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What is a relevant connective?*

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Abstract There appears to be few, if any, limits on what sorts of logical connectives can be added to a given logic. One source of potential limitations is the motivating ideology associated with a logic. While extraneous to the logic, the motivating ideology is often important for the development of formal and philosophical work on that logic, as is the case with intuitionistic logic. One family of logics for which the philosophical ideology is important is the family of relevant logics. In this paper, I explore the limits of what a relevant connective is, showing how some basic criteria motivated by the ideology of relevant logicians provide robust limits on potential connectives. These criteria provide some plausible necessary conditions on being a relevant connective.

Keywords relevant logic; connectives; philosophy of logic

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1 Introduction

Logics come in many varieties. While one can study formal systems for their own sake, one motivation for studying a particular logic is the philosophical view associated with it. Take the three-valued logic LP, for example. One of the things that makes it attractive to many logicians is that it purports to offer a unified response to the semantic paradoxes.[1] Alternatively, consider intuitionistic logic, J, with its attendant philosophical views of constructivism and verificationism.[2] A logic may have a technically interesting feature but an attendant ideology can turn an interesting feature into a compelling one. The fact that some relevant logics lack the contraction axiom may not antecedently be a draw, but that fact becomes compelling when coupled with the bold vision laid out by Routley [1980], for example.[3]

Logics tend to be studied with a narrow assortment of connectives. There are the old favorites: $\to$ (implication/conditional), $\wedge$ (conjunction), $\vee$ (disjunction), and $\sim$ (negation).[4] Depending on the context, one may see some other connectives, such as $\leftrightarrow$ (biconditional), $\Box$ (necessity), $\Diamond$ (possibility), and $\circ$ (fusion). There are also constants, or 0-ary connectives, that pop up, such as $\top$ (verum), $\bot$ (falsum), and t (Ackermann truth constant). While this assortment of connectives offers a lot for the curious logician, there is more logical space out there. Logicians can boldly go where no one has gone before.

Are there any limits to the bounds of where they can go? In a sense, no, not when any logical principle or rule is open to reconsideration. If *everything* is potentially up for grabs, one can surely move the boundary lines. There is, however, a sense in which limits can be set, and that is by attention to the philosophical ideology associated with a particular logic. Broadly, one can use the philosophical views associated with a logic to draw some distinctions among potential connectives and direct research efforts in different directions. The process can feed back into the philosophical views, as exploring their consequences for the associated logic can reveal features of the view that require clarification or revision as well as leading to distinctions in a view that were not visible before. Revising a philosophical view may in turn have formal consequences, and the cycle can repeat. This sort of reflection need not reach a stable equilibrium.

There are many places one can go when considering unusual connectives. To start with a mundane example, there is no problem adding a binary nor connective, $\Downarrow$, to an array of classical connectives. Depending on what other connectives are in the language, one may even be able to *define* it using the other connectives. What about a binary connective $\pitchfork$ (tonk) that obeys the

1 Priest [1994, 1995]

2 See Dummett [1975] and Prawitz [1977]. See van Atten [2017] for more on the formal and philosophical development of intuitionistic logic.

3 See Meyer et al. [1979], Routley et al. [1982], Restall [1993a], Rogerson and Restall [2004], Bimbó [2006], Rogerson [2007], and Shapiro and Beall [2018], among others, for discussions of contraction.

4 See the table of contents of Humberstone [2011].

following rules?[5]

$$A \vdash A \pitchfork B \qquad\qquad A \pitchfork B \vdash B$$

On one understanding of what it is to be a classical connective, that's not a legitimate connective.[6] On another understanding, tonk is totally fine.[7] We can see, then, that the philosophical views associated with a logic can determine what connectives are legitimate.

In this paper, I will explore the question of when connectives cohere with the philosophical view associated with a logic. The best way into this question is to look at a relatively clear case, namely classical connectives and classical logic. In §2 will provide some general background for the paper and then in §3 I will consider the question of what a classical connective is. I will then turn, in §4, to consider the question of what an intuitionstic connective is. The point of these case studies is to see the sorts of issues that are raised, how they are addressed, and what consequences they have. Following this, in §5 I will provide some background on relevant logics, which will be the main target of the paper. In §6, I will present three criteria to be used to start to address the question of what a relevant connective is and I will draw out consequences of these criteria. The discussion proceeds without considering propositional constants, although I will briefly consider some issues they raise in §7. Following this, I will briefly consider how the question looks with respect to other sorts of models (§8). Finally, in §9, I will draw out some general morals.

2 Background

This paper will discuss logics and connectives, and in this section I will explain what those things are. I will begin with connectives. Connectives are items in a formal language that are used to build new formulas out of existing formulas. Connectives have a meaning, whether stipulated via verification conditions in a model, stipulated via valuations, or determined by rules or axioms in a proof system. This explanation leaves some things open, but for present purposes it seems acceptable for there to be some vagueness and underspecification in the notion of a connective.

There are different frameworks for approaching logics over a language $\mathfrak{L}$. Fully addressing the question of whether a connective in some logic is acceptable or legitimate would seem to require a general setting, such as SET-FMLA consequence relations or SET-SET generalized consequence relations.[8] For the main sections of this paper, namely §5–6, however, it will be enough to focus

[5] This connective was introduced by Prior [1960]. As Wansing [2006] shows, there are connectives that are yet stranger than tonk, against which the standard connectives considered here will appear more mundane.

[6] See Stevenson [1960]. See Belnap [1962] for a different response that does not depend on classicality but still rules out tonk.

[7] Ripley [2015]

[8] See ch. 1 of Humberstone [2011] for consequence relations and p. 103 ff. for more on general frameworks, such as SET-SET.

on the logical truths, i.e. the framework FMLA. This can be viewed as a special case of the more general consequence relations.

Having settled the issue of the framework for addressing our questions of interest, there is the matter of the structure of the language. To keep things simple, I will focus almost exclusively on propositional languages without any variable-binding connectives. This will, perhaps, leave out some important issues, but some omissions are required at this stage, which is setting out the start of an answer to the question of what a relevant connective is.

Next, there is a general issue of what sort of answer we are after, or could be after. There are three apparent places to look for formulating answers: models, proofs, and consequence relations (even in FMLA). The first two have the benefit of being comparatively concrete. We may be able to specify conditions on models, or on verification conditions in models, that will distinguish some connective as legitimate or broadly cohering with the underlying philosophical viewpoint. Similarly, we may be able to specify features of rules and axioms in proof systems that will guarantee that connectives whose rules have those features cohere with the philosophical viewpoint. Both of these options run the risk of being too presentation-dependent, distracting us from questions properly about the *logic* with issues properly about the *models* or properly about the *proof systems*. Ideally, a comprehensive answer to our question will tell us about the logic, while also addressing features of the models and proof systems we use to understand and study the logic. Reality may, however, turn out to be less than ideal, in which case care and caution will be required.

Before getting to some of the responses to the question of when a connective is legitimate, I will look at an area where the question of when a connective broadly coheres with a philosophical view associated with a logic comes up regularly, namely in responses to the semantic paradoxes. An example is the discussion of revenge phenomena.[9] In revenge phenomena, one typically presents a solution to the semantic paradoxes and a critic of the solution will propose a new connective whose addition to the language reinstates paradox. The proponent of the view then argues that the proposed connective is in some way illegitimate. A good response will appeal to reasons apart from paradox so as not to appear *ad hoc*. I will briefly consider two examples.

The first example is Boolean negation in the context of Priest's paraconsistent solution to the semantic paradoxes based on the logic LP.[10] LP is a three-valued logic where its negation, de Morgan negation, permits there to be contradictions, formulas of the form $A \wedge {\sim}A$, that take a designated value, and so de Morgan-contradictions do not imply arbitrary formulas. Boolean negation, which is not the usual negation of LP, results in contradictions that cannot take a designated value, and so yield Boolean-contradictions that do imply arbitrary formulas. Priest [1990] argues that Boolean negation is ille-

[9] See the introduction of Beall [2007], for an overview of revenge phenomena in the context of paradoxes.

[10] See Priest [1979] for more on LP.

gitimate and so there is no problem rejecting that connective. The alleged problem is, in fact, illusory.

The second example comes from Field [2008, ch. 21-22]. Field considers several connectives that one might add to his preferred logic for truth, any one of which results trivializing paradoxes. In Field's logic, one can define a determinateness operator, $\mathcal{D}$, that allows one to express a sense in which the liar λ is defective, which is to say $\sim\mathcal{D}\lambda$. The construction of determinateness operators and liar-like sentences can be iterated to the transfinite. Field considers the objection that one can introduce a super-determinateness operator, $\mathcal{S}$ into the language that "obeys the main laws for determinate truth, but is a maximal notion."[11] Field shows that $\mathcal{S}$ is inconsistent without appeal to the truth predicate, and that slightly weakened versions of the proposed connective fail to capture all possible iterations of $\mathcal{D}$. Field then concludes that the connective should be rejected.

In these examples, we see philosophers arguing against the legitimacy of connectives. Evaluating these arguments is not the point here. However compelling these arguments might be, neither philosopher sets out criteria by which to evaluate putative connectives and uses those to reject the connectives. One may be able to reconstruct plausible candidate criteria, and in a more general discussion, it would be good to have principled criteria by which to evaluate connectives. In the main part of this paper, §6, I will lead with criteria and use those to evaluate connectives, primarily to show that different connectives will need to be rejected by a certain sort of relevant logician. The sort of relevant logician is one who adopts the "tough relevant position" of Routley et al. [1982, 376] that connectives that violate relevantist criteria are not to be countenanced.[12]

The examples above, from the debates over the semantic paradoxes, can be multiplied. These sorts of debates are not, however, limited to the semantic paradoxes. Rather than spend more time on the semantic paradoxes, I will look at responses to the question of when a connective is legitimate, or broadly coherent with a view, that arose for three different areas, and then I will turn to the main topic of this paper, how the relevant logician should response to this question.

3 What is a classical connective?

We begin with the question of when a putative propositional connective is classical. This will be a good starting point because there is widespread agreement. Classical logic is the logic of Boolean valuations in the sense of Humberstone

[11] Field [2008, 344]

[12] The question of how a relevant logician who adopts a different view, whether the intermediate position of the just cited passage or the extremely permissive view of Meyer and Routley [1974, 193] that accepts even Boolean negation, should evaluate the connectives considered below is left to the proponents of such positions. Indeed, the proponents of those views may feel that the criteria proposed here are overly restrictive, in which case they would need to propose their own.

[2011].[13] In classical logic, there are exactly two truth values, 0 and 1, and in a valuation, every formula is assigned exactly one of those two. Classical connectives are extensional and truth-functional. A natural criterion for being a classical connective is that it be interpretable using a truth table with 2^n rows whose values are from the set $\{0, 1\}$, where n is the arity of the connective.[14] This is the idea worked out by Gabbay [1978]. Gabbay's proposal goes beyond just that, as he was interested in the further question of when a connective presented via a Hilbert-style axiom system is classical, a question which is not immediately answered by the truth table proposal.

The condition that for a connective # to be *classical*, # must be interpretable via a truth table is a plausible starting point. One can, no doubt, offer refinements, , but for present purposes, it will be enough to use the truth table criterion as an illustration.

To determine whether a connective # is classical, given a distinguished set of logical truths using #, one needs to find a truth table that will result in exactly that set of logical truths lacking counterexamples. Alternatively, if the verification condition for # was given as a truth table, then we are done. To show that a connective is not classical, one shows no adequate truth table is possible.

The truth table criterion includes many familiar connectives, such as the nor connective, $\Downarrow$, and material equivalence, $\equiv$. By design, the criterion excludes intensional connectives, such as the modal connectives necessity, $\Box$, and possibility, $\Diamond$, as well as the intuitionistic implication.[15] Although we are not concerned with variable binding connectives here, notice that first-order quantifiers do not count as classical on this criterion. One cannot give a truth table for $\forall$ that captures exactly the first-order logical truths.

What has been presented here is perhaps best described as the hardline view of what counts as a classical connective. One might think that the hardline view is too stringent and propose various weakenings of it. Such details, however, will be left to the proponent of classical logic to provide.[16]

We can see that even in the simplest case, classical propositional logic, there are some limits to the criterion that do not fit neatly with some common demarcations. There are natural extensions of the logic that do not satisfy the criterion. Let us move away from the domain of classical logic to see how the question is addressed in the setting of intuitionistic logic.

[13] Cobreros et al. [2012, 2013] and Ripley [2012, 2013a,b] develop an understanding of classical logic in terms of sequents. This uses tri-valuations in an important way.

[14] To be a little more precise, in a valuation, each connective is interpreted via a row in that connective's truth table. This is to rule out connectives whose evaluation appeals to the entire truth table for its components, such as an operator $\Box$ such that $v(\Box A) = 1$ iff for every valuation v', $v'(A) = 1$. I would like to thank Greg Restall for this point.

[15] Some interpretations of $\Box$ will be permitted, namely those that collapse into truth functions.

[16] The interested reader should consult Gabbay [1978] for some ideas concerning the standard quantifiers.

4 What is an intuitionistic connective?

Intuitionistic logic, J, is perhaps the best known non-classical logic.[17] The question of what an intuitionistic connective is has received some attention, and looking at those proposals will be helpful in seeing how to address the related question regarding relevant connectives.

There are some developed philosophical viewpoints connected with intuitionistic logic, particularly in connection with the philosophy of mathematics.[18] An important feature of these viewpoints, for our purposes, is that intuitionistic logic is supposed to be *constructive*. As Dummett says, “What everyone who has heard of intuitionism knows is that intuitionists want their proofs to be constructive.”[19] There are different understandings of what, exactly it means for a proof to be constructive. There are two features that Dummett highlights, the Disjunction and Existence Properties, of which the former will appear in connection to the criteria of Gabbay [1981] below.

What are legitimate intuitionistic connectives? What are the connectives that broadly cohere with the philosophical views typically associated with intuitionism? There has been a lot of interest in these questions, and new connectives have been suggested in many places, such as Humberstone’s strongest anticipator connective, empirical negation, or intuitionistic actuality.[20] One can see why there would be interest in the question. Unlike with classical logic, there is no apparent sense in which the standard intuitionistic vocabulary can be used to define all the plausible intuitionistic connectives. In classical logic, truth-functional completeness means that any of the usual sets of connectives will ensure that the plausible options are all definable.

Gabbay [1981, ch. 7.4] made a proposal, presenting five conditions on intuitionistic connectives.[21] For the presentation of Gabbay’s conditions, E will be an extension of the basic intuitionistc language and logic J with some new connective $\mathfrak{c}$.

(G1) E is a conservative extension of J: no new theorems entirely in the old vocabulary.[22]
(G2) E has the Disjunction Property: if $\vdash_E A \vee B$ then $\vdash_E A$ or $\vdash_E B$.
(G3) Adding Peirce’s law to E results in classical logic and formulas using $\mathfrak{c}$ are provably equivalent to a formula in the usual classical vocabulary.
(G4) The axioms of E determine $\mathfrak{c}$ uniquely.
(G5) The connective $\mathfrak{c}$ is not already definable in J.

[17] See van Atten [2017] and Moschovakis [2018] for overviews of intuitionistic logic.
[18] See Iemhoff [2020].
[19] Dummett [2000, 6]
[20] For the first, see Humberstone [2001]. For the second, see Solomon and DeVidi [2006]. For the third, see Niki and Omori [2020].
[21] Gabbay [1977] includes a sixth condition, but that drops out in the later presentation.
[22] I will note one complication for the question of what an intuitionistic connective regarding this condition. There are connectives that are conservative over the propositional logic that are not conservative over first-order intuitionistic logic. See López-Escobar [1985] and Humberstone [2011, 547, 626] for examples.

Some of the conditions, such as (G1) and (G4), are pretty standard in discussions of introducing new connectives, echoing the proposal of Belnap [1962] for explaining what is wrong with tonk. Condition (G3) is an outlier in the sense that the proponent of intuitionistic logic may plausibly reject it. It is not clear why the stated connection to classical logic should be a constraint on connectives in this non-classical logic. Indeed, a particularly revolutionary proponent of intuitionistic logic, such as Brouwer himself, would think that classical logic gets things fundamentally *wrong* in freely appealing to excluded middle or double negation elimination. While the revolutionary intuitionist may find it interesting whether formulas using the new connectives collapse into classical formulas, there is no need for them to take this as a requirement on their connectives any more than it should be a requirement that intuitionistic mathematical theories collapse into familiar classical theories with the addition of Peirce's law.[23,24]

There are some other proposals and additions out there. Humberstone [2011, 615,1233] proposes the condition that new connectives should preserve synonymy according to the base logic in the sense that if $A \leftrightarrow B$ is valid in the logic prior to the extension, then for any formula context $C(\cdot)$ in the extended language, $C(A)$ is equivalent to $C(B)$. A new connective should not allow one to distinguish formulas that were previously indistinguishable from the point of view of the logic. Humberstone [2011, 1236] points out that strong negation, a connective often discussed in the context of intuitionistic logic, violates this condition, along with condition (G3).

Kaminski [1988] argued that Gabbay's conditions were not correct. Among other things, Kaminski took issue with (G4), arguing that the condition rules out the possibility of modal operators.[25] Kaminski provides alternative criteria that involve both models and proof systems.

We have, so far, looked at answers to the question of what connectives are legitimate, from three different points of view, classical logic, (classically based) modal logic, and intuitionistic logic. In each case we saw that some particular philosophical views were operative in answering the question. Even in the straightforward case of classical logic, there was still room for disagreement and competitors to, or developments of, the hardline view presented in §3. When one moves to modal and intuitionistic logics, one finds more room for competing answers to the question. One would expect that the situation with relevant logics would be more similar to that of intuitionistic logic than that of classical logic. As such, I will be presenting the start of the answer to the

[23] See Shapiro [2014, ch. 3] for some discussion of consistent intuitionsitic theories that become inconsistent with the addition of Peirce's law.

[24] Caicedo and Cignoli [2001] show that all connectives that satisfy Gabbay's conditions apart from (G3) satisfy (G3) as well. Thus, the revolutionary proponent of intuitionistic logic can reject (G3) without issue. Indeed, that position appears to be an intensional improvement on Gabbay's that is extensionally equivalent.

[25] See Božić and Došen [1984], Došen [1985], Ewald [1986], Bierman and de Paiva [2000], and Braüner and de Paiva [2006] for some work on modal extensions of intuitionistic logic.

question, a fuller answer needing more development than can be given here. We will next turn to the main area of interest for this paper, relevant logics.

5 What is a relevant logic?

Relevant logics are a family of non-classical logics.[26] The distinguishing mark of a relevant logic, for our purposes, is the logical behavior of its implication connective, which requires a strong connection between the antecedent and consequent of valid implications. This idea motivates rejecting many familiar axioms, principal among them the axiom of weakening, $A \to (B \to A)$.

A central idea of relevant logics is that if $A \to B$ is a logical truth, there has to be a substantive connection between A and B. The nature of this substantive connection gets spelled out in different ways by different relevant logicians. Anderson and Belnap [1975] explain the connection in terms of the *Use Criterion*, which says that one must really use A in obtaining B in order to conclude that $A \to B$. Routley [1980] explains the connection in terms of *absolute sufficiency*, which says that the truth of the antecedent must be sufficient for the truth of the consequent, no matter what the possibilities are. Brady [2006] explains the connection as one of *meaning containment*, according to which the validity of $A \to B$ requires that the meaning of B be contained in the meaning of A. As this brief overview of different views indicate, the connection at issue is one that is understood in terms of different primitive concepts by different relevant logicians.

While there is much disagreement even among relevant logicians, there is something everyone in the area agrees upon, Belnap's variable sharing criterion: if $A \to B$ is a logical truth, then A and B share a propositional variable.[27] The variable sharing criterion is taken as a necessary condition for a logic to be a relevant logic.[28] It is clear that both classical logic and intuitionistic logic, J, violate the variable sharing criterion and so are not relevant logics.

There is a family of relevant logics, and I will need to demarcate the extent of the family. For the purposes of this paper, relevant logics will be any logic containing B and contained in R, both of which will be presented below.[29] This demarcation excludes some weak logics that might reasonably be included, those lacking some of the B principles or Brady's non-distributive logic MC.[30]

[26] Dunn and Restall [2002] and Bimbó [2007] provide accessible overviews of the field.

[27] See Anderson and Belnap [1975, 252-254], Robles and Méndez [2011, 2012], and Logan [2021, 2022] for more on the variable sharing criterion.

[28] Other criteria have been proposed. See Øgaard [2020] for a discussion of a less accepted criterion, suppression freedom, and its relation to the variable sharing criterion.

[29] Avron [2014] provides two different accounts of what a relevant logic is, both of which differ from the present demarcation. The first of Avron's accounts excludes many of the logics considered here, requiring the logic extend the implicational fragment of RW, and the second account excludes most of those considered here, as the $\{\to, \sim\}$ fragment of R is the minimal relevant logic on that account.

[30] Brady and Meinander [2013] argue for modifying Brady's logic DJ, defended by Brady [2006], by rejecting distribution. DJ does fall within the demarcation proposed, while MC falls outside it.

It also excludes some stronger logics that might reasonably be included, such as some logics extended with the mingle axiom, $A \to (A \to A)$, or the logic of the semilattice frames of Urquhart [1972].[31]

It will be useful to have the axioms and rules for B and R for reference.

(A1) $A \to A$
(A2) $(A \wedge B) \to A$, $(A \wedge B) \to B$
(A3) $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$
(A4) $A \to (A \vee B)$, $B \to (A \vee B)$
(A5) $((A \to C) \wedge (B \to C)) \to ((A \vee B \to C))$
(A6) $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$
(A7) $\sim\sim A \to A$

(B1) $A, A \to B \Rightarrow B$
(B2) $A, B \Rightarrow A \wedge B$
(B3) $A \to B \Rightarrow (C \to A) \to (C \to B)$
(B4) $A \to B \Rightarrow (B \to C) \to (A \to C)$
(B5) $A \to \sim B \Rightarrow B \to \sim A$
(B6) $A \to (B \to C) \Rightarrow (A \circ B) \to C$
(B7) $(A \circ B) \to C \Rightarrow A \to (B \to C)$

Those are the axioms and rules for the logic B. To obtain R, one adds to the above axioms and rules the following.[32]

(C1) $(A \to (B \to C)) \to (B \to (A \to C))$
(C2) $(A \to (A \to B)) \to (A \to B)$
(C3) $(A \to B) \to ((B \to C) \to (A \to C))$
(C4) $(A \to B) \to ((C \to A) \to (C \to B))$
(C5) $(A \to \sim B) \to (B \to \sim A)$

Later it will be useful to discuss stronger and weaker relevant logics, as groups. For the purposes of this paper, the stronger logics will be ones that include (C3), (C4), and (C5) as well as either (C2) or $(A \wedge (A \to B)) \to B$. This definition is somewhat arbitrary, but it marks out a boundary, namely the logic C discussed by Routley et al. [1982], a slight weakening of Anderson and Belnap's logic T.[33]

Relevant logics have a frame semantics based on ternary relational frames, also known as Routley-Meyer frames.

[31] While RM, which is R plus the mingle axiom, violates the variable sharing criterion, some nearby logics with the mingle axiom satisfy the criterion, as shown by Robles et al. [2010] and Méndez et al. [2012]. The logic $\mathsf{BN4}$ of Brady [1982] violates the variable sharing criterion but does satisfy the same semi-relevance criterion as RM, for which see Anderson and Belnap [1975, 448]. Since neither of those logics satisfy the variable sharing property, they are excluded from further consideration here. Some of the logics studied by Robles and Méndez [2020] do have the variable sharing property, but they are not contained in R, which will exclude them. Semilattice logic was shown to have the variable sharing property by Weiss [2019].

[32] For non-redundant axiomatizations of R and many other logics in the family, see Brady [1984b].

[33] C is of interest because it is the weakest relevant logic shown to be complete with respect to its reduced frames by the methods of Routley et al. [1982]. Slaney [1987] showed how to obtain completeness results for weaker logics. Standefer [2021] shows that many natural modal extensions of C, and stronger logics, are incomplete with respect to their reduced frames, and Standefer [202x] shows that many modal extensions of sublogics of C are complete with respect to their reduced frames.

Definition 1 *A Routley-Meyer frame F is a 4-tuple $\langle K, N, R, ^* \rangle$, where $K \neq \emptyset$, $N \subseteq K$, $^* : K \mapsto K$, $R \subseteq K^3$, $a \leq b =_{Df} \exists x \in N, Rxab$, and the following conditions are satisfied.*

- $a \leq b \Rightarrow b^* \leq a^*$
- $a^{**} = a$
- $\leq$ *is a partial order*
- *If* $a \leq b$ *and* $a \in N$, *then* $b \in N$
- *If* $d \leq a$, $e \leq b$, $c \leq f$, *and* $Rabc$, *then* $Rdef$

Definition 2 *A Routley-Meyer model M is a frame F with a valuation V such that if $V(p, a) = 1$ and $a \leq b$ then $V(p, b) = 1$. M is said to be a model on the frame F.*

A valuation extends to a verification relation for the whole language as follows.

$$
\begin{array}{lll}
a \Vdash p & \textit{iff} & V(p, a) = 1 \\
a \Vdash \sim B & \textit{iff} & a^* \nVdash B \\
a \Vdash B \wedge C & \textit{iff} & a \Vdash B \textit{ and } a \Vdash C \\
a \Vdash B \vee C & \textit{iff} & a \Vdash B \textit{ or } a \Vdash C \\
a \Vdash B \to C & \textit{iff} & \forall b, c(Rabc \textit{ and } b \Vdash B \Rightarrow c \Vdash C) \\
a \Vdash B \circ C & \textit{iff} & \exists b, c(Rbca \textit{ and } b \Vdash B \textit{ and } c \Vdash C)
\end{array}
$$

With the definition of a model, we can define validity.

Definition 3 *A formula A holds in a model iff for all $a \in N$, $a \Vdash A$.*

A formula A is valid on a frame F iff A holds in all models on F.

A formula A is valid in a class of frames $\mathcal{C}$ iff A is valid on all frames $F \in \mathcal{C}$.

There is a feature of holding and validity on which I will comment. Holding requires that a formula is verified on all the normal points, which may fall short of being all the points. Consequently, validity does not require that the formula be verified at all points. This is in contrast to usual definitions of validity on Kripke frames for classically based modal logic, which is to say modal logic over a classical base logic.[34] Below, I will show that some formulas are verified at all points in all models, which may not seem remarkable to the modal logician working with classical logic as the base logic. I flag it here to underscore that that is not the usual way of proceeding for the relevant logician. Rather, one expects that once one ventures outside of the normal points, any formula might fail somewhere.[35]

The logic B is sound and complete with respect to the class of all Routley-Meyer frames. One can obtain classes of frames appropriate for stronger logics

[34] See Blackburn et al. [2002, 24].

[35] Copeland [1979, 1980, 1983] criticized many aspects of the Routley-Meyer frames, including the distinction between normal and non-normal points. The discussion of ubiquitous formulas below indicates some of the limitations of the distinction. I would like to thank a referee for suggesting Copeland's critical work in this regard.
Friends of the relevant logic enterprise have been critical of the use of Routley-Meyer frames, as well. Objections have been raised by Girard and Weber [2015] and Brady [2017b], for example. They might view the results of this paper in a different light.

by imposing frame conditions. For this paper, I will focus on the class of all frames, so there will be no need to consider the additional conditions.[36]

There are two standard lemmas that will be implicitly appealed to in what follows, the Heredity and Verification Lemmas.

Lemma 4 (Heredity) *If* $a \Vdash A$ *and* $a \leq b$*, then* $b \Vdash A$*.*

This is proved by an induction on the complexity of A.

Lemma 5 (Verification) $B \to C$ *holds in a model* M *iff for all* $a \in K_M$*, if* $a \Vdash B$ *then* $a \Vdash C$*.*

This is a corollary of the Heredity Lemma. In the next section, I will appeal to the right-to-left direction in showing that a given conditional $B \to C$ holds in a model. The argumentation showing that $B \to C$ holds in a model, appealing to the Verification Lemma, is importantly different from the argumentation showing that the conditional holds at all points in the model. For the former, one assumes B holds at an arbitrary point and shows that C also holds at that point. This does not establish that the conditional holds at all points in the model. To show that, one assumes, for arbitrary points, $a, b, c \in K$, $Rabc$ and $b \Vdash B$ and one shows that $c \Vdash C$. I flag this distinction because in relevant logics, one does not often show that a certain conditional holds at all points in a model, as holding in a model suffices for most standard purposes.

With that background in place, I will turn to the criteria for relevant connectives and discussion.

6 What is a relevant connective?

In the previous section I gave an overview of the Hilbert-style axiom systems for B and R and the model theory for B. In this section, I will provide an initial answer to the question of what a relevant connective is. I will provide two main criteria that are framed in terms of the logic. The investigation will proceed primarily by appeal to the frame theory, so I will need a third condition involving models to make things work.

The following three criteria provide an initial answer to the question of when a connective is a relevant connective.

- New connectives do not lead to violations of variable sharing: $\models A \to B$ only if A and B share a variable.
- New connectives do not generate valid weakening-like formulas, $C(A) \to (B \to D(A))$, where $C(\cdot)$ and $D(\cdot)$ are formula contexts and none of $C(\cdot)$, $D(\cdot)$, and A share a propositional variable with B
- Formulas using the new connectives must obey Heredity: $a \Vdash A$ and $a \leq b$ implies $b \Vdash A$.

Some comments are in order.

The first condition requires that new connectives do not lead to violations of Belnap's variable sharing criterion. This seems to get at an essential feature

[36] The interested reader should see Routley et al. [1982, ch. 4], Restall [2000, ch. 11], or Goldblatt and Kane [2009], among others, for surveys of conditions and logics.

of what it is to be a relevant logic. This condition could be refined. In particular, there are strengthened versions of Belnap's variable sharing criterion one could opt for, such as Brady's depth relevance condition.[37] Depth relevance requires that not only do the antecedent and consequent share a propositional variable, but, further, each has a common variable at the same depth, that is, nested under the same number of conditionals. The depth relevance condition is satisfied by many of the weaker relevant logics but not by the stronger ones. For this paper, I will work with Belnap's criterion rather than Brady's.

The first condition is primary. The second, in a sense, depends upon the first. In the implicational fragment of the logic, Anderson and Belnap [1975] note that the axiom of weakening, $A \to (B \to A)$, should be rejected in a relevant logic. Clearly, the extension of the logic with a new connective should not result in weakening becoming available. Say that a formula is *weakening-like* if it is of the form $C(A) \to (B \to D(A))$, where $C(\cdot)$ and $D(\cdot)$ are formula contexts and none of $C(\cdot)$, $D(\cdot)$, and A share a propositional variable with B.[38] In weakening-like formulas, B is merely along for the ride, so to speak.[39] Examples of weakening-like formulas are $C(A) \to (B \to C(A))$ or $C(A) \to (B \to A)$, where $C(\cdot)$ is a formula context and neither $C(\cdot)$ nor A contain any propositional variables in common with B.

In some work on relevant logics, logicians will point to valid instances of weakening-like as being uncomfortably close to weakening, which everyone in the area agrees is to be avoided.[40] Valid weakening-like formulas loosen the connection between antecedent and consequent of valid implications a bit too much, so they should be avoided.

Different logics in the relevant family handle weakening-like contexts differently. For logics that contain both

– $(A \to B) \to ({\sim}B \to {\sim}A)$ and
– $(A \to (A \to B)) \to (A \to B)$,

the weakening-like validities lead to violations of variable sharing. From $C(A) \to (B \to D(A))$, we get $C(A) \to ({\sim}D(A) \to {\sim}B)$, and then we get $C(A) \land {\sim}D(A) \to {\sim}B$. Logics that contain $(A \to (B \to C)) \to (B \to (A \to C))$ also lead to violations of variable sharing, since $C(A) \to (B \to D(A))$ leads to $B \to (C(A) \to D(A))$. A proponent of, say, R has strong reason to reject connectives that produce weakening-like validities, while the proponent of B need not have these reasons. The proponent of B may find such validities distasteful, but they may not lead to violations of the variable sharing criterion.

[37] See Brady [1984a] Robles and Méndez [2014b,a], Salto et al. [2018], and Logan [2022] for some discussion of depth relevance.

[38] Note that any propositional variable occurring in A will occur in $C(A)$ and in $D(A)$.

[39] Cf. the "no loose pieces" idea of Anderson and Belnap [1975, 254-255], which is discussed by Robles and Méndez [2012]. A condition banning the generation of loose pieces in validities would be a natural addition to the conditions above, since it seems to get at a different idea than the three conditions above. I would like to thank an anonymous referee for pressing this point.

[40] See Dunn [1987], Mares [1992], and Kremer [1999], for example.

There are two views one might take to what a relevant connective is. One is based on how the criteria play out amongst the whole family, everything between B and R, inclusive. The other view is to identify a distinguished logic or subfamily of logics and say that the connectives will be evaluated according to the criteria only with respect to that logic or that subfamily. The distinction matters for weakening-like validities, because the weaker logics may be able to incorporate them without resulting violations of variable sharing. This leads to a distinction among connectives. Say that a connective $\mathbb{C}$ is *strongly relevant* iff the addition of $\mathbb{C}$ does not lead to violations of variable sharing and does not generate weakening-like validities. Say that a connective $\mathbb{C}$ is *weakly relevant* iff the addition of $\mathbb{C}$ does not lead to violations of variable sharing but does generate weakening-like validities. For many of the stronger logics, there will be no weakly relevant connectives, as weakening-like validities entail violations of variable sharing. The distinction is of interest when the focus is on the weaker relevant logics.

The third condition, and the only one dealing with models, is a Heredity condition, named after the standard lemma from the meta-theory of frame semantics. The usual relevant logical vocabulary has the property encapsulated in the third condition. That fact is, it turns out, important for the study of frame semantics for relevant logics. Since the models do not work well without that condition, it makes sense to require it as long as we are using models in our investigation. If we were proceeding via proof theory, or an alternative frame theory, that condition could be dropped. Towards the end of this paper, I will briefly comment on alternative frame theories, a feature of which is that the third condition can be dropped without loss.

It is worth noting that Brady [2006, 34-39] discusses some connectives to explain why he does or does not include them in his study. This discussion appeals to a project of formalizing natural language. Brady rejects the connectives that he concludes do not have a natural language interpretation, which connectives include fusion, taken here as among the basic connectives. Fusion is admitted as offering technical uses. In his discussion, Brady does appeal to a violation of relevance, which presumably means variable sharing, although that is used to argue that a certain connective should be rejected because it would mean that natural language entailment would not have the features Brady takes it to have. Brady, then, is answering a similar question to the one being addressed, but his scope is narrower and his criteria are bound up with his particular philosophical project of developing a logic of meaning containment.[41]

Before moving on to the consequences of these criteria, I will note that one could try to adapt Gabbay's conditions for intuitionistic connectives directly to a relevant logical key. The results, however, are not terribly compelling, as one

[41] Mares [2021, §6] discusses barring connectives on informational grounds for logics similar to those Brady is interested in, although the reasons are not the same as Brady's.

might expect since those conditions were specified with an eye to intuitionistic logic, which has a different philosophical background view.[42]

The analog of condition (G3), which requires a collapse to classical logic under certain conditions, is particularly uncompelling and should be rejected by the relevant logician. It is an important part of the views of most relevant logicians that classical logic goes wrong in incorporating the paradoxes of material implication.[43] This is so even for proponents of relevant logics, like Anderson and Belnap, who concessively note that their preferred logics contain all classical tautologies in the vocabulary $\{\wedge, \vee, \sim\}$. It is hard to see why the die-hard relevantist should care whether formulas using the new connective collapse into classical formulas upon the addition of many axioms they regard as fundamentally flawed.[44] The relevant logician has less reason to adopt the analog of (G3) than the intuitionistic logician.

Analogs of Gabbay's conditions for relevant logics are not particularly attractive, so they will not be considered further. I will, then, proceed to consider some connectives using the three criteria presented at the start of this section. I will begin with one connective that should be out, if anything is, namely Boolean negation.

Boolean negation, $\neg$, has the following verification condition.

– $a \Vdash \neg A$ iff $a \nVdash A$

According to my criteria, this is not acceptable, as it leads to violations of variable sharing. It is easy to see that $p \vee \neg p$ will be true at every point in every model, so $q \rightarrow p \vee \neg p$ will come out valid.

The star connective must also be rejected.[45]

– $a \Vdash A^*$ iff $a^* \Vdash A$

This connective can be used with de Morgan negation to define Boolean negation, so it must be rejected.

$$a \Vdash (\sim A)^* \text{ iff } a^* \Vdash \sim A \text{ iff } a^{**} \nVdash A \text{ iff } a \nVdash A$$

To get Heredity, one must postulate that $a \leq b$ implies $a^* \leq b^*$. Given the other conditions on Routley-Meyer frames, this requires that $\leq$ be identity. While requiring that $\leq$ be identity is not a standard condition on Routley-Meyer frames, it is a condition that will appear a few more times below.

[42] The analog of condition (G2), that the extended logic has the Disjunction Property, may be appealing to proponents of weaker relevant logics that are metacomplete, since those base logics do have the disjunction property. The relevant logical justification, due to Slaney [1984, 161], is that for certain metacomplete relevant logics, all logical truths are fundamentally implicational, and so they can be seen as recording a valid inference. This is a different justification for the Disjunction Property than we saw with intuitionistic logic, where the appeal was due to the constructive nature of disjunction. For more on the technique of metavaluations, see Brady [2017a].

[43] See Anderson and Belnap [1975], Routley et al. [1982], or Read [1988], for example.

[44] Perhaps Routley [1980] or Brady [2006] would count as die-hard relevantists.

[45] This connective is considered by Meyer and Routley [1973].

At this point, it is worth pausing to note that Meyer and Routley [1973, 1974] present Boolean negation as being a natural and important part of the relevant logical vocabulary. The latter paper ends with, "But the case is now pretty strong that $\neg$ was just left out of Anderson-Belnap formulations of their logics, and evidence is building that the entire project of relevant logic is unified and simplified when the *semantic* $\neg$, with a different function from the *deduction-theoretic* [$\sim$] that has been present from the start, is added. This paper is part of that evidence."[46] While the presentation of R extended with Boolean negation in those papers is, indeed, simplified, it is not clear that it is the way that relevant logics should be developed. On the view suggested here, Meyer, who likely authored those lines, has gone astray from the relevant logic project in giving up on Belnap's variable sharing criterion. Of course, Meyer was a self-avowed non-relevantist, so this may not be a surprising evaluation.[47] If one is, like Meyer, open to Boolean negation, then one will need to adopt alternative criteria for what is a relevant connective, and, arguably, what is distinctive about relevant logics, as many of their interesting features fall by the wayside in the presence of Boolean negation.

As goes Boolean negation, so goes "material implication", $\supset$, given the following verification condition.

– $a \Vdash A \supset B$ iff $a \nVdash A$ or $a \Vdash B$.

Note that this connective is not the defined material implication that is sometimes seen in work on relevant logics. That connective is defined as $\sim A \vee B$, and its addition does not upset the usual variable sharing criterion. As with Boolean negation, we will, for the time being, impose the condition that for the frames, $\leq$ is identity. With that assumption in place, the Heredity condition is trivialized. We can see that $a \Vdash A \supset A$ will hold for all $a \in K$. It will then be the case that $p \to (A \supset A)$ will be true at all points in in all models. Selecting an atom that is not in A will result in a violation of variable sharing.

Some reflection on the problem with Boolean negation brings out general lessons regarding putative relevant connectives. The formula $p \vee \neg p$ ends up verified everywhere, so one cannot find any counterexamples to a conditional with it as consequent. Fuhrmann [1990, 509], using a suggestion attributed to Humberstone, says that a formula is *ubiquitously true* in a model iff it is verified at all points in the model. Similarly, we can say that a formula is *ubiquitously false* in a model iff it is not verified at any point in a model. Formulas that are either ubiquitously true or ubiquitously false in a model will simply be called *ubiquitous*. It is immediate that if A is ubiquitously true in a model, then so is $p \to A$, and so if A is ubiquitously true in all models, $p \to A$ will be valid. It does not matter which atom is chosen for the antecedent, so violations of variable sharing are readily available. In model-theoretic terms, connectives that generate ubiquitous truths in every model must be rejected. The same point holds for ubiquitous falsehoods, as the ubiquitous falsity of A in all models will yield the validity of $A \to p$.

[46] Meyer and Routley [1974, 193], emphasis in the original.

[47] See Meyer [1985].

We can step back and consider some conditions on connectives, satisfaction of which will mean that they violate the earlier criteria, in particular the variable sharing criterion. A connective whose verification condition does not display the point of evaluation will lead to ubiquitous truths. One way to generate such connectives is to use verification conditions that do not display any unbound points in K, in particular the point of evaluation.

– $a \Vdash \mathbb{Z}A$ iff ϕ, where no unbound point appears in ϕ.

To see that these connectives will obey heredity, we can argue as follows. $a \Vdash \mathbb{Z}A$ iff ϕ iff $b \Vdash \mathbb{Z}A$. The result is that either $\mathbb{Z}A$ is ubiquitously true or it is ubiquitously false. These formulas whose main connective is $\mathbb{Z}$ will satisfy the Heredity criterion, as their assumed truth will imply their truth everywhere, including further up the $\leq$-ordering.

Such a connective will generate formulas that are ubiquitously true or ubiquitously false in a model. They will generate formulas that are ubiquitously true in all models or ubiquitously false in all models as well. For an example involving implication, consider $\mathbb{Z}A \to \mathbb{Z}A$. From the preceding argument, we see that $\mathbb{Z}A$ is either ubiquitously true or ubiquitously false in a given model. Suppose it is ubiquitously true. Then $\mathbb{Z}A \to \mathbb{Z}A$ is also ubiquitously true as there are no counterexamples. Suppose, on the other hand, that $\mathbb{Z}A$ is ubiquitously false. Then $\mathbb{Z}A \to \mathbb{Z}A$ is ubiquitously true, as there are no counterexamples, which require a point where the antecedent is true. For an example not involving implication, consider $\mathbb{Z}A \wedge {\sim}\mathbb{Z}A$, which is ubiquitously false in all models. To see this, suppose that $a \Vdash \mathbb{Z}A$. Then, $a^* \Vdash \mathbb{Z}A$, from which it follows that $a \nVdash {\sim}\mathbb{Z}A$. Therefore, the conjunction is false at every point. Similarly, $\mathbb{Z}A \vee {\sim}\mathbb{Z}A$ is ubiquitously true. If $\mathbb{Z}A \wedge {\sim}\mathbb{Z}A$ is ubiquitously false, then ${\sim}(\mathbb{Z}A \wedge {\sim}\mathbb{Z}A)$ is ubiquitously true, so by de Morgan equivalences, $\mathbb{Z}A \vee {\sim}\mathbb{Z}A$ is ubiquitously true.

Sticking a formula that is ubiquitously true in all models into the consequent of a conditional will result in violations of variable sharing. Similarly, sticking a formula that is ubiquitously false in all models into the antecedent of a conditional will result in violation of variable sharing. Therefore, the relevant logician has reason to reject any connective that generates ubiquitous truths or ubiquitous falsehoods in too many models. Let us consider some connectives that have this feature.

Take the universal modality:

– $a \Vdash \mathbb{U}A$ iff $\forall b \in K, b \Vdash A$.

The verification condition does not use the initial point of evaluation, instead containing only bound variables over points. A formula of the form $\mathbb{U}A$ is either ubiquitously true or ubiquitously false in a model. Therefore, extensions with $\mathbb{U}$ result in violations of variable sharing. Therefore, the relevant logician has reason to reject the universal modality.[48]

[48] For further exploration of the universal modality in the context of relevant logics, see Standefer [2022b].

Instead of looking at truth at *all* points in a model, one could instead look at truth at a single, distinguished point in the model. For the next connective, augment each frame with a distinguished point $g \in K$. We can add a connective $\mathbb{A}$ whose verification condition is that of the actuality connective of classically based two-dimensional modal logic.

– $a \Vdash \mathbb{A}B$ iff $g \Vdash B$

This connective yields validities that violate variable sharing, such as $\sim\mathbb{A}B \wedge \mathbb{A}B \to p$ and $q \to (\mathbb{A}B \to \mathbb{A}B)$. That the relevant logician should reject this connective, which is a simple-minded version of the actuality operator, may be surprising. All is not lost with respect to actuality, however, as the relevant logician has options for adding an actuality connective, options which are explored by Standefer [2020].

The universal modality and the actuality operator are two connectives that yield violations of variable sharing. In fact, any connective whose verification condition is stated purely in terms of set-theoretic relations among sets of points will yield violations of variable sharing. There will be many such connectives, a few of which I will list. I will use the notation $|B|_M$ for the set $\{a \in K_M : a \Vdash B\}$.

– $a \Vdash A \Rightarrow B$ iff $|A|_M \subseteq |B|_M$.
– $a \Vdash A \doteq B$ iff $|A|_M = |B|_M$.
– $a \Vdash A \bowtie B$ iff $|A|_M \cap |B|_M = \emptyset$.
– $a \Vdash \#A$ iff $|A|_M \neq \emptyset$.
– $a \Vdash \mathbb{X}A$ iff $\forall x \in X, x \Vdash A$, where $X \subseteq K$ is a distinguished set.
– $a \Vdash \mathbb{N}A$ iff $N \subseteq |A|_M$.

It is straightforward to see that these generate ubiquitous truths and ubiquitous falsehoods. The last two on this list deserve some further comment.

The penultimate connective, $\mathbb{X}$, is studied by Pietruszczak [2009] and Pietruszczak et al. [2020] as the box operator of some simplified frames for some modal logics.[49] Those frames have the form $\langle K, X\rangle$, using a distinguished subset X to interpret the modal, rather than using a binary accessibility relation. While modal extensions of relevant logics can satisfy the variable sharing criterion, frames for those extensions use a binary accessibility relation rather than a distinguished subset of points. While the distinction does not have serious consequences in the classically based setting, it marks a deep divide when considering extensions of relevant logics.

The last one on this list deserves comment, since it is close to something more familiar from relevant logics. Define the Ackermann constant, t, as follows.

– $a \Vdash t$ iff $a \in N$

It turns out that $\mathbb{N}A$ is not the same as $t \to A$. The latter holds at a point a iff for all b, c such that $Rabc$ and $b \in N$, $c \Vdash A$, or $c \in |A|$. Note that this

[49] Humberstone [2016, 205-208] calls these frames "semi-simplified".

verification condition displays the point of evaluation in the initial ternary relation, $Rabc$, whereas the verification condition for $\mathbb{N}$ does not. As one would expect given the ubiquity reasoning, $q \to \mathbb{N}(p \to p)$ is ubiquitously true in all models while $q \to (t \to (p \to p))$ is not.[50] The addition of t to the language complicates the notion of variable sharing, since t lacks propositional variables while having widespread implication connections to other theorems. I will return to constants, including t, in §7. The addition of $\mathbb{N}$, on the other hand, provides simple violations of variable sharing, since the formulas it attaches to will contain propositional variables.

No connectives whose verification conditions are only set-theoretic relations between points will be acceptable by lights of my criteria. Instead of mining those connectives for more examples, let us turn to connectives that do display the point of evaluation to eliminate the suspicion that the problems can be pinned simply on that feature of the verification conditions. To begin, let us look at connectives I will call *projections*. Consider the following connectives

- $a \Vdash \mathbb{L}A$ iff $\exists b, c(Rbca$ and $b \Vdash A)$
- $a \Vdash \mathbb{M}A$ iff $\exists b, c(Rbca$ and $c \Vdash A)$

These are projections of fusion, which requires that both of its fuse-juncts be verified at related points. Both of these connectives yield violations of the criteria, although not the same violations. The first connective yields a weakening-like validity, namely $p \to (q \to \mathbb{L}p)$. The second results in a formula ubiquitously true in all models, $p \to \mathbb{M}p$. To see this, suppose there is a point a in a model such that $a \nVdash p \to \mathbb{M}p$. Then there are points b, c such that $Rabc$, $b \Vdash p$, and $c \nVdash \mathbb{M}p$. The last of these requires that there be no points d, e such that $Rdec$ and $e \Vdash p$, which contradicts the assumptions. This suffices for the claimed ubiquity. It then follows that $q \to (p \to \mathbb{M}p)$ is ubiquitously true in all models.

For some logics, such as R, $\mathbb{L}$ and $\mathbb{M}$ will be equivalent. For some, they will not be equivalent but they will both violate the initial criteria. Proponents of some weaker logics may be able to accept $\mathbb{L}$, assuming there are no other routes to violations of variable sharing, but rejecting $\mathbb{M}$ pushes against accepting $\mathbb{L}$ on symmetry grounds.

There are other connectives in the area of the projections that one might consider.

- $a \Vdash \mathbb{O}A$ iff $\forall b, c(Rabc \Rightarrow c \Vdash A)$

This satisfies Heredity. This yields weakening-like validities, such as $\mathbb{O}A \to (p \to A)$. To see this, suppose $a \Vdash \mathbb{O}A$. To establish $a \Vdash p \to A$, suppose $Rabc$ and $b \Vdash p$. From the initial assumption, we have that $c \Vdash A$, which suffices for the desired conclusion.

Another connective in the vicinity of the projections is the following, whose verification condition adds a small permutation to that of $\mathbb{O}$.

- $a \Vdash \mathbb{J}A$ iff $\forall b, c(Rbac \Rightarrow c \Vdash A)$

[50] To see that the latter is invalid, it is recommended that the reader use John Slaney's program MaGIC to generate a falsifying model.

This satisfies Heredity. It also generates violations of variable sharing. As an illustration, suppose that $Rabc$ and $b \Vdash \mathbb{J}A$. From the verification condition, $c \Vdash A$, which suffices to establish $a \Vdash \mathbb{J}A \to A$. This means the conditional $\mathbb{J}A \to A$ is ubiquitously true in all models, from which violations of variable sharing follow right away.

To complicate the picture a bit, let us consider a connective with a higher-order verification condition. Define the following notation: $Rac =_{Df} a \leq c$, $Rabcd =_{Df} \exists x(Rabx \wedge Rxcd)$, $Rabcde =_{Df} \exists x(Rabcx \wedge Rxde)$, and so on. We can use this notation to define an R-chain connective, whose intuitive gloss is that the given formula holds no matter how many steps down the R-relation one goes.

– $a \Vdash \mathbb{R}A$ iff for all $\langle c_1, \ldots, c_n \rangle$, with $n \geq 1$, if $Rac_1, \ldots, c_n$, then $c_n \Vdash A$.

This obeys Heredity. It generates weakening-like validities. For example, we get the following as valid: $\mathbb{R}p \to (q \to p)$ and $\mathbb{R}p \to (q \to \mathbb{R}p)$. Further, for some logics, it generates ubiquitous truths, in particular, for logics whose frames obey the permutation condition $Rabc \Rightarrow Rbac$, such as R. To see this, suppose that $Rabc$ and $b \Vdash \mathbb{R}p$. It then follows by permutation that $Rbac$, so by the verification condition for $\mathbb{R}$, $c \Vdash p$. Therefore, $a \Vdash \mathbb{R}p \to p$, for all points $a \in K$. By the usual reasoning, $q \to (\mathbb{R}p \to p)$ is valid on such frames. In contrast, $\mathbb{R}p \to p$ is valid, although not necessarily ubiquitous, on all frames. This follows from the fact that $a \leq a$, or Raa, so if $a \Vdash \mathbb{R}p$, then $a \Vdash p$.

The connective $\mathbb{R}$ looks down the R-relation in the same direction as in the verification condition for the implication. One might ask about the connective that stands in the analogous relation to fusion, looking backwards down the R-relation.

– $a \Vdash \mathbb{Q}A$ iff there is $\langle c_1, \ldots, c_n \rangle$, with $n \geq 1$ such that $Rc_n, \ldots, c_1 a$, and $c_n \Vdash A$

This generates formulas ubiquitously true in all models. To see this, note that for all $a \in K$, there are $b, c \in N$ such that $Rbaa$ and $Rcbb$, in light of the fact that $a \leq a$ and $b \leq b$. It then follows that $Rcbaa$. As $c \in N$, $c \Vdash p \to p$, so $\mathbb{Q}(p \to p)$ is ubiquitously true.

A notable feature of $\mathbb{L}, \mathbb{M}, \mathbb{O}, \mathbb{R}$, and $\mathbb{Q}$ not shared by some of the earlier connectives is that their verification conditions use the point of evaluation in a substantive way. That was something that we did not see with $\mathbb{U}$, for example. The verification conditions for each of these connectives involves the ternary relation as well, although in two of the cases, not all the points feature in the condition in a substantive way. In the other two cases, there are higher-order quantifications, namely quantification over sequences of points. Both of these features weaken the force of the ternary relation.

Next, consider a connective from the model theory of intuitionistic logic.[51]

– $a \Vdash \downarrow A$ iff $\exists b \leq a, b \nVdash A$

[51] This connective is discussed by López-Escobar [1985].

This connective satisfies Heredity. It is worth examining the argument that it does so, in particular the salient case of the usual inductive argument. For that, suppose that $a \Vdash \downarrow A$ and $a \leq b$. From the initial assumption, there is a $c \leq a$ such that $c \nVdash A$. As $c \leq a$ and $a \leq b$, we have that $c \leq b$, whence $b \Vdash \downarrow A$. The connective satisfies Heredity, but the argument did not appeal to the inductive hypothesis. The reader may be concerned that being able to establish Heredity without appeal to the inductive hypothesis indicates that there may be problems lurking. While such arguments do not always indicate a problem, as showing that the conditional obeys Heredity appeals to a frame condition rather than the inductive hypothesis, in this case such concerns are justified. The connective also generates ubiquitous truths, such as $p \vee \downarrow p$. To see this, suppose that $a \nVdash p$. As $a \leq a$, $a \Vdash \downarrow p$. Thus, we can obtain a violation of variable sharing. One might wonder about connectives that look forwards down the $\leq$-relation. As far as I can tell, these either are unremarkable or do not satisfy the Heredity criterion.

Finally, one might consider alternative approaches to negation. One prominent alternative replaces the Routley star with a binary compatibility relation, C, obeying the condition that if $a \leq c$, $b \leq d$, and cCd, then aCb.[52] The verification conditions for negation using compatibility are

– $a \Vdash {\sim} A$ iff $\forall b(aCb \Rightarrow b \nVdash A)$.

Without further conditions, the resulting negation is weaker than the de Morgan negation obtained from the Routley star. One can obtain Boolean negation by adopting the condition aCa, for all $a \in K$. Just as the * connective generated ubiquitous truths when paired with de Morgan negation under certain conditions, namely that both are interpreted using the Routley star on points and the heredity relation is identity, so can one generate ubiquitous truths with natural connectives paired with negation interpreted using compatibility, with heredity as identity. Define $+$ using the following verification condition.

– $a \Vdash +A$ iff $\exists b(aCb$ and $b \Vdash A)$

One then obtains a formula, $+B \vee {\sim} B$, that is ubiquitously true in all models. This pair of connectives falls short of defining Boolean negation, but that is not needed for the current point. Requiring that the heredity relation be identity is, of course, a strong condition. One can obtain similarly problematic connectives, on the assumption that compatibility relation is serial, $\forall a \exists b(aCb)$. Define $\mathbb{P}$ using the following verification condition.

– $a \Vdash \mathbb{P}A$ iff $\forall b(aCb$ and $b \Vdash A)$

This connective satisfies Heredity, independent of the assumption of seriality. Given seriality, however,, one then obtains a formula, $\mathbb{P}B \wedge {\sim} B$, that is ubiquitously false in all models.

There is a connective that generates the same problems without the assumption of seriality. Let $\uparrow$ be defined via the following verification condition.

[52] See Dunn [1993], Restall [1999, 2000], Berto [2015], and Berto and Restall [2019].

– $a \Vdash \uparrow B$ iff $\forall b(a \leq b \Rightarrow \exists c(bCc \text{ and } c \Vdash B))$

This connective obeys Heredity.[53] The formula $\uparrow B \wedge {\sim}B$ is ubiquitously false in all models. If we suppose that there is a model with a point a such that $a \Vdash \uparrow B \wedge {\sim}B$, it follows that for all b such that aCb, $b \nVdash B$. Since $a \leq a$, there is a c such that aCc and $c \Vdash B$, which is a contradiction.

One can use conditions on compatibility to obtain an intuitionistic-type negation, which will generate violations of the variable sharing criterion. As Robles and Méndez [2018] show, there is an alternative way to modify the Routley-Meyer frames, minus the Routley star, to obtain intuitionistic-type negations that do satisfy the variable sharing criterion.[54] One can impose conditions on those frames to force violations of variable sharing. Indeed, one can add conditions on the Routley star or on compatibility to force violations of variable sharing. It would be enlightening to have some general characterization of what sorts of frame conditions on relations used to interpret connectives are compatible with variable sharing, at least, but that will have to wait for future work.

The criteria set out at the start of this section were motivated by ideas from the philosophy of relevant logics. They do not appear to be overly demanding. Despite this, they have some important consequences. Many connectives that one might consider, or that have arisen in other logical contexts, are ruled out. A few connectives have arisen incidentally above, propositional constants. I will now discuss them briefly.

7 What about constants?

The variable sharing criterion is most intuitive in the context of propositional logic without any constants. At the quantificational level, there is not widespread agreement on how it should be adapted. The presence of propositional constants, or 0-ary connectives, muddies the water at the propositional level.

There are three main constants to consider, although they are not the only constants one might want to consider.

– $a \Vdash t$ iff $a \in N$
– $a \Vdash \top$ iff $a \in K$
– $a \Vdash \bot$ iff $a \notin K$

The constants $\top$ and $\bot$ have the feature of being ubiquitous, being so by definition. They generate many validities that violate the variable sharing criterion, such as $q \to (p \to \top)$ or $\bot \to p$, as neither formula has a variable occurring in its consequent occurring also in its antecedent. One response to this problem is treat the constants as containing all propositional variables, so that both of

[53] Proof: Suppose $a \Vdash \uparrow B, a \leq b$, and $b \nVdash \uparrow B$. It follows that for some c such that $b \leq c$, for all d such that cCd, $d \nVdash B$. Since $b \leq c$ and $a \leq b$, $a \leq c$. As $a \Vdash \uparrow B$, for some e, cCe and $e \Vdash B$, which is a contradiction.

[54] I would like to thank an anonymous referee for pointing this out.

the displayed formulas satisfy variable sharing. This response is motivated by providing an infinitary gloss on the constants, with t being the "conjunction" of all logical truths, $\top$ being the "disjunction" of all formulas, and $\bot$ being the "conjunction" of all formulas. Note that, as the language is not infinitary, these glosses provide only suggestive ideas, not syntactic descriptions. The looser understanding of the relation between propositional variables and the constants just outlined provides a way of seeing the constants as satisfying the variable sharing criterion, albeit by trivializing it for formulas that contain constants.[55]

One can use $\top$ to define some of the problematic connectives of the previous section, such as $\mathbb{O}$, $\mathbb{L}$, and $\mathbb{M}$. The defined versions, unlike their primitive counterparts, will not violate variable sharing on the view just described. Since $\mathbb{M}A$ would be defined as $\top \circ A$, $q \to (p \to \mathbb{M}p)$ would in fact satisfy variable sharing, as every variable occurs, in the looser sense, in the consequent. Note that while some of the problematic connectives could be defined using these constants, not all of them can be. The connectives that cannot be defined in terms of constants generate ubiquitous that still violate variable sharing, and so there will still be reason to reject those connectives.

Do the criteria say anything about $\top$ or $\bot$? Those constants do generate weakening-like validities, such as $\top \to (p \to \top)$. That criterion, however, is dependent on the variable sharing criterion, which does not clearly cut against the constants. Some other concerns from relevant logic may be able to motivate a rejection of these constants. These constants are, by definition, ubiquitous. It is a feature of the constant-free language that for every formula, there is *some* point in *some* model at which it fails. This is an important feature of ternary relational models. Logical truths are not ubiquitously true in all models, although they are valid. This idea, that every formula fails *somewhere*, provides a reason to reject these constants. It is a rejection of ubiquitous formulas at the propositional level.

One might think that in the models, the constants $\top$ and $\bot$ are too natural to omit, much as Meyer thought about Boolean negation. Unlike Boolean negation, it seems that there is room for relevant logicians to differ regarding the constants $\top$ and $\bot$. The die-hard relevantist would deny that there should be truths ubiquitous in all models. It is the nature of the content assigned to formulas that every formula must fail somewhere or other, so the constants should be rejected as illegitimate.[56] The more accepting relevant logician would

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[55] Yang [2013] provides a detailed discussion of the variable sharing property, there called "the relevance principle", in the presence of the propositional constants. Yang's preferred formulation of the property is that if $A \to B$ is a theorem, then either A and B explicitly share a propositional atom or they implicitly share an atom in virtue of the infinitary glosses on the constants as conjunctions or disjunctions of certain kinds. I thank an anonymous referee for suggesting Yang's work on this topic.

[56] Propositional quantifiers have been a part of the relevant logic tradition for a long time, with Anderson et al. [1992] opening with a study of propositional quantifiers and, more recently, Goldblatt and Kane [2009] providing adequate models for a range of relevant logics with propositional quantifiers. The constants $\top$ and $\bot$ can be defined using propositional quantifiers. What should the die-hard relevantist say about propositional quantification?

permit the constants, and the defined versions of some of the problematic connectives, adopting the more permissive conception of variable sharing to accommodate the constants. Note that even if the constants are accepted, the *primitive* versions of, say, $\mathbb{M}$ would still be out, as it leads to violations of variable sharing, even on the more permissive understanding of the notion.

Let us turn briefly to the other constant, the Ackermann constant, t. This constant does not generate ubiquitous truths. It does, however, muddy the waters of the variable sharing criterion. To make it so that, for example, $t \to (p \to p)$ is acceptable, one must say that t counts as containing every propositional variable. It does, after all, entail every theorem, and so can be viewed as a kind of infinitary conjunction.[57] It is less straightforward to see how even a die-hard relevantist would object to t. One might deny that *any* formula bears substantive connections to formulas with a disjoint set of propositional variables, at least connections substantive enough to support valid implications, while insisting that the Ackermann constant contains no propositional variables. I will leave defense of such a notion of substantive connection to relevant logicians who wish to reject t. It appears to pass the criteria I have provided, so, at least here, it seems like a connective the relevantist could accept.

Let us turn from constants to a brief discussion of other frames for relevant logics.

8 What about other frames?

In studying intuitionistic logic, one often uses Kripke frames, and some of the work cited above uses Kripke frames in an important way. Kripke frames are not, however, the only sort of frame used to study intuitionistic logic.[58] The situation is similar with relevant logics. The discussion of relevant connectives so far has employed Routley-Meyer frames. While these are generally the preferred frames for relevant logics, there are other options out there, such as the semilattice frames of Urquhart [1972], the operational frames of Fine [1974, 1988], the simplified frames of Priest and Sylvan [1992], and the collection frames of Restall and Standefer [20xx], as well as the American plan models of Routley [1984]. There is a worry that the assessment of connectives in a given frame theory will be too parochial. Indeed, other kinds of frames might suggest different sorts of connectives to consider. These are, I think, reasonable worries about presentation-dependent criteria, but I will note that the criteria

That is not clear. Neither is it clear what the analog of the variable sharing criterion should be in that context.

[57] For the logic R, there is a way of replacing the Ackermann constant. Anderson and Belnap [1975, 362-363] says that, when the language has only finitely many propositional atoms, $p_1, \ldots, p_m$, then one can use $(p_1 \to p_1) \wedge \ldots \wedge (p_m \to p_m)$ instead of t. This does not hold in weaker logics. For example, $(q \to q) \to (p \wedge q \to q)$ is a theorem of B but $((p \to p) \wedge (q \to q)) \to ((q \to q) \to (p \wedge q \to q))$ is not, as can be shown using MaGIC to obtain a countermodel.

[58] See, for example, Bezhanishvili and Holliday [2019].

proposed in the previous section are mostly not presentation-dependent. While I will not be able to assuage these worries in generality, it will be instructive to briefly look at one other sort of frame, namely Urquhart's semilattice frames, to see how different connectives might fare with respect to our criteria.[59]

Definition 6 (Semilattice frame, model) *A semilattice frame is a triple* $\langle K, \sqcup, 0\rangle$, *where* $0 \in K$, $\sqcup : K \times K \mapsto K$, *satisfying the following conditions.*

- $0 \sqcup x = x$
- $x \sqcup x = x$
- $x \sqcup y = y \sqcup x$
- $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$

A semilattice model M *is a pair of a semilattice frame* F *with a valuation* $V : \mathsf{At} \times K \mapsto \mathbf{2}$. M *is said to be a model on the frame* F.

A valuation is extended to a verification relation on the whole language inductively as follows.

$x \Vdash p$	iff	$V(p, x) = 1$
$x \Vdash A \wedge B$	iff	$x \Vdash A$ and $x \Vdash B$
$x \Vdash A \vee B$	iff	$x \Vdash A$ or $x \Vdash B$
$x \Vdash A \to B$	iff	$\forall y(y \Vdash A \Rightarrow x \sqcup y \Vdash B)$
$x \Vdash A \circ B$	iff	$\exists y, z(x = y \sqcup z$ and $y \Vdash A$ and $z \Vdash B)$

Definition 7 (Holding, validity) *A formula* A *holds in a model* $\langle K, \sqcup, 0, V\rangle$ *iff* $0 \Vdash A$.

A formula A *is valid iff it holds in all models on all frames.*

UR *is the set* $\{A \in \mathfrak{L} : A$ *is valid*$\}$.

The criteria for being a relevant connective in the context of the semilattice logic UR will be much the same as in the previous section.[60] One simplification in the context of semilattice frames is that no Heredity criterion is needed, so we can focus on the two presentation-independent criteria. Much of our earlier discussion is still in force. In particular, the points about ubiquity remain, so connectives that generate too many ubiquitous formulas in too many models must be rejected.

There are some connectives that are particular to semilattice frames.[61]

- $x \Vdash \mathbb{B}A$ iff $\forall y \in K, x \sqcup y \Vdash A$
- $x \Vdash \mathbb{C}A$ iff $\forall y \in K, y \sqcup x \Vdash A$

It turns out that both of these must be rejected, despite the fact that the point of evaluation is used in a substantive way in both. The problem is that $\mathbb{C}$ yields a violation of variable sharing, namely $q \to (\mathbb{C}p \to p)$. To see this, suppose that $0 \nVdash q \to (\mathbb{C}p \to p)$. Then for some x, $x \Vdash q$ and $0 \sqcup x \nVdash \mathbb{C}p \to p$. This implies that there is a y such that $y \Vdash \mathbb{C}p$ and $x \sqcup y \nVdash p$. The former

[59] See Standefer [2022a] for an overview of recent work on semilattice frames.

[60] UR properly extends the positive fragment of R, so it technically falls outside the demarcation set in §5. We can, as shown by Humberstone [1988], modify the semilattice frames to bring the resulting logic into alignment with R by using an additional operation and adjusting the verification condition for disjunction. This modification has important connections to other areas of logic, some of which are explored by Humberstone [2018]. We will stick with the unmodified semilattice frames here, for the simplified presentation.

[61] I thank Lloyd Humberstone for proposing these connectives to me.

implies that for all z, $z \sqcup y \Vdash p$, which results in a contradiction, $x \sqcup y \Vdash p$. The connective $\mathbb{B}$ is equivalent to $\mathbb{C}$ in light of the frame conditions.

There are weaker operational frames obtained by dropping some of the frame conditions. In those frames, $\mathbb{C}$ still generates violations of variable sharing, as can be seen by inspecting the proof above. $\mathbb{B}$ generates weakening-like validities, such as $\mathbb{B}p \to (q \to p)$.

There are analogs of the projection connectives in the semilattice frame setting.

- $x \Vdash \mathbb{G}A$ iff $\exists y, z \in K(x = y \sqcup z$ and $y \Vdash A)$
- $x \Vdash \mathbb{H}A$ iff $\exists y, z \in K(x = y \sqcup z$ and $z \Vdash A)$

As $0 \Vdash p \to p$ and $0 \sqcup x = x$, for all $x \in K$, $\mathbb{G}$ yields violations of variable sharing. The formula $\mathbb{G}(p \to p)$ is ubiquitously true in all models, so $q \to \mathbb{G}(p \to p)$ is as well. Given the permutation condition, $\mathbb{H}$ is equivalent to $\mathbb{G}$, but it is worth noting that even without that condition, $\mathbb{H}$ will generate violations of variable sharing. Suppose $0 \nVdash q \to (p \to \mathbb{H}p)$. Then there is an x such that $x \Vdash q$ but $0 \sqcup x \nVdash p \to \mathbb{H}p$. This implies that there is a y such that $y \Vdash p$ but $x \sqcup y \nVdash \mathbb{H}p$, but the former contradicts the latter in light of the verification condition for $\mathbb{H}$.

There is one more connective to consider in this section. The difference operator, $\mathbb{D}$, from classically based modal logic.[62]

- $x \Vdash \mathbb{D}A$ iff $\exists y \in K(x \neq y$ and $y \Vdash A)$

This verification condition does not, in general, work in the context of Routley-Meyer frames because it violates Heredity. In some variant frames, such as for the simplified semantics, one can do away with the heredity ordering for many logics, and for those frames, one can use this verification condition.[63] The difference operator yields ubiquitous truths, such as $\mathbb{D}(p \to p) \vee (p \to p)$. To see this, note that $0 \Vdash p \to p$, so for every point x, either $x = 0$, in which case $x \Vdash p \to p$, or $x \neq 0$, in which case $x \Vdash \mathbb{D}(p \to p)$. Violations of variable sharing then follow. One might expect from the meaning of the difference operator that it should produce some irrelevancies, and this result bears out that expectation.

As we can see, alternative frames permit the interpretation of some new connectives. Many are analogs of connectives that we saw on the Routley-Meyer frames. While I will not consider the other types of frames in detail here, I expect that the situation with them will be similar. The considerations of the preceding two sections have, for the most part, proceeded with few additions to the frames. Further augmentation of the frames, such as equipping them with binary modal accessibility relations, neighborhood functions, or a host of other standard logical devices would open the gates to more connectives still.

62 See Blackburn et al. [2002, 421 ff., 480-481].

63 See Priest and Sylvan [1992], Restall [1993b], or Priest [2008] for more on simplified semantics without the ordering. See the latter or Restall and Roy [2009] for more on which logics appear to need the ordering.

9 Conclusions

In §6, I introduced three criteria for being a relevant connective. Two of these were presentation-independent, with the third being particular to the frame theory adopted. These criteria are comparatively *simple* and distinctively *relevant*, not being plausible, for example, from the point of view of the classical logician. These criteria are fairly minimal, acceptable to most any relevant logician. Nonetheless, they are far from trivial, and in fact provide substantial constraints, as they rule many plausible connectives as being in violation of the relevant logical philosophical views. Many connectives do not respect the requirement that there be a substantive connection between the antecedent and consequent of a valid conditional. These connectives will not even be weakly relevant.

Let us summarize some of the features of connectives that get in the way of them being relevant connectives. The first thing is that the verification condition for a connective must involve the initial point of evaluation in a substantive way, which we can understand here as meaning that the condition is not equivalent to one that does not feature the initial point. There are familiar ways of using logically equivalent forms that can ensure that the initial point occurs, albeit in an inessential way.[64]

One feature of verification conditions that can signal that something is amiss, even when the point of evaluation is used in an essential way, is the argument that the connective satisfies Heredity. For the many of standard connectives, one appeals to the inductive hypothesis in showing Heredity is satisfied. Failure to cite the inductive hypothesis is not dispositive, however, as the salient case for the conditional does not and that connective is a paradigm relevant connective. For some connectives, such as actuality and $\downarrow$, we see that the argument that the connective obeys Heredity does not appeal to the inductive hypothesis in an essential way. While it may seem like a small point, failure to appeal to the inductive hypothesis can be an indication that something is amiss. It can indicate that the truth of the whole does not depend, in appropriate ways, on the truth of the parts at suitably related points. If that dependence is lacking, there is a danger that the connectives generate ubiquitous truths.

Another idea suggests itself from a survey of the results so far. If the verification condition for a connective features quantification into the ternary relation, or a chain of ternary relations, then for each quantified point there should be some subformula evaluated there. This distinguishes fusion from the projection connectives, for example. We saw some cases where violating this stricture led to violations of variable sharing, as with $\mathbb{J}$, but there are cases where it seems only to lead to weakening-like validities, as with $\mathbb{O}$.

The preceding idea does not address the issue with $\mathbb{R}$. The verification condition quantifies over *sequences of points* rather than just points. The condition

[64] Equivalent forms that use inessential variables comes up elsewhere in relevant logics, such as Dunn [1987, 351-352]

does not evaluate a subformula at every of those quantified points, but the condition could be rephrased to do so conjunctively without changing the issue. A similar point holds for $\mathbb{Q}$, albeit disjunctively rather than conjunctively. This suggests that connectives with infinitary or higher-order verification conditions will violate the sort of locality constraints that are important for capturing relevance in the models.

Restall [2000, 259 ff.] and Badia [2016] present relevant directed simulations on Routley-Meyer models to obtain some preservation and expressibility results similar to results in the area of classically based modal logics. Badia uses a basic relevant language that includes $\bot$ and negation, and so $\top$.[65] Connectives definable in the language will be preserved as a consequence of the general preservation results proved. In particular, $\mathbb{L}$ and $\mathbb{M}$ will be preserved, as they are definable in the language. Therefore, directed simulations do not preserve the sense of substantive connection at the heart of relevant logics. Consequently, relevant directed bisimulations do not seem like a good tool for delineating the relevant connectives.[66] They can do many things, but they do not isolate the connectives that satisfy my criteria.

Many of the considerations raised for connectives interpreted on Routley-Meyer models hold as well for connectives interpreted on other sorts of models. There may, however, be some important differences, as illustrated by the semilattice models. One important difference is that they lack a heredity ordering, which provides some additional flexibility in interpreting connectives. There are variations on and subclasses of Routley-Meyer frames that also lack a heredity ordering, such as the flat frames used for Boolean negation or some forms of simplified semantics.[67] Another important difference is that a semilattice frame has a single distinguished point that has distinctive features. The distinctive logical behavior of the distinguished point features prominently in the problem with $\mathbb{D}$. These features lead to violations of variable sharing from the projection connective $\mathbb{G}$, although that one violates the earlier requirement that a subformula be evaluated at each quantified point. Upon closer inspection, that is plausibly the problem with $\mathbb{B}$ and $\mathbb{C}$. While a subformula of $\mathbb{B}A$ is evaluated at $x \sqcup y$, no subformula is evaluated at y, the absence of which is perhaps obscured by the operational notation. An analogous point holds for $\mathbb{C}A$.

I will close with two promising strategies for introducing connectives on Routley-Meyer frames. The first, following Bimbó and Dunn [2008], is to interpret an n-ary connective using quantifiers and an $(n+1)$-ary relation. Following the ideas set out so far, one of the $(n+1)$ spots in the relation should be the initial point of evaluation and each of the remaining quantified points

[65] Although Badia does not consider fusion in the language, Restall does consider fusion in the language when presenting the simulations. Restall's results for fusion will carry over to Badia's setting.

[66] Cf. the discussion and role of bisimulation in modal logic, as presented by, e.g. Blackburn et al. [2002].

[67] For the former, see Meyer and Routley [1973, 1974]. For the latter, see Priest and Sylvan [1992].

should be used to evaluate a subformula.[68] Once the relation moves outside of the normal points, even logical truths can fail. This possibility of failure, mentioned earlier in the context of the constants, seems to be important in modelling the substantive connections imposed by relevant implications. While good as a heuristic, this is not sufficient to ensure the resulting connective is a relevant one, since both $\downarrow$ and $\mathbb{D}$ have the desired features but violate variable sharing.

The second strategy follows that of the simplified semantics. For this, some detail on the simplified semantics will be useful. A simplified frame is a quadruple $\langle K, N, R, ^* \rangle$, where $K \neq \emptyset$, $N \subseteq K$, $^* : K \mapsto K$, $a^{**} = a$,and R is a ternary relation on K.[69] A model is a pair of a frame and a valuation $v : \mathsf{At} \times K \mapsto \mathbf{2}$, which extends to a verification relation on the whole language much as with Routley-Meyer models, although we will ignore fusion here. The twist comes with the verification condition for the conditional, which splits into two parts, one for normal points and one for the rest.

- $a \Vdash B \to C$ iff for all $b \in K$, if $b \Vdash B$ then $b \Vdash C$, provided $a \in N$.
- $a \Vdash B \to C$ iff for all $b, c \in K$, if $Rabc$ and $b \Vdash B$ then $c \Vdash C$, provided $a \notin N$.

One can interpret an n-ary connective with whatever verification condition one wants on the normal points and on the non-normal points, one can use an $(n+1)$-ary relation and quantifiers in a different condition to ensure that no violations of variable sharing arise. As an example, one could split the verification condition for the universal modality as follows, where S is a binary relation on K.

- $a \Vdash \mathbb{U}B$ iff for all $b \in K$, $b \Vdash B$, provided $a \in N$.
- $a \Vdash \mathbb{U}B$ iff for all $b \in K$, if Sab then $b \Vdash B$, provided $a \notin N$.

One can then construct a model for which $\mathbb{U}p \wedge {\sim}\mathbb{U}p$ holds at a point. For this, we must augment the simplified frame with a binary relation S. Let $K = \{a, b, c\}$, $N = \{a\}$, $a^* = a$, $b^* = c$, $c^* = b$, $Raaa$, $Rabb$, $Racc$, Sbb, and Scc. Set $v(p, a) = v(p, b) = 1$ and $v(p, c) = 0$. In this model, $b \Vdash \mathbb{U}p$ and $b \Vdash {\sim}\mathbb{U}p$. So, the formula $\mathbb{U}p \wedge {\sim}\mathbb{U}p$ is not ubiquitously false, so the particular violation of variable sharing from §6, $\mathbb{U}p \wedge {\sim}\mathbb{U}p \to q$ can be invalidated. This strategy seems promising, although, if the Ackermann constant is in the language, one can still find formulas ubiquitously false in all models. An example of such is $t \wedge \mathbb{U}p \wedge {\sim}\mathbb{U}p$. The viability of either of these strategies will, however, have to wait on future work.

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[68] This idea is suggested, in a restricted form, by Brady [2003, 390-392] and briefly on p. 36 in the context of adding intensional connectives.

[69] This presentation follows that of Priest [2008, ch. 10] in permitting a non-singleton set of normal points.

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