A METALINGUISTIC AND COMPUTATIONAL APPROACH TO THE PROBLEM OF MATHEMATICAL OMNISCIENCE

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Abstract

In this paper, I defend the metalinguistic solution to the problem of mathematical omniscience for the possible-worlds account of propositions by combining it with a computational model of knowledge and belief. The metalinguistic solution states that the objects of belief and ignorance in mathematics are relations between mathematical sentences and what they express. The most pressing problem for the metalinguistic strategy is that it still ascribes too much mathematical knowledge under the standard possible-worlds model of knowledge and belief on which these are closed under entailment. I first argue that Stalnaker's fragmentation strategy is insufficient to solve this problem. I then develop an alternative, computational strategy: I propose a model of mathematical knowledge and belief adapted from the algorithmic model of Halpern et al. which, when combined with the metalinguistic strategy, entails that mathematical knowledge and belief require computational abilities to access metalinguistic information, and thus aren't closed under entailment. As I explain, the computational model generalizes beyond mathematics to a version of the functionalist theory of knowledge and belief that motivates the possible-worlds account in the first place. I conclude that the metalinguistic and computational strategies yield an attractive functionalist, possible-worlds account of mathematical content, knowledge, and inquiry.

1 Introduction

According to the possible-worlds account of propositions, there is only one necessarily true proposition. Assuming that mathematical truths are necessary, it seems to follow that whoever knows any necessary truth is mathematically omniscient, i.e., knows every mathematical truth. But this is clearly false: everyone is ignorant of or mistaken about some mathematical truth. The possible-worlds account thus faces the problem of mathematical omniscience. The metalinguistic strategy for solving this problem, developed by Robert Stalnaker (1976; 1984), is to characterize the nature of mathematical ignorance as metalinguistic: when an agent seems to be ignorant of some mathematical truth, what they are ignorant of is that some mathematical sentence expresses the one necessarily true proposition.

The metalinguistic strategy faces a seemingly fatal problem: If one assumes the standard possible-worlds model of knowledge and belief on which these are closed under entailment, it still ascribes too much mathematical knowledge to many people. This is because the relevant metalinguistic propositions are entailed by other propositions that these people know. I call this 'the closure problem'. Given that the metalinguistic strategy is considered to be the main and most developed strategy available to defenders of the possible-worlds account, many have taken its apparent failure to give reason to abandon the possible-worlds account in favor of more fine-grained accounts of propositions, such as structured or impossible-worlds accounts.

My aim in this paper is to solve the closure problem for the metalinguistic strategy. I first argue that Stalnaker's fragmentation strategy for solving the closure problem is insufficient (§4). I then develop an alternative approach, which I call 'the computational strategy' (§5). The computational strategy starts from the observation—made, among others, by Rohit Parikh (1987) and Joseph Y. Halpern et al. (1994)—that standard possible-worlds models fail to capture the computational aspects of knowledge and belief. As I explain, this is a particularly salient problem when modeling mathematical thought. The algorithmic models developed by Halpern et al. (1994), however, have some shortcomings, and they are incompatible with the possible-worlds account of propositions. I thus develop a model of mathematical knowledge and belief that combines key elements of the algorithmic models with the possible-worlds account. When added to the metalinguistic strategy, this model entails, in outline, that mathematical knowledge and belief require computational abilities to access metalinguistic information, and thus aren't closed under entailment. This solves the closure problem for the metalinguistic strategy. As I explain, the computational model can be generalized beyond mathematics. Moreover, the generalized view that I develop is a version of the functionalist theory of knowledge and belief that motivates the possible-worlds account in the first place. As I hope to show, the case of mathematics turns out to be a fruitful starting point for developing a general possible-worlds, functionalist account of content, belief, and knowledge.

There is another other kind of problem that is standardly raised against the metalinguistic strategy, which is that it seems to yield a counterintuitive account of mathematical thought and inquiry.[1] For instance, mathematical ignorance just doesn't seem to be ignorance of meaning.[2] I won't address these counterintuitiveness problems in this paper. But I think they can be shown to be misguided from a certain Carnapian perspective that has been making a comeback in the philosophy of mathematics.[3] From this perspective, mathematical truths are analytic and mathematical inquiry consists in discovering whether certain mathematical sentences follow from meaning postulates or axioms, and thus whether they express a necessarily true proposition. Mathematical inquiry is thus in an important sense metalinguistic. Mathematical ignorance, in turn, is also metalinguistic

[1] Those who raise these counterintuitiveness problems include Stalnaker (1984, 74) himself, Field (1986, 111), Robbins (2004, 62), Nuffer (2009), and Stanley (2010, fn. 9).

[2] Those who raise this specific counterintuitiveness problem include Robbins (2004, 64), Jago (2014, 61f.), and Berto and Jago (2019, 164f.).

[3] The Carnapian perspective is defended by Carnap (1937; 1939), and more recently by Azzouni (2006); Gabbay (2010); Rayo (2013); Donaldson (2015); Warren (2020); Ruffino, San Mauro, and Venturi (2021); and Soysal (2021).

in the sense that it is ignorance of whether a given sentence is analytic. At the same time, mathematical ignorance is compatible with basic linguistic competence, which can be characterized as knowledge of meaning postulates (and which, given the computational strategy, doesn't entail mathematical omniscience). I won't defend or rely on this Carnapian perspective in this paper, but it motivates my defense of the metalinguistic strategy. No existing account of content works very well when applied to mathematics. The metalinguistic strategy combined with the possible-worlds account yields mathematical propositions that are roughly as finely grained as they are on the fine-grained accounts. But in my view, the metalinguistic construal, especially when combined with the computational strategy, has deeper motivations from the philosophy of mathematics and, as this paper will argue, from the psychology of mathematical inquiry.

2 The possible-worlds account, the standard model, and functionalism

On the possible-worlds account, propositions are identified with functions from possible worlds to truth-values, or, equivalently, with sets of possible worlds. In this paper, I adopt the latter convention. Accordingly, a proposition, P, is true at some possible world, w, just in case $w \in P$. I call '$\mathscr{W}$' the one necessarily true proposition, i.e., $\mathscr{W} = \{w \mid w \text{ is a possible world}\}$. The possible-worlds account has been highly influential both within and outside of philosophy in linguistics and computer science, among others, for the elegant formal semantics that it yields, and the unified account of mental, linguistic, and informational content that it provides. As Jens Christian Bjerring and Wolfgang Schwarz (2017) recently argued, it is far from clear that the fine-grained alternatives to the possible-worlds account have any of its theoretical attractions. The fine-grained accounts also face problems of their own, notably, that they are overly fine-grained. For instance, no two sentences ever express the same proposition on standard impossible-worlds accounts, and many sentences that are intuitively synonymous are assigned different propositions by the structured-propositions accounts because they have different structures.[4]

Proponents of the possible-worlds account such as David Lewis (1979; 1986; 1994) and Stalnaker (1976; 1984) standardly combine the possible-worlds account with a model of knowledge and belief as truth in all accessible worlds, which I will call 'the standard model'. More specifically, the standard model characterizes the relationship between an agent's individual beliefs and her *belief state*, i.e., her total set of beliefs. On the standard model, the belief state of an agent, S, at a time (henceforth '$\mathscr{D}_S$') is the set of worlds that are "doxastically accessible" to S, i.e., worlds that might, for all S believes, be her world. S believes some particular proposition, P (henceforth also '$B_S(P)$') just in case P is true in all the worlds that are doxastically accessible to S, i.e., $B_S(P)$ if and only if $\mathscr{D}_S \subseteq P$. Knowledge is modeled similarly. The *information state* of an agent, S, at a time (henceforth '$\mathscr{I}_S$') is the set of worlds that are "epistemically accessible" to S, i.e., worlds that might, for all S knows, be her world. $\mathscr{I}_S$ can be understood as S's total information at a time. S knows some particular proposition, P (henceforth also '$K_S(P)$') just in case P is true in all the worlds that are epistemically accessible to S, i.e., $K_S(P)$ if and only if $\mathscr{I}_S \subseteq P$.

The standard model, too, is widely applied within and outside of philosophy, and it is part of standard epistemic logics. But it has two highly counterintuitive consequences, which are part of what is called 'the problem of logical omniscience'. First, on the standard model, every agent knows $\mathscr{W}$, since for any S, $\mathscr{I}_S \subseteq \mathscr{W}$. Second, if an agent knows all of the propositions in some set, Π, and Π *entails* Q (i.e., Q is true in all worlds in which every $P \in \Pi$ is true, or $\bigcap \Pi \subseteq Q$) then the agent also knows Q, since $\mathscr{I}_S \subseteq \bigcap \Pi \subseteq Q$. I call this second consequence 'closure under entailment'.[5] Both of these consequences also hold for belief. The problem of mathematical omniscience, then, is a special case of the first consequence: assuming that mathematical truths are necessary, there is only one mathematical proposition, $\mathscr{W}$, and it is known and believed by everyone. Even if one rejects the standard model, as I will in §5, there is a weaker version of the problem of

[4]For the former criticism, see, e.g., (Bjerring & Schwarz 2017, 30–32), for the latter, e.g., (Lewis 1970, 31f.).

[5]This principle is also sometimes called 'full logical omniscience' (Fagin et al., 1995, 335).

mathematical omniscience that only assumes that the possible-worlds account holds and that mathematical truths are necessary and thus identical to $\mathscr{W}$: namely, whoever knows $\mathscr{W}$ knows any true mathematical proposition (this is the version of the problem stated in §1).

For Lewis and Stalnaker, the standard model isn't an optional add-on to the possible-worlds account of propositions; rather, both the possible-worlds account and the standard model are motivated from their functionalist theory of knowledge and belief. In this paper I focus on Stalnaker's views, but much of the discussion applies to Lewis's (1979; 1986) as well. For Stalnaker, propositional attitudes such as beliefs and desires are part of a theory of rationality, i.e., a theory that is intended to explain the behavior of rational agents (Stalnaker 1976, 80f.; 1984, 4). On this view, we attribute beliefs and desires to make sense of agents' behavior, including (but not limited to) their linguistic behavior, such as making assertions or answering questions. Beliefs and desires thus play certain functional roles in explaining and characterizing action. The role of belief, specifically, is as follows:

> To believe that P is to be disposed to act in ways that would tend to satisfy one's desires, whatever they are, in a world in which P (together with one's other beliefs) were true. (Stalnaker 1984, 15)

This is the "pragmatic" component of Stalnaker's account of belief. Stalnaker's account also has a "causal" component according to which belief states indicate or carry information about a state of the world under normal conditions, which is to say that, under normal conditions, one has the belief that P only if P (Stalnaker 1984, 13f., 18). The complete account, which is called the 'causal-pragmatic account', is as follows:

The causal-pragmatic account: An agent, S, believes that P iff:

1. S is in a state that would carry the information that P in normal conditions, and,
2. S is disposed to act in ways that would tend to satisfy her desires in a world in which P, together with S's other beliefs, is true.

According to Stalnaker, the causal-pragmatic account carries over to knowledge, for "belief is what would be knowledge if the relevant normal conditions obtained, or [...] knowledge is full belief when it is non-defective" (Stalnaker 2019, 2). Thus, on the causal-pragmatic account, knowing that P entails being in a state that carries the information that P (with no restriction to normal conditions), and having the capacity to make one's actions depend on one's information (Stalnaker 1991, 437; 1999b, 260f.; 2019, 1f.).

Stalnaker argues that the causal-pragmatic account motivates the possible-worlds account of propositions and commits him to closure principles (Stalnaker 1976, 80–82; 1984, 23f., 82f.). These arguments have been criticized,[6] and, as Stalnaker (2010, 145) acknowledges, the causal-pragmatic account doesn't entail either the possible-worlds account or the standard model. But it does entail that belief and knowledge are closed under entailment. To see this, let us assume the possible-worlds account of propositions (as we will throughout the paper). On the causal-pragmatic account, belief distributes over intersection, i.e., if an agent believes $\bigcap \Pi$, then she believes P for all $P \in \Pi$. (Assume that S believes $\bigcap \Pi$, and let $P \in \Pi$. S is thus in a state she would be in only if $\bigcap \Pi$ were the case (in normal conditions). Since $\bigcap \Pi \subseteq P$, S is also in a state she would be in only if P were the case (in normal conditions). Moreover, since S is disposed to act in ways that would tend to satisfy her desires in worlds in which $\bigcap \Pi$, together with her other beliefs, is true, she is also disposed to act in ways that would tend to satisfy her desires in worlds in which P, together with her other beliefs, which include $\bigcap \Pi$, is true.) Given that if Π entails Q (i.e., if $\bigcap \Pi \subseteq Q$) then $\bigcap \Pi = \bigcap \Pi \cap Q$, if an agent believes every proposition in Π, then she also believes Q. A parallel argument works for knowledge. Therefore, if we would like to maintain the possible-worlds account of propositions, but not have knowledge and belief be closed under entailment (for reasons that will become clear in §3), then we can't accept the letter of Stalnaker's functionalist theory of knowledge and belief. In §5, I will argue that we can nonetheless keep its spirit.

[6]For criticisms, see, e.g., (Speaks 2006) and (Stanley 2010).

3 The metalinguistic strategy and the closure problem

Let me use 'mathematical proposition' to refer to the objects of mathematical belief or doubt and to the contents usually expressed by and communicated with mathematical sentences. The metalinguistic strategy for solving the problem of mathematical omniscience is based on the claim that mathematical propositions are propositions about the relationship between some mathematical sentence and the one necessarily true proposition, $\mathscr{W}$ (Stalnaker 1976, 87f.; 1984, 73–76).[7] For example, when Ola doesn't seem to know that $5,801$ is prime, what Ola is ignorant of isn't $\mathscr{W}$, but rather, that the sentence '$5,801$ is prime' expresses $\mathscr{W}$. So, on the metalinguistic strategy, mathematical propositions are propositions of the form $\{w \mid \phi \text{ expresses } \mathscr{W} \text{ at } w\}$, where ϕ is a mathematical sentence.[8] Given that, plausibly, for any distinct sentences ϕ and ψ, $\{w \mid \phi \text{ expresses } \mathscr{W} \text{ at } w\} \neq \{w \mid \psi \text{ expresses } \mathscr{W} \text{ at } w\}$, the metalinguistic strategy entails that although there are only two necessary propositions, $\emptyset$ and $\mathscr{W}$, there are as many distinct mathematical propositions as there are distinct mathematical sentences. The problems of mathematical omniscience of §2 are thus averted: it isn't the case that everyone knows every true mathematical proposition, or that whoever knows $\mathscr{W}$ knows every true mathematical proposition.

As I have stated it, the metalinguistic strategy primarily gives an account of the nature of the information that is believed or doubted in mathematics—it says that mathematical information is metalinguistic. But of note is that the metalinguistic strategy also has consequences for the semantics of ascriptions of propositional attitudes, i.e., accounts of the truth-conditions of sentences of the form 'S believes that P'. On Stalnaker's (1984, 73f.; 1999a; 1999c) view, the semantics of ascriptions of propositional attitudes is more complex

[7]Others who endorse the metalinguistic strategy include Lewis (1986, 36; 1996, 552) and Braddon-Mitchell and Jackson (2007, 200f.).

[8]It wouldn't make a difference for my arguments to take mathematical propositions to be of the form $\{w \mid \phi \text{ expresses a truth at } w\}$, along the lines of the "diagonal" propositions Stalnaker defines in (1978, 81), but here I follow Stalnaker's exposition in (1976; 1984).

and context-dependent than a simple theory of the form: 'S believes that P' is true if and only if the referent of 'S' believes the proposition that is expressed by 'P'. In particular, when 'P' is a mathematical sentence, the ascription 'S believes that P' is true usually when the object of belief is the metalinguistic proposition $\{w \mid$ 'P' expresses $\mathscr{W}$ at $w\}$. I won't discuss theories of propositional attitude ascriptions in this paper (except some in §5.2), but focus on the metalinguistic strategy as a theory of the nature of the subject matter of mathematical inquiry.[9]

The metalinguistic strategy faces the closure problem: There are cases in which an agent knows certain (contingent) propositions that entail that some true mathematical sentences express $\mathscr{W}$, and the metalinguistic strategy deems these metalinguistic propositions to be the objects of the agents' ignorance. Such cases (henceforth 'closure-problem cases') are incompatible with the standard model of §2. Thus, although proponents of the metalinguistic strategy don't have to say that everyone (or whoever knows $\mathscr{W}$) knows every true mathematical proposition, their account still entails that there are many people who have much more mathematical knowledge than they intuitively seem to have. The closure problem has been raised by many, including by Stalnaker (1984, 76), who presents the following kind of case.[10] Let A be a set of axiom sentences (henceforth simply 'axioms'), and let R be a set of rules of inference. Suppose Ola knows that every $a_i \in A$ expresses $\mathscr{W}$. Suppose Ola also knows that if every $a_i \in A$ expresses $\mathscr{W}$, then every sentence that

[9]This is also how Stalnaker (1984, 73f.) frames the metalinguistic strategy.

[10]Field (1986, 111), Speaks (2006, 448–450), and Williamson (2016, fn. 1) are among those who raise the closure problem, but they raise it in a slightly different way. On their view, whoever is linguistically competent with, e.g., arithmetical expressions, will believe the metalinguistic proposition expressed by ($*$): "'$197 \times 49 = 9,653$' means that $197 \times 49 = 9,653$." This, together with the proposition expressed by '$197 \times 49 = 9,653$'—viz. $\mathscr{W}$, which everyone knows—entails the metalinguistic proposition that was supposed to capture the speaker's ignorance of the multiplication. However, as Stanley (2010, fn. 14) and Stalnaker (2021, 187f.) note, defenders of the metalinguistic strategy should deny that such a speaker believes the proposition expressed by ($*$)—at best they believe that $\{w \mid (*)$ is true at $w\}$. In any case, the computational strategy developed in §5 also solves this version of the closure problem.

is derivable in $\langle A, R\rangle$ (henceforth '$\vdash_{\langle A,R\rangle}$') also expresses $\mathscr{W}$. Now assume that $\vdash_{\langle A,R\rangle} \psi$. Since this is necessarily true, Ola knows it. But then, what Ola knows entails that ψ expresses $\mathscr{W}$. Thus, by closure under entailment, Ola knows that ψ expresses $\mathscr{W}$. This holds for any derivable sentence ψ, and hence contradicts that Ola doesn't know that ψ expresses $\mathscr{W}$, for some theorem ψ.

4 The fragmentation strategy

Stalnaker (1984, 79–88; 1991; 1999b; 2021) and others (such as Lewis (1982; 1986, fn. 27), Agustín Rayo (2013), Adam Elga and Rayo (2021a; 2021b)) have proposed to solve the closure problem by means of the fragmentation strategy. The first step of this strategy is to claim that agents can be "fragmented" in the sense of having more than one belief state at the same time. On this view, each belief state corresponds to a context the agent is in, or a task that the agent is engaged in (Stalnaker 1984, 83, 86). Functionalism motivates the possibility of fragmentation, since different behavioral dispositions can be displayed in different kinds of contexts or for different kinds of tasks (Stalnaker 1984, 83). The possibility of fragmentation entails that information, in turn, can be accessible for some purposes or in some contexts, but inaccessible for others. Elga and Rayo (2021a, 39f.; 2021b, 3f.) give the example of a puzzle-solver for whom the information that 'dreamt' is a word of English with six letters and ending in 'mt' is inaccessible for the purpose of solving a cross-word puzzle, but accessible for the purpose of answering the question "Is 'dreamt' a word of English with six letters and ending in 'mt'?"[11] The final step of the fragmentation strategy is to claim that deduction is the process of integrating one's different belief states by changing "one's dispositions so that the actions one is disposed to perform in the two kinds of situations are appropriate relative to the same belief state" (Stalnaker 1984, 84). The "local reasoning" model of knowledge and belief, developed by Fagin and Halpern (1987), is standardly associated with the fragmentation strategy. It

[11]Stalnaker (1991, 438) gives a similar example.

differs from the standard model in that, among others, agents have multiple belief states, and a notion of ("local") belief is defined as truth in all possible worlds in some belief state.[12]

It is highly plausible that agents can be fragmented. Many clear cases of fragmentation have been described, including the case of the puzzle-solver above, or the case of the "disavowed racist" whose actions towards people of a certain racial identity suggest she has prejudices that she sincerely professes not to have (Stalnaker 2021, 190).[13] The problem with the fragmentation strategy is that it is highly implausible that every case in which one is ignorant about some entailment of one's beliefs is a case of fragmentation. One can easily construct cases in which an agent clearly has all the relevant beliefs in one fragment—the beliefs explain the one task the agent is involved in and there is only one relevant context—and yet the agent doesn't know all the entailments of these beliefs. Consider, for instance, the case of Ola from §3. To simplify it, assume that there are only two axioms, a_1 and a_2, in A. We have the following situation:

1. $K_{\mathrm{Ola}}(\{w \mid a_1$ expresses $\mathscr{W}$ at $w\})$ [*assumption*],
2. $K_{\mathrm{Ola}}(\{w \mid a_2$ expresses $\mathscr{W}$ at $w\})$ [*assumption*],
3. $K_{\mathrm{Ola}}(\{w \mid$At w, for any ϕ, if a_1 expresses $\mathscr{W}$ and a_2 expresses $\mathscr{W}$ and $\vdash_{\langle A,R\rangle} \phi$, then ϕ expresses $\mathscr{W}\})$ [*assumption*],
4. $K_{\mathrm{Ola}}(\{w \mid \vdash_{\langle A,R\rangle} \psi$ at $w\})$ [*since* $\{w \mid \vdash_{\langle A,R\rangle} \psi$ at $w\} = \mathscr{W}$],
5. $K_{\mathrm{Ola}}(\{w \mid \psi$ expresses $\mathscr{W}$ at $w\})$ [*by 1–4 and closure under entailment*].

The fragmentation strategy for this closure-problem case denies 5 by claiming that it must be the case that 1–3 aren't all believed in the same belief state. In other words, the

[12] Another notion of ("implicit") belief is defined as truth in all possible worlds in all belief states (Fagin & Halpern 1987, 59).

[13] This example is introduced in (Schwitzgebel 2010) and discussed in (Elga & Rayo 2021a). For other cases, see, e.g., (Lewis 1986, 31f.), or (Elga & Rayo 2021a; 2021b).

fragmentation strategy says that Ola must be like the disavowed racist and act as though, say, she believes that a_1 expresses $\mathscr{W}$ in one context or for one kind of task, and that a_2 expresses $\mathscr{W}$ in another context, but never both in the same. But why must this be so? Let us assume that Ola's task is to figure out whether ψ expresses $\mathscr{W}$. Ola knows all the axioms and rule of inference of $\langle A, R\rangle$, perhaps she even writes them all down as she begins her task. Ola knows that derivations in $\langle A, R\rangle$ preserve expression of $\mathscr{W}$—assume Ola states that she is looking for a proof of ψ because she knows that ψ expresses $\mathscr{W}$ if it is derivable. Ordinarily, we wouldn't hesitate to say that all of 1–3 are true in this context. Furthermore, there is no intuitive sense in which the axioms and rules of inference aren't accessible to Ola for the purpose of proving ψ—this is in contrast with the puzzle-solver case where, intuitively, one piece of information is available for a simpler task but isn't available for a more complicated task. Here, there is no intuitive sense in which Ola no longer knows the axioms and rules of inference when her task is to prove ψ.

More generally, it seems that for the one task of solving mathematical problems—or, to take another example, for the one task of winning a chess game—thinkers have, or at least can have, all the rules and axioms or board positions in one belief state. Just as playing chess doesn't make one cease to know the rules of the game and the board positions, proving theorems doesn't make one cease to know the axioms and rules of inference. Solving the closure problem by diagnosing every closure-problem case as a case of fragmentation is ad hoc—it isn't backed by the intuitions that motivate the clear cases of fragmentation.[14]

[14]Field (1986, 110f.) and Jago (2013, 1154f.) also raise this kind of problem. Jago (2014, 58f.) and Berto and Jago (2019, 167f.) raise a similar problem: they argue that the fragmentation strategy cannot explain that the difficult part of deduction isn't putting the relevant premises together, but moving from those premises to the conclusion.

5 The computational strategy

We thus need a different approach for solving the closure problem for the metalinguistic strategy. In the following, I propose what I call 'the computational strategy'. Here is the motivating idea. In many cases of failure of logical omniscience, the problem, intuitively, is that the agent is unable to perform or to efficiently perform a certain kind of computation. For instance, if Ola had an algorithm that she could run and that would output a proof of ψ after a few seconds of reflection, we would be inclined to say that Ola knows that ψ is true. Similarly, if Ola had a procedure to search through a set of theorems reliably stored in her memory (for instance, via expert testimony) and that would output ψ, we would be inclined to say that Ola knows that ψ is true. But if Ola has no algorithm for finding a proof of ψ, no method to retrieve ψ from her memory, or if she has a method for finding a proof of ψ but it would take her too many resources—say, months of work—to find it, then we wouldn't say that she knows that ψ is true. To take another example, assume that Pars knows that $1,001$ is composite. It seems plausible that Pars has some kind of algorithm that underlies this knowledge, for instance, an algorithm to quickly factorize numbers; or an algorithm to search through some memorized list of composite numbers that includes $1,001$, or through some memorized list of numbers divisible by 11 that includes $1,001$. Many non-mathematical examples can also be given. Holmes and Watson have the same evidence about a case. Holmes knows who the culprit is but Watson doesn't, because Holmes has an efficient way of computing who the culprit is based on the information that he has, whereas Watson doesn't. Ola and Pars have the same information about the rules of chess and the current board position. Ola knows that Black mates in three but Pars doesn't, because Ola has an efficient way of computing the mate, whereas Pars doesn't.[15]

The idea that lack of computational abilities is the proper diagnosis of many cases of failure of logical omniscience isn't new. Stalnaker himself voices it in numerous places.[16]

[15]Example like these and others are given by Parikh (1987).

[16]See, e.g., (Stalnaker 1991, 436; 1996, 201f.; 1999b, 260–262; 2021, 190–192). However, for Stalnaker compu-

Furthermore, it is the basis of a formal model of knowledge and belief called the 'algorithmic' model, developed by Halpern et al. (1994).[17] In line with the functionalist account of §2, the algorithmic model is motivated by the idea that knowledge and belief are capacities or dispositions to act (Halpern et al. 1994, 256). According to Halpern et al. (1994, 256), the ability to act on the information that ϕ requires the ability to compute ϕ, which should thus be captured in an adequate model of knowledge and belief. Formally, the algorithmic model supplements the standard model of §2: in addition to epistemically accessible worlds, agents have *knowledge algorithms* that take as input a sentence, ϕ, and return either 'Yes', 'No', or '?'.[18] There are different ways of interpreting these outputs. As Halpern and Riccardo Pucella (2011, 222) put it, the algorithm returns 'Yes' if the agent can compute that ϕ is true, 'No' if the agent can compute that ϕ is false, and '?' otherwise. In Halpern et al. (1994, 257f.) and Fagin et al. (1995, 394), an answer of 'Yes' instead means that the agent "can compute that she knows ϕ," where 'know' here is understood in the standard model's sense (i.e., one knows ϕ in this sense just in case ϕ is true in all epistemically accessible worlds), and an output of 'No' means that the agent can compute that she doesn't know ϕ in this sense. In either case, the algorithmic model defines knowledge (or 'algorithmic knowledge') as follows: an agent (algorithmically) knows ϕ just in case the agent's knowledge algorithm outputs 'Yes' on input ϕ.[19]

Motivated by the idea that one's knowledge should be part of one's information, Halpern et al. propose the constraint that knowledge algorithms should be *sound*, i.e., an output of 'Yes' should entail that ϕ is known in the standard model's sense (and

tation is always a kind of defragmentation (which I reject).

[17] Similar approaches include that of Konolige (1986), Parikh (1987), and Aaronson (2013). For comparisons, see (Fagin et al. 1995, 411–413). For comparisons between the algorithmic approach and other approaches, see (Halpern & Pucella 2011).

[18] In (Halpern et al. 1994), knowledge algorithms also take as input a variable representing the agent's local state, but since we are concerned with single-agent and static knowledge and belief throughout the paper, we can set it aside (as Halpern and Pucella (2011) do in their exposition).

[19] For exposition, see also (Fagin et al. 1995, ch. 10).

thus also that ϕ is true), and an output of 'No' should entail that ϕ isn't known in the standard model's sense (Halpern et al. 1994, 259). According to Halpern et al. (1994, 259), algorithmic knowledge without the requirement of soundness can be understood as a model of belief: one believes ϕ just in case one has an algorithm that outputs 'Yes' when queried about ϕ, but ϕ needn't be true or part of one's information. On the algorithmic model, one can make finer-grained distinctions between different kinds or grades of knowledge and belief by considering differences in the algorithms and in the resources used by the different algorithms. But because there are no general constraints on algorithms, (sound) algorithmic knowledge doesn't suffer from problems of logical omniscience. In particular, having an algorithm that outputs 'Yes' given ϕ doesn't entail having an algorithm that outputs 'Yes' given ψ, even if ϕ entails ψ or is equivalent to it.

The algorithmic model looks promising for defenders of the metalinguistic strategy: it doesn't entail closure under entailment, it gives plausible diagnoses of failures of mathematical omniscience in closure-problem cases, and it fits with the functionalist motivations of the possible-worlds account. But it also has some shortcomings. For one, the algorithmic model is under-specified as an account of knowledge and belief. The algorithmic model states that an agent knows ϕ if and only if the agent's knowledge algorithm outputs 'Yes' given ϕ. To turn this into a functionalist account of knowledge, we should specify a realistic interpretation of knowledge algorithms that draws a connection between knowledge algorithms and agents' capacities to act on the basis of relevant information. The algorithmic model is also simplistic in a number of ways. For example, on the algorithmic model, knowledge is always and only manifested in linguistic behavior, which is implausible. Knowledge should be applicable in response to a broader range of inputs (i.e., not only if one is asked to determine the truth-value of some sentence), and it should yield a broader range of manifestations or outputs (i.e., not only 'Yes'). The algorithmic model also ignores the relevance of desires to action: one will only output a sentence's truth-value if one desires to give a correct answer. These shortcomings aren't surprising: the algorithmic model, unlike, for instance, the causal-pragmatic account, is a formal model that isn't in the business of providing a realistic functionalist account of knowledge and belief.

A much more important problem with the algorithmic model given our purposes is that as it stands, it is incompatible with the possible-worlds account of propositions. On the algorithmic model, the algorithms in question operate on sentences and not on propositions. The algorithmic model can thus only be understood as either providing a theory of the truth-conditions of sentences of the form 'S knows that P' (i.e., a theory of knowledge ascription as discussed in §3), or as modeling knowledge of sentences as opposed to (possible-worlds) propositions.[20] This isn't an accidental feature of the algorithmic model. For instance, a desideratum of the algorithmic model is that a knowledge algorithm can output 'Yes' given ϕ but either 'No' or '?' given ψ even when ϕ and ψ are necessarily equivalent (Fagin et al. 1995, 398f.), which works precisely because ϕ and ψ are sentences and not contents (in which case $\phi = \psi$).[21] Furthermore, the algorithmic model is designed to capture an agent's ability to answer questions about sentences—the agent (or her algorithm) is given a sentence and responds 'Yes', 'No', or '?'.[22] The algorithmic model thus in effect captures a linguistic ability, which is why the algorithms operate on sentences. From this perspective, it isn't clear what it would even mean to give the agent (or the algorithm) sets of possible worlds instead of sentences. As I explained in §3, my primary concern in this paper is to give a theory of the nature of mathematical propositions and not a semantics of ascriptions of propositional attitudes. Furthermore, it would be unfortunate to have a disjunctive theory of knowledge and belief where their objects are sometimes sentences, and at other times propositions. As it stands, the algorithmic model thus can't serve our purposes.

[20]For discussion and the related notion of "linguistic" knowledge and belief, see (Parikh 1987; 2008). See also (Pucella 2004, Appendix A) for a related notion of computational knowledge of sentences.

[21]Parikh (1987; 2008) also notes this desideratum for his notion of linguistic knowledge.

[22]For these kind of motivations for the algorithmic model, see, e.g., (Fagin et al. 1995, 392), (Pucella 2004, 4f., 23). Parikh's (1987) notion of linguistic knowledge explicitly captures verbal abilities to answer questions.

In the following, I develop a version of the algorithmic model that is adapted to the possible-worlds account and that also provides a more realistic account of knowledge and belief. I call it the 'computational' account. As I explain, the computational account is a close variant of the causal-pragmatic account of §2 that also improves upon it for reasons independent of closure under entailment. Conjoining the computational account with the metalinguistic strategy will solve the closure problem and yield an attractive, functionalist account of mathematical thought and inquiry. I proceed as follows. I first motivate a first-pass computational account that is restricted to metalinguistic propositions (§5.1). The first-pass account is still simplistic in some respects, but it is instructive, and it captures much of the original motivating idea concerning mathematical knowledge and belief. I then explain how to improve the first-pass account and to generalize it beyond metalinguistic propositions (§5.2).

5.1 A first-pass computational account

I henceforth take on board the causal component of the causal-pragmatic account of §2: knowledge requires information possession, and belief requires information possession under normal conditions. Since knowledge on the standard model is often used interchangeably with information possession,[23] it is natural to further assume that the standard model is a correct model of information possession (under normal conditions). Thus, I assume that the standard model is partially correct about knowledge and belief, i.e., $K_S(P)$ only if $\mathscr{I}_S \subseteq P$, and $B_S(P)$ only if $\mathscr{D}_S \subseteq P$.[24]

The pragmatic component of the causal-pragmatic account encapsulates the idea that knowledge and belief are capacities or dispositions to act. It is this component, I claim, that

[23]Epistemically accessible worlds are standardly taken to encapsulate one's information; see, e.g., (Halpern et al. 1994, 256f.).

[24]This assumption has the same consequence as adopting the soundness requirement on knowledge algorithms discussed in §5.1, viz., (algorithmic) knowledge entails knowledge in the standard model's sense, and therefore also truth.

should be modeled in terms of algorithms. This is a natural idea. It is highly plausible that whenever one is disposed to exhibit some behavior, this is because one has an (internal) algorithm that produces this behavior. Replacing 'S is disposed to act ...' by 'S has an algorithm that outputs some action ...' in clause 2 of the causal-pragmatic account would thus plausibly yield extensionally equivalent accounts. But, as I will explain, taking seriously the idea that knowledge and belief require having certain algorithms involves rejecting clause 2 of the causal-pragmatic account, as well as the standard model that it motivates.

I start by focusing exclusively on metalinguistic propositions. This is because adapting the algorithmic model to the possible-worlds account is easiest in the case of metalinguistic propositions. Since metalinguistic propositions are about sentences, we can simply follow the algorithmic model in taking the algorithms in question to operate on sentences, while departing from the algorithmic model in taking the objects of knowledge and belief to be possible-worlds propositions. So, we can take knowledge and belief of a metalinguistic proposition $\{w \mid \phi \text{ expresses } \mathscr{W} \text{ at } w\}$ to require an algorithm that operates on ϕ. The next step is to specify what exactly these algorithms should do. Plausibly, and in line with the original motivating idea above, an algorithm underlying one's knowledge of a metalinguistic proposition should efficiently compute that the relevant sentence is true when the agent needs to act on this information, for instance, when the agent is asked to determine the truth-value of the sentence. It seems plausible that if Ola knows that ψ is true, then if she is asked to determine whether ψ is true, she should be able to give an affirmative answer. Moreover, the answer she gives shouldn't be a random guess, but be based on some reliable process, such as calculating, proving, or retrieving something from (reliably stored) memory. Knowledge should thus require the algorithm in question to be reliable. Belief shouldn't have the reliability requirement. If Ola believes that ψ is true, then if she is asked to determine whether ψ is true, she will give an affirmative answer, but she may have no proof, computation, or reliable memory backing this answer. The following first-pass accounts are meant to capture these intuitive ideas. Starting with knowledge:

Mathematical Knowledge 1: An agent, S, knows that $\{w \mid \phi \text{ expresses } \mathscr{W} \text{ at } w\}$ iff:

1. S is in a state that carries the information that $\{w \mid \phi \text{ expresses } \mathscr{W} \text{ at } w\}$ (i.e., $\mathscr{I}_S \subseteq \{w \mid \phi \text{ expresses } \mathscr{W} \text{ at } w\}$), and,
2. S has an algorithm that reliably outputs 'True' if S is asked to determine ϕ's truth-value, using at most resources R.

Mathematical Knowledge 1 yields a notion of mathematical knowledge that isn't closed under entailment. Even if one knows propositions (including perhaps metalinguistic ones) that entail that some ϕ expresses $\mathscr{W}$, and one thus possesses the information that ϕ expresses $\mathscr{W}$ (since information possession is as before closed under entailment), it doesn't follow that we have an algorithm that reliably outputs ϕ. For instance, Ola may have an algorithm that reliably lists the axioms and rules of inference, but none that lists theorems, or none that lists theorems within some reasonable bound R on resources. The closure problem thus doesn't arise if one adopts Mathematical Knowledge 1.

Clause 2 of this account is its key algorithmic component. Let us examine it closer. First, one might worry that mathematical knowledge requires actually having performed some calculation or found a proof, and thus find clause 2 to be too weak. One could for this reason modify clause 2 to require also that the algorithm in question has terminated, for instance, by adding "and the algorithm has output 'True'."[25] However, on a dispositional understanding of knowledge in line with functionalism, it is more natural to take mathematical knowledge to require having the ability to output some metalinguistic information within certain bounds: as Stalnaker (1991; 1999b) discusses, on a functionalist conception, we would like to make sense of the idea that knowledge is "available" for the

[25]This would be closer to Parikh's (1987) account: on his definition, S *linguistically-knows* ϕ if and only if S has said 'Yes' to the question whether ϕ is true, and whenever she says 'Yes' to ϕ she knows ϕ in the sense of the standard model, and whenever she says 'No' to ϕ she knows $\neg\phi$ in that sense (Parikh 1987, 4).

purpose of determining action, or that it can "easily be used to determine an output" (Stalnaker 1999b, 261)—such as answering 'True'—without having actually output it.

Second, consider constraints one might put on the algorithms. In algorithmic models, the algorithms in question are only assumed to be effectively computable and to terminate (Pucella 2004, 23). Here, one can make these assumptions as well—after all, it is highly plausible that cognition in general can be explained in terms of effectively computable algorithms.[26] Otherwise, and in line with algorithmic models, we should have a permissive understanding of algorithms so as to account for different ways in which one might have mathematical knowledge. For instance, an agent might have an algorithm that reliably outputs 'True' if asked to determine ϕ's truth-value because she has an algorithm that finds either a proof or a refutation of all sentences of some particular type (such as basic numerical equations) and outputs 'True' if the sentence has a proof, and ϕ is of that type. Alternatively, the agent can have an algorithm that searches through some set of sentences that are stored in her memory via a reliable process (such as, the process of accepting the testimony of experts mathematicians) and outputs 'True' if the input sentence is in that set, and ϕ is in that set.

Third, consider the bound on resources, R. I mean 'resource' here to be understood broadly and to include, for instance, time, attention, or energy. If we understand knowledge to be a certain kind of capacity to act, then it is plausible that there are bounds on the resources one can use to have knowledge. As before, one will intuitively not count as knowing that $38,629$ is prime if one only has a highly inefficient algorithm to check for primality; for instance, if it would take one months of work to figure this out. If one holds that ascriptions of knowledge are in general context-dependent, one can add here that in different situations, there are going to be different bounds that are relevant for attributing mathematical knowledge.[27]

[26]These assumptions aren't essential, however; e.g., Pucella (2004, 188) considers using relativized Turing machines.

[27]Stalnaker (1991, 437) suggests this possibility.

More generally, if knowledge is a certain kind of capacity, then it makes sense that knowledge should be evaluable along a number of different dimensions and degrees, and not be a binary matter of either having or lacking information. Mathematical Knowledge 1 yields a notion of mathematical knowledge that lends itself to such fine-grained evaluations, and thus has another advantage over both the standard model and the fragmentation account. For instance, if Ola is very proficient at mental arithmetic, her knowledge of $197 \times 49 = 9,653$ differs in some ways from Pars's, who only (reliably) memorized a number of multiplications, of which $197 \times 49 = 9,653$ happens to be one. In particular, Ola's knowledge unlike Pars's displays mathematical abilities. Perhaps Ola is also faster at answering such questions, and her knowledge is thus more useful for certain kinds of tasks. These kinds of differences between Ola's knowledge and Pars's knowledge will be captured by the differences in both the algorithms and the resources that they use.

Finally, consider the reliability requirement. This requirement yields a broadly reliabilist account of knowledge, which is fitting given the functionalist outlook I am adopting here. At the same time, it isn't essential to the account, for one can modify clause 2 to make different restrictions on the processes that output 'True' given ϕ. In general, the reliability of an algorithm should be measured with respect to a class of sentences of some type. For instance, S might know that '90 is composite' is true, because she has an algorithm which, when asked to determine the truth-value of sentences of the form '$d_0 ... d_n$ is composite' where $d_n \in \{0, 2, 4, 6, 8\}$, outputs 'True'. But she might have no algorithm that reliably outputs 'True' whenever she is asked to determine the truth-value of 'n is composite', for an arbitrary n.[28]

Along the same lines as Mathematical Knowledge 1, we can outline a first-pass account of mathematical belief:

Mathematical Belief 1: An agent, S, believes that $\{w \mid \phi \text{ expresses } \mathscr{W} \text{ at } w\}$ iff:

[28]Pucella (2004, 47–60) and Halpern and Pucella (2005) formulate a relevant weaker notion of soundness in terms of randomized algorithms that is meant to formally represent a probabilistic reliability requirement on knowledge algorithms.

1. S is in a state that would carry the information that $\{w \mid \phi \text{ expresses } \mathscr{W} \text{ at } w\}$ in normal conditions (i.e., $\mathscr{D}_S \subseteq \{w \mid \phi \text{ expresses } \mathscr{W} \text{ at } w\}$), and,
2. S has an algorithm that outputs 'True' if S is asked to determine ϕ's truth-value, using at most resources R.

On this account, believing that $\{w \mid \phi \text{ expresses } \mathscr{W} \text{ at } w\}$ can be understood to require having a disposition to assent to ϕ. This is in line with the general accounts of belief of Hartry Field (1978) or Jeff Speaks (2006), only restricted to metalinguistic propositions. For the same reasons as for Mathematical Knowledge 1, the notion of belief that Mathematical Belief 1 yields isn't closed under entailment, and thus the closure problem doesn't arise if one adopts Mathematical Belief 1.

These first-pass accounts of mathematical knowledge and belief are promising additions to the metalinguistic strategy. They provide a plausible, functionalist account of mathematical knowledge and belief. They enable defenders of the metalinguistic strategy to avoid the closure problem. And they capture the specifically computational aspect of failures of logical omniscience in closure-problem cases. But the first-pass accounts still need some improvements. Just like the algorithmic model, they are simplistic in their account of the inputs and outputs of knowledge and belief, and in ignoring the relevance of desires to action. They are also restricted to metalinguistic propositions. In the following, I outline a proposal that builds upon the first-pass accounts to address these issues.

5.2 Improving and generalizing the first-pass accounts

Taking cues from the causal-pragmatic account, a natural idea is to construe the algorithms in the first-pass accounts more generally as outputting desire-satisfying behavior instead of just 'True'. However, we can't simply replace clause 2 of the causal-pragmatic account for knowledge with, say:

2′. S has an algorithm that reliably produces desire-satisfying behavior in worlds in which $\{w \mid \phi \text{ expresses } \mathscr{W} \text{ at } w\}$, together with S's other beliefs, are true, and using at most resources R.

Such an account would entail closure under entailment for the same reasons as the causal-pragmatic account does, as seen in §2.

To avoid closure under entailment, we should consider why exactly the first-pass accounts avoid it. The first-pass accounts put constraints on the kinds of algorithms one must have in order to know a given mathematical proposition. Knowledge and belief that $\{w \mid \phi \text{ expresses } \mathscr{W} \text{ at } w\}$ require having a specifically ϕ-involving algorithm: one that can answer questions about the truth-value of ϕ. The first-pass accounts thus in effect put constraints on the kinds of tasks one must be able to achieve (if one so desires) in worlds in which some mathematical proposition (together with one's other beliefs) is true. These accounts differ in this respect from the causal-pragmatic account, where there is no restriction on the relevant actions or tasks associated with the knowledge or belief that P. This, as I will explain, is the key virtue of the first-pass accounts that we ought to preserve in both improving and generalizing them.

Accordingly, here is an improved proposal for mathematical knowledge:

Mathematical Knowledge 2: An agent, S, knows that $\{w \mid \phi \text{ expresses } \mathscr{W} \text{ at } w\}$ iff:

1. S is in a state that carries the information that $\{w \mid \phi \text{ expresses } \mathscr{W} \text{ at } w\}$ (i.e., $\mathscr{I}_S \subseteq \{w \mid \phi \text{ expresses } \mathscr{W} \text{ at } w\}$), and,
2. S has an algorithm that reliably produces desire-satisfying behavior in worlds in which $\{w \mid \phi \text{ expresses } \mathscr{W} \text{ at } w\}$, together with S's other beliefs, are true, with respect to ϕ-related tasks, and using at most resources R.

For the same reasons as those seen in §5.1, Mathematical Knowledge 2 yields an account of knowledge that isn't closed under entailment. For instance, Ola may have algorithms that produce desire-satisfying behaviors with respect to axiom-related tasks, but none that produce desire-satisfying behaviors with respect to theorem-related tasks. The closure problem thus doesn't arise if one adopts Mathematical Knowledge 2. A computational account of mathematical belief can be developed similarly, changing clause 1 to indication under normal conditions, and changing clause 2 to 'would produce desire-satisfying behavior'. In the following I focus on knowledge, for the discussion carries over exactly to belief.

Mathematical Knowledge 2 doesn't have the limitations of Mathematical Knowledge 1. The relevance of desires to action is taken into account, and the relevant actions are construed more broadly. In particular, one can take ϕ-related tasks to go beyond answering questions about the truth-value of ϕ. For instance, it is plausible that mathematical knowledge of a theorem also requires being able to use it in further (mathematical) arguments, for instance, as an assumption in further proofs. Using ϕ in mathematical arguments should thus also count as a ϕ-related task. It may be that which and how many tasks count as "ϕ-related" will vary depending on context. I will set aside such context-dependence here, but note that the account can accommodate it either via the fragmentation strategy (by endorsing that agents can have different information sets in different context), or by adding that what specific ϕ-related tasks count as relevant for knowledge that $\{w \mid \phi \text{ expresses } \mathscr{W} \text{ at } w\}$ depends on context.

Having improved the first-pass account of mathematical knowledge, the next step is to generalize it. Once again, I propose to do this by putting constraints on the kinds of tasks one must be able to achieve in order to know that P. This modification is an independently plausible amendment of the causal-pragmatic account. Recall that on the causal-pragmatic account, the functional role of the belief that P is being disposed to act in ways that would tend to satisfy one's desires in worlds in which P, *together with one's other beliefs*, are true, i.e., in worlds in which *all* of one's beliefs are true. Assume that S believes that P and that S believes that Q, where $P \neq Q$. On the causal-pragmatic account, the functional roles of both beliefs turn out to be the same: believing that P requires acting in ways that satisfy S's desires in a world in which P and Q and the rest of S's beliefs are true, and believing that Q requires the same. But it is highly counterintuitive that every belief in an agent's belief state has the same functional profile. For instance, my belief that there is dessert in the fridge surely has a different functional role than my belief that someone was elected

President: the former explains my dispositions to perform actions of one type—such as eating the dessert or saying that there is dessert in the fridge—while the latter explains my dispositions to perform actions of an entirely different type. Eric Schwitzgebel (2002, 251) captures this idea using the notion of a belief's *dispositional stereotype*, i.e., dispositions that people would ordinarily regard as characteristic of having that belief. On this view, believing that there is dessert in the fridge has a dispositional stereotype which includes the disposition to say that there is dessert in the fridge (in appropriate circumstances) and the disposition to go to the fridge (if one wants dessert), whereas the dispositional stereotype of the belief that someone was elected President is entirely distinct. Most broadly dispositionalist accounts of belief capture these differences in the functional roles of an agent's different beliefs.[29] It is a problem for the causal-pragmatic account that it doesn't.

Putting constraints on the kinds of tasks one must be able to achieve in order to know or believe that P is one way to remedy this problem. For instance, one can maintain that knowledge that there is dessert in the fridge requires having algorithms that produce desire-satisfying behaviors specifically with respect to dessert- and fridge-related tasks, such as consuming dessert; answering questions such as 'Is there dessert in the fridge?'; or emptying the fridge. And these dessert- and fridge-related tasks will be different from the tasks relevant to the belief that someone was elected President. There are thus independent reasons to think that the functional role of belief given by the pragmatic component of the causal-pragmatic account should have an implicit constraint on relevant tasks, just like first- and second-pass computational accounts of mathematical knowledge and belief. Accordingly, here is a generalized proposal:

Computational Knowledge: An agent, S, knows that P iff:

1. S is in a state that carries the information that P (i.e., $\mathcal{I}_S \subseteq P$), and,
2. S has an algorithm that reliably produces desire-satisfying behavior in worlds

[29]See, e.g., the discussion of dispositionalism in (Schwitzgebel 2019).

in which P, together with S's other beliefs, are true, with respect to tasks τ, and using at most resources R.

The key difference between the computational and the causal-pragmatic accounts is the restriction in clause 2 to the relevant tasks τ. We arrived at this amendment by taking seriously the idea that knowledge requires having certain algorithms or abilities to produce action, and from the computational account of mathematical knowledge where these algorithms plausibly operate on sentences or perform other specifically sentence-related tasks.

One difference between Computational Knowledge and Mathematical Knowledge 2 is that it is less straightforward to specify which tasks are relevant for which proposition. In the case of a metalinguistic proposition $\{w \mid \phi \text{ expresses } \mathscr{W} \text{ at } w\}$, the relevant tasks are ϕ-related tasks. But which are the relevant tasks τ in the case of a general proposition P? Because my focus here is on the case of mathematics, I leave a more extensive investigation of this question for future work. But it is plausible that the dispositional stereotypes we associate with different contents will at least help determine which tasks τ are relevant for which contents P. Using the example above, the dispositional stereotype for the belief that there is dessert in the fridge will determine that the tasks relevant for knowledge that there is dessert in the fridge are tasks related specifically to the dessert in the fridge, such as eating it; giving it to someone; or throwing it in the trash. And these are distinct from the tasks relevant for knowledge that someone was elected President, because we associate a different dispositional stereotype with that belief.

Once again, because of the restriction on relevant tasks, the notion of knowledge defined by Computational Knowledge isn't closed under entailment. If P entails Q, and one knows that P, one thereby possesses the information that Q, but one needn't know that Q, because the tasks associated with knowledge that P might (and likely will) differ from the tasks associated with knowledge that Q whenever $P \neq Q$. For instance, let B be the set of worlds in which the axioms are true and the rules are truth-preserving (i.e., worlds in which "the basics" are true), and T the set of worlds in which some complicated theorem ψ is true. Let us assume that B entails T (i.e., $B \subseteq T$). The tasks associated with knowledge that B are tasks related to axioms and rules of inference, such as listing them, or perhaps using them in basic derivations. But having the ability to produce desire-satisfying behavior with respect to these tasks doesn't entail having the ability to produce desire-satisfying behavior with respect to ψ-related tasks, such as giving the right answer to the question of whether ψ is true.

Given that Computational Knowledge isn't restricted to metalinguistic propositions (none of which are necessarily equivalent), it yields an account of knowledge that is closed under necessary equivalence. For instance, given that $B = B \cap T$ in our example, one knows B if and only if one knows $B \cap T$, even though one doesn't thereby know T. Accordingly, knowledge doesn't distribute over intersections, unlike on the causal-pragmatic account, as seen in §2. Of course, given that we have adopted the metalinguistic strategy, closure under necessary equivalence doesn't entail mathematical omniscience. It also doesn't entail logical omniscience, or knowledge of all necessary truths or equivalences, since, as Stalnaker (1976, 86–91; 1984, 72–75) explains, the metalinguistic strategy applies beyond mathematical propositions to all (seemingly) necessary and necessarily equivalent propositions: in each case, we interpret the objects of belief or ignorance to be relations between certain sentences and what they express.

The fact that knowledge doesn't distribute over intersections given Computational Knowledge does, however, have consequences for one's theory of propositional attitude ascriptions. In particular, one will have reason to reject that sentences of the form 'S knows that A and B' are true if and only if the referent of 'S' knows the proposition that is the intersection of the propositions expressed by 'A' and 'B' (for simplicity, let me call this set '$A \cap B$'). For example, if Ola knows the basics, i.e., $K_{\text{Ola}}(B)$, and $B = B \cap T$, then Ola knows the proposition $B \cap T$, i.e., $K_{\text{Ola}}(B \cap T)$. But we shouldn't be able to conclude from this that 'Ola knows the basics and the theorem' is true. As I explained in §3, I am not concerned with developing a theory of propositional attitude ascriptions in this paper. However, I do think that proponents of the metalinguistic and computational strategies have clear motivations for denying a theory of attitude ascriptions on which 'S knows that A and B' is true if and only if $K_S(A \cap B)$. In particular, if knowledge that P is a capacity to satisfy one's desires with respect to certain tasks τ, then we should expect knowledge ascriptions to be sensitive to this task-relativity. It seems clear, for instance, that an ascription such as 'Ola knows the basics and the theorem' conveys that Ola has both basics-related abilities and theorem-related abilities. But knowledge of a given proposition, whether or not it is an intersection of sets, will require abilities that are related to only one salient set of tasks τ. This gives a clear motivation for denying the simple theory of conjunctive attitude ascriptions. The theory of propositional attitude ascriptions on the computational strategy will thus be complex and probably also context-dependent, which, as seen in §3, is in line with accepting the metalinguistic strategy in the first place.

6 Conclusion

My concern in this paper has been the closure problem for the metalinguistic solution to the problem of mathematical omniscience: if we assume the standard possible-worlds model of knowledge and belief under which they are closed under entailment, people who know some basic facts about meaning and mathematics end up knowing too much mathematics. To solve this problem, I proposed to replace the standard possible-worlds model of knowledge and belief with a computational account. This computational account is a close variant of Stalnaker's functionalist account, and it states that knowing mathematical propositions requires having algorithms that achieve desire-satisfaction with respect to sentence-related tasks. The computational account doesn't entail closure under entailment. As I argued, it gives plausible diagnoses of mathematical achievement and ignorance especially in closure-problem cases, and it can be generalized to all propositions.

I focused on the case of mathematics in part because the problem of mathematical omniscience is often taken to be the most serious problem for the possible-worlds account, and in part because my background motivation is to develop an account of mathematical content fitting with a Carnapian perspective on the nature of mathematical truth. However, the results of this paper apply beyond mathematics to all necessary propositions, and, as we saw in §5.2, they can be even further generalized. At the most general level, the account we reached says that ignorance is always in part informational and in part computational. In a priori necessary domains, the information in question is metalinguistic, and the computations are sentence-involving. Moreover, one can have all the relevant metalinguistic information and still be ignorant of some a priori truth. In such cases, as Stalnaker notes:

> The information which one receives when one learns about deductive relationships does not seem to come from outside of oneself at all. It seems to be information which, in some sense, one has had all along. What one does is to transform it into a usable form [...]. (Stalnaker 1984, 86)

Stalnaker goes on to claim that "transforming information into a usable form" is a matter of putting different belief states or fragments together. I proposed instead that "transforming information into a usable form" is a matter of having certain computational abilities—specifically, algorithms that produce desire-satisfying behavior relative to certain tasks. This general view of content, thought, and inquiry is to be developed further, but thinking about the problem case of mathematics already brought us a long way forward.[30]

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