means that mathematics does not exist in itself, independent of the doer, but also that it does not exist independent of the observer, who assesses these actions as mathematical. Hence, assessing students' mathematics is as much assessing one's own mathematics. The sentence could then read: "[…] investigating students' mathematical knowledge, *through investigating one's own mathematical knowledge.*"

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Methodological Issues of Second-order Model Building

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> Upshot • I argue that radical constructivism poses a series of deep methodological constraints on educational research. We focus on the work of Ulrich et al. to illustrate the practical implications of these constraints.

« 1 » The target article "Constructivist Model Building: Empirical Examples from Mathematics Education" by Catherine Ulrich et al. is a fine piece of educational radical constructivist research. In particular, it covers several relevant aspects regarding the application of radical constructivism (RC) to an actual teaching-learning situation. On the other hand, due to its applied character, the authors do not pay attention to the theoretical assumptions underlying their approach nor to the far-reaching methodological implications that stem from it. In my commentary, I will focus on these basic issues. It must be stressed that while the target article deals with mathematics education, the aspects I am going to address can be extrapolated to any field of educational research – provided, of course, that the obvious differences are taken into account.

« 2 » Ulrich et al. advocate the construction of *second-order models* of the mathematical thinking of students as an optimal way to bridge the gap between radical constructivist theory and the practical understanding of actual teaching-learning settings (§1). A second-order model (SOM) is an explanatory reconstruction of a student's thought on the basis of his or her observed behaviour in the didactic interaction, together with a conceptual analysis of his or her mathematical thinking (§10). In this commentary, I focus on the implicit psychological theory that necessarily underlies the construction of a SOM.

« 3 » In order to deal with these questions, let me introduce some concepts from analytical philosophy of mind, in particular the concepts of *psychological state* and *narrow psychological state* introduced by Hilary Putnam in his 1975 paper "The Meaning of 'Meaning.'" According to Putnam, a psychological state is a two-place predicate of a psychological property whose arguments are an individual and a time (Putnam 1975: 136). Brushing the analytical jargon out of this definition, a psychological state can be defined as an array of three elements, namely a subject, a temporal specification, and, of course, a psychological property.

« 4 » One kind of psychological states is those which are said to include a representation of the world. In common terms, the psychological property that typifies these states is *being aware of something*. These representations, the "something" that we are aware of, are usually called the *content* of a psychological state.

« 5 » A psychological state is said to be narrow if it can be defined without any reference external to the subject. The necessary and jointly sufficient conditions for a psychological state to be narrow are immediate:

- The subject of the state must be an individual.
- The psychological property that is attributed must be definable without any reference external to this individual.

For obvious reasons, the content of a narrow psychological state is usually called narrow content.

« 6 » In sum, in analytical terms an SOM is a model of a psychological state of a pupil whose content is related to a school topic. Hereafter we shall use the acronym SOM in this sense, and in particular we shall speak of the content of an SOM.

« 7 » It is easy to see that within RC, psychological states must be narrow (see Glasersfeld 1995b: 1 for an absolutely clear, if synthetic, programmatic statement in this sense). This implies that the *locus* of educational radical constructivist research must be the individual student, not any form of collective subject, such as, for example, the classroom group. Besides, in order to set the content of an SOM, nothing external to the mental realm of the student to whom the state is attributed must be required. Since the work of Ulrich et al. explicitly complies with the requirement of individualism, we shall focus on the latter condition.

« 8 » The question of whether there is such a thing as a narrow mental content is one of the open questions in analytic philosophy. Putnam (1975) rejected the possibility of a narrow content. Other philosophers, however, have defended that at least some mental contents are narrow. These authors have proposed several ways of understanding what narrow content is, and have proposed different strategies for gaining access to it (for a recent revision, see Brown 2011). For our purposes, we shall focus on a crucial methodological issue implicit in these strategies.

« 9 » In order to determine the narrow content of the mental state of an individual, the researcher must gather signs of this content out of his or her behavior. In other words, and using the well known Saussurean distinction, some actions of the subjects must be interpreted as *signifiers* of a fully internal *signified* narrow content. In many cases, the action that is taken to be the *via regia* to these contents is the individual's speech, but other ways are also possible, at least in some cases. In the work of Ulrich et al., several examples of these semiotic inferences can be found. In particular, the reader is presented with pieces of the behaviour of the subjects under study that are *significant*, as they convey information about some mental mathematical procedures. This, which is rather clear in §§13, 14, 21 and 22, becomes transparent in §32, where the

reader is even presented with a graphic sign (Figure 3) of the mathematical reasoning of an individual.

« 10 » Assuming that language (or any other semiotic system) can express a narrow content has some severe semantic implications. In particular, it implies that for a piece of speech, or any other *signifier*, to be taken to be a sign of a narrow content, its meaning must be assumed to be fully determined in the inner sphere of the speaker. Or, using the Putnamian terminology, the determination of narrow content by semiotic means necessarily rests on an internalist semantics.

« 11 » Internalism imposes a stark methodological restriction to the works that are useful from a radical constructivist point of view. Only transcriptions of students' speech, and other qualitative methods that present the ideas of individual pupils untouched, can be acceptable since it is assumed that they convey the narrow content of their mental states in their own terms. Thus, only qualitative studies of students' thought would be valid to make hypotheses about the content of an SOM of a pupil. I cannot see any way to escape this limitation. RC implies a hard-line qualitative educational research program when it comes to studying students' thought. The paper of Ulrich et al. is an example of such a qualitative methodology.

« 12 » From this qualitative educational point of view, and in accordance with the tenets of RC, SOMs are idiosyncratic. Each SOM is in principle distinct. While acknowledging this fact, the authors claim that substantial commonalities have been found between SOMs attributed to different students in equivalent educational settings (§27). That is, and this is a crucial point, the possibility of an extension[2] of an SOM is presented as supported by empirical data, not by means of an argument mounted on basic radical constructivist ideas. The authors find it natural that researchers who share a common theoretical and methodological background attribute equivalent SOMs to students that experience similar physical, biological, and social constraints. This seems a sensible stance, but it is in fact an *ex post* justification of the extension of an SOM from a given student to another one. There is nothing in the theory of RC that *a priori* supports such an extension. Quite the opposite.

« 13 » In sum, the authors present the extension of a particular SOM to new students as warranted by its possible empirical adequacy for describing the psychological state of these new students. But assuming that such an extension is possible only on the basis of previous studies is an inferential generalization that is far from being justified. There are two possible ways to support such a generalization:

- A statistical survey that shows that equivalent researchers investigating equivalent students' thought regarding equivalent school topics produce equivalent SOMs.
- A convincing law-like radical constructivist argument in the direction of the previous point.

« 14 » To the best of my knowledge, neither has been published so far. In fact, it could be difficult to mount them, basically because of the obvious intrinsic difficulties that the definition of an operationally meaningful notion of equivalence between individuals poses to RC. And thus, the validity of the application of a given SOM to a new individual must be evaluated case by case.

« 15 » The extension of an SOM to new students is therefore severely hampered by the hard-line methodological qualitative commitments of radical constructivist educational research. The only meaningful way to understand the extension of SOMs would be based on a well-known philosophical dichotomy, the type-token distinction, as detailed in the following.

« 16 » It is evident that narrow psychological states must be regarded to as tokens. When we say, for example, that two different people share the same belief, we are not implying that they have the same narrow contents associated with this belief. In fact, expecting that any two people have exactly the same mental content is a rather unreasonable assumption, particularly, but not only, from a radical constructivist point of view. In our case, since SOMs are about particular students produced by individual researchers under given conditions at specific times, they must be regarded as a token models.

« 17 » The question as to whether SOMs can be generalized can be now readily reformulated in the terms of the type-token distinction. The point is whether different SOMs can be grouped under a type, and if so, whether this type can be used to extend the conclusions obtained out of an individual case. Brown (2011: 6.1) has studied this problem in the general case of narrow mental content. In order to delve into this point, he uses an example that we have adapted for the sake of the clarity of our argument here. Our ability to determine the weight of a token "dog" depends, of course, on our ability to identify a dog among other animals and things. It must be stressed that not even an ostention is enough for that purpose. For if we were asked, for example, to "*weigh that*" by a person pointing at a dog before us, what would we *exactly* be assumed to weigh? The whole animal, or maybe just a part of it, let us say, its head? The only way to ask somebody properly to weigh a particular dog is to make sure that he or she knows in advance what a dog is. In other words, he or she must be able to use the type "dog" in order to individualize a token dog.

« 18 » Let us now think of an investigation of one student's thought on a particular school topic. In order to identify the narrow content of an SOM attributed to a pupil, it is necessary to know in advance the type of content we are interested in. At the very least, we must know the issue the content is about. In other words, even the purest qualitative study of the student's thought rests on a previous knowledge that cannot be reduced to particular specimens of the pupil's speech. Without this *typical* previous knowledge, the discourse of an individual student is a formless stream of information that cannot be linked to psychological states with a discrete content.

« 19 » Giving up on any form of previous type-like knowledge in educational radical constructivist research implies rejecting a discretized model of the psyche of students. Students' thought should be re-

2| I use "extension" since it is the term that the authors employ in their paper. It is a clear enough word to describe the application of a SOM to new individuals. The term that is used in analytic philosophy of science to describe the extrapolation of a model to a new context is "generalization." In my commentary, the relation between "extension" and "generalization" is implicit in §13: the extension of a SOM must be regarded as an inferential generalization.

garded as an unstructured whole, as a mental continuum. While this perspective may be acceptable from certain points of view, it is definitely not the approach of the different subject-specific didactics, such as those of mathematics or the experimental sciences. A holistic vision of the mental content of the students is arguably a defendable position, but it simply rules out any model of the students' thought based on school topics or disciplinary contents. Educational research would be reduced to a pure psychological discipline, where curricular contents would play little, if any, role.

« 20 » It is through the type that the wealth of disciplinary knowledge enters into educational radical constructivist research. The two-faced character, type and token, of SOMs is what actually permits the researcher to keep one foot in RC as a theory of knowledge and the other in educational disciplinary research. If this twofold nature is contested, this delicate equilibrium fails.

« 21 » To sum up, the work of Ulrich at al. is a fine example of several traits that necessarily accompany radical constructivist educational research:

Individualism: The *locus* of the research is a specific student.

Methodological solipsism: The properties attributed to the student must be definable without any reference external to him or her.

Semantic internalism: The meaning of a word, or any other sign, uttered by students is fully determined in their inner sphere. This, together with the previous point, is the analytic philosophy counterpart of the *operational closure* that has been outlined as an attribute of RC (Riegler 2001: 4).

Qualitative methodologies: Adapting the well-known Foucaultian saying, students' documents must be treated as *monuments*. In an ideal limit, the materials on which the investigation is based should be presented untouched to the eventual reader. While this is obviously a utopian limit, a hard-line qualitative program is the methodological framework of educational radical constructivist research.

Type-token dichotomy of SOMs: SOMs must be regarded as tokens, since they refer to individual students. On the other hand, the only meaningful way to extend them is by grouping them in types. The obvious way to define these types in educational research is by means of the school contents.

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Second-order Models of Students' Mathematics: Delving into Possibilities

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> Upshot • I look at the different possibilities offered by the trajectory of second-order models in mathematics education. It seems to me that although possibilities are extended as models become more elaborate, this is only the case if teacher/researchers remain cognisant of a radical constructivist perspective. I also suggest that broad-ranging research on the models affords insight into the "big" mathematical ideas to which all students should have access.

« 1 » Before I begin this commentary, I need to declare my own interests in constructivism. My doctoral research (Dooley 2010) concerned the construction of mathematical insight by individual children in the context of whole-class conversation. My theoretical stance was *emergent*, that is, an approach in which an account of individual students' learning is coordinated with the means by which the learning is supported. Paul Cobb and Erna Yackel suggest that "(f)rom this vantage point, individual activity is seen to be situated within the practices of a local community such as that constituted by the teacher and students in the classroom" (Cobb & Yackel 1996: 188). In taking this viewpoint, I did not dismiss the broader social and cultural frameworks of mathematics teaching and learning but, in terms of the aims of my research, the emergent approach as a theoretical position afforded me the requisite "grain size" (Cobb et al. 2001). From a psychological perspective, the level of analysis is that of the teacher and pupils as individuals constructing knowledge as they interact; from a social perspective, it is the class as a local community establishing communal norms and practices.

« 2 » I conducted research in three different schools. All classroom interactions were audio-taped; additional data, such as children's written artefacts, photographs of materials, etc., were also gathered. For each lesson, I identified "knowledge elements" (Ron, Dreyfus & Hershkowitz 2010), that is, the constructs that pupils might be expected to develop; these informed a hypothetical learning trajectory. Initially I garnered these knowledge elements from relevant research but, as the project continued, I incorporated constructs that emerged in previous lessons. For example, I taught the "Handshakes" problem (Slavitt 1999) in two different classes – analysis of the first of these lessons informed the hypothetical trajectory that I formulated for the second. As was the case with some of the teacher/researchers in this article, I became more aware of possible ways that children might engage in tasks as I progressed through the study.

« 3 » A particular challenge that I encountered was attending simultaneously to the personal nature of constructions and the ways knowledge was distributed among various individuals in the group. In order to address this challenge, I first explored the new mathematical ideas (with reference to knowledge elements) that were created by the class members over the course of a lesson or series of lessons, e.g., development of an explicit rule to develop the number of handshakes for n (number of individuals) in the "Handshakes" problem. Later, I isolated individuals' contributions and traced