Logic and Logical Philosophy
Volume 32 (2023), 53–80
DOI: 10.12775/LLP.2022.011

**Shawn Standefer**

# Varieties of Relevant S5

**Abstract.** In classically based modal logic, there are three common conceptions of necessity, the universal conception, the equivalence relation conception, and the axiomatic conception. They provide distinct presentations of the modal logic S5, all of which coincide in the basic modal language. We explore these different conceptions in the context of the relevant logic R, demonstrating where they come apart. This reveals that there are many options for being an S5-ish extension of R. It further reveals a divide between the universal conception of necessity on the one hand, and the axiomatic conception on the other: The latter is consistent with motivations for relevant logics while the former is not. For the committed relevant logician, necessity cannot be the truth in all possible worlds.

**Keywords**: relevant modal logic; S5; universal necessity; conceptions of necessity

## 1. Introduction

According to a familiar idea, the *universal* conception of necessity, necessity is truth in all possible worlds, giving the logic that has become known as S5. The universal conception takes all worlds to be equally possible, which is a special case of partitioning the set of worlds into equivalence classes, in fact it is the case in which there is only a single equivalence class. This *equivalence* conception of necessity permits situations in which the space of possibilities is partitioned into equivalence classes, and necessity at a particular world is truth in all worlds in its equivalence class. Both of these conceptions are equally well axiomatized by a standard Hilbert-style axiomatization.[1] This *axiomatic* conception

[1] This is not to say that a Hilbert-style axiomatization is the only way to approach S5 proof-theoretically. There are other options, such as [Braüner, 2000; Indrzejczak,

*Received* January 30, 2021. *Accepted* February 16, 2022. *Published online* March 8, 2022

generates the same logic as the other two. These conceptions provide equivalent presentations of the logic S5 and they offer three presentations of the logic: two model-theoretic and one proof-theoretic [see Blackburn et al., 2002; Garson, 2018].

- Kripke models with a universal accessibility relation
- Kripke models with an accessibility relation that is an equivalence relation
- Axiomatic form

The situation is different when one changes the setting from classically based modal logic to relevant modal logic, that is, when the base logic is changed from classical logic to the relevant logic R.[2] Although the point is a general one, in this paper, I will focus on showing that this equivalence does not hold when R is used as the base logic. This means that there are multiple options for being an S5-ish extension of R.[3] It further reveals a divide between the universal conception of necessity on the one hand, and the axiomatic conception on the other: The latter is consistent with motivations for relevant logics while the former is not.[4]

1996, 1998; Mints, 1992; Ohnishi and Matsumoto, 1957; Poggiolesi, 2008; Prawitz, 1965; Restall, 2007] for example. See [Bednarska and Indrzejczak, 2015] for a survey of some options, and see [Indrzejczak, 2010] or [Poggiolesi, 2011] for general overviews. The Hilbert-style axiomatization is better suited to the aims of this paper.

It should be noted that there is still some subtlety in identifying the proof-theoretic conceptions of S5, which would be better grouped into equivalence classes inducing the same consequence relation. For more on this latter point, see [Porte, 1981]. I would like to thank Lloyd Humberstone for the reference and for discussion of this point.

[2] For more on relevant logics, see [Bimbó, 2007; Dunn and Restall, 2002] for overviews. For more in-depth treatments, see [Anderson and Belnap, 1975; Anderson et al., 1992; Brady, 2003; Mares, 2004; Read, 1988; Restall, 2000; Routley et al., 1982].

[3] Much work on relevant modal logics has focused on S4-ish logics. [Mares and Meyer, 1993; Meyer, 1966; Meyer and Mares, 1993; Routley and Meyer, 1972] all investigate S4-ish extensions of R. As noted by Anderson and Belnap [1975, ch. 1], the logic E can define a kind of logical necessity, with $\Box A$ defined as $(A \to A) \to A$, the logic of which is similar to that of S4. S5-ish extensions, by comparison, have received little attention. See [Mares and Standefer, 2017] for an investigation of different modal analogues of E, including an S5-ish one.

[4] Parks and Byrd [1989] discuss universal necessity, which they call "Leibnizian necessity", noting that there was a "feeling that the Leibnizian view of necessity is incompatible with key tenets of relevance logic" (p. 180). The results below go some way towards substantiating that sentiment. Thanks to Andrew Tedder for directing me towards this article.

For the committed relevant logician, necessity cannot be truth in all possible worlds.

## 2. Background

In this section, I will present the standard (ternary relational) model theory for relevant logics and provide an axiomatization of the main logic to be considered. The basic relevant language, $\mathfrak{L}$, is built from a countable set of atoms $\mathsf{At}$ and the connectives and constant in the set $\{\rightarrow, \wedge, \vee, {\sim}, \mathsf{t}\}$.

DEFINITION 2.1. A ternary relational frame $F$ is a quadruple $\langle K, N, R, {}^*\rangle$, where $K \neq \emptyset$, $N \subseteq K$, $R \subseteq K^3$, ${}^*\colon K \longrightarrow K$, where

(B1) $a \leq b =_{\mathrm{Df}} \exists x \in N\ Rxab$,
(B2) $\leq$ is a partial order,
(B3) $a^{**} = a$,
(B4) $a \leq b$ only if $b^* \leq a^*$, and
(B5) if $d \leq a$, $e \leq b$, $c \leq f$, and $Rabc$, then $Rdef$.

Define $Rabcd =_{\mathrm{Df}} \exists x \in K(Rabx \wedge Rxcd)$ and $Ra(bc)d =_{\mathrm{Df}} \exists x \in K(Raxd \wedge Rbcx)$. An $\mathsf{R}$-frame $\langle K, N, R, {}^*\rangle$ is a ternary relational frame obeying the following conditions

(F1) $Rabc \Rightarrow Rac^*b^*$
(F2) $Rabc \Rightarrow Rbac$
(F3) $Rabcd \Rightarrow Rb(ac)d$
(F4) $Rabc \Rightarrow Rabbc$

It is a consequence of (B2), in particular the reflexivity portion, together with (B1) that the set of normal points, $N$, is non-empty.[5]

DEFINITION 2.2 (Model). A model $M$ is a pair $\langle F, V\rangle$ consisting of an $\mathsf{R}$-frame $F$, $\langle K, N, R, {}^*\rangle$, and a valuation function $V\colon \mathsf{At} \longrightarrow \wp K$ such that if $a \in V(p)$ and $a \leq b$, then $b \in V(p)$. Such a model is said to be built on $F$.

The valuation function is extended to a verification relation on the whole language as follows.

- $a \Vdash p$ iff $a \in V(p)$

[5] The normal points, sometimes called 'regular' or 'logical', are used to define validity below.

- $a \Vdash \mathsf{t}$ iff $a \in N$
- $a \Vdash {\sim}B$ iff $a^* \nVdash B$
- $a \Vdash B \wedge C$ iff $a \Vdash B$ and $a \Vdash C$
- $a \Vdash B \vee C$ iff $a \Vdash B$ or $a \Vdash C$
- $a \Vdash B \to C$ iff for all $b, c \in K$, if $Rabc$ and $b \Vdash B$, then $c \Vdash C$

This provides us with the resources needed to define counterexamples and validity.

DEFINITION 2.3 (Counterexample, holding, validity). A model $M$ is a counterexample to a formula $A$ iff for some $a \in N$, $a \nVdash A$. The point $a$ is said to be a counterexample point.

A formula $A$ holds in model $M$ iff $M$ is not a counterexample to $A$.

A formula $A$ is valid in a frame $F$ iff no model built on $F$ is a counterexample to $A$.

A formula $A$ is valid in a class of frames $\mathfrak{C}$ iff $A$ is valid on each frame $F \in \mathfrak{C}$.

Write $\models_{\mathfrak{C}} A$ when $A$ is valid in the class of frames $\mathfrak{C}$ and write $\models_{\mathsf{R}} A$ when $A$ is valid in the class of $\mathsf{R}$-frames.

It will be useful to have a notation for specifying the points at which a formula holds.

DEFINITION 2.4 (Propositions, Truth sets). In a frame $F$, a set $X \subseteq K$ is a *proposition* on $F$ iff if $a \in X$ and $a \leq b$ then $b \in X$.

In a model $M$, the *truth set* of $A$, $|A|_M$, is $\{a \in K : a \Vdash A\}$.

Where no confusion will arise, the subscript on the truth sets will be omitted.

There are two standard lemmas that I will state without proof.

LEMMA 2.1 (Heredity Lemma). *For all ternary relational models $M$ and all formulas $A$, if $a \Vdash A$ and $a \leq b$, then $b \Vdash A$.*

It follows from this lemma that all truth sets are propositions. The Heredity Lemma has an important consequence.

LEMMA 2.2 (Verification Lemma). *Let $M$ be a ternary relational model. For all formulas $A$ and $B$, $A \to B$ holds in $M$ iff for all $a \in K$, if $a \Vdash A$, then $a \Vdash B$.*

Appeal to the Verification Lemma will generally be left implicit.

The set of formulas valid in the class of $\mathsf{R}$-frames is the logic $\mathsf{R}$, an axiomatization of which we now give. The logic is the set of formulas inductively defined taking the following axioms and closing under the following rules.

(R1) $A \to A$
(R2) $(A \wedge B) \to A$, $(A \wedge B) \to B$
(R3) $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
(R4) $A \to (A \vee B)$, $B \to (A \vee B)$
(R5) $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
(R6) $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$
(R7) $(A \to {\sim}B) \to (B \to {\sim}A)$
(R8) ${\sim}{\sim}A \to A$
(R9) $(A \to (A \to B)) \to (A \to B)$
(R10) $A \to ((A \to B) \to B)$
(R11) $(A \to B) \to ((B \to C) \to (A \to C))$
(R12) $\mathsf{t}$
(R13) $A, A \to B \Rightarrow B$
(R14) $A, B \Rightarrow A \wedge B$
(R15) $A \Rightarrow \mathsf{t} \to A$

The notion of *proof* will be defined in the usual way, and $\vdash_{\mathsf{R}} A$ will be used to mean that $A$ has a proof from the axioms and rules of $\mathsf{R}$. Such a formula is a *theorem*. As is well known, this axiom system is sound and complete with respect to validity in the class of $\mathsf{R}$-frames.

THEOREM 2.3 (Soundness and Completeness). *For all formulas $A$, $A$ is a theorem of* $\mathsf{R}$ *iff $A$ is valid in the class of* $\mathsf{R}$*-frames.*

There are a few facts about the logic $\mathsf{R}$ and $\mathsf{R}$-frames that will be useful below. I will begin with the fact that $\mathsf{R}$ enjoys the variable sharing property.[6]

THEOREM 2.4 (Variable sharing). *For all formulas $A$ and $B$ that do not contain* $\mathsf{t}$, *if $\vdash_{\mathsf{R}} A \to B$, then $A$ and $B$ share a propositional atom.*

PROOF. See [Anderson and Belnap, 1975, pp. 252–254] or [Standefer, 2020] for details. $\dashv$

The variable sharing property is regarded as a necessary (but not sufficient) condition on a logic being a relevant logic [see Mares, 2020]. The variable sharing property captures part of a fundamental motivating

[6] See [Logan, 2021, 2022; Robles and Méndez, 2011, 2012] for more on the variable sharing property.

intuition for relevant logics, that there must be a substantive connection between the antecedent and consequent of a valid conditional.

Next, I will list some features of R-frames that will be useful below.

LEMMA 2.5. *All* R*-frames obey the following frame conditions.*

(i) $\forall x \in K$, $Rxxx$
(ii) $\forall x \in K$, $Rxx^*x$
(iii) $\forall x \in N$, $x^* \leq x$

PROOF. Let $\langle K, N, R, ^* \rangle$ be an R-frame.

For (i), we have $a \leq a$, for all $a \in K$, so for some $b \in N$, $Rbaa$, which, incidentally, shows that $N \neq \emptyset$. By condition (F4), $Rbaaa$, so there is some $c$ such that $Rbac$ and $Rcaa$. The former is $a \leq c$, so by condition (B5) on ternary relational frames, $Raaa$, as desired.

For (ii), we have $Ra^*a^*a^*$, by (i). Using (F1) and (B3) , we have $Ra^*aa$. Then (F2) yields the desired $Raa^*a$.

For (iii), we use (ii) on points in $N$, as for $a \in N$, $Raa^*a$ just is $a^* \leq a$. ⊣

From (iii) in the preceding lemma, it is the case that $\models_{\mathsf{R}} A \vee {\sim}A$. That is enough of the basic background. I will now turn to the modal extensions of R that will occupy the remainder of the paper.

## 3. Modal logics

The introduction noted that there were three different ways to present classically based S5. While in the setting of classical logic, it makes sense to consider a single connective, $\Box$, for all three concepts, for the present relevant-logical setting, that will be confusing. Rather than add a single connective, I will consider three extensions of the basic relevant language $\mathfrak{L}$ with singulary connectives, $\Box, \mathbb{E}$, and $\mathbb{U}$, denoted $\mathfrak{L}^{\Box}$, $\mathfrak{L}^{\mathbb{E}}$, and $\mathfrak{L}^{\mathbb{U}}$, respectively. I will use translations between the languages that change only the singulary modals, e.g. if $\tau$ goes from $\mathfrak{L}^{\mathbb{E}}$ to $\mathfrak{L}^{\mathbb{U}}$, $\tau(\mathbb{E}A) = \mathbb{U}\tau(A)$. These translations will be suppressed, unless explicitly writing them will aid clarity.

I will begin with the axiomatic logic RS5, which is an S5-ish extension of R.[7] Only $\Box$ will be taken as primitive, with $\Diamond$ defined as ${\sim}\Box{\sim}$. To

[7] There is another logic that plausibly would count as an S5-ish extension of R. This logic adds an additional axiom, which is discussed in §5.

define RS5, we add to R the following axioms and rules, adjusting the definition of proof and theorem accordingly.

(Agg) $(\Box A \wedge \Box B) \to \Box(A \wedge B)$
(T) $\Box A \to A$
(4) $\Box A \to \Box\Box A$
(B) $A \to \Box\Diamond A$
(K) $\Box(A \to B) \to (\Box A \to \Box B)$
(Nec) $A \Rightarrow \Box A$

Axiom (5), $\Diamond A \to \Box\Diamond A$, follows from the other axioms of RS5, as it does in the classically based modal logic with the above axioms and rules, taking the conditional in them to be the classical material conditional. Rule (Mono), $A \to B \Rightarrow \Box A \to \Box B$, can be obtained from (K) and (Nec). It is mentioned here because it will come up later and because in the setting of relevant modal logics defined via frames with a binary accessibility relation, (Mono) and (Agg) hold generally, while (K) and (Nec) are optional extras that must be enforced via frame conditions.

To provide models for this modal logic, we add a binary accessibility relation, $S$, to R-frames.

DEFINITION 3.1. An RS5-frame is a quintuple $\langle K, N, R, S, ^*\rangle$, where $\langle K, N, R, ^*\rangle$ an R-frame and $S \subseteq K^2$, obeying the following conditions:

(M1) If $Sbc$ and $a \leq b$, then $Sac$.
(M2) If $a \in N$ and $Sab$, then $b \in N$.
(M3) $Saa$.
(M4) If $Sab$ and $Sbc$, then $Sac$.
(M5) If $Sab$, then $Sb^*a^*$.
(M6) If $\exists z(Rabz \wedge Szc)$, then $\exists x\exists y(Sax \wedge Sby \wedge Rxyc)$.

The definitions of model, counterexample, holding, and validity are all adapted in a straightforward way. The verification condition for $\Box$ is the following.

- $a \Vdash \Box B$ iff for all $b$ such that $Sab$, $b \Vdash B$

The Heredity Lemma and the Verification Lemma carry over to these models. Fuhrmann [1990] showed that RS5 is sound and complete with respect to the class of RS5-frames.[8]

[8] To be slightly more precise, Fuhrmann did not show these results for the logic with t, although his results still hold after its addition. See [Seki, 2003] for results

Next, we turn to $\mathfrak{L}^{\mathbb{E}}$ and the equivalence modality. The logic will be defined via frames.

Definition 3.2. An equivalence relation $\approx$ on a set $X$ is a binary relation on $X$ that is reflexive, transitive, and symmetric.

An equivalence frame is a quintuple $\langle K, N, R, \approx, {}^*\rangle$, where $\langle K, N, R, {}^*\rangle$ is an R-frame and $\approx$ is an equivalence relation on $K$, obeying the condition that if $a \leq b$ and $b \approx c$, then $a \approx c$.

For a given equivalence frame $F$, the equivalence class of the point $a$, $[a]$, is $\{b \in K : a \approx b\}$.

The additional condition, closure of equivalence classes under $\leq$, is in force to ensure the Heredity Lemma remains provable. The definitions of model, counterexample, holding, and validity are adapted in the straightforward way.

The verification condition for $\mathbb{E}$ in an equivalence model is the following.

- $a \Vdash \mathbb{E}B$ iff $\forall b \approx a, b \Vdash B$

To ensure the Heredity criterion is satisfied, we need to require that the $\approx$-classes closed under $\leq$, which is to say that if $a \in [c]$ and $a \leq b$, then $b \in [c]$.

The final logic to be considered here is the logic of the universal modality, $\mathbb{U}$. For this logic, we will use the subclass of equivalence frames where $\approx$ is the universal relation, i.e. $\approx\, = K \times K$. Such frames will be called *universal frames*. Universal frames can also be viewed simply as R-frames, since each R-frame can be uniquely extended to a universal frame and each universal frame has a unique underlying R-frame, obtained by omitting the universal relation. The verification condition for $\mathbb{U}$ is the same as that for $\mathbb{E}$, but since the equivalence relation is trivial, it will be omitted, yieding the following verification condition:

- $a \Vdash \mathbb{U}B$ iff for all $b \in K$, $b \Vdash B$.

---

for an extended language.

Apart from the first condition, which is required for any modal expansion of ternary relational frame with a binary accessibility relation, all the conditions correspond to axioms and rules in the sense that the class of frames obeying that condition validates the axiom and the canonical model for the logic with the axiom obeys that condition. Condition (M2) goes with (`Nec`), (M3) with (`T`), (M4) with (`4`), (M5) with (`B`), and (M6) with (`K`). The reader might have expected the condition paired with (`B`) to be symmetry rather than (M5). This point will be discussed more in the final section.

The Heredity Lemma and Verification Lemma still hold when $\mathbb{U}$ is in the language.

For ease of reference, I will now define the following classes of frames:

- $\mathfrak{S5}$ is the class of $\mathsf{RS5}$-frames.
- $\mathfrak{E}$ is the class of equivalence frames.
- $\mathfrak{U}$ is the class of universal frames.

These classes gives us three *prima facie* distinct logics, the sets of formulas valid in each of the three classes of frames just mentioned:

- $\mathsf{RS5}$ is the set of $\mathfrak{L}^{\Box}$-formulas valid in $\mathfrak{S5}$.
- $\mathsf{Eq}$ is the set of $\mathfrak{L}^{\mathbb{E}}$-formulas valid in $\mathfrak{E}$.
- $\mathsf{U}$ is the set of $\mathfrak{L}^{\mathbb{U}}$-formulas valid in $\mathfrak{U}$.

As just stated, these are all distinct logics since they are not even in the same language. Using the translation mentioned at the start of this section, I will talk as though these logics were formulated in the same language, so the question of their sameness and difference can be settled in a non-trivial way.

## 4. Universal modality

In this section, I will demonstrate the relation between the logic $\mathsf{U}$ and the logics $\mathsf{Eq}$ and $\mathsf{RS5}$. As mentioned above, the universal relation can be viewed as an equivalence relation, so $\mathfrak{U}$ has a simple embedding in $\mathfrak{E}$. Therefore, $\mathsf{Eq} \subseteq \mathsf{U}$. Below I will show that this inclusion is proper.

It will be useful to have a bit of terminology for the discussion to come. The terminology will be introduced for arbitrary models, to apply across the different notions of model introduced.

DEFINITION 4.1 (Ubiquity). Let $M$ be a model based on an $\mathsf{R}$-frame $\langle K, N, R, {}^*\rangle$.

A formula $A$ is *ubiquitously true* in $M$ iff $|A|_M = K$.

A formula $A$ is *ubiquitously false* in $M$ iff $|A|_M = \emptyset$.

A formula $A$ is *ubiquitous* in $M$ iff $A$ is either ubiquitously true or ubiquitously false.

The terminology of ubiquity is based on a remark by Fuhrmann in an explanation of a phenomenon in relevant modal logics.[9] In the un-

[9] Fuhrmann [1990] attributes the terminology to a suggestion by Humberstone. For further discussion of ubiquity and relevant logics, see [Standefer, 2022].

derlying relevant logic $\mathsf{R}$, and in fact in $\mathsf{RS5}$, no formulas are ubiquitous in all models, a fact to be proven shortly. Before that, I will introduce some more terminology.

DEFINITION 4.2 (Defining Falsum). Say that a language *defines Falsum* with respect to a class of frames iff there is a formula $A$ in the language such that $A$ is ubiquitously false in all models built on those frames. A witnessing formula $A$ is said to *define Falsum*.

If a language defines Falsum with respect to a class of frames and a witnessing formula does not contain $\mathsf{t}$, then the logic obtained from that class will violate the variable sharing property, as demonstrated by the following instance of the general idea.

LEMMA 4.1. *$\mathfrak{L}^{\mathbb{U}}$ defines Falsum with respect to $\mathfrak{U}$ and a witnessing formula is $\mathbb{U}p \wedge {\sim}\mathbb{U}p$.*

PROOF. Let $M$ be a model built on an $\mathsf{R}$-frame and let $a \in K$ be arbitrary. Suppose $a \Vdash \mathbb{U}p \wedge {\sim}\mathbb{U}p$. Then for all $b \in K$, $b \Vdash p$, so $a^* \Vdash \mathbb{U}p$. Therefore, $a \nVdash {\sim}\mathbb{U}p$, so $a \nVdash \mathbb{U}p \wedge {\sim}\mathbb{U}p$. So, $a \nVdash \mathbb{U}p \wedge {\sim}\mathbb{U}p$. ⊣

COROLLARY 4.2. *$(\mathbb{U}p \wedge {\sim}\mathbb{U}p) \to q$ is a validity of $\mathsf{U}$.*

PROOF. Suppose $(\mathbb{U}p \wedge {\sim}\mathbb{U}p) \to q$ is not valid, so there is counterexample. This requires that there be a point $a$ in a model such that $a \Vdash \mathbb{U}p \wedge {\sim}\mathbb{U}p$, but $|\mathbb{U}p \wedge {\sim}\mathbb{U}p| = \emptyset$, as that formula defines Falsum. Therefore, there is no counterexample. ⊣

In view of Corollary 4.2, the logic $\mathsf{U}$ does not have the variable sharing property. It should be noted that the involvement of negation and conjunction is not required to demonstrate the failure of variable sharing. One could define the notion of defining Verum, rather than Falsum, as a formula being ubiquitously true in all models on all frames in a class. Then the formula $\mathbb{U}p \to \mathbb{U}p$ defines Verum and $q \to (\mathbb{U}p \to \mathbb{U}p)$ is a theorem of $\mathsf{U}$, providing another witness to the failure of the variable sharing property.

There is a general feature exhibited the formula defining Falsum above. It will be worth making it explicit and pulling it out into a lemma.

LEMMA 4.3. *Suppose that $M$ is a model and $A$ is a formula such that for all $a \in K$, $a \in |A|_M$ iff $a^* \in |A|_M$. Then $A \wedge {\sim}A$ is ubiquitously false in $M$.*

PROOF. Suppose $M$ and $A$ are as in the statement of the lemma. Suppose that $A \wedge {\sim}A$ is not ubiquitously false in $M$, so for some $a \in K$, $a \Vdash A \wedge {\sim}A$. Then $a \Vdash A$. By assumption, $a^* \Vdash A$. But then $a \nVdash {\sim}A$, which contradicts the assumption. $\dashv$

If a formula's truth set has the displayed feature, containing a point if and only if it contains that point's star, then it provides a recipe for generating formulas that are ubiquitously false in a model. If the property can be guaranteed for all models, then it is possible to define Falsum.

The variable sharing property is one of the features separating $\mathsf{RS5}$ from $\mathsf{U}$.

THEOREM 4.4. $\mathsf{RS5}$ *has the variable sharing property.*

PROOF. To show this, I will use the matrix $M_0$ from [Anderson and Belnap, 1975, pp. 252–254], which was used to prove that $\mathsf{R}$ has the variable sharing property. On this matrix, interpret $\Box$ as the identity operator so that $v(\Box B) = v(B)$. All the axioms of $\mathsf{RS5}$ are designated on all such assignments and all rules preserve being designated. For any formula $A$ in $\mathfrak{L}^{\Box}$, define $A^-$ to be $A$ with all occurrences of $\Box$ removed. It can be shown by induction on the complexity of $A$ that $v(A) = v(A^-)$, for all formulas $A$. Further, as can be shown by induction on the length of a proof, if $B$ is a theorem of $\mathsf{RS5}$, then $B^-$ is a theorem of $\mathsf{R}$.

Suppose that $B \to C$ is a theorem of $\mathsf{RS5}$ that violates the variable sharing property. Then $(B \to C)^-$ is a formula in $\mathfrak{L}$. By the 'further' observation above, $(B \to C)^-$ is a theorem of $\mathsf{R}$ that violates the variable sharing property, which contradicts theorem 2.4. Therefore there is some assignment $v$ such that $v((B \to C)^-)$ is not designated, so $v(B \to C)$ is not designated. Therefore, $B \to C$ is not a theorem of $\mathsf{RS5}$, contradicting the assumption. $\dashv$

The details of this proof suffice to establish that $\mathfrak{L}^{\Box}$ cannot define Falsum with respect to $\mathfrak{S}5$, which is more than is needed for the desired corollary.

COROLLARY 4.5. *The formula* $(\Box p \wedge {\sim}\Box p) \to q$ *is not a thoerem of* $\mathsf{RS5}$.

COROLLARY 4.6. *It is not the case that* $\mathsf{U} \subseteq \mathsf{RS5}$, *that is,* $\mathsf{U} \nsubseteq \mathsf{RS5}$.

The second corollary is perhaps not surprising. Let us turn to the converse inclusion.

The rule (Nec) is not sound for $\mathfrak{U}$, which is to say that $\mathsf{U}$ is not closed under that rule.

Lemma 4.7. *The formula $p \to p$ is a theorem of $\mathsf{U}$ but $\mathbb{U}(p \to p)$ is not.*

Proof. To see that $p \to p$ is a theorem, note that it is valid on all $\mathsf{R}$-frames, which is to say valid on $\mathfrak{U}$. It is well known that there are models on $\mathsf{R}$-frames with (non-normal) points at which $p \to p$ fails, which must be the case in order to provide a counterexample to $q \to (p \to p)$, which is not a theorem of $\mathsf{R}$. Take such a model, and take any $a \in N$. Then, $a \not\Vdash \mathbb{U}(p \to p)$, so $\mathbb{U}(p \to p)$ is not a theorem of $\mathsf{U}$. ⊣

Corollary 4.8. *The logic $\mathsf{U}$ is not closed under* (Nec).

Corollary 4.9. *It is not the case that $\mathsf{RS5} \subseteq \mathsf{U}$, that is, $\mathsf{RS5} \not\subseteq \mathsf{U}$.*

That $\mathsf{U}$ and $\mathsf{RS5}$ are incomparable as logics is, I think, somewhat surprising. I will return to the significance of this result in the final section.

The proof of lemma 4.7 can be used to show a similar failure of (Nec) with respect to $\mathsf{Eq}$. The problem, in both cases, arises from the fact that (Nec) requires a special frame condition for its soundness, namely condition (M2) in the definition of an $\mathsf{RS5}$-frame. Both classes $\mathfrak{U}$ and $\mathfrak{E}$ contain frames that violate that condition. Imposing that condition on $\mathfrak{U}$ means taking the proper sublcass of frames where $K = N$. Such a move would be disastrous from the point of view of the relevant logician for the following reason.

Lemma 4.10. *Let $\mathfrak{U}^N$ be the subclass of $\mathfrak{U}$ such that each frame in $\mathfrak{U}^N$ obeys the condition $K = N$. Then $\sim(p \to p) \to q$, $q \to (p \to p)$, and $p \to (q \to p)$ are valid in $\mathfrak{U}^N$.*

Proof. Since $p \to p$ is valid in $\mathfrak{U}$, $p \to p$ holds at each point $a \in N$ in a given model. Since all models built on frames in $\mathfrak{U}^N$ have only normal points, $p \to p$ holds at every point in such a model. So, there is no point $b^*$ such that $b^* \not\Vdash p \to p$, so $\sim(p \to p)$ is false throughout all such models. Therefore, $\sim(p \to p) \to q$ has no counterexamples. Similar reasoning establishes that $q \to (p \to p)$ is valid as well.

Finally, since $q \to (p \to p)$ is valid, $p \to (q \to p)$ is valid. The reason is that every $\mathfrak{U}$ frame is an $\mathsf{R}$-frame, (R10) is valid, which together with (R11) and (R13), suffices for the dreaded conclusion.[10] ⊣

[10] Thanks to an anonymous referee for asking whether this last part was provable.

| $R$ | $0$ | $a$ | $b$ | $^*$ | $\approx$ |
|---|---|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ | $0$ | $0$ |
| $a$ | $a$ | $ab$ | $0ab$ | $b$ | $a$ |
| $b$ | $b$ | $0ab$ | $ab$ | $a$ | $b$ |

| | $\Vdash$ |
|---|---|
| $0$ | $p$ |
| $a$ | $p$ |
| $b$ | $\emptyset$ |

Table 1. Countermodel on equivalence frame

Attempting to regain (Nec) for U by imposing the corresponding frame condition results in disaster, since it means that the extension of R with the new connective $\mathbb{U}$ is no longer a conservative extension of the base logic. In fact, it upsets the variable sharing property enjoyed by the base logic. Let us now turn to the relation between U and Eq.

As mentioned earlier, Eq is contained in U. This inclusion is proper.

LEMMA 4.11. *The formula* $(\mathbb{E}p \wedge {\sim}\mathbb{E}p) \to q$ *is not a theorem of* Eq.

PROOF. To see that $(\mathbb{E}p \wedge {\sim}\mathbb{E}p) \to q$ is not valid, we present a simple countermodel, provided in table 1. For this countermodel, let $K = \{0, a, b\}$ and let $N = \{0\}$.[11] In the $R$ portion of the table, entries with multiple points indicate multiple $R$ facts. For example, the entry in the $a$-row and $a$-column means that $Raaa$ and $Raab$. The frame in table 1 is an equivalence frame.[12] To show that $(\mathbb{E}p \wedge {\sim}\mathbb{E}p) \to q$ is not valid, it suffices to find a point such that the antecedent holds there but the consequent does not. For that, $a$ will work. First, note that $a \Vdash \mathbb{E}p$, since $[a] = \{a\}$. Next, $a \Vdash {\sim}\mathbb{E}p$ iff $a^* = b \nVdash \mathbb{E}p$. It is the case that $b \nVdash p$, so, as $[b] = \{b\}$, $b \nVdash \mathbb{E}p$. Thus, $a \Vdash \mathbb{E}p \wedge {\sim}\mathbb{E}p$. As $q$ holds nowhere, it follows that this is a countermodel to $(\mathbb{E}p \wedge {\sim}\mathbb{E}p) \to q$. $\dashv$

COROLLARY 4.12. Eq $\subsetneq$ U

The logic of the universal modality, over R, does not coincide with that of either of the other two formulations of S5, over R. We can say a bit more about the logic, although the question of Completeness, or indeed even axiomatizability, will be left open.

THEOREM 4.13. (1) *Axioms* (Agg), (T), (4), (B), (K) *are theorems of* U.

[11] Thanks to Greg Restall for suggesting the underlying R-frame.

[12] Note that the table for $\Vdash$ only displays the relevant atoms verified at a point. The row with $b$ and $\emptyset$ says that $b \nVdash p$; it does not say that $b$ verifies no formulas.

(2) *Axiom* ($\mathtt{U}$), $\mathbb{U}(\mathbb{U}A \to A)$, *is a theorem of* $\mathsf{U}$.[13]
(3) $\mathsf{U}$ *is closed under* ($\mathtt{Mono}$).

Proof. For (1), it is obvious that ($\mathtt{Agg}$), ($\mathtt{T}$), and ($\mathtt{4}$) are theorems.

For ($\mathtt{B}$), suppose $a \Vdash A$, for an arbitrary point $a$ in an arbitrary model based on a frame in $\mathfrak{U}$. Suppose $a \nVdash \mathbb{U}{\sim}\mathbb{U}{\sim}A$. Then there is a point $b$ such that $b \nVdash {\sim}\mathbb{U}{\sim}A$. So, $b^* \Vdash \mathbb{U}{\sim}A$. So for all points $c$, $c \Vdash {\sim}A$, so $c^* \nVdash A$. Let $c = a^*$, so, as $a^{**} = a$, $a \nVdash A$, which is a contradiction. Therefore, ($\mathtt{B}$) does not have a counterexample in $\mathfrak{U}$.

For ($\mathtt{K}$), suppose $a \Vdash \mathbb{U}(A \to B)$, for an arbitrary point $a$ in an arbitrary model based on a frame in $\mathfrak{U}$. Suppose $a \nVdash \mathbb{U}A \to \mathbb{U}B$, so there are $b, c \in K$ such that $Rabc$, $b \Vdash \Box A$, but $c \nVdash \Box B$. From the latter, there is a point $d$ such that $d \nVdash B$. From $b \Vdash \Box A$, it follows that $d \Vdash A$. As $Redd$, for some $e \in N$ by (B2), and $e \Vdash A \to B$. But then $d \Vdash B$, which is a contradiction. Therefore, ($\mathtt{K}$) does not have a counterexample in $\mathfrak{U}$.

For (2), let $a$ be an arbitrary point in an arbitrary model based on a frame in $\mathfrak{U}$. Suppose $b, c \in K$ are such that $Rabc$ and $b \Vdash \mathbb{U}A$. From the verifiation condition for $\mathbb{U}$, it follows that $c \Vdash A$, so $a \Vdash \mathbb{U}A \to A$. As $a$ is arbitrary, $a \Vdash \mathbb{U}(\mathbb{U}A \to A)$ as well.

For (3), assume that $A \to B$ is a theorem and suppose, for some model, $a \Vdash \Box A$. Then for every $b \in K$, $b \Vdash A$, whence $b \Vdash B$ from the initial assumption. Thus, $a \Vdash \Box B$, which suffices for the validity of $\Box A \to \Box B$ with respect to $\mathfrak{U}$. $\dashv$

There is more to explore concerning $\mathsf{U}$, but let us instead turn to $\mathsf{Eq}$.

## 5. Equivalence relations

The logic $\mathsf{Eq}$ differs from both $\mathsf{U}$ and $\mathsf{RS5}$. The relationship to $\mathsf{U}$ was established in the preceding section, so we will focus on the connection to $\mathsf{RS5}$.

As noted in the preceding section, $\mathsf{Eq}$ is not closed under the rule ($\mathtt{Nec}$). We can consider what happens if we enforce the condition that $a \in N$ and $a \approx b$ only if $b \in N$. Let $\mathfrak{E}^{(\mathrm{M2})}$ be the subclass of $\mathfrak{E}$ such that every frame in $\mathfrak{E}^{(\mathrm{M2})}$ obeys the condition $a \in N$ and $a \approx b$ only if

[13] The axiom ($\mathtt{U}$) should not be mistaken as characteristic of the logic $\mathsf{U}$, which mistake may be suggested by its name. The name of the axiom is fairly standard in work on modal logic [see, e.g. Humberstone, 2016, p. 33]

$b \in N$. As before, there are some consequences that violate relevantist principles.

LEMMA 5.1. *Let $F$ be an equivalence frame in $\mathfrak{E}$ and suppose $a \in N$. Then $[a^*] = [a]$.*

PROOF. Let $F$ be an equivalence frame in $\mathfrak{E}$ and suppose $a \in N$. By lemma 2.5, $a^* \leq a$. Therefore, $a^* \approx a$, so $a^* \in [a]$. So for every $b \in [a]$, $a^* \approx b$. By the fact that $\approx$ is an equivalence relation, $[a^*] = [a]$ ⊣

COROLLARY 5.2. *$\mathfrak{L}^{\mathbb{E}}$ defines Falsum over $\mathfrak{E}^{(\mathrm{M2})}$ with $(p \to p) \wedge {\sim}(p \to p) \wedge \mathsf{t}$ as a witness.*

PROOF. Suppose $a \in |(p \to p) \wedge {\sim}(p \to p) \wedge \mathsf{t}|_M$, for some model $M$ on an equivalence frame in $\mathfrak{E}^{(\mathrm{M2})}$. Then, $a \in N$, since $a \Vdash \mathsf{t}$. But then, $a \Vdash p \to p$, as $p \to p$ is a theorem of $\mathsf{R}$. From the lemma and condition (M2), it follows that $a^* \Vdash p \to p$, so $a \nVdash {\sim}(p \to p)$, contradicting the assumption. ⊣

COROLLARY 5.3. *$((p \to p) \wedge {\sim}(p \to p) \wedge \mathsf{t}) \to q$ has no counterexamples in $\mathfrak{E}^{(\mathrm{M2})}$.*

Thus, over the restricted class of frames, the addition of $\mathbb{E}$ is not conservative over the base logic, as the displayed formula is not a theorem of $\mathsf{R}$.

While the frame conditions for $\mathsf{R}$-frames have not featured much in the paper so far, they are essential to arguments of this section. The trouble stems from the condition that, for normal points $a$, $a^* \leq a$. This condition holds in frames for some weaker relevant logics, indeed any logic validating excluded middle, $A \vee {\sim}A$.[14] Let us turn to the more general class of frames, $\mathfrak{E}$, to see what further trouble this condition causes.

Lemma 5.1 tells us that, over $\mathfrak{E}$, $\approx$ is not, in general, the identity relation, i.e. $a \approx b$ iff $a = b$. In frames where $\approx$ is the identity relation, then for all normal points $a$, $a = a^*$, which limits their ability to provide counterexamples to contradictory claims. Indeed, the formula displayed in the preceding corollary is valid in the class of frames where $\approx$ is the identity relation on the normal points.

[14] See [Routley et al., 1982], especially chapter 4, for discussion of some of the virtues and charms of weaker relevant logics and the details of their frames. Excluded middle is the source of a surprising amount of difficulty for relevant logics, See [Slaney, 1987] for some discussion, as well as [Standefer, 2021, 202x].

| $\to$ | $0$ | $\frac{1}{2}$ | $1$ | $\sim$ | $\Box$ |
|---|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ | $0$ |
| $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $1$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $1$ | $0$ | $0$ | $1$ | $0$ | $1$ |

Table 2. Matrix for algebraic countermodel

Falsum is definable in the language with $\mathbb{E}$ and $\mathtt{t}$, with the latter's significant role suggested by the earlier discussion.

LEMMA 5.4. *$\mathfrak{L}^{\mathbb{E}}$ defines Falsum over $\mathfrak{E}$ and $\mathtt{t} \wedge \mathbb{E}p \wedge {\sim}\mathbb{E}p$ is a witnessing formula.*

PROOF. Let $M$ be an equivalence model. Suppose for some point $a$, $a \Vdash \mathtt{t} \wedge \mathbb{E}p \wedge {\sim}\mathbb{E}p$. The first conjunct implies $a \in N$, which by lemma 5.1 means that $[a^*] = [a]$. Reasoning much as in the proof of lemma 4.3, $a^* \Vdash \mathbb{E}p$, so $a \nVdash {\sim}\mathbb{E}p$. Therefore, $\mathtt{t} \wedge \mathbb{E}p \wedge {\sim}\mathbb{E}p$ is verified at no point in $M$, which was arbitrary. Therefore, $\mathtt{t} \wedge \mathbb{E}p \wedge {\sim}\mathbb{E}p$ defines Falsum over $\mathfrak{E}$. ⊣

COROLLARY 5.5. *$(\mathtt{t} \wedge \mathbb{E}p \wedge {\sim}\mathbb{E}p) \to q$ is a theorem of* $\mathsf{Eq}$.

That leads us to another difference with $\mathsf{RS5}$. The formula displayed in the corollary is not a theorem of $\mathsf{RS5}$.

LEMMA 5.6. *$(\mathtt{t} \wedge \Box p \wedge {\sim}\Box p) \to q$ is not a theorem of* $\mathsf{RS5}$.

PROOF. A matrix argument similar to the one in the proof of theorem 4.4 shows that this formula is not a theorem of $\mathsf{RS5}$. The following algebra supplies a countermodel. Let $V = \{0, \frac{1}{2}, 1\}$ with $0 < \frac{1}{2} < 1$ and $0$ the only undesignated element. Valuations assign elements of $V$ to atoms, with $v(\mathtt{t}) = \frac{1}{2}$. Conjunction and disjunction are interpreted as minimum and maximum on the ordering, with the other connectives interpreted via table 2. Every axiom of $\mathsf{RS5}$ is designated on every valuation, and all rules of $\mathsf{RS5}$ preserve the property of being designated on all valuations.[15] For the counterexample, choose a $v$ such that $v(p) = \frac{1}{2}$ and $v(q) = 0$, so $v(\mathtt{t} \wedge \Box p \wedge {\sim}\Box p) = \frac{1}{2}$. Then $v((\mathtt{t} \wedge \Box p \wedge {\sim}\Box p) \to q) = 0$, so it is not a theorem of $\mathsf{RS5}$. ⊣

The fact that $\mathsf{RS5}$ has the variable sharing property does not suffice, on its own, to show that $(\mathtt{t} \wedge \mathbb{E}p \wedge {\sim}\mathbb{E}p) \to q$ is not a theorem of $\mathsf{RS5}$. The

[15] This algebra is a simple extension of $\mathsf{RM3}$, for which $\mathsf{R}$ was shown sound by Meyer. See [Anderson and Belnap, 1975, p. 470] for more on this algebra.

| $R$ | $0$ | $a$ | $b$ | $*$ | $\approx$ |
|---|---|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ | $0$ | $0, b$ |
| $a$ | $a$ | $ab$ | $0ab$ | $b$ | $a$ |
| $b$ | $b$ | $0ab$ | $ab$ | $a$ | $0, b$ |

| | $\Vdash$ |
|---|---|
| $0$ | $\emptyset$ |
| $a$ | $p, q$ |
| $b$ | $q$ |

Table 3. Countermodel to (K) on equivalence frame

| $R$ | $0$ | $a$ | $b$ | $*$ | $\approx$ |
|---|---|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ | $0$ | $0, b$ |
| $a$ | $a$ | $ab$ | $0ab$ | $b$ | $a$ |
| $b$ | $b$ | $0ab$ | $ab$ | $a$ | $0, b$ |

| | $\Vdash$ |
|---|---|
| $0$ | $p$ |
| $a$ | $p$ |
| $b$ | $\emptyset$ |

Table 4. Countermodel to (B) on equivalence frame

Ackermann constant excludes that formula from the force of the variable sharing property.[16] The formula violates the spirit, if not the letter, of the variable sharing property, which idea is partially vindicated by the countermodel. The lemma establishes the final relation left open among our logics. Thus, we obtain $\mathsf{Eq} \not\subseteq \mathsf{RS5}$.

There are some further divergences between $\mathsf{Eq}$ and $\mathsf{RS5}$ that are worth bringing out. The former lacks some validities provable in the latter, including some used as axioms in the basis for $\mathsf{RS5}$ of §3, as the following observation shows.

Lemma 5.7. *Axiom* (K), $\mathbb{E}(A \to B) \to (\mathbb{E}A \to \mathbb{E}B)$, *is not valid in* $\mathfrak{E}$.

Proof. The instance $\mathbb{E}(p \to q) \to (\mathbb{E}p \to \mathbb{E}q)$ is not valid. The countermodel, presented in table 3, is obtained from a few small tweaks to the countermodel from table 1. Suppose that $a \Vdash \mathbb{E}(p \to q)$. We have $Raab$, and $a \Vdash \mathbb{E}p$, as $[a] = \{a\}$. As $0 \approx b$ and $0 \nVdash q$, $b \nVdash \mathbb{E}q$, which suffices to show that $a \nVdash \mathbb{E}p \to \mathbb{E}q$. ⊣

Lemma 5.8. *Axiom* (B), $A \to \mathbb{E}{\sim}\mathbb{E}{\sim}A$, *is not valid in* $\mathfrak{E}$.

Proof. The instance $p \to \mathbb{E}{\sim}\mathbb{E}{\sim}p$ is not valid. The countermodel is obtained by altering the valuation from the previous countermodel, and it is displayed in table 4. Since $a^* = b$ and $b \nVdash p$, $a \Vdash {\sim}p$. It follows that $a \Vdash \mathbb{E}{\sim}p$, so $b \nVdash {\sim}\mathbb{E}{\sim}p$. Since $b \approx 0$, $0 \nVdash \mathbb{E}{\sim}\mathbb{E}{\sim}p$. By assumption, $0 \Vdash p$, so this is a countermodel to the $p \to \mathbb{E}{\sim}\mathbb{E}{\sim}p$. ⊣

[16] For further discussion of the variable sharing property in the presence of propositional constants, see [Yang, 2013].

| $R$ | $0$ | $a$ | $b$ | $*$ | $\approx$ |
|---|---|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ | $0$ | $0, b$ |
| $a$ | $a$ | $ab$ | $0ab$ | $b$ | $a$ |
| $b$ | $b$ | $0ab$ | $ab$ | $a$ | $0, b$ |

| | $\Vdash$ |
|---|---|
| $0$ | $q$ |
| $a$ | $\emptyset$ |
| $b$ | $p$ |

Table 5. Countermodel to (MC) on equivalence frame

There is another axiom to consider, modal confinement.[17]

(MC) $\Box(A \vee B) \to (\Box A \vee \Diamond B)$

(MC) is a theorem of classically based S5, taking the conditional to be the material one, but it is not a theorem of RS5.[18] The addition of (MC) enables a straightforward embedding of classically based modal logics into relevant modal logics [see Mares and Meyer, 1993; Meyer and Mares, 1993]. As one would expect, there is an additional frame condition that needs to be imposed on RS5-frames to validate the axiom, which condition is

(M7) $Sab \Rightarrow \exists x(Sax \wedge Sa^*x^* \wedge x \leq b)$ [see Mares, 1992, 1994; Mares and Meyer, 1993; Meyer and Mares, 1993].

(MC) is not a theorem of Eq but it is a theorem of U.

LEMMA 5.9. *Axiom* (MC), $\mathbb{E}(A \vee B) \to (\mathbb{E}A \vee {\sim}\mathbb{E}{\sim}B)$, *is not valid in* $\mathfrak{E}$.

PROOF. The instance $\mathbb{E}(p \vee q) \to (\mathbb{E}p \vee {\sim}\mathbb{E}{\sim}q)$ has a countermodel. It is obtained from adjusting the valuation from table 4 and it is displayed in table 5. In this model, $b \Vdash \mathbb{E}(p \vee q)$, as $[b] = \{0, b\}$ and exactly one disjunct is true at each point in $[b]$. We have $b \nVdash \mathbb{E}p$, as $0 \nVdash p$ and $b \approx 0$. As $b \nVdash q$ and $a^* = b$, $a \Vdash {\sim}q$. Since $[a] = \{a\}$, $a \Vdash \mathbb{E}{\sim}q$, so $b \nVdash {\sim}\mathbb{E}{\sim}q$. This suffices to show that $b$ is a counterexample point for the instance of (MC). $\dashv$

While (MC) is not a theorem of Eq, it is valid when we strengthen the force of the necessity operator to be universal.

LEMMA 5.10. *Axiom* (MC), $\mathbb{U}(A \vee B) \to (\mathbb{U}A \vee {\sim}\mathbb{U}{\sim}B)$, *is valid in* $\mathfrak{U}$.

---

[17] This axiom, and a related principle, are discussed by Dunn [1995] and by Bimbó and Dunn [2008, §10.3]. Restall [2000, 265ff.] calls them the "Dunn conditions". I would like to thank an anonymous referee for suggesting the first two references.

[18] Slaney's program MaGIC generates a six element countermodel.

Proof. In universal frames, $\approx$ is the universal relation, so letting $x$ be $b$, the condition holds. This is sufficient to validate (MC). ⊣

That is enough about what is *not* a theorem of Eq. I will summarize which modal axioms of RS5 *are* theorems of Eq. The proof of the following lemma is straightforward and so will be omitted.

Lemma 5.11. *Axioms* (Agg), (T), *and* (4) *are validities of* Eq *and the logic is closed under rule* (Mono).

As with U, the question of Completeness will be left open.[19] Before turning to the concluding discussion, I will discuss the role of frame conditions in Eq. The definition of Falsum over $\mathfrak{E}$ offered in this section depended on the frame conditions for R-frames for its validity. Since Falsum is definable, RS5 and Eq are incomparable. It seems plausible that if the base logic is weakened to a logic L whose frames do not have to obey the problematic frame condition $a^* \leq a$, for $a \in N$, the L-analog of Eq will be contained in the L-analog of S5. That question, however, will be left open here. As the countermodels offered to (B), (K), (MC), and (Nec) were based on R-frames, they will remain available as countermodels in the transition to a weaker base logic.

## 6. Conclusion

While the three conceptions of S5 coincide over classical logic, from the vantage point of relevant logics, we can see that they conflate distinct ideas. This is similar to the phenomenon in which distinctions in a concept can be drawn when one moves to a weaker logic, or, to put it in terms more amenable to proponents of the weaker logics, one stops

[19] I will note a particular difficulty, raised in a different context by Mares [1994], with proving Completeness via the usual canonical model construction. The closure conditions on equivalence classes ensures that one can show that the models obey the condition

$$b \Vdash \Box A \wedge a \leq b \Rightarrow a \Vdash \Box A.$$

A consequence of this condition is that in the canonical frame, the heredity relation cannot be the subset relation, which is perhaps the standard way to define it. This is not to say that no alternative definition will work, only that the standard methods will not do. An alternative, such as that adopted by Mares [1992], might work, or alternative frame conditions may be need, such as in [Mares, 1994].

conflating distinct concepts.[20] This happens, for example, with additive and multiplicative conjunction in the move from classical logic to $\mathsf{R}$. In the present case, it is the different senses of necessity that are being distinguished, and with them, the different possibilities for the meaning of the phrase "$\mathsf{S5}$ extension of $\mathsf{R}$".[21]

In a sense, that the different formulations of $\mathsf{S5}$ come apart over a relevant base logic such as $\mathsf{R}$ should not be too surprising, since even in classically based modal logic, the universal modality and equivalence relation conception of necessity come apart in the presence of an actuality operator [see Crossley and Humberstone, 1977, p. 20]. In the present setting, the difference appears without the addition of actuality to the language.[22] Perhaps more interesting is the failure of the axiomatic form of the logic to coincide with either of the other formulations. In the relevant setting, the De Morgan negation, used to formulate possibility, $\Diamond$, as $\sim\Box\sim$, in ($\mathtt{B}$), is weaker than the boolean negation of classical logic.[23] This shows up in the frame condition (M5) corresponding to ($\mathtt{B}$), which fails to be symmetry for $S$, securing only $Sab \Rightarrow Sb^*a^*$. In some cases the condition will yield symmetry, namely when $a^* = a$ and $b^* = b$, but not in general. This means that the frame conditions fall short of forcing $S$ to be an equivalence relation on $K$.

In $\mathsf{RS5}$-frames, the binary relation $S$ is not forced to be an equivalence relation. In equivalence frames, on the other hand, the equivalence relations are not forced to coordinate in the appropriate ways with the ternary relation and star operator. These two features of the frames can shift a point of evaluation between equivalence classes, in much the same way that in classically based modal logics, the actuality operator can move one between equivalence classes. This shifting issue is somewhat ameliorated in the universal frames at the cost of introducing violations of variable sharing.

[20] For more discussion of this phenomenon, see [Humberstone, 2005]. I thank Lloyd Humberstone for discussion of these issues.

[21] The number of options increases if ($\mathtt{MC}$) is included in the mix.

[22] The relevant logician can add an actuality operator to the language, although that idea will not be pursued here. For details on actuality in relevant logics, see [Standefer, 2020].

[23] Bimbó and Dunn [2008, §1.3], especially p. 35ff., discusses the role of boolean negation in linking necessity and possibility. I would like to thank an anonymous referee for this reference.

The closure condition on equivalence classes in equivalence frames forces an additional amount of symmetry, but that symmetry does not suffice for the validity of (B). Adding the frame condition (M5) to equivalence frames would suffice for the validity of (B), and it would further validate (MC). That frame condition has the further consequence that $\Diamond$ would have a verification condition in line with that of $\Diamond$ in classically based modal logics:[24]

- $a \Vdash \Diamond A$ iff $\exists x \in K(Sax \wedge x \Vdash A)$.[25]

Imposing condition (M5) on equivalence frames would, then, make $\mathbb{E}$, and its defined dual, closer in many ways to the operators of classically based S5. It is not clear that adding condition (M6), the frame condition for (K), has such large effects on the resulting logic. Adding frame conditions to the equivalence frames, however, does undermine the otherwise neat characterization of the logic in terms of equivalence relations. As (Nec) would remain unavailable to anyone who wants to maintain the spirit of relevant logics, there is a question of what the rationale would be for strengthening Eq in that way. The rationale for such a strengthening will be left for future work.

To conclude, I will return to the fact that U and RS5 are incomparable as logics. The incomparability shows that necessity, for the relevant logician, is a delicate matter. It reveals that the relevant logician should reject the universal conception of necessity, as that contains the seeds of (disastrous) irrelevance. Something can be true, even necessarily so, even true by the lights of logic, without it being implied by everything else. Such a fine-grained notion of implication is at the heart of the relevant logic enterprise. Further, for the paraconsistent logician attracted to relevant logics, contradictions should not imply everything, and contradictions among modal claims should be no different. The universal conception of necessity is, in at least these respects, too permissive. Rejecting the universal conception of necessity raises a serious question for the relevant logician: What other options are there? Answering this question requires getting clear on what about the universal conception

[24] Such logics are "sufficiently classical", in the terminology of [Ferenz, 2022].

[25] Note that if $a \approx b$, then by (M5), $b^* \approx a^*$. By the symmetry of $\approx$, $a^* \approx b^*$. As shown by Mares [1994], the condition that $Sab \Rightarrow Sa^*b^*$, which holds replacing the prefixed $S$ with an infix $\approx$, suffices for the displayed verification condition to hold. This argument also shows that in universal models, $\Diamond$ obeys the classical verification condition.

of necessity is attractive. Are we primarily interested in a *model theory* with a particular sort of simple verification condition or are we primarily interested in the resulting *logic*?

If the primary interest is in a certain sort of model theory, then options seem to be limited.[26] There is little wiggle room if the verification condition for $\mathbb{U}$ must be maintained.[27] If some flexibility is permissible, there are some salient alternatives. One such alternative is restricting the universal quantification to the normal points, for which we will use a different symbol:

- $a \Vdash \mathbb{N}A$ iff $\forall b \in N, b \Vdash A$.

This connective suffers from some of the same defects as $\mathbb{U}$. A formula of the form $\mathbb{N}A$, if true anywhere in a model, will be ubiquitously true. This means that $q \to (\mathbb{N}p \to \mathbb{N}p)$ will be ubiquitously true in all models. Additionally, due to the quantification over the normal points, $\mathbb{U}$ will have some features of $\mathtt{t}$, such as the validity of $\mathbb{N}p \to (q \to q)$. Apart from violations of the variable sharing property, there are some further problems. The connective will not deliver some of the target principles, notably ($\mathtt{T}$).

Other alternatives split the verification condition, in the manner of classically based, non-normal modal logics or the simplified semantics for relevant logics.[28] That sort of verification condition is split between normal and non-normal points, such as the following:

- $a \Vdash \mathbb{S}A$ iff (i) if $a \in N$, then $\forall b \in N, b \Vdash A$, and (ii) if $a \notin N$, then $\forall b \in K, Sab \Rightarrow b \Vdash A$.

There are other options for both clauses, such as quantification over all points for the first clause and simply false for the second, but it is not clear that any other combination is a great improvement. Some of the same issues as we have seen can be reproduced even with the

[26] The development of a suitably non-classical model theory, of the sort suggested by Girard and Weber [2015] and Weber et al. [2016], might help.

[27] One route, which will not be explored further here, is defining a special set of points in the frame as worlds, as is done by [Meyer and Mares, 1993] or [Sedlár, 2015]. It is natural to define such points in a way that makes negation more classical there, which, we expect, will reproduce some of the problems highlighted with the other modalities considered in this paper.

[28] For non-normal modal logics, see Priest [2008, ch. 4]. For simplified semantics, see [Priest and Sylvan, 1992; Restall, 1993; Restall and Roy, 2009], as well as [Priest, 2008, ch. 10].

split conditions, by conjoining $\mathsf{t}$ with a formula of the form $\mathbb{S}A$. For this reason, further investigation of this option will be left for future research.

Suppose that the interest is not primarily in the models, but rather, in the *logic*. Then the axiomatic conception, something in the vicinity of RS5, is the natural option. The logic RS5 gives the desired modal principles, while maintaining many of the features of the underlying relevant logic that relevant logicians find appealing.[29]

The universal and axiomatic conceptions of necessity coincide against the backdrop of classical logic. We can see from the vantage point of relevant logic, that far from coinciding, one conception is, perhaps surprisingly, deeply in tension with the motivations for relevant logics. The other conception, by contrast, aligns well with those motivations.

**Acknowledgments.** I would like to thank Greg Restall, Rohan French, Lloyd Humberstone, Fabio Lampert, Andrew Tedder, Shay Logan, Heinrich Wansing, Hitoshi Omori, George Metcalfe, Marta Bilkova, S. M. Love, and the audiences at the Buenos Aires Logic Group, Bochum Research Colloquium on Logic and Epistemology, the Florence Logic Seminar, and the Seminar on Applied Mathematical Logic at the Czech Academy of Sciences for comments and discussion that have greatly improved this paper. I would also like to thank the anonymous referees whose feedback significantly improved this paper.

## References

Anderson, A. R., and N. D. Belnap, 1975, *Entailment: The Logic of Relevance and Necessity, Vol. I*, Princeton University Press.

Anderson, A. R., N. D. Belnap, and J. M. Dunn, 1992, *Entailment: The Logic of Relevance and Necessity, Vol. II*, Princeton University Press.

[29] A natural idea to pursue is whether there is any sort of connection between RS5 and the one-variable fragment of a first-order version of R, similar to the connection between classically based S5 and the one-variable fragment of classical first-order logic, as illustrated by Mints [1992, 40ff.], or corresponding logics over Corsi logic, as explored by Caicedo et al. [2019]. There are some questions that arise in the case of quantified R, namely which models, since there are different ways to handle the quantifiers, as explored by Brady [1988, 1989]; Fine [1988]; Logan [2019]; Mares and Goldblatt [2006]. The straightforward, Tarskian verification condition overgenerates, leading to incompleteness, as shown by Fine [1989], so one of the cited alternatives will need to be used. This will be left for future work. I would like to thank George Metcalfe, Hitoshi Omori, and Andrew Tedder for discussion of this idea.

Bednarska, K., and A. Indrzejczak, 2015, "Hypersequent calculi for S5: The methods of cut elimination", *Logic and Logical Philosophy*, 24 (3): 277–311. DOI: 10.12775/LLP.2015.018

Bimbó, K., "Relevance logics", 2007, pages 723–789 in D. Jacquette (ed.), *Philosophy of Logic*, volume 5 of *Handbook of the Philosophy of Science*, Elsevier. DOI: 10.1016/B978-044451541-4/50022-1

Bimbó, K., and J. M. Dunn, 2008, *Generalized Galois Logics: Relational Semantics of Nonclassical Logical Calculi*, Center for the Study of Language and Information.

Blackburn, P., M. de Rijke, and Y. Venema, 2002, *Modal Logic*, Cambridge University Press. DOI: 10.1017/CBO9781107050884

Brady, R., editor, 2003, *Relevant Logics and Their Rivals, Volume II: A continuation of the work of Richard Sylvan, Robert Meyer, Val Plumwood and Ross Brady*, Ashgate.

Brady, R. T., 1988, "A content semantics for quantified relevant logics. I", *Studia Logica*, 47 (2): 111–127. DOI: 10.1007/BF00370286

Brady, R. T., 1989, "A content semantics for quantified relevant logics. II", *Studia Logica*, 48 (2): 243–257. DOI: 10.1007/BF02770515

Braüner, T., 2000, "A cut-free Gentzen formulation of the modal logic S5", *Logic Journal of the IGPL*, 8 (5): 629–643. DOI: 10.1093/jigpal/8.5.629

Caicedo, X., G. Metcalfe, R. Rodríguez, and O. Tuyt, 2019, "The one-variable fragment of Corsi logic", pages 70–83 in R. Iemhoff, M. Moortgat, and R. de Queiroz, editors, *Logic, Language, Information, and Computation*, Berlin, Heidelberg: Springer Berlin Heidelberg. DOI: 10.1007/978-3-662-59533-6_5

Crossley, J. N., and L. Humberstone, 1977, "The logic of "actually"" *Reports on Mathematical Logic*, 8: 11–29.

Dunn, J. M., 1995, "Positive modal logic", *Studia Logica*, 55 (2): 301–317. DOI: 10.1007/BF01061239

Dunn, J. M., and G. Restall, 2002, "Relevance logic", pages 1–136 in D. M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 6, 2nd edition, Kluwer.

Ferenz, N., 2022, "Quantified modal relevant logics", *Review of Symbolic Logic*, pages 1–32. DOI: 10.1017/s1755020321000216

Fine, K., 1988, "Semantics for quantified relevance logic", *Journal of Philosophical Logic*, 17 (27–59). DOI: 10.1007/BF00249674

Fine, K., 1989, pages 205–225, “Incompleteness for quantified relevance logics”, in J. Norman and R. Sylvan, editors, *Directions in Relevant Logic*, Kluwer, Dordrecht. Reprinted in [Anderson et al., 1992, §52]. DOI: 10.1007/978-94-009-1005-8_16

Fuhrmann, A., 1990, “Models for relevant modal logics”, *Studia Logica*, 49 (4): 501–514. DOI: 10.1007/BF00370161

Garson, J., 2018, “Modal logic”, in E. N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*, Metaphysics Research Lab, Stanford University, fall 2018 edition. https://plato.stanford.edu/entries/logic-modal/

Girard, P., and Z. Weber, 2015, “Bad worlds”, *Thought: A Journal of Philosophy*, 4 (2): 93–101. DOI: 10.1002/tht3.162

Humberstone, L., “Logical discrimination”, 2005, pages 207–228 in J.-Y. Beziau, editor, *Logica Universalis*, Birkhäuser Basel. DOI: 10.1007/3-7643-7304-0_12

Humberstone, L., 2016, *Philosophical Applications of Modal Logic*, College Publications, London.

Indrzejczak, A., 1996, “Cut-free sequent calculus for S5”, *Bulletin of the Section of Logic*, 25 (2): 95–102.

Indrzejczak, A., 1998, “Cut-free double sequent calculus for S5”, *Logic Journal of the IGPL*, 6 (3): 505–516. DOI: 10.1093/jigpal/6.3.505

Indrzejczak, A., 2010, *Natural Deduction, Hybrid Systems and Modal Logics*, vol. 30 of *Trends in Logic*, Springer. DOI: 10.1007/978-90-481-8785-0

Logan, S. A., 2019, “Notes on stratified semantics” *Journal of Philosophical Logic*, 48 (4): 749–786. DOI: 10.1007/s10992-018-9493-y

Logan, S. A., 2021, “Strong depth relevance”, *The Australasian Journal of Logic*, 18 (6): 645–656. DOI: 10.26686/ajl.v18i6.7081

Logan, S. A., 2022, “Depth relevance and hyperformalism”, *Journal of Philosophical Logic*. DOI: 10.1007/s10992-021-09648-y

Mares, E. D., 1992, “The semantic completeness of RK”, *Reports on Mathematical Logic*, 26: 3–10.

Mares, E. D., 1994, “Mostly Meyer modal models”, *Logique et Analyse*, 146: 119–128.

Mares, E. D., 2004, *Relevant Logic: A Philosophical Interpretation*, Cambridge University Press.

Mares, E. D., 2020, “Relevance logic”, in E. N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*, Metaphysics Research Lab, Stanford University, winter 2020 edition. https://plato.stanford.edu/entries/logic-relevance/

Mares, E. D., and R. Goldblatt, 2006, “An alternative semantics for quantified relevant logic”, *The Journal of Symbolic Logic*, 71 (1): 163–187. DOI: 10.2178/jsl/1140641167

Mares, E. D., and R. K. Meyer, 1993, “The semantics of R4”, *Journal of Philosophical Logic*, 22 (1): 95–110. DOI: 10.1007/BF01049182

Mares, E. D., and S. Standefer, 2017, “The relevant logic E and some close neighbours: A reinterpretation”, *IfCoLog Journal of Logics and Their Applications*, 4 (3): 695–730.

Meyer, R. K., 1966, “Topics in modal and many-valued logic”, PhD thesis, University of Pittsburgh.

Meyer, R. K., and E. D. Mares, 1993, “Semantics of entailment 0”, pages 239–258 in P. Schroeder-Heister and K. Došen, editors, *Substructural Logics*, Oxford Science Publications.

Mints, G., 1992, *A Short Introduction to Modal Logic*, Center for the Study of Language and Information.

Ohnishi, M., and K. Matsumoto, 1957, “Gentzen method in modal calculi”, *Osaka Mathematical Journal*, 9 (2): 113–130.

Parks, Z., and M. Byrd, 1989, “Relevant implication and Leibnizian necessity”, pages 179–184 in J. Norman and R. Sylvan, editors, *Directions in Relevant Logic*, Kluwer Academic Publishers. DOI: 10.1007/978-94-009-1005-8_13

Poggiolesi, F., 2008, “A cut-free simple sequent calculus for modal logic S5”, *Review of Symbolic Logic*, 1 (1): 3–15. DOI: 10.1017/S1755020308080040

Poggiolesi, F., 2011, *Gentzen Calculi for Modal Propositional Logic*, volume 32 of *Trends in Logic*, Springer. DOI: 10.1007/978-90-481-9670-8

Porte, J., 1981, “The deducibilities of S5”, *Journal of Philosophical Logic*, 10 (4): 409–422. DOI: 10.1007/BF00248735

Prawitz, D., 1965, *Natural Deduction: A Proof-Theoretical Study*, Almqvist and Wicksell.

Priest, G., 2008, *An Introduction to Non-Classical Logic: From If to Is*, Cambridge University Press. DOI: 10.1017/CBO9780511801174

Priest, G., and R. Sylvan, 1992, “Simplified semantics for basic relevant logics”, *Journal of Philosophical Logic*, 21 (2): 217–232. DOI: 10.1007/BF00248640

Read, S., 1988, *Relevant Logic: A Philosophical Examination of Inference*, Blackwell.

Restall, G., 1993, "Simplified semantics for relevant logics (and some of their rivals)", *Journal of Philosophical Logic*, 22 (5): 481–511. DOI: 10.1007/BF01349561

Restall, G., 2000, *An Introduction to Substructural Logics*, Routledge.

Restall, G., 2007, "Proofnets for S5: Sequents and circuits for modal logic", pages 151–172 in C. Dimitracopoulos, L. Newelski, and D. Normann, editors, *Logic Colloquium 2005*, Cambridge: Cambridge University Press.

Restall, G., and T. Roy, 2009, "On permutation in simplified semantics", *Journal of Philosophical Logic*, 38 (3): 333–341. DOI: 10.1007/s10992-009-9104-z

Robles, G., and J. M. Méndez, 2011, "A class of simpler logical matrices for the variable-sharing property", *Logic and Logical Philosophy*, 20 (3): 241–249. DOI: 10.12775/LLP.2011.014

Robles, G., and J. M. Méndez, 2012, "A general characterization of the variable-sharing property by means of logical matrices", *Notre Dame Journal of Formal Logic*, 53 (2): 223–244. DOI: 10.1215/00294527-1715707

Routley, R., and R. K. Meyer, 1972, "The semantics of entailment – II", *Journal of Philosophical Logic*, 1 (1): 53–73. DOI: 10.1007/BF00649991

Routley, R., V. Plumwood, R. K. Meyer, and R. T. Brady, 1982, *Relevant Logics and Their Rivals*, volume 1, Ridgeview.

Sedlár, I., 2015, "Substructural epistemic logics.", *Journal of Applied Non-Classical Logics*, 25 (3): 256–285. DOI: 10.1080/11663081.2015.1094313

Seki, T., 2003, "A Sahlqvist theorem for relevant modal logics", *Studia Logica*, 73 (3): 383–411. DOI: 10.1023/A:1023335229747

Slaney, J. K., 1987, "Reduced models for relevant logics without WI", *Notre Dame Journal of Formal Logic*, 28 (3): 395–407. DOI: 10.1305/ndjfl/1093637560

Standefer, S., 2020, "Actual issues for relevant logics", *Ergo*, 7 (8): 241–276. DOI: 10.3998/ergo.12405314.0007.008

Standefer, S., 2021, "An incompleteness result for modal relevant logics", *Notre Dame Journal of Formal Logic*, 62 (4): 669–681. DOI: 10.1215/00294527-2021-0035

Standefer, S., 2022, "What is a relevant connective?", *Journal of Philosophical Logic*. Forthcoming.

Standefer, S., 202x, "Completeness via metacompleteness", in K. Bimbó, editor, *Essays in Honor of J. Michael Dunn*, College Publications. Forthcoming.

Weber, Z., G. Badia, and P. Girard, 2016, "What is an inconsistent truth table?", *Australasian Journal of Philosophy*, 94 (3): 533–548. DOI: 10.1080/00048402.2015.1093010

Yang, E., 2013, "R and relevance principle revisited", *Journal of Philosophical Logic*, 42 (5): 767–782. DOI: 10.1007/s10992-012-9247-1

SHAWN STANDEFER
Department of Philosophy
National Taiwan University
Taipei, Taiwan
standefer@ntu.edu.tw
https://orcid.org/0000-0002-3032-5290