extrinsic tie mentioned above.

Given that manifolds are groups obeying the principle of extensionality, manifold theory is powerless to describe the constitution of groups not obeying this principle, just as mereology is powerless to explain the nature of an individual which may gain or lose parts.[37] Nevertheless, it will not be wasted effort to develop the formal theory of manifolds, any more than it is a wasted effort to develop a mereology. Groups are, or are usually, 'many-fold', and a formal theory of pluralities will serve to show something of the logic of plural reference, as well as linking up more obviously with traditional set theory, where extensionality is always obeyed. To this end further aspects of the use of plural terms, those expecially relevant to the basic notions of such a formal theory, should be mentioned.

Firstly, there is identity. We have spoken rather glibly of the identity of groups, but we need to be assured that there can be genuine identity predications involving plural terms. Sentences like the following:

The men in this room are John and Henry

resemble singular identity sentences in two important respects. Firstly, like singular identities, and unlike copulative sentences, the terms flanking the verb may be commuted without loss of sense, indeed without loss of truth (or falsity). Secondly, the logical properties of identity: reflexivity, symmetry, transitivity and intersubstitutibility in all extensional contexts *salva veritate* apply in the plural case also. There are apparent counterexamples to this last claim. Suppose John and Henry are the men in this room. Then while we may say

(1) The men in this room are few
(2) Max is not one of the men in this room,

the following sentences are less acceptable:

(3) ?John and Henry are few.
(4) ?Max is not one of John and Henry.

These facts do not however amount to a refutation of the proposition that intersubstitutility applies to plural terms. Sentence (1) is somewhat idiomatic as it stands: it would be far more acceptable to say the same thing by

(5) There are few men in this room.

In this case, there is no plural to be substituted, and the problem vanishes. On the other hand, if by 'few' we mean something fairly definite, say, 'less than ten in number', then even if we accept (1) at its face value as containing a plural term, and tantamount to something like

(6) The men in this room number less than ten (men),

then substitution gives

(7) John and Henry number less than ten (men).

The readiness to drop the second occurrence of 'men' in (6) but not (7) may be explained by its having already occurred once in (6). (7) seems to me no less acceptable than (6). In case (2), again, if (2) is tantamount to something like

(8) Max is not a man in this room

then the problem vanishes, whereas if we accept (2) at face value as containing a plural term, as I am more inclined to than with (1), then we may look on (4) as merely pragmatically or conversationally deviant, in that it is not usual to use different names for one person in close proximity, so that the need to make assertions like (4) does not often arise. Nevertheless, cases when assertions like (4) would be both apt and true are not hard to imagine: for instance

(9) Tully is one of Vergil and Cicero, but not one of Plautus and Livy.

If sentences like 'The men in this room are John and Henry' are not plural identities, it is hard to see what they could be.

I thus take it as established that identity has a sense which is shared

between singular and plural identity propositions, involving the syntactic and logical properties mentioned above. Plural identities need not entail singular identities either: for instance

(10) John's parents are the two oldest inhabitants of the village

entails nothing about which parent is oldest and which is second oldest.[38]

Next, there is membership and inclusion. We must distinguish between sentences like

(11) John is a man in this room
(12) These cows are brown

where the predicate does not involve a plural term, from those such as

(13) John is one of the men in this room
(14) These cows are among the cows owned by Brown

where the predicate does contain a plural term. The copulas 'is', 'are' (and their equivalents in other tenses) cannot be considered candidates for the vernacular equivalents of the '$\in$' and '$\subseteq$' of set theory, which are binary predicates, flanked by terms. For '$\in$' the nearest equivalent in English is 'is one of' or 'is (one) among', e. g. 'John is among the winners of Olympic Medals of 1964'. The nearest equivalent of '$\subseteq$' similarly appears to be 'are among' or 'are some of'.[39]

Now the difference between 'is one of' and 'are some of', or between 'is among' and 'are among', appears to be no greater in principle than that between 'is' and 'are' or 'runs' and 'run': one of grammatical number. While this is only a linguistic point, and does not bear directly on set theory, it is worth recalling that the Peano-Frege distinction between membership and singular inclusion was not always regarded as commonplace. Some of the most notable logicians of the last century such as Schröder and Dedekind did not make the distinction, while in Leśniewski's Ontology the distinction between singular inclusion '$a \in b$' and strong inclusion '$a \sqsubset b$' is merely that the former is false if 'a' is not a singular name,[40] otherwise the two are equivalent. It is worth recalling also that the Peano-Frege argument rests on the assumption that sets can be *members* of other sets even when they contain more than one member, a view which Russell was, at first, not ready to accept at face value, and in which we agree with him.[41] The case for there being a distinction

of *type* between individuals and pluralities thereof rested for Russell on there being certain predicates which applied to individuals which did not apply to pluralities, and vice versa.[42] But 'apply to' is ambiguous. It can mean that the predicate may be predicated *truly* of the subject, or that it can be predicated *significantly* of the subject. Only the second yields evidence for a distinction of type. The first suggests, trivially, only that there are some predicates true of individuals and not true of pluralities, and, if incorporated into a logical system yields a type-free system like that of Leśniewski.

There are indeed predicates which are, at least, never true of individuals. Most obviously, there are the plural number-predicates, like 'are seven in number'.[43] Less obviously, there are predicates such as 'meet', 'disperse', 'surround', and those derived from relational predicates, like 'are shaking hands', 'are similar', 'are cousins'. (I have put these in the plural: it is of course trivially true that a predicate in the singular cannot *correctly* follow a plural subject and vice versa, but the underlying verbs 'be shaking hands' when used in the singular sometimes have the derelativised sense 'be shaking someone's hand', although this can hardly be said of all the predicates mentioned here.) Some of the predicates, like 'disperse' and 'surround', may be used in a grammatical singular number, but in such cases they apply not to individuals but to masses of stuff, as 'The fog is dispersing', 'Water surrounds the house'.

The existence of such predicates might be used to justify the introduction of type distinctions. But if one prefers to say that sentences like

?John surrounded the fort
?The cow dispersed to various parts ot the field

are not nonsense but simply and necessarily *false*, as I confess I am inclined to do, although I have not usually succeeded in getting agreement on this, then the same examples may be used to stake a stronger claim for the legitimacy of plural reference. Whether plural reference is always eliminable in favour of singular, or singular reference together with quantification, is not in any case the main point. I certainly believe that even if plural reference is in principle eliminable, it would be at least highly inconvenient to actually eliminate it, and maybe practically impossible. However, I am not claiming its ineliminability or its practical indispensability, merely its existence and usefulness. It is not as if eliminating plural reference brings ontological economy. Manifolds do not exist over and above, or even alongside, individuals. A manifold is simply

one or many individuals. A manifold exists if and only if at least one individual exists.[44]

Looking at the question of membership and inclusion from the point of view of plural reference, the semantic condition:

> True iff everything designated by the subject term is also designated by the object term (where the subject term is the one before, and the object term the one after, the relational predicate)

applies equally to membership, inclusion, and indeed identity, which can be considered a limiting case of inclusion. Were we to replace 'everything' by 'anything' in the above condition, this would also let in the case where the subject term is empty.[45]

§ 3 Problems of Formalisation

The phenomenon of agreement or concord in syntax arises whenever expressions in certain syntactic categories fall into subcategories in such a way that even when two expressions are of compatible categories, that is, categories such that when expressions from them are combined, the result is syntactically connected,[46] (as for instance adjective and common noun, term and verb), there are still restrictive rules, usually called *selection restrictions*, governing which combinations are to count as well-formed. When such rules are violated, we get the most obvious examples of bad grammar, as *'This books', *'They smokes' etc. Many languages utilise selection restrictions in connection with grammatical number, distinguishing singular from plural, among terms, verbs, nouns, adjectives etc., and sometimes a three-way distinction between singular, dual and plural (more than two).

It is interesting that in typed languages like that in *Principia Mathematica* the restrictions on forming formulas may be regarded as selection restrictions, with each category of expression: term, predicate etc. being divided into denumerably many different syntactic subcategories. The extreme inconvenience of such restrictions may be seen by the frequent resort made in describing typed languages to the device of typical ambiguity.

Despite our introduction of plural terms, it would be similarly inconvenient for us to have a formal language employing selection restrictions with respect to grammatical number. Suppose we had a language

with singular terms $s, s', \ldots$ and plural terms $t, t', \ldots$ and predicates $P, P', \ldots$ (one-place) and $R, R', \ldots$ (two place), and we require further that predicates always occur modified for number, so that if P, R, etc. are the unmodified predicates, then $\dot{P}$, $\dot{R}$, etc. are in the singular and $\bar{P}$, $\bar{R}$, etc. are in the plural. Suppose further that selection restrictions operate as follows: the number of a predicate is to be the same as that of its first argument. This procedure would resemble closely the practice in many natural languages. Now consider how we should state that a binary relation R is symmetric: in more orthodox formal languages it would go

$$(\forall xy)(xRy \supset yRx)$$

whereas in the suggested language it would go

$$(\forall\ s\ s'\ t\ t')\ ((s\dot{R}s' \supset s'\dot{R}s)\ \&\ (s\dot{R}t \equiv t\dot{R}s)\ \&\ (t\dot{R}t' \supset t'\dot{R}t))$$

where we have shortened it somewhat by using a biconditional in the middle conjunct. Similar encumbrances would accompany all other generalisations: consider how formidable the formula stating the transitivity of R would look, for instance. This problem has not arisen hitherto because formal languages have invariably employed only singular terms.

One possible weakening would be to make the modification of predicates optional rather than compulsory, allowing concord to be used for highlighting certain predications. This would however necessitate the postulation of equivalences like $(s\dot{R}t \equiv sRt)$, and would not result in the total disappearance of selection restrictions anyway, since $s\dot{R}t$ would still be ill-formed. Once modification of predicates becomes optional, it seems arbitrary to stop there; better to drop modification altogether. Predicates would then be *neutral* as between singular and plural, and there are no longer any selection restrictions. This does not stop us from continuing to divide the category of terms into singular and plural. We can reflect the difference between the singular 'is' of identity and the plural 'are' of identity by having two different identity predicates, '$=$' for singular identity and '$\approx$' for plural identity. Then rather than extract a *syntactic* penalty when a singular term flanks the plural predicate or vice versa, and declare the result ill-formed, we shall extract a *semantic* penalty, and declare the result *false*. This is preferable to declaring it meaningless, for we should then have to decide how to deal with well-formed formulae lacking truth-values in compounds, and this is a messy

affair[47] which we can quite easily avoid. However, in line with the view put forward in the previous section that there is a sense to the identity concept which is independent of the distinction between singular and plural, we shall employ an identity predicate '$\simeq$' which is *neutral* as between singular and plural, in that it may be flanked by either singular or plural terms and still be true.[48]

Suppose that s and s' are singular terms designating the same individual. What is the status of the ostensibly plural term 's and s''? It has the form of a list, yet, if it designates anything, it designates just one thing. Two policies are open here I think, only one of which will be pursued in the next section. We could take a list like this to be a plural term, as its syntactic form suggests, but because it is not *semantically* plural, regard it as empty, having no referent. On the other hand, we could regard it as having as referent the same individual as it subdesignates with both its subterms. This then poses the question whether a redundant list of this kind should be counted singular or plural: syntactically it looks plural, whereas semantically it looks singular. However, as in the case of predicates, there is no reason to regard the singular/plural distinction as exhaustive of all the kinds of terms there might be. It is highly expedient to employ *neutral terms*, which are neither singular nor plural syntactically, but can be either singular or plural semantically. In practice such neutral terms are far more useful than strictly plural ones, and in the formal language developed in § 4, neutral terms will be employed extensively. I do not know of any neutral terms in natural language, though there is no reason in principle why natural languages should not employ them. In the language of § 4, all term lists will be neutral: plural term lists could be used, but are not.

We must next decide policy on empty terms. There is a vast literature on the problem, since Russell first proposed the theory of descriptions. It would be impossible to review in detail which course is the best to adopt, although I believe that the course I shall adopt is both the best and the most natural. Russell's procedure for singular definite descriptions is to allow them as terms only where existence and uniqueness have been proved. Against this it has been objected that this complicates the notion of term in that there is then no general decision procedure as to which expressions are terms, and involves a further complication in case the uniqueness and existence formulas are not categorically, but only conditionally derivable.[49] Frege on the other hand proposed arbitrarily assigning a referent to every term which would otherwise be emp-

ty.[50] This procedure seems, indeed is, artificial, and a better and more natural alternative is available. While recognising, with Frege, the syntactic affinity between names and descriptions, that is, regarding them all as terms, we regard empty terms as being simply terms which do not designate anything. It has sometimes been said that an expression may denote without denoting anything. This seems absurd to me. Provided only that we are able to handle sentences containing empty terms in our logic, then there is no need to assign artificial referents to empty terms. It is the great merit of free logic that it allows empty terms to be handled for logical purposes without requiring the development of three-valued truth-tables for the connectives. Of the free logics available, the one which appears to me to have the best philosophical justification is one which allows some formulas containing empty terms to be true, others to be false, and yet others to have no fixed truth-value on an interpretation. This is the sort of free logic developed by Lambert and van Fraassen.[51]

The possibility will be admitted of all terms, whether singular, neutral or plural, being empty. This indeed could hardly be otherwise if the logic is to be free of existence assumptions in remaining valid for the empty domain, when all terms are perforce empty. There will in fact be a standard empty term '$\wedge$', and this will be neutral. Furthermore, the neutral identity predicate '$\simeq$' will be allowed to hold truly when flanked by empty terms. Indeed, to keep the system extensional, we shall require that if 'a' and 'b' are both empty, that '$a \simeq b$' is true. Consideration of the intended semantics of '$\simeq$' shows why this is so: '$a \simeq b$' is to count as true iff whatever is designated by 'a' is designated by 'b', and vice versa, and this is vacuously so when both terms are empty. The standard empty term '$\wedge$' therefore plays a role like that of '$\varnothing$' in standard set theory, except that $\varnothing$ is usually taken to exist: '$(\exists x)(x = \varnothing)$' is a theorem in standard set theory, while we shall have as a theorem '$\sim E \wedge$', where 'E' is the existence predicate. This appears to me to be a considerable intuitive advantage of the present theory of manifolds: set theorists were once wont to deny that there was an empty set, or apologize for it as a 'convenient fiction',[52] though in latter days they have become more brazen about asserting its existence. Systems of pure set theory, without *Urelemente*, admit indeed nothing but $\varnothing$ and the various sets compounded therefrom according to the axioms, which, from the point of view of intuitive considerations, is a total retreat from reality. We can, with a sensible logic allowing the manipulation of empty terms, gloss the notion of a "convenient fiction" not by reluctantly admitting the entity as having a shad-

owy sort of existence, but by allowing that a term may be highly useful and yet still be empty. To sum up: there is no empty manifold. But, skirting paradox, we might say, extensionality ensures that there are not two distinct empty manifolds.

When empty terms are admitted, but bivalence is retained, there are various ways in which quantifiers and variables may be employed. The first is to take variables as ranging over both actual and possible objects, with universal and particular quantifiers meaning roughly 'for all (actual and) possible', 'for some (actual or) possible', and an existence predicate separating the actual from the merely possible.[53] This line appears not only blatantly extravagant ontologically, but also rests on the dubious notion of a purely possible individual. A second possibility would be to follow Leśniewski and allow what has been called 'unrestricted quantification'[54], so that for instance both '$\forall x Px \supset Pa$' and '$Pa \supset \exists x Px$' would be true even where 'a' is an empty term. I shall not follow this line here, since I believe it gives a non-standard meaning to the quantifiers.[55] However, I shall return in § 5 to a comparison with Leśniewski's ontology, which could be readily interpreted as a calculus of neutral terms. The approach followed here maintains allegiance to the maxim that to be is to be the value of a bound variable, allowing all parameters to be empty, but not variables. This is the approach of free logic, as exemplified by Lambert and Van Fraassen.[56]

Since we shall hold to Quine's maxim on existence, and manifolds with more than one member may exist, it is only consistent to allow variables to range not only over individuals, but also over manifolds of individuals. We shall accordingly employ three kinds of variable, corresponding to the three kinds of term: singular, plural and neutral. We shall allow these all to be bound by quantifiers and the description operator, so the intended meaning of sentences and terms where variables other than singular ones are bound must be spelt out in the next section. It would be possible to dispense altogether with singular and plural terms and variables without loss of expressive power, but we have not done so in the present exposition, since the motivation for the introduction of neutral terms etc. was that there could be plural ones. Having established that plural terms do indeed play a role in natural languages, it would be somewhat ungrateful to banish them completely from our formal language, although the price to be paid for keeping in the three subcategories of term and variable is, as we shall now see, a certain complication of the formalism.

§ 4 Axiomatisation of Manifold Theory

In this section we shall be concerned to present axioms for a theory of manifolds, remarking as we go on the intended interpretation of the axioms. No formal semantics will be set out, nor will any metamathematical results concerning the system be proved. Such tasks lie in the future. The first task is to make the basic ideas more familiar.

We shall speak about an object language without being too concerned as to what it actually looks like: all axioms and rules will be characterised metalinguistically, using schematic meta-axioms. Definitions will be regarded as semantically motivated metalinguistic abbreviations.[57] So, if a and b are terms such that '$a := b$' is a definition, a and b are automatically intersubstitutible, and '$a \simeq b$' is a metatheorem. Similarly, if A and B are formulas such that '$A := B$' is a definition, A and B are automatically equivalent, and '$A \equiv B$' is a metatheorem.

Primitive Symbols of the Metalanguage

The following constant symbols are used:

Connectives: 1-place: $\sim$; 2-place: $\supset$.
Quantifier: $\forall$.
Determiner: ι.
Predicates: 2-place: $\simeq$; $\in$.
Punctuators: (;).

The following metavariables range over expressions of the kind listed:

Terms:	Singular: $s, t, s', t', s'', \ldots$ etc.
	Plural: $m, n, m', \ldots$
	Neutral: $q, r, q', \ldots$
	All terms: $a, b, c, a', \ldots$
Variables:	Singular: $w, x, w', \ldots$
	Plural: $h, k, h', \ldots$
	Neutral: $u, v, u', \ldots$
	All variables: $y, z, y', \ldots$
Predicates:	1-place: $P, P', \ldots$
	2-place: $R, R', \ldots$
	(predicates of greater adicity will not be considered.)
Well-Formed Formulas:	$A, B, C, A', \ldots$

Formation Rules

Those expressions which are terms and well-formed formulae (wffs) are specified by a double recursion.

A *term* is either a singular term, or plural term, or neutral term.

Singular Terms comprise singular parameters (if there are any in the object language), singular variables, and singular descriptions, and nothing else.

Plural Terms comprise plural parameters, if any, plural variables, plural descriptions, and nothing else.

Neutral Terms comprise neutral parameters, if any, neutral variables and neutral descriptions, and nothing else.

Descriptions have the following forms: singular: ιxA
plural: ιhA
neutral: ιuA
where A is any wff. Descriptions in general therefore have the form ιzA.

Terms may therefore be divided into singular, plural and neutral or into parameters, variables and descriptions.

Wffs comprise atomic and compound wffs, and nothing else.

Atomic wffs have the forms: Pa; aRb, where a and b are any terms.

Compound wffs have the forms: $\sim(A)$; $(A \supset B)$; $\forall zA$, where A and B are any wffs, atomic or compound.

The usual definition of free and bound occurrences of variables within terms and formulae will be understood. An *open formula* is one containing at least one free variable occurrence. A *closed formula* is a formula in which all occurrences of variables are bound. Assuming that the variables are given some linear alphabetic ordering, then if A is any wff, the *universal alphabetic closure* of A is that wff obtained from A by binding all the free variables remaining within it with universal quantifiers, working outwards in alphabetic order. If A is closed, then it is its own closure. In the following, the expression '$\vdash A$' will mean 'the universal alphabetic closure of A is a theorem'.[58]

'$A(b/a)$' will designate that formula obtained from A by substituting occurrences of b for all occurrences of a, while '$A(b//a)$' will range over all formulae obtainable from A by substituting occurrences of b for occurrences of a in all, some or none of the places where a occurs. In each of these definitions it is assumed that if A contains a well-formed

part of the form $\forall aB$ in which the term b occurs free, this part is rewritten with a variable not otherwise occurring in A. We shall also dispense with parentheses wherever possible, following the conventions of Church.[59] Thus

$$A \supset B \supset C := ((A \supset B) \supset C)$$
$$A \supset . B \supset C := (A \supset (B \supset C))$$

and we shall continue this practice when other connectives are introduced.

The constants '&', 'v', '$\equiv$' and '$\exists$' are defined in the usual way in terms of '$\supset$', '$\sim$' and '$\forall$'.

Meta-axioms for Predicate Logic

a1 If A is a tautology of propositional calculus, $\vdash A$.
a2 $\vdash \forall z(A \supset B) \supset . \forall zA \supset \forall zB$
a3 $\vdash A \supset \forall zA$, where z is any variable not free in A.
a4 $\vdash \forall zA \supset A(y/z)$, where z is free in A, and y is of the same subcategory as z.
a5 If A is a theorem and $A \supset B$ is a theorem then B is a theorem. (Modus ponens)

These axioms are of a form which is familiar in free logics. They differ from axioms for predicate calculus with existence assumptions by not having such theorems as '$\forall xPx \supset Ps$' or '$Ps \supset \exists xPx$', since the *dictum de omni* axiom a4 is restricted to the case where a variable is replaced by another variable.

The difference between these axioms and those for normal free logic lies of course in the fact that we have three kinds of variables. It should be made clear how these work. If D is any non-empty domain of interpretation, then an assignment of values to variables in D assigns individuals to singular variables, manifolds with at least two members to plural variables, and manifolds with at least one member (i.e. manifolds in general) to neutral variables. Of course, if D is a singleton, no values can be assigned to plural variables, and only individuals to neutral variables. In similar fashion, if there are any parameters in the object language, an interpretation over D assigns individuals or nothing to singular parameters, pluralities (by which I mean manifolds with at least two members)

or nothing to plural parameters, and manifolds or nothing to neutral parameters. The difference between parameters and variables thus consists in the possibility of parameters being empty even on non-empty domains.

So '$\forall xPx$' means that the predicate P applies to all individuals, '$\forall hPh$' means that P applies to all pluralities, and '$\forall uPu$' means that P applies to all manifolds. From this it will be seen that care should be taken not to mix variables of different categories carelessly. This is catered for by the restrictions in a4. It also seems evident that both $\forall uA(u/a) \supset \forall xA(x/a)$ and $\forall uA(u/a) \supset \forall hA(h/a)$ should be metatheorems, since whatever is true of all manifolds should also be true of all individuals and of all pluralities. Such a metatheorem, $\forall uA(u/a) \supset \forall zA(z/a)$ is indeed forthcoming, but in order to prove it further axioms are needed which will serve to link the roles of the various subcategories of variables.

Meta-axioms for Identity

a6 $\vdash a \simeq a$
a7 $\vdash a \simeq b \supset . A \supset A(b//a)$

The predicate '$\simeq$' is the neutral identity predicate, holding between terms a and b just when they designate the same manifold. The familiar properties of symmetry and transitivity are readily derivable. On the other hand, the extensional property, that $a \simeq b$ when both a and b are empty, is not derivable from a1–a7, and has to be ensured by further axioms. It must be noticed that $a \simeq a$ holds for *all* terms: in this lies its usefulness. However, with identity and quantification on hand, we could readily define an existence predicate and various other identity predicates.

In free logic, existence is usually defined in terms of identity and quantification, and we could proceed thus:

$$\mathrm{E}a := \exists u(u \simeq a)$$

with singular and plural existence defined as follows:

$$\mathrm{E}!a := \exists x(x \simeq a)$$
$$\mathrm{E}!!a := \exists h(h \simeq a)$$

Notice that here the distinction between the three subcategories of variable allows us to define three closely related predicates. It will turn out that these are not the only ways in which existence could be defined, but they are intuitively appealing to some, in that they represent the maxim: to be is to be identical with something.[60] It should be here noted that there are indeed systems of free logic in which the existence predicate is present but the identity predicate is lacking. In such systems, not only is existence not defined in the usual way; it can be shown to be indefinable (I owe notice of this to Karel Lambert).

A neutral identity predicate which does not hold between empty terms may be defined thus:

$$a \cong b := \mathrm{E}a \,\&\, (a \simeq b)$$

and singular and plural identities as follows:

$$a = b := \mathrm{E}!a \,\&\, (a \simeq b)$$
$$a \approx b := \mathrm{E}!!a \,\&\, (a \simeq b)$$

while yet further predicates would cater for the cases where we allow singular-or-empty, and plural-or-empty terms. The predicate '$\simeq$' is however taken as basic here because of the familiar properties represented by a6–a7, preserving the analogy with singular identity in our chosen system of free logic.

Inclusion

As indicated in § 3, membership is to be regarded as singular inclusion. To reflect this, we choose as primitive the predicate '$\in\!\!\!\!-$' of non-empty neutral inclusion. Its intended interpretation is as follows: $a \in\!\!\!\!- b$ is true just in case (i) a is non-empty and (ii) every individual designated by a is also designated by b.[61] This is captured by the first axiom a8 below. In addition we now introduce the principle of extensionality in a9: manifolds are the same if they have the same members (the converse follows from a7 and the quantification axioms). For technical reasons, I prefer to define the existence predicate E not in terms of identity, as in the previous subsection, but in terms of inclusion as follows:

$$\mathrm{E}a := \exists x(x \mathrel{\epsilon\!\!\!/} a)$$

and E! and E!! similarly (but these predicates will not be used.) This could be expressed as: to be is to comprise at least one individual. It applies to individuals as well as to pluralities. That it amounts to the same thing as the previously suggested definition can be seen only if we grant that variables range over things that exist, singular variables ranging in ones, plural variables in twos or more, neutral variables in ones or more. This is the import of a10, which comes in three instalments for the three subcategories of variable. To formulate the condition for singular variables, we need to be able to say when exactly one individual satisfies a given condition. In fact we shall give a more comprehensive definition, which enables us also to say what it means for exactly one individual, exactly one plurality, or exactly one manifold, to satisfy a condition.

First, we define 'at least one' trivially as follows:

$$\exists_1 zA := \exists zA$$

and now we define 'at most one':

$$\exists^1 zA := \forall z \forall y(A \mathbin{\&} A(y/z) \supset y \simeq z)$$

where it is a condition that y and z belong to the same subcategory. This sort of definition, without the complication about subcategories, is in any case already familiar from ordinary first-order predicate logic with identity. We now simply define 'exactly one' as usual as 'at least and at most one'.

$$\exists^1_1 zA := \exists_1 zA \mathbin{\&} \exists^1 zA$$

The sense of $\exists^1 xA$ will be familiar already, but what of $\exists^1 hA$? This says: there exists exactly one plurality such that A, whereas $\exists^1 uA$ means: there exists exactly one manifold (whether singular or plural) such that A. Suppose for tax purposes an apartment block is divided into *households*, some of which are individuals, others *families*. Then $\exists^1_1 xA$, $\exists^1_1 hA$ and $\exists^1_1 uA$ respectively correspond to saying something like: there is exactly one individual/family/household in the block such that . . .

a8 $\vdash a \mathrel{\in\!\!\!\!\!-} b \equiv \mathrm{E}a \,\&\, \forall x(x \mathrel{\in\!\!\!\!\!-} a \supset x \mathrel{\in\!\!\!\!\!-} b)$
a9 $\vdash \forall x(x \mathrel{\in\!\!\!\!\!-} a \equiv x \mathrel{\in\!\!\!\!\!-} b) \supset a \simeq b$
a10a $\vdash \exists_1^1 x(x \mathrel{\in\!\!\!\!\!-} w)$
b $\vdash \exists_2 x(x \mathrel{\in\!\!\!\!\!-} h)$
c $\vdash \exists x(x \mathrel{\in\!\!\!\!\!-} u)$

To understand a10b the numerical quantifier $\exists_2$ must be defined. This is done in the obvious way:

$$\exists_2 zA := \exists z \exists y(A \,\&\, A(y/z) \,\&\, \sim (y \simeq z))$$

where y and z must be of the same subcategory.

We can define different inclusion predicates in terms of the notions introduced up to now. Of these, the most interesting are singular inclusion, or membership, and inclusion which holds even when the subject terms is empty:

$$a \in b := \mathrm{E}!a \,\&\, a \mathrel{\in\!\!\!\!\!-} b$$
$$a \subset b := \sim \mathrm{E}a \vee a \mathrel{\in\!\!\!\!\!-} b$$

Some ready metatheorems following from these axioms and definitions tell us e. g. that existence and self-inclusion come to the same thing: $\mathrm{E}a \equiv a \mathrel{\in\!\!\!\!\!-} a$, that everything is emptily or genuinely self-included: $a \subset a$, and that, when singular terms are in question as subjects, inclusion and membership amount to the same thing: $s \in a \equiv s \mathrel{\in\!\!\!\!\!-} a$. A metatheorem which will be of interest in the next section is the following

$$\vdash a \in b \equiv .\exists u(u \in a) \,\&\, \forall u(u \in a \supset u \in b)$$
$$\& \,\forall uv\,(u \in a \,\&\, v \in a \supset u \in v)$$

where it will be noted that instead of using a singular variable and neutral inclusion predicate to express existence, we may equivalently use a neutral variable and the singular inclusion predicate. This suggests that we could manage with slimmer resources: neutral terms alone. That this is so is shown by these metatheorems, which express the 'ubiquity' of neutral terms:

$\vdash \exists u(u \simeq z)$
$\vdash \mathrm{E}a \equiv \exists u(u \simeq a).$

Descriptions

When descriptions are introduced into a system with plural and neutral terms we must consider how their sense is to be specified. With singular descriptions we already know how to gloss 'the x such that', which (when completed, e.g. by 'x is in this room') is the nearest equivalent in a language without common nouns to a natural language description like 'the man in this room'. If we look at plural descriptions in natural language, such as 'the men in this room', then it is clear that something is comprised in the manifold of such men as a member if and only if it is a man in this room. However some predicates, unlike 'man in this room', apply to pluralities. Consider 'meet' for instance. A sentence of the form 'a met' can only be true if 'a' is a plural term. Corresponding to this verb we get as plural description something like 'those who met'. But clearly an individual can truly be said to be among those who met: the manifold designated by 'those who met' is a plural manifold, but like all manifolds is comprised of individuals, even if the predicate used does not itself apply to the individuals individually, so to speak. Consider a complicated plural description like 'those who met either in the dining room or in the lounge'. Clearly an individual belongs to the manifold so designated iff he is one of those who met in either (or both) of those places: we could specify which manifold is designated here by giving a list of *individual names*. Suppose, for example, that John, Fred and Jim met in the dining room, while Mike, Sam and Fred met (later) in the lounge. Then those who met in either the dining room or the lounge are John, Fred, Jim, Mike and Sam. (This is a good example of a plural identity sentence at work.)

Neutral descriptions may be understood then as covering both the individuals such that . . . and the pluralities such that . . .. The list of English monarchs comprises not only those who ruled alone, but also William and Mary, the joint monarchs.

A pocket-size example will show the different kinds of description at work. Consider various collections of dots drawn on the page, and envisage them as being in a procession proceeding from left to right across the page. Let the one-place predicate 'R' be interpreted as 'forms a rank

in the procession.' Any one or more dots in line abreast form a rank. Names will be assigned to the ranks: a token of each name appears below the rank in question. Where a rank consists of only one dot, the name is singular, and where it consists of more than one dot the name is plural. Then in the first procession

$$\begin{array}{cc} \multicolumn{2}{c}{\longrightarrow} \\ \bullet & \\ \bullet & \bullet \\ a & b \end{array}$$

$\iota x \mathrm{R} x \simeq b$, while $\iota h \mathrm{R} h \simeq a$, and $\iota u \mathrm{R} u \simeq a$ and b. We shall in fact write lists with terms between braces: a precise definition will follow. Here $\iota u \mathrm{R} u \simeq \{a,b\}$. The term $\iota x \mathrm{R} x$ is not empty, because there is a unique singleton rank, namely b. The term $\iota h \mathrm{R} h$ is likewise not empty, because there is at least one rank with more than one member. The term $\iota u \mathrm{R} u$ embraces all those things which are in any rank.

In the second procession

$$\begin{array}{ccc} \bullet & & \\ \bullet & \bullet & \bullet \\ a & b & c \end{array}$$

the term $\iota x \mathrm{R} x$ is empty, because there is no unique singleton rank, while $\iota h \mathrm{R} h \simeq a$ and $\iota u \mathrm{R} u \simeq \{a,b,c\}$. It must not be thought here that because the *term* '$\{a,b,c\}$' contains three atomic subterms that the manifold thereby designated contains three individuals: here it contains four. It is also useful to have an expression marking out the manifold consisting of *all* individuals falling under R. We shall use the familiar notation $\{x|A\}$, for this purpose also, to stress further the analogy with normal set theory. The definition is this:

$$\{x|A(x/a)\} := \iota u(\exists\{x(x \in u) \,\&\, A(u/a)\})$$

and in the second procession $\{x|\mathrm{R}x\} \simeq \{b,c\}$.

In the third procession

$$\begin{array}{cc} \bullet & \bullet \\ a & b \end{array}$$

bot $\iota x \mathrm{R} x$ and $\iota h \mathrm{R} h$ are empty, since there are no plural ranks and there is no unique singleton rank. Here $\iota u \mathrm{R} u \simeq \{x|\mathrm{R}x\} \simeq \{a,b\}$.

In the fourth procession

$$\begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ a & b \end{array}$$

both $\iota x \mathrm{R} x$ and $\{x|\mathrm{R}x\}$ are empty, while $\iota h \mathrm{R} h \simeq \iota u \mathrm{R} u \simeq \{a,b\}$. If there could be such a thing as a null procession, all the descriptions would then be empty.

Descriptions, especially neutral descriptions, add greatly to the expressive power of the language. They enable us to define a great many constants in a way which makes the resulting theory begin to resemble more familiar set theory.

Firstly we define the universal manifold:

$$\vee := \iota u(u \simeq u).$$

and then, by analogy, we can define

$$\wedge := \iota u \sim (u \simeq u)$$

It will transpire that for every empty term a, $a \simeq \wedge$ is true. This is the extensionality principle mentioned earlier. Another metatheorem will be $\mathrm{E}a \equiv a \in \vee$: to be is to be comprised among the things there are. Only in the trivial interpretation over the empty domain is $\wedge \simeq \vee$. Like the night in which all cows are black, in the empty domain all terms, including the most comprehensive one $\vee$, are empty. The terms $\wedge$ and $\vee$ play a role similar to that of 0 and 1 in a Boolean algebra. The difference is that in Boolean algebras the zero element exists. However there are interesting analogies with Boolean algebras, which will be touched on briefly in the next section.

It is now possible to go ahead and define the usual Boolean operators of union, intersection and complement.

$$\begin{aligned} a \cup b &:= \iota u(u \in a \vee u \in b) \\ a \cap b &:= \iota u(u \in a \,\&\, u \in b) \\ a - b &:= \iota u(u \in a \,\&\, \sim \exists v(v \in u \,\&\, v \in b)) \\ -a &:= \vee - a \end{aligned}$$

These definitions could alternatively have been given using the notation $\{x|A\}$ just introduced. In this guise they look more familiar, especially if we use the singular inclusion predicate. As it is, the following are forthcoming as metatheorems:

$$\vdash a \cup b \simeq \{x|x \in a \vee x \in b\}$$
$$\vdash a \cap b \simeq \{x|x \in a \,\&\, x \in b\}$$
$$\vdash a - b \simeq \{x|x \in a \,\&\, x \notin b\}$$

From these definitions we may form arbitrary finite unions and intersections, and, because of extensionality, the usual Boolean identities and equivalences hold. The use of lists in normal discourse corresponds to expressions like $\{a,\ldots,c\}$ in ordinary set theory, and we shall have an equivalent. We define term lists inductively as follows: if a is any term, then a is a term list, and if d is any term list, then a,d is a term list. We get terms from term lists by surrounding by braces, and the resulting terms are defined as follows:

$$\{a\} := a$$
$$\{a,d\} := \{a\} \cup \{d\}$$

This may seem like cheating, but it isn't. Given the motivation of the previous section, lists designate the individuals designated by each term in the list, whether it be singular or plural. We shall follow the convention that all term lists are neutral terms. Finite lists turn out to be indistinguishable from finite unions. This is in contrast to orthodoxy in set theory, but is motivated by the phenomenology of plural reference. It means that there are in the present theory no manifolds of manifolds distinct from manifolds of individuals. This point was defended as intuitively justified in the previous essay. In practice what it means is that manifolds do not stack up in an infinite hierarchy of types or ranks, but remain single-storied. This ought to appeal to the lovers of desert landscapes. So any expression formed out of terms by nesting lists to any finite depth may be replaced by a one-dimensional list, erasing all the braces except the outermost. Other cherished distinctions from orthodox set theory are casualties also. Firstly there is the distinction between a and $\{a\}$, as the last definition shows. In general it is only true that $a \in \{a\}$ when $E!a$, a is a singleton. Where a is a singleton, not all set theorists

distinguish the element from the singleton set. As we mentioned before, Dedekind did not, and Cantor was not firm either way, while, in recent times, Quine has regarded the distinction as dispensable.[62] The view that $a \in \{a\}$ only when a is a singleton embodies what I believe is right about Russell's distinction between classes as one and classes as many: that the only classes as one that exist are singleton classes!

One of the most powerful devices for generating sets in Zermelo's theory was the power set axiom. However, if we look at what must be our equivalent, the power set of a is $\iota u(u \mathrel{\in\!\!\!\!-} a)$. This turns out to be nothing but a, again, as our informal motivation would suggest. It is a metatheorem that $\vdash a \simeq \iota u(u \mathrel{\in\!\!\!\!-} a)$. For example if a is the pair $\{s,t\}$, then the power set of a is $\{\{s\}, \{t\}, \{s,t\}\}$. (There is no null manifold: even if we defined the power set in terms of '$\subset$' rather than '$\mathrel{\in\!\!\!\!-}$' the result would be the same, in any case.) Now, recalling that where braces are nested, we may remove all but the outermost, this manifold is revealed as $\{s, t, s, t\}$, which is simply an unnecessarily long way of designating $\{s, t\}$.

It might be thought that we are now crippled in terms of expressive power. How, for instance, can Russell's combinatorial problems be stated, and what is the status of the assertion that if a manifold has g members, then it has $2^g - 1$ submanifolds (minus 1 because there is no null manifold)? Firstly, we can talk about all the manifolds that satisfy a certain condition, rather than all the manifolds belonging to a (higher-order) manifold. There is nothing to stop us from making assertions about all pairs, for instance. But if we try to assemble the manifolds together into a single manifold in order to be able to 'handle' them (surely a manifestation of prejudice in favour of the singular), we shall find that we lose the original manifolds, getting landed simply with their union manifold. The manifold of all pairs is (assuming at least two things exist) simply $\vee$. The use of conditions instead of higher-order manifolds does bring a loss of expressive power however if one is not prepared to quantify over conditions. It may be said, however, that any ontological commitments incurred in quantifying over predicates is not lost when one trades predicates for sets: it simply reappears in a different form.[63] In any case, the paradoxes show that not every condition can earmark a distinctive individual as the corresponding set: this was where Bernays entered the fray in 1937.

We shall not develop number predicates or numerical quantifiers in detail, although it is clear from the definitions of $\exists_1^1$, $\exists_2$ etc. and the discussion of the previous paper how finite number predicates and numeri-

cal quantifiers can be defined. But it is important to realize that we may define similar looking but different numerical quantifiers by using different subcategories of variable. For instance, $\exists_2 x \mathrm{R} x$ and $\exists_2 h \mathrm{R} h$ do not mean the same thing, and neither means the same as $\exists_2 u \mathrm{R} u$. Consider the processions examples again. $\exists_2 x \mathrm{R} x$ means that two individuals are ranks: this is true in the second and third cases, false in the first and fourth. $\exists_2 h \mathrm{R} h$ means that exactly two pluralities are ranks: this is only true in the fourth case. $\exists_2^2 u \mathrm{R} u$ means that exactly two manifolds are ranks. This is only false in the second case. So we are quite able to say that

$$\exists_g^g\, x\,(x \mathrel{-\!\!\!\in} a) \equiv \exists_{2g-1}^{2g-1}\, u\,(u \mathrel{-\!\!\!\in} a)$$

and indeed, given recursive definitions of the numerical predicates, the result could be proved as a metatheorem by mathematical induction.

Combinatorial problems such as would have warmed the cockles of Russell's heart could drop out of the system as metatheorems. For instance, a football manager with a squad of thirteen players has to pick an eleven to take the field. He can select any one of 78 different possible teams: but it would be surely only a matter of patience to prove the following as a metatheorem of the calculus of manifolds:

$$\exists_{78}^{78}\, u\,(u \mathrel{-\!\!\!\in} a \;\&\; \exists_{11}^{11}\, x\,(x \mathrel{-\!\!\!\in} u) \;\&\; \exists_{13}^{13}\, x\,(x \mathrel{-\!\!\!\in} a))$$

(I have not the patience.)

Having different styles of variable and accordingly different senses for numerical quantifiers also enables us to put a firmer gloss on the contention, made in the previous essay, that number predicates, when applied to individuals, have senses analogous to the sense they have when applied to pluralities or manifolds in general. The analogy comes out in the common form of the definitions of numerical quantifiers despite the use of different subcategories of variable. In this way the informal motivations of the previous essay link up with the formal treatment of this one. This distinction enables us easily to do the work which Stenius suggests requires a procedure he calls "second-order counting".[64] Indeed, it is more flexible, since it allows us to count arbitrary finite numbers of manifolds, not just those which are submanifolds of a given manifold.[65]

But suppose, to revert to our example, that the football manager, not content with knowing how many teams he can pick, wishes to know how many ways he can slot his selected players into the eleven available positions, and arrives at the (I hope) correct answer of 39,916,800. How can this be expressed in terms of the 11 players, maybe the 2047 submanifolds thereof, without sets of sets? Surely it is here that we need sets of sets, or, as Stenius uses, arbitrary representative individuals to go proxy for sets. I am not convinced. Certainly simply considering the 11 players and submanifolds of them will never advance us to the relatively astronomical figure of 11!; but I do not think we are in this case counting men, or groups of men, at all. We are computing possible *ways* of slotting eleven men into eleven positions. This is the same as the number of different ways we may pair any two disjoint collections of eleven, or, speaking mathematically, the number of different bijections between disjoint sets of eleven. I would suggest that expressing combinatorial problems in terms of sets of sets, or sets of sets of sets, is merely a convenient device, and does not represent the ontology of combinatorial problems at all.

Meta-axioms for Descriptions

a11a $\vdash \mathrm{E}\iota xA \equiv \exists^{!} xA$

b $\vdash \mathrm{E}\iota hA \equiv \exists hA$

c $\vdash \mathrm{E}\iota uA \equiv \exists uA$

a12 $\vdash \mathrm{E}\iota zA \supset . \forall y(A(y/z) \supset y \in \iota zA)$, where z is either neutral or of the same subcategory as y.

a13 $\vdash s \in \iota zA \supset \exists u(s \in u \,\&\, A(u/z))$

a14 $\vdash \sim \mathrm{E}\iota zA \supset \iota zA \simeq \iota u(u \not\simeq u)$

a15 $\vdash a \simeq \iota z(z \simeq a)$, where z is either neutral or of the same subcategory as a.

The three instalments of a11 present the conditions on the existence of manifolds designated by descriptions. a12 and a13 tell us about the membership of manifolds designated by such descriptions when they exist, while a14 "identifies" all empty descriptions in the way suggested by § 3. This treatment is most suited for mathematical applications, though the possibility of varying the axioms for other applications, e.g. in considering the logic of fiction, is not to be ruled out without further

consideration. The final axiom a15 states an identity not otherwise derivable. This in fact makes a6 derivable as a metatheorem.[66]

The axiom a11 c is quite a powerful one, and simulates union axioms in orthodox set theory after this fashion: If for instance Pu states a condition in one free variable on manifolds, then so long as at least one manifold satisfies the condition, the union of all such manifolds exists. Conversely, if such a union exists then at least one manifold satisfies the condition. That ιuPu is in effect a union can be seen by considering its membership conditions, using a12–a13. By a12, if ιuPu exists, then any manifold satisfying the condition is included in it, and by a13, any individual which is a member of ιuPu is a member of some manifold satisfying the condition. Notice that the individual need not itself satisfy the condition: this should be clear from the examples given before. In general, we cannot infer either that the union ιuPu itself satisfies the condition: if P is the predicate 'is a pair' then in any world containing three or more individuals, the manifold of pairs is not a pair.

This sort of consideration may put one in mind of Russell's paradox. It is worth seeing how it fails to arise in the present theory. All singleton manifolds are self-membered, and all pluralities are not. The manifold of non-self-membered manifolds is $\iota u(u \notin u)$. In a domain with less than two members, this does not exist. In one with two or more members, it exists, and is identical with $\vee$. Now in such domains certainly $\vee$ exists and $\vee \notin \vee$, but this does not entitle us to infer that $\vee \in \vee$: merely, and harmlessly, that $\vee \not\in \vee$. $\vee \in \vee$ only when the domain has only one member, and then $\iota u(u \notin u) \approx \wedge$. The paradox simply does not arise, for precisely the reason originally suggested by Russell: there is a gulf between one and many.

In general P(ιzPz) only in the case where ιzPz is a singular description which is not vacuous, although cases arise with other subcategories: for instance, in domains with at least two members, the manifold of manifolds with more than one member is itself a manifold with more than one member, viz. $\vee$. We can indeed prove as a metatheorem the following general principle of comprehension:

$$\vdash s \in \{x|A(x/a)\} \equiv A(s/a)$$

though again the manifold $\{x|A(x/a)\}$ only satisfies the condition $A(\xi)$ when exactly one individual satisfies it, and this individual *is* the manifold.

While the foregoing meta-axioms delineate a system with deceptive power, the following principle appears to be independent of them, and yet intuitively satisfactory, especially in the form given.

a16 $\vdash \exists u A(u/a) \,\&\, \forall u \forall v (A(u/a) \,\&\, A(v/a) \,\&\, u \not\simeq v \supset u \cap v \simeq \Lambda)$
$\supset \exists u \forall v (A(v/a) \supset \exists_1^! x (x \in (u \cap v)))$

What the axiom amounts to is this: if $A(\xi)$ is any condition in one variable satisfied by at least one manifold, and such that any two distinct manifolds satisfying it are pairwise disjoint, then there exists a manifold intersecting each manifold satisfying the condition in a single element. This is sometimes known as the weak or disjoint choice principle. It is hard to see how it could be questioned. In this form, the axiom is not really about choice or selection in any real sense: it is about the existence of certain manifolds.

In his 1908 paper on set theory,[67] Zermelo used the principle in just this disjoint form, although the pairwise disjoint sets were not those satisfying a condition, but those belonging to a set of sets. Notice that in our case we do not need to state that the sets be non-empty: this is taken care of by the variables, which range only over manifolds that have members. Zermelo's original 1904 proof of well-ordering[68] uses not this disjoint principle but the principle which he called in 1908 the General Choice Principle: that any set of non-empty sets possesses a choice function. In his 1908 paper on well-ordering he again uses the General Principle as premiss, but, as if by way of placation, assures that the General Principle is but a consequence of the Disjoint Principle,[69] while in his paper on the foundations of set theory the General Principle is derived as a consequence of his axioms. Now Zermelo gives the appearance of regarding the Disjoint Principle as more likely to secure acceptance from the sceptical, while proving that it is just as strong as the General Principle. But what he in fact shows is that the General Principle follows from the Disjoint Principle together with the other axioms of Zermelo's set theory. These include assumptions about set existence, especially the power set and infinity axioms, which are much stronger than we have employed. By these means, Zermelo is enabled to trade in arbitrary sets of sets for equinumerous but pairwise disjoint sets of sets, using pairs consisting of one element and one set. Such means are not

here available, nor indeed can the strong General Principle be formulated as one of set existence, which makes it appear rather different in kind from the Disjoint Principle. It has indeed been suggested that Zermelo's axiomatisation was motivated less by a desire to avoid paradoxes as to gain acceptance of the well-ordering theorem, in which the axiom of choice serves of course as premiss.[70]

The interesting question left unanswered by this is whether, in the presence of weaker though still intuitively justifiable assumptions as to set existence, the General Choice Principle does not turn out to be stronger than Disjoint Choice.[71]

§ 5 Some Comparisons

The following remarks assess in broad outlines the affinities of the system presented in the previous section. In many respects the ideas, despite some obvious departures from current practice, represent a return to an older tradition, not fully distinguished in its time from general logic, namely that tradition running from Leibniz through Boole, Peirce and Schröder to Husserl, Löwenheim and Leśniewski, a tradition to be distinguished sharply from that running from Frege, Peano and Russell through to modern predicate logic on the one hand and from Cantor and Frege through Zermelo to modern axiomatic set theory on the other. Despite Russell's initial clarity about classes, he soon forsook that path in favour of a reduction of classes to propositional functions.

In many ways the present system is similar to Schröder's application of his calculus of identity and subsumption to domains taken in extension. Schröder developed a type theory of sorts.[72] Church has suggested that this was essentially a substitute for a difference between set membership and set inclusion.[73] But Schröder introduces the type-like distinction rather to avoid paradoxes. These arise, in my opinion, through a lack of adequate understanding of the difference between a predicate's being applicable to a thing and a thing's being included in a domain,[74] together with an inability to handle empty terms. Schröder uses the symbol '$\in$' for subsumption: there are in his earlier, type-free system in addition the two domains 0 and 1 such that $0 \in a \in 1$ for every domain a. But Schröder does not distinguish between every element of one domain being an element of another, i. e. subsumption, and a subject's be-

ing characterised by a predicate. Thus he regards '$0 \mathrel{\in\!\!\!\!-} a$' simply as signifying that '0 is subject to every predicate a'. Hence, if a predicate determines a domain a, then since $0 \mathrel{\in\!\!\!\!-} a$, that predicate applies to 0. So, in considering the predicate 'is equal to 1', Schröder regards the class of classes (domain of domains) satisfying this predicate as comprising just 1 and 0. But since 0 is subject to this predicate it follows that $0 = 1$, and all distinctions collapse: the night in which all cows are black.[75] Hence, Schröder concludes that classes of classes should be distinguished according to level from classes of individuals. That is indeed one way out, although it did not appeal to Frege. Frege suggested that Schröder's *Gebietenkalkül* was really only a theory of part and whole, and that in such a case there could be no null entity 0. I agree with the view that *if* this is how we interpret Schröder's system, as a mereology, then an empty individual indeed is out of place. But our system, like Schröder's, is intended not as a theory of part and whole but as a theory of extensions of terms. In such an extensional approach to classes, there should likewise be no null class, as Russell saw.[76] But we can retain the usefulness of Schröder's 0 without regarding it as an entity, by the scrupulous use of empty terms. Schröder's paradox does not arise in our system, even though we do not distinguish classes of classes from classes of individuals, because while $\wedge \subset a$ for every a it does not follow that $\wedge$ exists, nor, if a is $\iota u \mathrm{P} u$ or $\{x|\mathrm{P}x\}$, that $\mathrm{P}\wedge$. *Pace* Frege,[77] the extension of a concept *does* consist of the things falling under it in the same way as a wood (as manifold, not group), consists of trees. Having on hand the concept of a manifold means that we can treat the extension of a concept as what it is: one or more individuals. An empty concept then is not a concept with an empty extension, if by this we mean that there is something, its extension, which happens to comprise no individuals. Rather, it is a concept without an extension.[78]

The system of § 4 is a first-order system: we do not quantify over predicates. The differences, and complications, all arise from the introduction of plural terms and variables, with quantification over all manifolds, plural as well as singular. If the expressions involving terms other than singular are not employed, the remaining fragment is simply equivalent to a normal free logic, with '$\mathrel{\in\!\!\!\!-}$' equivalent to '$=$', where '$=$' differs from '$\simeq$' in not holding between terms which are empty.[79] On the other hand, the system cannot be proved consistent simply by interpreting all terms as singular (or empty), because it would be inconsistent when so interpreted: the axiom a10b would be interpreted as

$$\vdash \exists xw(x = h \,\&\, w = h \,\&\, x \not\simeq w)$$

which is inconsistent.

The existing system of logic which our system most nearly resembles is Leśniewski's Ontology, sometimes called the calculus of names. In Leśniewski, names, like our terms, can designate one or more than one or they can fail to designate at all. On the other hand it is clear that Leśniewski's "names" comprise both what I should call terms, and common nouns. I had previously thought that the only possible interpretation of Ontology which made sense in terms of the sort of expressions to be found in natural languages was as a calculus of common nouns.[80] But it now seems to me that it can equally well be interpreted as a calculus of terms, whether these be singular, plural or empty. Leśniewski's calculus could be regarded as one involving solely neutral variables, with somewhat different principles and axioms governing quantification. In some Leśniewskian systems singular names are informally marked by use of capital letters, but this does not affect their substitutivity, which is why variables are all de facto neutral.

If we had adopted quantifiers without existential import, say $\prod$ and Σ, such that $\Sigma uA := \sim \prod u \sim A$, subject to the axiom $\prod uA \supset A(r/u)$, where r is *any* term, empty or not, and axioms analogous to a2–a3, then we should have a ready way to interpret Leśniewskian expressions as follows:[81]

Usual Leśniewskian Form		*Interpretation*
[a] · A		$\prod uA$
$\in$		$\in$
ex(a)		Er, or $\exists u(u \in r)$
sol(a)		$\exists^1 u(u \in r)$
ob(a)		$\exists^1_1 u(u \in r)$
$\sqsubset$	(Strong inclusion)	$\Subset$
$\subset$	(Weak inclusion)	$\subset$
$=$	(Singular Identity)	$=$
$\bigcirc$	(Weak identity)	$\simeq$

I have preferred to develop the calculus of manifolds in such a way that it is recognisably an extension of the usual predicate calculus involving only singular and empty terms. The introduction of identity as a primi-

tive by a6–a7 seems especially preferable, since identity has stronger claims to be a logical relation than inclusion or membership, in terms of which it is usually defined in Leśniewskian systems. However, despite its unusual treatment of quantifiers, Ontology can be said to embody a theory of manifolds, although these cannot be construed as sets in the usual sense.[82] Ontology could claim to embody a skeletal theory of extensions of expressions, whether these be construed as common nouns or as terms, exhibiting the algebraic similarities between a calculus of nouns and a calculus of terms. In this it could be said also to belong in the Boole-Schröder tradition. I should be unwilling however to give up the view that there is a syntactic difference between terms and common nouns, despite their many semantic similarities. Such an identification erases many distinctions to be found in the syntax of natural languages, even though these distinctions may not be strictly necessary for logical purposes. An enlargement of the present theory could introduce common nouns and quantifiers and descriptors adjoining them.[83]

I have several times mentioned the quasi-Boolean properties of the calculus of manifolds. It is instructive to see how we can interpret the axioms in certain Boolean algebras. This has the advantage of enabling us to trade in some of the more unusual features, such as empty and plural terms, with quantifiers binding variables other than singular, for an interpretation in which all terms are singular and quantifiers are as in a normal first-order theory, without even empty terms to worry about. The system is also thereby shown to be consistent relative to the algebras in which it can be interpreted. As the simplest of these are finite, this is a heartening claim. Let us consider the particular case first, and then comment briefly on more general interpretations.

Consider any subset of the positive integers consisting of all the divisors of a number which is square-free, in the sense that it has no divisors of the form p^2, for $p > 1$. The smallest such set is $\{1\}$, but there is no largest, so we can have models of any finite cardinality 2^n, where $n \geqslant 0$. Let M denote any such set of divisors, with subscripts e. g. M_{30}, to denote particular cases. We may interpret predicate parameters as predicates defined over M, though we shall not in general be interested in arbitrary predicates. We interpret term parameters as follows:

Singular parameters are assigned either 1 or a prime number.

Plural parameters are assigned either 1 or a composite number.

Neutral parameters are assigned any number.

Variables are interpreted to range over M.

Quantifiers are interpreted as follows. If A is any condition in one free variable, suppose A' is the associated condition defined over M.

A formula $\forall xA$ is true on the interpretation iff A' is satisfied by all prime numbers in M.

A formula $\forall hA$ is true on the interpretation iff A' is satisfied by all composite numbers in M.

A formula $\forall uA$ is true on the interpretation iff A′ is satisfied by all numbers greater than 1 in M.

It is to be noticed that all universal quantifications are vacuously interpreted as true on interpretation over the domain M_1.

Descriptions are assigned values in M as follows:

If a single prime number in M satisfies A', then ιxA is assigned that number, otherwise it is assigned the number 1.

If at least one composite number satisfies A', then ιhA is assigned the lowest common multiple of all those composite numbers satisfying A' (which is in M, by choice of the sort of set M is), otherwise it is assigned 1.

If at least one number greater than 1 satisfies A', then ιuA is assigned the l. c. m. of all those numbers that do, whether prime or composite, otherwise it is assigned 1.

Notice that, as with parameters, 1 is playing the role of a null manifold, prime numbers are playing the role of individuals, and composite numbers the role of pluralities.

We interpret primitive formulas involving '$\simeq$' and '$\not\in$' as follows:

A formula $a \simeq b$ is true on the interpretation iff a and b are assigned the same number by the interpretation (so we are interpreting '$\simeq$' as '$=$').

A formula $a \not\in b$ is true on the interpretation just in case a and b are assigned numbers a' and b' such that (i) $a' \neq 1$ (ii) a' divides b'.

From these it follows that a term $\{x|A\}$ is assigned the product of all the prime numbers satisfying A' (which, by construction, is in M), or else 1.

It may then be checked that on any such interpretation all the axioms a1–a16 come out as true, indeed logically true. a1–a6 are quite straightforward, being valid according to the usual principles of quantification and identity in any first-order theory. a8–a16 get interpreted as follows:

a8: if a' divides b' (where we shall assume that when we say one number 'divides' another, that it is also $\neq 1$), then $a' \neq 1$ and every prime factor of a' is a prime factor of b'.

a9: if a' and b' have the same prime factors, they are equal.

a10: prime numbers have exactly one prime factor, composite numbers have at least two, and numbers greater than 1 have at least one prime factor.

a11: these are conditions for the number assigned to a description to be $\neq 1$: that they are met can be seen by checking the conditions for assigning numbers to descriptions given above.

a12: if a number corresponding to a description is greater than 1, then every prime factor of any number meeting the associated condition A' divides this number. This is so by construction of the number as prime or l.c.m.

a13: if a prime number divides the number assigned to a description, then it divides some number satisfying the associated condition.

a14: if the number assigned to a description has no prime factors, then it is equal to 1.

a15: every number is equal to the product of all the numbers equal to itself, or, if prime, then equal to itself, or, if 1, then equal to 1.

a16: if some number greater than 1 satisfies a given condition A', and all the numbers that satisfy A' are pairwise relatively prime, then there exists a number in M such that its common factor with every number satisfying A' is a prime number. That this is so is easily seen. Since the numbers satisfying A' are relatively prime in pairs, if we select one prime factor from each, say the smallest, then no prime is selected twice, and the product of all these primes is in M and satisfies the condition by construction.

The sets M_n form Boolean algebras under division as the partial ordering. This suggests that we could model the calculus of manifolds generally in any Boolean algebra. However, the algebra must satisfy certain conditions for the interpretation analogous to that given above for the finite algebras M_n to go through. This interpretation is a particularly straightforward and appealing one. Let B be any Boolean algebra, with distinguished elements 1 and 0, under the partial ordering $\leqslant$. Let us suppose further

(1) that B is *atomic*, i. e. for all elements $b \in B$, there is an element $a \leqslant b$ such that for all $c \in B$, $c \leqslant a$ implies $c = 0$ or $c = a$.

(2) that B is *complete*, i. e. for any non-empty subset A of B, a supremum sup A exists relative to $\leqslant$, that is, an element $s \in B$ such that (i) for all $a \in A$, $a \leqslant s$, and (ii) for all $s' \in B$ such that $a \leqslant s'$ for all $a \in A$, $s \leqslant s'$.

(3) that B is *distributive*, i. e. for every subset A of B which is not emp-

ty, and for every element $b \in B$, $b \cap \sup A = \sup\{b \cap a | \text{such that } a \in A\}$.

We can now sketch how interpretation in any such algebra B will go: since the details are similar to the finite cases M_n, we can be brief. Predicate parameters are assigned predicates defined over B, term parameters are assigned elements of B: singulars to atoms or 0, plurals to non-atoms or 0, neutrals to anything. Universal quantifications are true in these cases: singular variable bound: true iff every atom satisfies the associated condition, plural: true iff every non-atom $\neq 0$ satisfies it, neutral: true iff every element $\neq 0$ satisfies it. Descriptions are assigned elements of B as follows: if A'' is the set of elements of B satisfying the associated condition, then singulars are assigned 0 unless A'' is a singleton whose element is an atom, when this is assigned to the description. Plurals are assigned the supremum of the set of all non-atoms satisfying the condition, or else 0, and neutrals are assigned $\sup A''$ if $A'' \neq \varnothing$, or else 0. The completeness property assures that such a supremum exists where the set is not empty. The distributive property assures that suprema behave nicely in formulas. It corresponds to the following metatheorem of the calculus:

$$\vdash b \cap \iota u A(u/a) \simeq \iota v(\exists u(A(v/a) \,\&\, v \simeq b \cap u))$$

The axioms for manifolds can then be verified to be valid for all such Boolean algebras. The Axiom of Choice is interesting, because while its proof was trivial in the finite case, to prove the validity of its interpretation in the general case, where B may be infinite, requires – unsurprisingly – the disjoint choice principle. For the interpretation comes to this: if $A \subseteq B$ is a subset not containing 0, such that for any distinct elements $a, b \in A$, $a \cap b = 0$, then there is an element $c \in B$ such that for all $a \in A$, $a \cap c$ is an atom. To see how it is proved, consider any such set A whose elements are pairwise relatively atomic. For each element $a \in A$, let $A(a)$ be the set of atoms $\leqslant a$. Since $0 \in A$, $A(a)$ is non-empty in each case, and, since if $a \neq b$ are both in A, $a \cap b = 0$, so $A(a) \cap A(b) = \varnothing$. Applying the disjoint choice principle to the $A(a)$, we select an atom from each. Let the resulting set of atoms be C. By completeness, $\sup C$ exists, and has the property that $a \cap \sup C$ is the selected atom in $A(a)$ for all $a \in A$, proving the result.

It is known that all Boolean algebras may be represented by an isomorphic algebra of subsets of some set, but in addition, if the Boolean

algebra is atomic, complete and distributive, in the senses given above, it is isomorphic to the algebra of all subsets of the set of atoms.[84] With this we come full circle.

I have also recently discovered that it is possible to interpret manifold calculus in ordinary whole-part theory. We simply interpret all terms as singular, and the relation '$\simeq$' as ordinary singular identity in a free logic, and the relation '$\not\in$' as the ordinary part-whole relation, so interpreted that only existents can be parts. The resulting calculus of individuals differs from that of Leonard-Goodman only in that it allows empty terms: a perfectly laudable difference, and that it is (according to axiom a10) *atomistic*, which is not necessarily so laudable. We can then interpret singular terms as designating *atoms*, plural terms as designating *non-atoms*, and neuter terms as designating all individuals, atomic or not. The only difficulty concerns the description operator, which does not readily generalise to the normal description operator. In fact, for plurals and neuters, the description operator represents the Leonard-Goodman *sum* or *fusion* operator. This difficulty can be removed by defining a new operator: let us confine ourselves solely to neutral terms here:

$$\mathrm{J}uA := \iota u(A \,\&\, \forall v(A(v/u) \supset v \simeq u))$$

It is then the operator J which generalises under the mereological interpretation to the normal description operator. We can give axioms rather for J than ι, which are symbolically exactly analogous to those for van Fraassen and Lambert's system FD_2, and then define ι as follows:

$$\iota uA := \mathrm{J}u\forall x(x \in u \equiv \cdot\, \exists v(A(v/u) \,\&\, x \in v))$$

where we assume we have already defined '$\in$' through '$\not\in$'. This now preserves perfectly the parallel with the fusion operator of the normal calculus of individuals. As to be hoped and expected, under the present interpretation, '$\wedge$' remains an empty term, unlike the case when we interpreted the calculus in Boolean algebras. This agrees naturally with the intuition that there are no null heaps, as Frege pointed out in his Schröder review, and the difference is perfectly congruous with Tarski's demonstration that mereology is Boolean algebra save for a Boolean zero. Of course this heartening symbolic parallel between the axiom systems in no way reduces manifolds to heaped individuals: far from it. In

an enriched language having both plural terms and a part-whole predicate, there would be things we should wish to say that we could not say if that were so, e. g. that no plurality is an individual, and that no mereological sum is a plurality. All the reasons I adduce in "Number and Manifolds" for rejecting the group theory of number here rise up again to refute the identification of manifolds with heaped individuals. In particular, the unheapability of such items as incompossible possibilities, and the generally wider applicability of the notion of manifold than that of mereological sum, applications of which are predominantly confined to the physical sphere, speak loudly against such an identification. So the subsumption relation and the whole-part relation, whatever their algebraic similarities, must always be distinguished. A square built up out of four other squares has each of the four component squares as parts: it is their sum. But it is not identical with the *squares*, for there are four of them, and only one of it. Nor is a part of one of the squares (a proper part) one of the four squares, while it is part of the one square. So the relations 'is one of' and 'is part of' are quite different. Whoever appreciates this will have no problems about the one and the many. The main *axiomatic* difference between manifold theory and whole-part theory consists in the self-evidence of the that fact all manifolds consist of individuals, and the lack of self-evidence of the proposition that all individuals consist of atoms, i. e. Axiom a10. It is worth recalling in this connection the independence of the atomic hypothesis from general mereology in Leśniewski, while the requirement that manifolds always reach back to individuals recalls the necessity felt for Miriamoff's grounding axiom in ordinary set theory.

§ 6 Sets as Representatives of Classes[85]

Stenius[86] suggests that the most plausible way to regard sets-as-things, as he calls them, or classes as one, is to regard them as individuals arbitrarily assigned to serve as representatives of, go proxy for, classes. He develops the idea that the relation of representation can be seen as a genuine relation between individuals in the domain, with the membership relation $\in$ being considered as the converse of representation. In this way the formal results of the theory of sets may be preserved, without engendering the problems of the trinitarian conception of sets. The idea is

appealing: if all the mathematician wants is *some* object to do the job of sets, why not let him have an individual as proxy-object, subject simply to certain conventions on how to assign such proxies.

The idea is not new, however. Frege's *Wertverläufe* are precisely individuals which do service for functions, and have the added advantage of being saturated entities.[87] Frege's realism induced him to worry about what such *Wertverläufe* were: he was unable to take the conventionalist step of letting them be arbitrarily assigned subject to conventions. That some restrictions were necessary Russell found to Frege's cost. In a late paper of 1940, Löwenheim[88] suggested the 'Schröderisation' of mathematics by using individuals to represent classes, subject to restrictions analogous to those of axiomatic set theory to avoid paradoxes. Bernays reviewed the article quite favourably, which is not too surprising, since, as we have seen, his classes can be regarded as representatives of *predicates*, and some of these classes may themselves be represented by sets. The axioms of set theory would then take the form of conditions on how individuals may represent classes.[89]

It is interesting to see how such representation may be combined with a formal theory of manifolds as already presented. As will become clear, there are various possible ways in which representatives might be assigned. Looked at in this light, the different axiomatic set theories could be looked on not as different speculations as to what there is, but as alternative conventions, choice among which would be a matter of expediency rather than metaphysical anguish.

We shall not treat representation in detail, but sample a few of the leading ideas which would need to be developed in order to further the concept of sets as representatives.

The first point to note is that, assuming that the domain of individuals contains some fixed number α of individuals, by Cantor's diagonal argument, we should never have sufficient individuals at our disposal to represent, all distinctly, all the manifolds of individuals there are except in the trivial case when $\alpha = 1$, when it is true that $\alpha = 2^{\alpha} - 1$. So either every manifold gets an individual, but sometimes distinct manifolds get the same individual, as representative, or else not all manifolds are represented. This applies most obviously to finite domains: in a domain of 2 individuals there are 3 distinct manifolds, for instance.

Let us then introduce a new primitive relation '$\lhd$', where $a \lhd b$ is to be understood as meaning that *a represents b*. Now if any manifolds could represent others, we should trivially be able to use each manifold

as its own representative. But more interesting is the case where only individuals are representatives.

How is representation to be arranged? One obvious suggestion is that no manifold should have more than one representative:

$$\text{r1.}\ \forall xw\forall u(x \lhd u \,\&\, w \lhd u \supset x = w)$$

while a second is that no individual should represent two distinct manifolds:

$$\text{r2.}\ \forall x\forall uv(x \lhd u \,\&\, x \lhd v \supset u \simeq v)$$

These are in no sense metaphysical truths: they are stipulations. It would not be *false* for either of these not to hold, any more than it is false that there are two Senators to every State of the Union, or that the Queen is Head of more than one State. But we cannot combine r2 with universal representation, r3:

$$\text{r3.}\ \forall u\exists x(x \lhd u)$$

(except in the case of the one-member domain). For consider the manifold r defined as follows: $r := \{x|\exists u(x \lhd u \,\&\, x \notin u)$. Then on any domain with more than one member, r must exist, for suppose every representative were included in the manifold it represents. Then, since every manifold is represented, by r3, all three submanifolds of $\{s,t\}$ must have representatives in $\{s,t\}$, which they can only do if one of the representatives represents more than one manifold, contrary to r2. So r has at least one member. Suppose $s \lhd r$. By the theorem of comprehension, $s \in r \equiv \exists u(s \lhd u \,\&\, s \notin u)$. If $s \notin r$ then s satisfies the condition because r exists, so $s \in r$. But then s must satisfy the condition of being a representative of some manifold it is not a member of. By r2, this must be r, so $s \notin r$, a contradiction. This is an exact analogue of the Cantor-Russell diagonal argument, and makes the point made above without recourse to the cardinality of the domain except that it must be greater than 1.

So some restrictions on representation are necessary. It is usual to have representatives only for the smaller, more tractable classes. This is the way of ZF and NBG set theory. Or we could restrict representation of classes which are the extensions of conditions of the form $\{x|A\}$ to

cases where the syntactic form of the condition is of a certain simple kind. This is the way of Quine.[90] We might have an individual which does not represent any manifold. If we have *only* one such, then it could be regarded as an analogue of the empty set. This can be expressed thus:

$$\text{r4.}\ \exists x\,(\sim\exists u(x \lhd u)\ \&\ \forall w\,(\sim\exists u(w \lhd u) \supset x = w)$$

In such circumstances, every individual other than this one, which we shall call $\varnothing$, is a representative. This provides an analogy with so-called pure set theory, where there are no individuals which are not sets. It would not be too inappropriate to regard $\varnothing \lhd \wedge$ as true in such circumstances. Pure set theory seems an extraordinary artifice in normal set theory, but its analogue in representative theory is no more than a recipe for not wasting individuals by having them not represent.

We may now see what an analogue of a set of sets is. It is simply a representative of a manifold of representatives. A theory of types among representatives would be a recipe for partitioning representatives and other individuals so that there would be manifolds u_0, u_1, u_2, etc. with u_0 comprising individuals not representing anything *(Urelemente)*, u_1 comprising representatives of manifolds included in u_0, with perhaps an extra non-representative to serve as an empty representative, u_2 comprising representatives of submanifolds of u_1, and so on. If we wished to continue indefinitely we should need to be assured of an infinite supply of individuals. Such a proposal is quite restrictive: it does not allow mixing of types, and every representative of a singleton is distinct from, and one type higher than, the individual it represents. On a countable domain with finitely many *Urelemente* every representative allowed by the theory could be forthcoming: n for *Urelemente*, the next 2^n for first-order representatives, the next 2^{2^n} for second-order representatives, and so on. But notice that not every manifold of individuals in the domain gets represented: there are not enough individuals to go round. Even u_1 will have gaps in it if the domain and the *Urelemente* are both of the same transfinite cardinality.

Systems of set theory designed to serve as foundations for mathematics all have axioms of infinity. It is important to notice that no such axiom is included in our calculus of manifolds. If we require that $\lhd$ be irreflexive:

$$\text{r5.}\ \forall x(x \lhd a \supset x \not\simeq a)$$

and add the further recursive requirement

$$\text{r6.}\ \exists u \forall x (x \in u \supset \exists w (w \lhd x \,\&\, w \in u))$$

then this can only be satisfied on an infinite domain. In particular, if representation is single-valued, satisfying r1, then it may be considered a partial function, and for any manifold u which is represented, we may denote its unique representative by $[u]$. Then r6 may be expressed as

$$\text{r6a.}\ \exists u \forall x (x \in u \supset [x] \in u)$$

If u is any given manifold, let the manifold generated from u by taking representatives of its members, representatives of these representatives and so on, be designated $Z(u)$. Then if there is a null representative $\varnothing$, the manifold $Z(\varnothing)$ is the manifold $\{\varnothing, [\varnothing], [[\ \varnothing\ ]], \ldots\}$, which is of course Zermelo's model for the natural numbers, or rather, an analogue of it.

If u is represented, let us represent this fact by the predicate R:

$$\mathrm{R}u := \exists x (x \lhd u)$$

One sensible stipulation regarding representation is that it be closed under the taking of submanifolds:[91]

$$\text{r7.}\ \forall u (\mathrm{R}u \supset \forall v (v \lhd u \supset \mathrm{R}v))$$

Another is that whenever a number of manifolds are represented, so is their union:

$$\text{r8.}\ \mathrm{R}(\iota u (\mathrm{R}u \,\&\, A(u/a)))$$

where A is some condition on manifolds. In particular, selecting the condition $a \simeq a$, r8 yields the result that the union of represented manifolds is represented, $\mathrm{R}(\iota u \mathrm{R}u)$. This is a sort of closure condition. We can get another sort in the following way. Let S be the predicate 'is a representative':

$$\mathrm{S}x := \exists u (x \lhd u)$$

then we could require that all manifolds of representatives be represented:

$$\text{r9.}\ \forall u(u \not\in \{x|Sx\} \supset Ru)$$

We may set up relations among representatives analogous to those holding among sets in ordinary set theory. For instance let η, κ be relations defined as follows:

$$s\,\eta\,t := \exists u(t \lhd u \,\&\, s \in u)$$
$$s\,\kappa\,t := \forall x(x\,\eta\,s \supset x\,\eta\,t)$$

Then η and κ are analogues of the membership and subset relations respectively. However, $s\,\kappa\,t \,\&\, t\,\kappa\,s$ only entail $s = t$ if r1 and r2 are satisfied. We can formulate as a stipulation an analogue of the power set axiom as follows:

$$\text{r10.}\ \forall x(Sx \supset R(\{w|w\,\kappa\,x\}))$$

while an analogue of the axiom of regularity is

$$\text{r11.}\ \forall x(Sx \supset \exists w(w\,\eta\,x \,\&\, \sim\exists x'(x'\,\eta\,w \,\&\, x'\,\eta\,x)))$$

Given an infinite domain, single-valued representation and a null representative $\varnothing$, with this axiom we know that providing representatives are forthcoming at every stage, a manifold N such that (i) $\varnothing \in N$ (ii) $\forall x(x \in N \supset [\{x,[x]\}] \in N)$ and no other members besides those required by (i) and (ii), would furnish an analogue of von Neumann's version of the natural numbers. It would be the manifold $\{\varnothing, [\varnothing], [\{\varnothing, [\varnothing]\}], \ldots\}$, and the relation $s\,\eta\,t$ among its members would be the natural ordering $<$.

Enough has perhaps by now been said to suggest that mixing manifolds with representatives offers a reasonable promise for keeping distinct Russell's and Cantor's two concepts of class, while not incurring the burdens of a Platonic ontology.[92]

§ 7 Concluding Remark

"Sets", says Quine, "are classes... 'set' is simply a synonym of 'class' that happens to have more currency than 'class' in mathematical con-

texts".[93] Waiving the temptation to ask why Quine of all people should speak of synonyms, we might ask what underlies the claim. It is, I think, that there is identity, or at least continuity, between the mathematical concept of set and the familiar intuitive notion of a class. Modern set theory attempts to bite off as much of Cantor's Paradise as possible without biting off contradictions. It is worth asking whether in the process it has not forgotten what a class really is.[94]

Notes

[1] Black, 1971, Stenius, 1974. It was from Black's paper that I obtained the view that plural terms and sets are counterparts, although, as I later discovered, Russell had arrived at the same idea much earlier. The extent of my agreement and disagreement with Black and Stenius (who are by no means in complete accord) will become clear through this paper. While I find that on the whole, their destructive comments are more successful than their constructive proposals, it still seems to me that they have been somewhat unfair to the earlier tradition of set theory, strands of which, as I show, come close to solving the difficulties. It is perhaps a reflection on the ahistorical way in which set theory is read and taught today that such strands should have been so completely overlooked.

[2] Cf. the remarks on this in the previous essay.

[3] Abandoning the 'only if' part leads to Leśniewski's theory of 'collective classes', i. e. mereology. This kind of class is precisely Russell's class as one, for which see below. Leśniewski distinguishes collective from distributive classes. The latter do obey the extensionality principle. In Leśniewski however this is not a special set theory, but just the logic of names. It is interesting that Leśniewski was led to collective classes by consideration of Russell's paradox, and took a class as being most naturally conceived as the mereological sum. In view of the problems of the trinitarians, this is a natural attitude for anyone with nominalist inclinations. However, the calculus of manifolds, which I contend captures the notion of *class* rather than that of *whole*, bears affinities with Leśniewski's calculus of names, or 'ontology'. It also contains nothing a nominalist could find offensive.

[4] Hence Leonard and Goodman's version of mereology, which they call the "calculus of individuals", might be thought well-titled. I am not convinced however, that masses of stuff (including limitlessly dispersed masses), which are amenable to mereological treatment, indeed cry out for it, are most aptly called 'individuals', since this term seems to apply most naturally to things falling under count concepts, whereas stuff falls under mass concepts. If there were a special grammatical form for mass nouns, distinguishing them from singular count nouns, then we should I think be far less inclined to heap masses and individuals together. However, this is a point with far-reaching consequences and ramifications, and cannot be pursued here. It should be emphasised that 'manifold' is to be understood in this paper as comprehending both individuals and pluralities. There is no difference between an individual and a single-membered manifold: the member *is* the manifold.

[5] The predicate in manifold theory most closely analogous to '$<$' is not '$\in$' but '$\Subset$'. The manifold-theoretic notion of an individual is analogous to the mereological notion of

an atom. But manifold theory and mereology part company over this notion, for, if there are to be manifolds, there must be individuals (which might be called relative atoms) to comprise them, whereas the existence of composite entities does not, *pace* Leibniz, Wittgenstein etc., entail that there must be absolute atoms, i. e. entities without proper parts.

[6] The relation '$\in$' is one example of a predicate which is, in the terminology of the previous essay, not perfectly distributive. More precisely, the predicate '$a \in \xi$' does not distribute over manifolds, because from '$a \in b$' and '$c \Subset b$' it does not follow that '$a \in c$'. It is also clear that the relation '$\in$' is an ideal or *formal* relation, in the sense that 'exists' is a formal property, i. e. corresponds to no material property in the thing(s) concerned. In Kantian terms, '$\in$' is "no real predicate", arises simply from *a*s being among the things designated by '*b*', for instance.

[7] With Zermelo's axiomatisation, set theory became just another mathematical theory, albeit a very basic one. But the logicist intuition that in some sense 'class' is a fundamental *logical* notion, not a general mathematical one, deserves a better run for its money, provided, naturally, that the intuition can be separated from the familiar paradoxes.

[8] Letter to Dedekind, Cantor, 1899. So when Mostowski, 1966, p. 141 speaks of Cantor distinguishing between consistent and inconsistent *sets*, this is seeing Cantor through the eyes of von Neumann and Gödel. In fairness to Mostowski, Cantor occasionally talks of consistent pluralities *being* (rather than forming) sets, but it is also clear from the context that this is loose talk.

[9] The contrast with one such 'working mathematician', Felix Hausdorff, could not be greater. In his justly celebrated book, Hausdorff, 1914, he passes the paradoxes by with a cursory wave. In a recent article, Moore, 1978, G. H. Moore has shown convincingly how Zermelo's attitude was also that of a working mathematician, and that he was spurred to axiomatise set theory not to lay the ghost of the paradoxes but to provide a convincing proof of the well-ordering theorem using as weak a choice principle as possible, to gain the assent of the community of mathematicians, who had remained unconvinced by his earlier proof. For more on the weak principle, see § 4 below.

[10] In his letter to Dedekind, Cantor suggested the following principles:

(1) Two equinumerous pluralities are either both inconsistent or both consistent (Cantor in fact says, 'are both "sets"', which is an example of the sort of remark mentioned at n. 8 above).

(2) Wherever we have a set of sets, the elements of these sets again form a set (*not* loose talk). (Union principle.)

(3) Every sub-plurality of a set is a set.

The first property was made in von Neumann, 1925–6, a characteristic of the difference between sets and ultimate classes: an ultimate class (to use Quine's felicitous term) is one which is equinumerous with the class of all sets, which cannot be a set, by Cantor's diagonal argument (as Cantor recognized).

[11] Russell, 1903, § 74, p. 76.

[12] Ibid., § 104.

[13] Ibid., § 74. Russell here suggests that the class as one may be identified with the whole composed of the terms of the class, cf. § 139. This has the effect of allowing that more than one class as many may correspond to the same class as one. It also runs into difficulties about heaping together pluralities whose members come from widely different ontological domains.

[14] Russell changed his mind, between writing about classes in the body of the book, probably in 1900–1, and writing the Appendix on Frege, late in 1902, about the strength of the Peano-Frege argument. My sympathies are, as I hope is clear, with his first thoughts.

[15] Ibid., § 104.

[16] Ibid., § 74.

[17] Ibid., § 71. Note the widespread use of the concept of 'Und-Verbindung' by psychologists of the period, e.g. in Husserl, 1891a, p. 75f, or in the essay by Reinach below, § 15.

[18] Ibid., §§ 70, 74, 490. Russell however does I think distinctly favour the idea of there being propositions with more than one subject. It may be that there is interference between the linguistic idea of a single subject-*expression*, and the semantic idea of a proposition's being *about* one or many things. Even a relational predication is *about* more than one thing, but unless the relation is expressed conjunctively (cf. the previous essay) these things are not all designated by one and the same subject-expression.

[19] Ibid., § 70.

[20] Ibid., § 486.

[21] Ibid., § 489.

[22] Ibid., Appendix B. What is ironical about this is that the theory of types in the body of the book is motivated solely as a distinction between ones and many's, and rests on there being certain things which can be said of ones which cannot be said of many's and vice versa (§ 104). But it is of the essence of many's that they cannot be *members* of any class (ibid.), whereas all classes in the theory of types may be members of classes of the next higher type. So the theory of types enters at the expense of the one/many distinction, though it enters on the back of that distinction. There is therefore no justification for an *infinite* type hierarchy (§ 490), or even classes of classes.

[23] Cf. Miriamoff, 1917, Fraenkel, 1922.

[24] Von Neumann, 1925–6. The treatment is conducted entirely in terms of functions, but later commentators almost invariably present it more conventionally.

[25] In Bernays, 1937–54, these are symbolised '$\in$' for set membership, and 'η' for class membership. In § 6 we use the same pair of symbols in what is effectively the opposite way round.

[26] Bernays and Fraenkel, 1958, 41–2.

[27] Russell, 1903, § 489. The idea of representatives is further examined in § 6 below. Bernays also speaks of a set as *representing* a class in his 1937–54. A set a represents class A when $\forall x(x \in a \equiv x \eta A)$. It is a consequence of his axioms that every set represents a unique class, but of course not every class is represented by a set.

[28] This can be seen in part by the circumstance that Bernays does not quantify over classes, preferring always class parameters (free variables). Levy, 1973, p. 196 describes the move as one of replacing the metamathematical notion of a condition by the mathematical one of a class, while in the preface to his 1976, Müller reports that, unlike von Neumann, Bernays did not regard classes as real mathematical objects (p. vii). Levy describes this reluctance as 'not taking classes seriously', 1976, p. 205. That others have 'taken classes seriously', to the extent not only of quantifying over them and defining them impredicatively, but even considering their being elements of 'hyperclasses' – none of this can be laid at the feet of Bernays, who is always on stronger ground philosophically than those writers who block membership *solely* to prevent paradoxes from arising.

[29] Cf. my 1978. Other writers to "take common nouns seriously" include Lewis, 1970. A predicate is, after all, a sentence save some names (terms): if common nouns were predicates, then '*John man' should be a sentence, and if they were proper names, '*Tree is rotten' would be an acceptable sentence of English. The situation may not be so clear with other languages, but in English there is a clear syntactic difference between proper and common noun categories. Cf. the fuller remarks in the text below.

[30] It is interesting in this connection that Leśniewski's Ontology is often (and in my view preferably) called a calculus of names. Cf. § 5 below.

[31] Cf. my 1978. Although predicate logic was developed primarily to answer the sentence-forming requirements of mathematicians, it is noticeable that mathematical texts no more avoid common nouns than other natural-language works. But since the official

formal syntax of modern mathematics does not use common nouns, their role is in part assumed by set-theoretic expressions. After so many years of familiarity with formal languages there is no reason why a fully adequate formalisation of noun-using mathematical language cannot be devised. No attempt has been made in this paper to do so, for this would involve greater complexity and unfamiliarity. Also, the arguments for accepting manifolds are independent of the use of common noun expressions.

[32] Black, 1971, p. 104 in the reprint.

[33] 'Group' is to be taken here neither in the sense of McTaggart, 1921/7 nor that of Sprigge, 1970, nor, of course, in the mathematical sense.

[34] Stenius, 1974.

[35] Biologists evade the problem neatly by distinguishing between *Protozoa*, single-celled animals, *Metazoa*, multi-celled animals with two layers of cells, and sponges, which are set on one side as *Parazoa*.

[36] Such a view has to treat the identity of groups or individuals in flux as somehow second-rate. An obvious alternative, but one to be examined gingerly, is the view that there is also, or only, sortal-relative identity. Cf. Wiggins, 1967 or Griffin, 1977. In view of the distinctions between individuals, groups, wholes and classes made here I am hopeful that no such drastic expedient will be necessary.

[37] The view that the only genuine objects are those which can neither gain nor lose parts has a long history: it can be found in Leibniz and Hume, and has been defended most vigorously under the title of 'mereological essentialism' by Chisholm, e. g. 1976. For a rebuttal of this view, see Wiggins, 1979.

[38] It does of course entail a disjunction. Whether or not plural reference is eliminable, it is certainly useful. In any case, theoretical eliminability of certain kinds of expression, whether names, or variables, appears to me to carry ontological consequences only if it is supposed that ontology can be in some way "read off" linguistic facts.

[39] Cf. Russell, op. cit., §§ 68, 79, on 'is/are among'.

[40] Cf. § 5 below.

[41] Russell, ibid., § 74. But cf. his back-pedalling at § 491.

[42] Cf. the remarks at n. 22 above. It is arguable that what Russell understood under the term 'theory of types' underwent changes, apart from the obvious one of the introduction of ramification, between 1903 and 1908. In that time, Russell was not always enamoured of the type-theoretic way out, advocating, not always at different times, at least three alternatives: the 'limitation of size' theory, anticipating ZF and NBG axiomatics, the 'zigzag' theory, anticipating Quine's NF, 1937, and most radically of all, the 'no class' theory, which took class expressions as incomplete symbols (Russell, 1973). Nothing illustrates more vividly the fecundity of Russell's intellect during this period than the apparent ease with which he could throw off radically new ideas.

[43] Cf. Russell, op. cit., § 71.

[44] As noted in n. 4, the concept of manifold includes also individuals; the word *plurality* is used for manifolds with more than one member. No single term bridges the gap between individuals and pluralities very well so the term 'manifold' is as good as any. But, so long as it is understood that the term is not understood as in the recent philosophical logical tradition, the term 'class' may be substituted for 'manifold' by anyone who finds the latter term barbaric.

[45] On the difference between 'anything' and 'everything' cf. my 1978.

[46] On syntactic connection cf. Husserl, LU IV, Ajdukiewicz, 1935, and other texts on categorial grammar, such as Lewis, 1970 or Cresswell, 1973.

[47] Just how messy can be seen by consulting Routley and Goddard, 1973.

[48] There are in fact two possible neutral identity predicates, one carrying, the other not carrying, existential import. Cf. the definition of '$\cong$' in § 4 below.

[49] Cf. Bernays and Fraenkel 1958, p. 49. But Bernays' solution is as artificial as Frege's.

[50] Frege, 1893, § 11.

[51] That giving up bivalence may not be irredemiably problematical may be seen by consulting e. g. Humberstone and Bell, 1977. But complications of the sort their proposals involve ought to be resisted unless they are forced upon us.

[52] See e. g. the introduction of the null set in Hausdorff, 1914.

[53] See Routley, 1966.

[54] See e. g. Henry, 1972, Part II.

[55] On the interpretation of the quantifiers in Leśniewski see e. g. Küng, 1977. Orenstein, 1978 has disputed Küng's contention that Leśniewski's quantifiers are not substitutional (Appendix B), but it turns out that 'substitutional' has more than one possible meaning. At any rate, the quantifiers are certainly not objectual in Quine's sense.

[56] See Van Fraassen, 1966, Van Fraassen and Lambert, 1967.

[57] As e. g. in Thomason, 1970, Ch. V, § 5.

[58] See Quine, 1940, §§ 14, 16.

[59] See Church, 1956.

[60] For a convincing defence of this, cf. Hintikka, 1959.

[61] Schröder uses the symbol '$\Subset$' in his 1890–1905, and the symbol remained in use for some time afterwards, e. g. with Löwenheim and Zermelo, but then dropped out in favour of '$\subset$' or, more usually today, '$\subseteq$'. Schröder designed it as a combination of a sign for identity and one for proper inclusion. We do not use it in that sense, since for us '$a \Subset b$' is only true when 'a' is not empty. It can be seen more as a generalization of the sign '$\in$' for membership or singular inclusion to all cases of non-empty inclusion, proper or improper.

[62] Quine 1963, § 4.

[63] It will be noticed that the axiom of choice, a16 below, is in fact an axiom *schema*, since it uses predicate parameters. In this, the theory resembles ZF.

[64] Cf. Stenius, 1974. It seems to me that Stenius is here rather bent on preserving as much of orthodox set theory from the flames as possible. Black, too, seems to be too ready to allow orthodox set theory as a legitimate development of the naive theory suggested by plural reference, rather than as embodying distortions leading away from the original intuitions.

[65] Stenius, in his endeavour to pick up Cantor's result that to any set there are 2^n subsets if the set has n members, overlooks the other possible subsets of the power set, although more general "second-order" counting procedures could be added to his to allow for these.

[66] It is still preferable to treat identity separately first. A similar-looking metatheorem is $\vdash a \simeq \imath(u \Subset a)$, which identifies every set with its power set.

[67] Zermelo, 1908.

[68] Zermelo, 1904.

[69] Zermelo, 1908.

[70] See Moore, 1978.

[71] Hints that the disjoint choice principle, which Russell called the multiplicative axiom, might, in an environment of axioms for set theory weaker than, say, ZF, be strictly weaker than the full axiom of choice, arise out of various oddities in set theory. For example the proposition that every Boolean algebra has a maximal ideal, which is equivalent to Stone's representation theorem for Boolean algebras, has to date only been proved using the axiom of choice. But it is known (Halpern and Lévy, 1971) that the prime ideal theorem does not entail the principle of choice. It is notable that in our interpretation of what 'class' means, the general principle can only be stated using the concept of a function, while the weaker principle uses only the more general notion of a predicate or condition.

[72] Schröder, 1890–1905. Cf. Church, 1976.

[73] Church, ibid.

[74] There is indeed a considerable difference between a thing's falling under a concept and a thing's being included in a class. Frege was quite right to in insist that the latter must be separated from the subordination of one concept to another, but there is nothing wrong in treating membership as singular inclusion.

[75] As Schröder says (p. 245), 'hier wäre dann alles "wurst".'

[76] Russell, 1903, § 73.

[77] Cf. the final section of Frege's review of Schröder, 1890–1905, Frege, 1895.

[78] Interesting discussion comes from a perhaps unexpected quarter in G. E. Moore's *Commonplace Book* (Moore, 1962), pp. 13–4, where Moore discusses *class* and *extension*. He denies that with the ordinary meaning of 'class', classes could have less than two members, but that *if* we take '$\forall x(\varphi x \equiv \psi x)$' to imply '$\varphi$ and ψ have the same extension', then we must allow extensions having one or no members, so if we further identify classes with extensions, we must allow this for classes too. Moore seems very ready to throw over Russell's theory of classes on the strength of this somewhat grammatical point, and flirts with taking classes as pluralities, but in the end the discussion is inconclusive.

[79] In fact, because of the treatment of empty descriptions, it is the system FD_2 of Van Fraassen and Lambert, 1967.

[80] See Simons, 1978.

[81] For an exposition of Leśniewski's Ontology, including the notions here interpreted, see either Lejewski, 1958 or Henry, 1972.

[82] Asenjo, 1977 b, takes Leśniewski *not* to have a set theory, but our disagreement with this is only a matter of how 'set' is to be interpreted.

[83] As e. g. is outlined briefly in my, 1978. Cf. n. 31 above.

[84] See e. g. Stoll, 1974, p. 214. (In the 2nd edition Stoll drops as redundant the requirement of distributivity.)

[85] I am indebted for some of the stimulus to writing this section to Wolfgang Degen, who is developing in detail a family of formal systems embodying alternative strategies for representation. Where I have concentrated my attention on the representation of classes only, Degen's work provides for the representation of predicate-entities in general. The Schröderian tradition and the idea of representation put forward in Löwenheim's 1940 were brought to my attention by Barry Smith; cf. his 1978.

[86] Stenius, op. cit.

[87] Frege, 1893, § 3.

[88] Löwenheim, 1940.

[89] Cf. Bernays' review of Löwenheim, Bernays, 1940.

[90] Quine, 1937.

[91] Cf. the similar principle of Cantor, 1899, n. 10 above.

[92] Goodman, 1977, indeed *defines* Platonism as the acceptance of classes in one's ontology.

[93] Quine, 1963, pp. 1, 3.

[94] I should like to thank David Bell and Barry Smith, and an anonymous referee of the *Journal of Philosophical Logic*, for comments on an earlier effort which helped me to make many improvements embodied in this essay. In at least one respect, I am conscious that more needs to be said, for nothing in this essay deals with the problem posed by vague predicates. Zermelo, 1908, was criticised for employing the unclear notion of a definite property. It must be said that most of what I have said in this essay was said without thought for what difference it might make if some properties entering into the formalism are not, in a suitable sense, definite. Are there vague groups and manifolds, or is this simply an unwarranted transference of an idea from the linguistic to the ontological sphere? I am heartened by the fact that we talk about vague groups, or at least

talk vaguely about groups, all the time in ordinary discourse, e. g. 'the trees in Austria', 'the utensils in this room'. This question will need separate consideration, but it cannot be offloaded as 'not our problem', as effectively happened with Zermelo set theory, as modified by Fraenkel, 1922 or Skolem, 1929.

I should also like to thank Prof. Karel Lambert for stimulating discussion of my ideas at a later stage. I owe to him correction of certain factual errors regarding free logic.

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