Is the classical limit “singular”?

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Abstract

We argue against claims that the classical $\hbar \to 0$ limit is “singular” in a way that frustrates an eliminative reduction of classical to quantum physics. We show one precise sense in which quantum mechanics and scaling behavior can be used to recover classical mechanics exactly, without making prior reference to the classical theory. To do so, we use the tools of strict deformation quantization, which provides a rigorous way to capture the $\hbar \to 0$ limit. We then use the tools of category theory to demonstrate one way that this reduction is explanatory: it illustrates a sense in which the structure of quantum mechanics determines that of classical mechanics.

Contents

1 Introduction

One often finds skepticism about the significance of the classical $\hbar \to 0$ limit of quantum mechanics in the foundational literature. For example, Berry (1994), Batterman (2002), and Bokulich (2008) all claim that the classical limit is "singular" in a way that is supposed to frustrate an attempted intertheoretic reduction between classical and quantum physics. A central aspect of the issue appears to be that quantum mechanics (on its own) may not allow for the definition of the structure of classical physics. These authors seem to identify a barrier to taking an eliminativist attitude towards classical physics by arguing that any analysis of the recovery of classical from quantum physics requires prior reference to classical concepts. For example, Batterman (1995, 2002) emphasizes the importance of semi-classical approximations, claiming that they require extra information from classical physics beyond what one can glean from the representation of the system in quantum theory. In this paper, we argue—contra Berry, Batterman, and Bokulich—that there is a strong sense in which the $\hbar \to 0$ limit defines the structure of classical physics without making prior reference to it. Classical structure is, indeed, eliminable (although we stop short of advocating for its elimination!).

Of course, there is much more that may be of philosophical interest in the works of Berry, Batterman, and Bokulich beyond our point of contention here. All three authors draw attention to the ways semi-classical approximations may give rise to novel forms of explanation that draw on classical concepts. We do not wish to dispute those broader issues, and instead we only take up the specific claim that the $\hbar \to 0$ limit is "singular" in some sense that blocks more familiar forms of explanation of classical behavior. Quite generally, we are happy to take a pluralistic attitude and enjoy the understanding gleaned by many different kinds of explanation. Our purpose in this paper, however, is to investigate in more detail the kinds of reductive explanations whose impossibility Batterman takes to motivate considering the other explanatory forms.

Our argument builds on ideas from Butterfield (2011a), who claims that limiting operations often allow one to define the formalism of less-fundamental or "top-level" theories from that of more-fundamental or "bottom-level" ones. In particular, these definitions are often *explicit*: roughly, they allow one to eliminate reference to the top-level theory. One way of defining classical structure in terms of quantum structure is provided by Landsman (2013), who uses previous work in the framework of strict deformation quantization (e.g., Landsman, 1998a, 2007) to formulate a convergent $\hbar \to 0$ limit.[1] However, a staunch anti-eliminativist may saliently note that Landsman's construction of the limit makes prior reference to the classical theory that it seeks to derive. Our central contribution in this paper is to use the framework of strict deformation quantization to prove several results that show that this reference is eliminable. Specifically, we show that classical kinematics is definable from only the structure of quantum kinematics and the scaling behavior of physical quantities in quantum theories. We also present some preliminary results for dynamics but leave a detailed discussion for future work.

[1] The focus of the discussion in Landsman (2013) is the limiting behavior of symmetry-breaking states, but along the way he notes that the $\hbar \to 0$ limit can be used to recover the kinematics and dynamics of classical theories.

Our results lay the groundwork for a re-assessment of the explanatory role of the classical limit. Again, we adopt a pluralist attitude towards this role, one that has room enough for both eliminative reduction and Berry, Bokulich, and Batterman's coordination of theoretical structures. That is, we do not wish to critique these authors' positive contribution—we just seek to show that they are tossing out the champagne with the cork. Still, while our explicit definition of classical structure provides a kind of explanation of that structure (i.e., a deductive-nomological one), an opponent may reasonably demand a more telling explanation. In particular, it is not immediately clear whether such definitions determine the structure of classical physics from quantum physics alone. We argue that this kind of structural determination relation does hold by using the mathematical tools of category theory, building on previous philosophical applications of these tools to structural comparisons of theories (Halvorson, 2012; Weatherall, 2016a, 2017; Barrett, 2018; Hudetz, 2019b; Rosenstock et al., 2015). We use these tools to show how the physically significant structure of quantum physics determines the physically significant structure of classical mechanics.

The paper proceeds as follows. In §2, we discuss the philosophical debate over singular limits and pinpoint how we aim to use the $\hbar \to 0$ limit to defuse two challenges that they pose to the eliminativist. In §3, we introduce the tools of strict deformation quantization, which we will use to analyze the relationship between classical and quantum mechanics in the $\hbar \to 0$ limit. In §4, we summarize existence and uniqueness theorems for the $\hbar \to 0$ limit in this framework, theorems which we argue provide definitions of classical kinematics without making prior reference to classical concepts. §5 provides a discussion of the corresponding definitions for dynamics. The interlude in §6 summarizes the results thus far, and the significance of our explicit definitions for the debate concerning reduction. The remainder of the paper argues further that the classical limit is explanatory in the sense that it determines the structure of classical physics from that of quantum physics. In §7, we introduce the tools of category theory to prepare for our remarks on the explanatory status of the classical limit. In §8, we analyze a functor representing the classical limit from a category of quantum theories to a corresponding category of classical theories, characterizing features relevant to the central issue of structural determination. Finally, we conclude in §9 with further discussion of the philosophical interpretation of the technical results.

2 Inter-theoretic Reduction

This paper addresses whether classical mechanics reduces to quantum mechanics—in particular, whether quantum theory can recover classical physics without making prior reference to it. Such a reduction would provide one way to do classical physics using only quantum-mechanical tools. This project carries philosophical interest for *eliminativists*, i.e., those who follow Feyerabend (1981) in advocating that we ought to eliminate reference to less-fundamental theories (at least in some cases). The sort of eliminativist who interests us takes the terms of classical physics to be vague, ill-defined, or otherwise unsatisfactory within a quantum paradigm. Nonetheless, they recognize that the functional role of these terms in physical practice is too fruitful to abandon completely. We do not argue for eliminativism here—indeed, we remain skeptical of the view's ultimate tenability. But the view is a serious and interesting one, and mere skepticism is not an argument against it. So our aim in this paper is to assess whether two prominent arguments against eliminativism succeed.

These arguments arise from problems with so-called "singular limits" (Berry, 1994; Batterman, 2002; Bokulich, 2008). We isolate two related challenges that one might take such limits to pose. The first is that singular limits might render it impossible to *define* classical physics in a particular way—namely, as describing the behavior of quantum systems in an appropriate limit. The second is that such a definition (if possible) might never, on its own, impart a proper *understanding* of the physics. We will argue that neither challenge succeeds.

2.1 Definability

To address the first challenge, we follow the familiar Quinean strategy of explication (2013, pp. 239–240). This strategy aims to give an *explicit definition* of each classical term using only quantum terms. Such a definition amounts to an "if and only if" statement with a classical term on one side and some expression using only quantum terms on the other. In the language of philosophers, such a definition establishes that the two expressions co-refer.[2] As such, the quantum terms fully recover the functional use of the classical ones. With such a definition in hand, an eliminativist is free to claim that only quantum concepts have physical meaning; one can recover the usage of the older theory's terms while denying their physical significance.

As Butterfield (2011a) notes, such explicit definitions are precisely what we use to establish Nagelian reductions (Nagel, 1961, 1998). Suppose we have some "top-level" or less-fundamental theory T_t that we wish to reduce to a "bottom-level" or more-fundamental theory T_b. Recall that the Nagelian strategy seeks to deduce T_t from T_b in conjunction with bridge laws that link the terms of the two theories. These laws may be expressed as a set of explicit definitions, D, yielding the schema (Butterfield, 2011b, p. 1096)

$$T_b \wedge D \Rightarrow T_t. \tag{E}$$

Henceforth, we will refer to (E) as *the eliminativist's schema.* While we take this schema to provide a useful and general framework, we do not take it to give the final word on reduction.

In particular, (E) does not make clear how we ought to go about deriving the top-level theory. One fruitful strategy in mathematical physics aims to obtain the less fundamental theory as an asymptotic limit of the more fundamental theory. Drawing inspiration from Nickles (1973), we can view this strategy as an instance of the eliminativist's schema. Roughly, Nickles substitutes the set D above with the limit of some parameter ϵ in the bottom theory to obtain (Batterman, 2002, p. 18) what we call *Nickles' schema*:

$$\lim_{\epsilon\to 0} T_b = T_t. \tag{E'}$$

While Nickles argues that instances of the schemas (E) and (E') have different philosophical functions, these differences will not matter for what follows since both schemas may be used to capture explanations of T_t from T_b (Nickles, 1973, fn. 4, p. 185). In our case where T_b is quantum mechanics, T_t is classical mechanics, and $\epsilon = \hbar$, Nickles' schema says that the limit as $\hbar \to 0$ of quantum mechanics is classical mechanics. So long as the limiting procedure does not reference any classical terms, the schema ought to satisfy the eliminativist.

Batterman, however, argues that Nickles's schema "fails to hold" for classical and quantum physics (2002, p. 79). His argument goes roughly as follows. Consider one straightforward way of defining classical states in terms of quantum states: namely, by taking the $\hbar \to 0$ limit of wavefunctions that solve the Schrödinger equation. As Batterman notes in his motivating example of the WKB approximation, it turns out that semi-classical physics is often fruitfully approximated by wavefunctions of the form $\psi(x) = ae^{iS(x)/\hbar}$ for some real-valued function S on configuration

[2] Given that we ultimately aim to construct definitions using algebraic objects, the fit with the usual philosophical vocabulary—formulated in terms of the first-order predicate calculus—will be rough. Nonetheless, we find the notion of explicit definability instructive enough to make this imperfect fit worth using. Moreover, we take our liberal usage of logical terminology to mirror that of Butterfield (2011b). For a precise formulation of the notion in the usual logical language, see Butterfield (2011a, p. 949).

space (and complex constant a). Such functions *strictly diverge* in the $\hbar \to 0$ limit. So, according to Batterman, Nickles' schema fails to hold: for these limits, Batterman claims it is *not* the case that $\lim_{\epsilon \to 0} T_b = T_t$. On the contrary, only small-ϵ solutions (or approximations thereof) matter for the physics.

Batterman recognizes that, in the absence of convergence, he needs to provide some other explanation for why "bottom up" constructions like the $\hbar \to 0$ limit ought to link up in any meaningful way with classical physics. He suggests that reference to the classical theory ought to do the justificatory work—explicitly, via the "top down" construction of wavefunctions of a similar form directly on the classical phase space (Batterman, 2002, pp. 103–104).[3] If the recovery of classical concepts includes these divergent cases—cases that require reference to classical concepts for their justification—then it cannot provide the definitions the eliminativist seeks.

We do not wish to criticize this explanatory strategy. We are staunch pluralists about explanation: insofar as the co-ordination of top- and bottom-level structures yields a powerful (and fruitful) explanatory strategy, we are glad to add it to our toolkit! But when evangelizing the power of this schema, is Batterman too quick to reject Nickles' schema? There is good reason to think so, as the argument above only appeals to one popular limiting procedure. It seems well worth investigating whether other procedures yield convergent limits. Of course, convergence on its own is not enough for the eliminativist. We will also have to check whether such procedures succeed in avoiding reference to the classical theory.

It turns out that it is possible to construct strictly converging $\hbar \to 0$ limits, but it is not immediately clear that such constructions satisfy the eliminativist's needs. We have in mind Landsman's (1998a; 2007; 2013) definitions of convergent $\hbar \to 0$ limits using the framework of strict deformation quantization. Landsman interprets these limits as determining the kinematics and dynamics of classical theories from those of quantum theories.[4] But couched in the above dialectic, it is easy to spot a weakness in his approach that the anti-eliminativist may exploit: Landsman *refers to the classical theory* in his construction of the $\hbar \to 0$ limit. That is, a strict deformation quantization by definition (as we shall see next) already explicitly contains a classical algebra of quantities given by smooth functions on a Poisson manifold. So while Landsman's reduction seems to satisfy Nickles' schema (E'), it is not clear that it satisfies the eliminativist's schema (E), and hence it is not clear that it provides what is necessary for eliminating reference to classical physics.

Our first contribution in the present work is to show how to modify Landsman's $\hbar \to 0$ reduction such that it satisfies the eliminativist's schema (E). Using several results within the strict deformation quantization framework in Steeger and Feintzeig (2021), we show that reference to the classical theory is eliminable. Specifically, we show that classical kinematics is explicitly definable from only the structure of quantum kinematics and the scaling behavior of physical quantities in the quantum theory. We demonstrate some preliminary results for the recovery of dynamics, but we leave a more detailed investigation—as well as a discussion of states—for future work.[5]

[3] Landsman (2007) notes that this method has connections to an alternative to strict quantization known as geometric quantization, which we will not treat here. Note, as well, that we have switched Batterman's labels of "top down" and "bottom up" to match Butterfield's "top" and "bottom" theories. After all: "Sometimes down is up. Sometimes up is down. Sometimes, when you're lost, you're found."

[4] See especially the discussion of the "two questions" in Landsman (2013, p. 381).

[5] Indeed, as shown by the WKB solution, states must be understood differently to make sense of convergence in the $\hbar \to 0$ limit. As Landsman (2013) shows, a promising way forward is to understand states algebraically, i.e., directly on the dynamical C*-algebra of interest. His work suggests that convergence results for these states, where obtainable, will need to be formulated on a case-by-case basis. See also Landsman (2007) for a discussion of the classical limit of WKB states in terms of strict deformation quantization.

2.2 Understanding

But our response to the challenges of "singular limits" cannot end here. While Batterman (2002) does claim that Nickles' schema fails to hold, he may ultimately be less concerned with the schema's technical feasibility than with the understanding that it can provide. This attitude emerges most clearly in the conclusion of his argument against the schema:

> It seems that most investigators maintain a reductionist *and even eliminativist* attitude toward classical physics given the many successes of the quantum theory. Of course, it is just this attitude that I have been at pains to question. [...] Given that *the understanding* of various observed and observable universal behaviors "contained in" the finer theories ([e.g.,] quantum mechanics) *requires reference* to structures present only in the coarser theories to which they are related by asymptotic limits, it is indeed difficult to see how this attitude can coherently be maintained. (Batterman, 2002, p. 111, emphasis ours)

As we emphasize in the quote above, Batterman is chiefly concerned about achieving adequate *understanding* of the physics. So even when we find limiting relations and definitions satisfying the eliminativist's schema, Batterman would presumably think that it could not engender the right sort of understanding.

This point is helpfully clarified in a dialogue between Batterman and Belot. Belot (2005) also notes that the mathematical formalism of the less fundamental theory is often definable from the formalism of the more fundamental theory (although he focuses on a different example of Batterman's: geometrical and wave optics). Batterman (2005), however, asserts in his response to Belot that definability is not his central concern. He insists that what is at issue is the explanatory status of this relationship. The motivation for investigating the relationship in the first place, he claims, derives from the physical interpretation of extra information (e.g., initial or boundary conditions) coming essentially from the less fundamental theory. He writes:

> Thus, although Belot's pure mathematician can be given initial conditions and boundary conditions and be told to investigate the asymptotics of the relevant partial differential equation, those initial and boundary conditions are not devoid of physical content. They are 'theory laden'. And, the theory required to characterize them as appropriate for [the higher level phenomena] in the first place is the [less fundamental theory]. The so-called 'pure' mathematical theory of partial differential equations is not only motivated by physical interpretation, but even more, one cannot begin to suggest the appropriate boundary conditions in a given problem without appeal to a physical interpretation. In this case, and in others, such suggestions come from an idealized limiting older (or emeritus) theory. (Batterman, 2005, p. 159)

Batterman emphasizes here that the issue of whether the more fundamental theory yields understanding is distinct from the question of whether the less fundamental theory is definable from it. He insists that one requires a physical interpretation coming from the less fundamental theory in order to motivate and understand the significance of the appropriate conditions, operations, and analyses that lead one to the less fundamental theory. He ultimately draws the following moral:

> The deeper issue here concerns questions about the nature of applied vs. pure mathematics. I hold that the pure mathematician Belot envisages is mythical. In the context of partial differential equations, I doubt very much that it is possible to separate the pure mathematics from the physical interpretation. (Batterman, 2005, p. 163)

These responses help to identify common ground between Batterman and Belot. Both seek an answer to the why-question, "Why this top-level behavior?" Batterman demands that a relevant answer show how the *physically interpreted mathematical structure* of the bottom-level theory determines that of the top-level theory, and we take Belot to accept this demand (charges of "pure mathematics" notwithstanding). However, these responses also suggest a sense in which Batterman and Belot might be interested in different interpretative tasks.

Following Dewar (2017), we take physical interpretation to involve at least two tasks: the "internal" task of identifying which bits of mathematical structure are *physically significant*, and the "external" task of specifying what they are significant *for*. The external task is so-called because it requires specifying some connections between the theory and the world, between the theory and experiments, or between the theory and particular sets of data.

We take the distinction between internal and external aspects of interpretation to be a fuzzy one. Sometimes, what seems like an external interpretive task can be partly accomplished through internal means. (We will give an example of this later by encoding the units of physical quantities, which one might well take to be a part of the external interpretation of a theory, in terms of the internal structure of a mathematical object, viz., a bundle of C*-algebras.) Nonetheless, this fuzzy distinction is useful for sharpening the question of whether an eliminative classical-to-quantum reduction is possible.

There is reason to believe that the internal aspects of interpretation are what are at issue in Batterman's opposition to classical-quantum reduction, but we do not wish to make the strong claim that Batterman only has internal aspects of interpretation in mind. Indeed, one might plausibly take external tasks to be the root of Batterman's disagreement with Belot over the case of geometrical and wave optics. For example, one might couple the idea that observations are "theory laden" to a causal theory of reference, thereby rendering reference to the classical theory an *analytic consequence* of terms at play. Very roughly, a reader with this view might note that the "spherical boundary conditions" imposed on raindrops originated in the classical theory and conclude that that theory is an ineliminable part of term's reference. These philosophical commitments squarely concern what the mathematical structures of the relevant theories are physically significant *for*. We do not wish not to challenge such commitments. We merely wish to note that they are substantial and largely separable from internal tasks.[6] By focusing on internal aspects of interpretation, we ask whether reduction is possible while avoiding one way of ruling it out from the start—namely, with philosophical views that are (plausibly) relegated to external aspects.

At the very least, we hope that this sharpening of the question pinpoints a part of Batterman's view that needs further elaboration: precisely which aspects of the interpretation of classical physics he thinks are necessary for the explanation of interest. Regardless of how that view shakes out, it at least motivates the question of whether the internal aspects of the interpretation of classical physics can be determined by the internal aspects of the interpretation of quantum physics. That question is where our present interest lies.

Note well that on our pluralist approach, an affirmative answer to this question would afford one good explanation of classical behavior among many. On our approach, the reader is free both to view quantum theory as fundamental and to accept that classical theory offers explanatory value. We are thus sympathetic to a view that Bokulich articulates in response to criticisms from Belot and Jansson (2010) (that are similar to the ones that Belot levies against Batterman, above):

> Even though classical mechanics is not the true fundamental theory, there are impor-

[6] Another example of an external commitment that might obscure the internal question is provided by a (broadly) Bohrian approach measurement—i.e., one that requires macroscopic observables to be rendered in classical terms. For a modern discussion of such an approach, see Landsman (2007, 2017).

tant respects in which it gets things right, and hence reasoning with fictional classical structures within the well-established confines of semiclassical mechanics, can yield explanatory insight and deepen our understanding. (Bokulich, 2017, p. 206)

We agree that classical structures can yield unique explanatory insight. But we are omnivorous (and gluttonous). We would be loathe to leave a good reduction on the table. So if Bokulich endorses the claim that Nickels's schema cannot provide an adequate understanding of the classical limit (as one might read her (2008) as doing), then our views diverge.

To show that it can, we make the internal aspects of interpretation explicit with the language of category theory. As Dewar (2017) notes, internal tasks can be aided, if not accomplished, by specifying when apparently mathematically distinct models encode the same physically significant information (in an appropriate sense). The choice of category theory for this task is inspired by fruitful recent work applying the tool to a host of questions in philosophy of science. Halvorson (2012) and Weatherall (2017), for example, have argued that we should often understand physical theories not as mere collections of mathematical models, but rather as coming with further information encoded in relations between models. Their proposal amounts to representing physical theories by categories, which include information about the embedding and equivalence relationships among models.[7]

Category theoretic tools for encoding interpretations of physical theories also provide the information needed to settle questions concerning definability relations and structural comparisons between theories (Barrett, 2018). As such, they offer a precise and general framework in which to pursue the questions raised by the Batterman–Belot debate. However, there has not yet been extensive work applying these mathematical tools to questions concerning intertheoretic reduction.[8] In what follows, we hope to provide just a start to this project of bringing the literature on reduction together with that on the structure and interpretation of theories using the tools of category theory.

We wish to offer two caveats before proceeding to our contributions. First, it will suffice for our purposes to note that "explicit definitions" in the language of propositional logic and "structural determination" in the language of category theory are somehow similar but have different strengths. Namely, the former makes clear how our Quinean substitutions work, and the latter makes clear a precise sense in which these substitutions amount to removing structure from the quantum theory. However, the precise relationship between these approaches is fascinating in its own right, and it has been explored thoroughly elsewhere (e.g., by Barrett (2018) and Hudetz (2019a)). For example, our notion of "structural determination" fits rather closely with Hudetz's (2019a) notion of a *definable functor* (although we will not put our results in these terms). As Hudetz shows (2019a, Theorems 1-3), in some cases, explicit definability of one theory from another either implies or is implied by a functorial relation between the theories' categories of models. In our case of the $\hbar \to 0$ limit, it is likewise true that all of the structural and functorial comparisons we make between classical and quantum physics follow from our explicit definitions of classical concepts from quantum concepts. Rather than exploring this relationship, however, we instead want to highlight that the language of category theory helps address a substantive further philosophical question: namely, whether the

[7]For further general discussion, see Halvorson and Tsementzis (2017) and Halvorson (2019). We also mention two specific applications of category-theoretic tools for internal aspects of interpretation. First, these tools have been used in specifying the representational capacities of models (Weatherall, 2018; Fletcher, 2020). Second, and relatedly, such interpretive information helps us settle questions concerning when two formulations of a theory are equivalent (Rosenstock et al., 2015; Weatherall, 2016a; Rosenstock and Weatherall, 2016; Barrett, 2019). These investigations have now given rise to an entire segment of the literature discussing proposed analyses of theoretical equivalence when using categories to represent scientific theories (Barrett and Halvorson, 2016; Hudetz, 2019b; Weatherall, 2020).

[8]Notable recent exceptions include List (2019); Dewar et al. (2019); Feintzeig (2019).

definitions of classical concepts are somehow unnatural. One might worry that the definitions might somehow depend implicitly on structure beyond that of quantum theory in a way that is hidden in their syntactic presentation. Categorical language helps to emphasize that, on the contrary, the internal task of specifying which classical structures are physically significant need only depend on the structure of the quantum theory—and so this worry is unfounded.

Second, while we will show that the general structure corresponding to the interpretation of classical physics can be determined by the structure of quantum physics, we will make no effort to connect this general classical structure to the features of chaotic quantum systems that Batterman is primarily concerned with. Specifically, Batterman suggests that what is needed to explain semiclassical trace formulae for such systems is knowledge of the periodic solutions to the classical equations of motion (Belot, 2005, §4 and online Appendix). Recall, however, that we do not seek to invalidate Batterman's explanatory strategy, but rather to identify one place where his negative argument against reduction fails. We hope that a reductive explanation can also be found for chaotic systems, but it would require a significant digression to a very different technical setting. In the spirit of biting off an amount that can be reasonably chewed, we restrict our attention to the general question about the structural relation between classical and quantum physics—a question which is still at play in Batterman's work and which is interesting in its own right. We believe it is worthwhile to treat this general structural relation, even without knowing its precise bearing on the chaotic systems Batterman cares most about.

Those caveats out of the way, we turn to the mathematical preliminaries needed to state our contributions precisely.

3 Mathematical Preliminaries

To respond to Batterman's claims about the classical limit, we must look in detail at its mathematical formulation. Our approach to the $\hbar \to 0$ limit uses the modern mathematical tools of strict deformation quantization. Before defining this concept, we recall that the background for strict deformation quantization is the algebraic formulation of physical theories. On this approach, we represent the physical quantities of a system with elements of a C*-algebra.[9] A C*-algebra $\mathfrak{A}$ is an associative, involutive, complete normed algebra satisfying the C*-identity: $\|A^*A\| = \|A\|^2$ for all $A \in \mathfrak{A}$. The canonical examples of C*-algebras are commutative algebras of bounded functions on a locally compact topological space and possibly non-commutative algebras of bounded operators on a Hilbert space. We employ commutative algebras of functions on a classical phase space in classical physics and non-commutative algebras of operators satisfying a version of the canonical (anti-) commutation relations in quantum theories. Thus, using C*-algebras provides a unified framework for investigating the relationship between classical and quantum theories. With this background in place, we can use C*-algebras to analyze the classical limit.

3.1 Strict Deformation Quantization

A strict quantization provides extra structure to "glue together" a family of C*-algebras indexed by the parameter $\hbar$, as follows.

Definition 1 (Landsman, 1998a)**.** A *strict quantization* of a Poisson algebra $(\mathcal{P}, \{\cdot,\cdot\})$ consists in a locally compact topological space $I \subseteq \mathbb{R}$ containing 0, a family of C*-algebras $(\mathfrak{A}_\hbar)_{\hbar\in I}$ and a family

[9] We will not review the basic theory of C*-algebras here. Instead, we refer the reader to Sakai (1971), Dixmier (1977), Kadison and Ringrose (1997), and Landsman (2017). See Bratteli and Robinson (1987, 1996) and Haag (1992) for the C*-algebraic approach to quantum physics. See also Clifton and Halvorson (2001), Halvorson (2007), and Ruetsche (2011) for introductions intended for a philosophical audience.

of linear *quantization maps* $(\mathcal{Q}_\hbar : \mathcal{P} \to \mathfrak{A}_\hbar)_{\hbar \in I}$. We require that $\mathcal{P} \subseteq \mathfrak{A}_0$, $\mathcal{Q}_0$ is the inclusion map, and for each $\hbar \in I$, $\mathcal{Q}_\hbar[\mathcal{P}]$ is norm dense in $\mathfrak{A}_\hbar$. Further, we require that the following conditions hold for all $A, B \in \mathcal{P}$:

(i) *Von Neumann's condition.* $\lim_{\hbar\to 0}\|\mathcal{Q}_\hbar(A)\mathcal{Q}_\hbar(B) - \mathcal{Q}_\hbar(AB)\|_\hbar = 0$;

(ii) *Dirac's condition.* $\lim_{\hbar\to 0}\|\frac{i}{\hbar}[\mathcal{Q}_\hbar(A), \mathcal{Q}_\hbar(B)] - \mathcal{Q}_\hbar(\{A, B\})\|_\hbar = 0$;

(iii) *Rieffel's condition.* the map $\hbar \mapsto \|\mathcal{Q}_\hbar(A)\|_\hbar$ is continuous.

A strict *deformation* quantization is a strict quantization that satisfies the additional requirement that for each $\hbar$, $\mathcal{Q}_\hbar[\mathcal{P}]$ is closed under multiplication and is nondegenerate, i.e., $\mathcal{Q}_\hbar(A) = 0$ if and only if $A = 0$.[10]

We can use a strict deformation quantization to represent the classical limit of a quantum theory whose kinematics is represented by $\mathfrak{A}_\hbar$ for some $\hbar > 0$. In many of the examples of physical interest, $\mathfrak{A}_\hbar$ is *-isomorphic to $\mathfrak{A}_{\hbar'}$ for any $\hbar, \hbar' > 0$ and so one could take any of the algebras away from $\hbar = 0$ to represent the kinematics of the quantum theory. Feintzeig (2020) argues that given a classical quantity $A \in \mathcal{P}$, we can understand $\mathcal{Q}_\hbar(A)$ and $\mathcal{Q}_{\hbar'}(A)$ for $\hbar \neq \hbar' > 0$ to represent the *same* physical quantity in the quantum theory in different systems of units, namely units in which Planck's constant takes on either the value $\hbar$ or $\hbar'$, respectively. It is in this sense that a strict deformation quantization carries not only the structure of the quantum kinematics (encoded in the algebra $\mathfrak{A}_\hbar$), but also information about what we call the *scaling behavior* of the quantum system. One can understand this scaling behavior as arising from a partial interpretation of the abstract elements of a C*-algebra that assigns them to *unitful* physical quantities. A strict deformation quantization provides substantive and physically significant relationships between algebras at different values of $\hbar$—information encoded in the quantization maps, which "glue" the algebras together.

This interpretation motivates us to understand the classical limit of a quantity $\mathcal{Q}_\hbar(A)$ in $\mathfrak{A}_\hbar$ in the quantum theory to be the classical quantity $A \in \mathcal{P}$. Similarly, we can take classical limits of states by defining a *continuous field of states* as a family of states $\omega_\hbar \in \mathcal{S}(\mathfrak{A}_\hbar)$ for each $\hbar \in I$ such that the map $\hbar \mapsto \omega_\hbar(\mathcal{Q}_\hbar(A))$ is continuous for each $A \in \mathcal{P}$. In this case, the classical limit of such a continuous field of states is understood to be the classical state $\omega_0 \in \mathcal{S}(\mathfrak{A}_0)$. Thus, a strict deformation quantization provides enough structure to represent the classical limits of states and quantities in quantum theories. In the present paper, we will mostly ignore classical limits of states and restrict our focus to constructing explicit definitions of kinematical observables and their associated dynamics. We leave a more thorough treatment of states for future work.[11]

In general, there may be different strict deformation quantizations of the same Poisson manifold. If two strict quantizations $\mathcal{Q}_\hbar$ and $\mathcal{Q}'_\hbar$ of a given Poisson algebra $\mathcal{P}$ employ the same family of C*-algebras, but possibly differ in their quantization maps, then the quantizations are called *equivalent* just in case

$$\lim_{\hbar\to 0}\left\|\mathcal{Q}_\hbar(A) - \mathcal{Q}'_\hbar(A)\right\|_\hbar = 0 \tag{1}$$

[10] Rieffel (1989, 1993) uses this approach to define a "deformed" product on the classical algebra, which he uses in turn to specify the quantization conditions. Our goal is to work in the opposite direction, defining the elements of the classical algebra using information away from $\hbar = 0$. We will mostly ignore the deformation condition, but still refer to strict deformation quantizations to distinguish them from other approaches like geometric quantization.

[11] Note, however, that there is a close connection between the satisfaction of Rieffel's condition in a strict quantization and the existence of a continuous field of states converging to each classical state. See the proof of Thm. 4 in Landsman (1993, p. 33)

for all $A \in \mathcal{P}$. Equivalent quantizations share the same behavior in the limit as $\hbar \to 0$. One can encode this common behavior in a further object, variously called a continuous bundle or field of C*-algebras, which itself has enough structure to understand the classical limits of quantities and states. We will work with a recent treatment of bundles of C*-algebras due to Steeger and Feintzeig (2021), designed to analyze the cases of interest in this paper.

3.2 Bundles of C*-Algebras

The maps $[\hbar \mapsto \mathcal{Q}_\hbar(A)]$ for each $A \in \mathcal{P}$ are clearly significant in the structure of a strict deformation quantization. However, for many purposes it suffices to deal with any map from values of $\hbar \in I$ to observables in algebras $\mathfrak{A}_\hbar$ that is suitably continuous. A bundle of C*-algebras takes these continuous maps as primary (themselves forming a C*-algebra) and forgets the particular quantization map that generates them. We define these bundles now. In what follows, $UC_b(I)$ denotes the collection of uniformly continuous and bounded functions on a metric space I.

Definition 2 (Steeger and Feintzeig (2021))**.** A *uniformly continuous bundle of C*-algebras*[12] over a metric space (I, d) is a family of C*-algebras $(\mathfrak{A}_\hbar)_{\hbar\in I}$, a C*-algebra $\mathfrak{A}$ called the collection of *uniformly continuous sections*, and a family of *-homomorphisms $(\phi_\hbar : \mathfrak{A} \to \mathfrak{A}_\hbar)_{\hbar\in I}$ called *evaluation maps*, which we require to satisfy the following conditions:

(i) *Fullness.* Each evaluation map $\phi_\hbar$ is surjective and the norm of each $a \in \mathfrak{A}$ is given by $\|a\| = \sup_{\hbar\in I} \|\phi_\hbar(a)\|_\hbar$.

(ii) *Uniform completeness.* For each $f \in UC_b(I)$ and $a \in \mathfrak{A}$, there is an element $fa \in \mathfrak{A}$ such that $\phi_\hbar(fa) = f(\hbar)\phi_\hbar(a)$.

(iii) *Uniform continuity.* For each $a \in \mathfrak{A}$, the function $N_a : \hbar \mapsto \|\phi_\hbar(a)\|_\hbar$ is in $UC_b(I)$.

In general, we will restrict our attention to uniformly continuous bundles of C*-algebras whose base space is a locally compact metric space. We will often think of the C*-algebras $\mathfrak{A}_\hbar$ as fibers above the values $\hbar \in I$, hence forming a bundle structure over I. A uniformly continuous bundle of C*-algebras determines the continuity structure of the bundle by specifying the collection $\mathfrak{A}$ of uniformly continuous sections through the bundle.

It follows from a result of Landsman (1998a, Theorem 1.2.4, p. 111) that, given a strict quantization, there is a unique uniformly continuous bundle of C*-algebras containing among its sections the curves traced out by the quantization maps as $\hbar$ varies. More formally: given a strict quantization $((\mathfrak{A}_\hbar, \mathcal{Q}_\hbar)_{\hbar\in I}, \mathcal{P})$, there is a unique uniformly continuous bundle of C*-algebras $((\mathfrak{A}_\hbar, \phi_\hbar)_{\hbar\in I}, \mathfrak{A})$ such that for each $A \in \mathcal{P}$, there is a continuous section $a \in \mathfrak{A}$ with $\phi_\hbar(a) = \mathcal{Q}_\hbar(A)$ for all $\hbar \in I$. We can thus speak of the uniformly continuous bundle generated by a strict quantization. Moreover, Landsman's theorem shows equivalent quantizations generate the same bundle. In this sense, the bundles encode an invariant structure among different quantization maps capturing the same behavior in the $\hbar \to 0$ limit.[13]

The association of a C*-algebra of uniformly continuous sections in this definition allows one to use many familiar tools to analyze such bundles. In the next section, we will formulate our central

[12] Steeger and Feintzeig (2021) establishes an equivalence between a category of uniformly continuous bundles of C*-algebras and categories of models specified by other related definitions of bundles of C*-algebras, including continuous fields of C*-algebras (Dixmier, 1977) and (vanishingly) continuous bundles of C*-algebras (Kirchberg and Wasserman, 1995).

[13] See also Steeger and Feintzeig (2021) for more detail on the functorial determination of a bundle of C*-algebras from a strict deformation quantization.

question about quantum mechanics recovering its classical limit as follows. Suppose one knows the quantum kinematics for $\hbar > 0$ but has no knowledge of the corresponding classical kinematics, i.e., suppose one has a bundle of C*-algebras $((\mathfrak{A}_\hbar, \phi_\hbar)_{\hbar\in(0,1]}, \mathfrak{A})$ over the base space consisting only of parameter values $\hbar > 0$. Under what conditions is there a unique algebra $\mathfrak{A}_0$ that, when appropriately glued to the given bundle of algebras, provides an extended continuous bundle?

Before proceeding to make this question precise and provide an answer, we mention one more definition we will use. In what follows, we will also need to use a notion of morphism, or structure-preserving map, between uniformly continuous bundles of C*-algebras, which we define as follows. First, recall that a metric map $\alpha : I \to J$ is one satisfying $d_J(\alpha(x), \alpha(y)) \le d_I(x, y)$ for all $x, y \in I$, where d_I and d_J are the metrics on I and J, respectively. A *homomorphism* $\sigma : \mathcal{A}_I \to \mathcal{B}_J$ *between uniformly continuous bundles of C*-algebras* $\mathcal{A}_I = ((\mathfrak{A}_\hbar, \phi^I_\hbar)_{\hbar\in I}), \mathfrak{A})$ and $\mathcal{B}_J = ((\mathfrak{B}_\hbar, \phi^J_\hbar)_{\hbar\in J}), \mathfrak{B})$ is a pair of maps

$$\sigma = (\alpha, \beta), \tag{2}$$

where $\alpha : I \to J$ is a metric map, $\beta : \mathfrak{A} \to \mathfrak{B}$ is a *-homomorphism, and the following condition of compatibility is satisfied: for all $a_1, a_2 \in \mathfrak{A}$ and $\hbar \in I$, if $\phi^I_\hbar(a_1) = \phi^I_\hbar(a_2)$, then $\phi^J_{\alpha(\hbar)}(\beta(a_1)) = \phi^J_{\alpha(\hbar)}(\beta(a_2))$. The condition of compatibility between α and β ensures that β preserves fibers in the sense that it defines, for each $\hbar \in I$, a *-homomorphism from the fiber $\mathfrak{A}_\hbar$ to the fiber $\mathfrak{B}_{\alpha(\hbar)}$. A homomorphism of bundles $\sigma = (\alpha, \beta)$ is an *isomorphism* if α is an isometry and β is a *-isomorphism.

In the next section, we will use uniformly continuous bundles of C*-algebras to analyze the existence and uniqueness of limits of families of C*-algebras, of which the classical limit is one example.

4 Existence and Uniqueness of the Classical Limit

Recall from §2.1 that one aspect of the question of whether classical mechanics reduces to quantum mechanics concerns whether classical physics is explicitly definable from quantum physics. One way to formulate this issue mathematically is to ask after the existence and uniqueness of a classical limit given only a quantum theory for $\hbar > 0$.

We formulate the problem of the existence and uniqueness of the classical limit as follows. Suppose one has a strict deformation quantization of a Poisson manifold over the base space given by the closed interval $I = [0, 1]$. This structure, and its associated uniformly continuous bundle of C*-algebras $\mathcal{A}_{[0,1]}$, can be used to represent the $\hbar \to 0$ limit of the quantum theory whose kinematics is represented by the algebra $\mathfrak{A}_\hbar$ at the fiber $\hbar = 1$ of $\mathcal{A}_{[0,1]}$. But we seek to answer Batterman's challenge (or one aspect of it) to provide a formulation of the classical limit that does not make prior reference to the classical theory. As such, we want to consider the case where we begin with only information about the quantum theory and no information about the corresponding classical limit. In other words, suppose we are given only the restriction of our uniformly continuous bundle of C*-algebras to a bundle over the base space given by the half open interval $(0, 1]$. One can always canonically restrict a bundle of C*-algebras over the base space J to a base space I if one is given a metric embedding $\alpha : I \to J$. The restriction simply removes the fibers outside of $\alpha[I]$ and truncates continuous sections from J to $\alpha[I] \subseteq J$. (See Steeger and Feintzeig (2021) for more details.) In particular, one can use the natural inclusion map $\alpha : (0, 1] \to [0, 1]$ to define a restricted bundle $\mathcal{A}_{(0,1]}$ over the base space $(0, 1]$ resulting from a strict deformation quantization as above. Such a restricted bundle represents only the information in the quantum theory for $\hbar > 0$ with no reference to the corresponding classical theory at $\hbar = 0$.

Given such a restriction, our questions are: can one reconstruct the C*-algebra of classical quantities $\mathfrak{A}_0$ from this restricted continuous bundle? And can one continuously glue $\mathfrak{A}_0$ to the restricted bundle in a way that recovers the original information about the $\hbar \to 0$ limit? We answer both questions in the affirmative. The result is a two-step procedure for constructing explicit definitions of classical concepts from quantum ones, which we call "extension-and-restriction". Starting with a bundle representing only quantum theory and scaling information for $\hbar > 0$, we (uniquely) extend the bundle to one containing information at the privileged accumulation point $\hbar = 0$; then we (uniquely) restrict this new bundle to the fiber algebra $\hbar = 0$ to exactly recover the classical theory.

We pause here to respond to a potential objection to the philosophical relevance of this extension-and-restriction procedure for intertheoretic reduction. A supporter of the anti-reductionist view might object that understanding a quantum theory to come with the structure of a uniformly continuous bundle of C*-algebras already smuggles in the structure of classical physics. After all, as discussed in §3, the algebra of uniformly continuous sections is typically determined by a quantization map, whose definition itself relies on prior knowledge of a classical theory and its algebraic structure.

We concede that the algebra of continuous sections, and hence the continuity structure of the associated bundle, are *typically* defined via a quantization map, and that if the continuity structure *relied* on the quantization map, then this would pose a problem for our view of the reductive classical limit. However, we believe this objection fails because the continuity structure does *not* rely on a quantization map. Another way of putting the point is to recall our earlier remarks that we are focused on recovering internal aspects of the interpretation of classical physics from internal aspects of the interpretation of quantum physics. To accomplish this task, we are free to stipulate that the continuity structure of the bundle is physically significant straightaway.

Nonetheless, we would like to motivate this choice, and we would like to do so without referencing the classical theory. At the very least, we can motivate it by noting that the particular quantization map (out of a family of equivalent quantization maps) used to take the classical limit is not physically significant as long as one has the continuity structure of a bundle of C*-algebras. But, ideally, we would prefer to avoid reference to quantization maps altogether. Thus, we now argue for one sense in which the invariant bundle structure is physically significant within quantum theory itself. Specifically, we show that the continuity structure of a bundle of C*-algebras over $(0, 1]$ representing quantum theories for $\hbar > 0$ can be physically motivated (externally) and defined (internally) with no reference to or dependence on a classical theory or quantization map.

The basic point comes from the interpretation provided by Feintzeig (2020) of the classical limit. As such, we will use the example employed in that paper of the classical limit of the Weyl algebra[14] for a system with classical phase space $\mathbb{R}^{2n}$. We start with an external task: following Feintzeig, we interpret different numerical values $\hbar \neq \hbar' \in (0, 1]$ as the values of Planck's constant in different systems of units. On this interpretation, the numerical values of quantities like position Q and momentum P are likewise rescaled in different systems of units. The Weyl algebra has a basis constituted by elements $W(a, b)$ for $a, b \in \mathbb{R}^{2n}$ understood (at least in regular representations) as related to position and momentum by

$$W(a, b) \sim e^{i(a \cdot Q + b \cdot P)}.$$

This relationship motivates the commutation relations for the Weyl algebra from the canonical commutation relations for Q and P. And more importantly for our purposes, this relationship allows

[14]See, e.g., Clifton and Halvorson (2001) or Petz (1990) for background on the Weyl algebra, which is one particular C*-algebra used to represent quantum theories.

us to understand how the quantities $W(a,b)$ are rescaled in different systems of units. Feintzeig (2020) argues from dimensional analysis that changing from units in which Planck's constant takes on the value $\hbar' \in (0,1]$ to units in which Planck's constant takes on the value $\hbar \in (0,1]$ yields the change

$$e^{i(a\cdot Q+b\cdot P)} \mapsto e^{i(a\sqrt{\hbar/\hbar'}\cdot Q+b\sqrt{\hbar/\hbar'}\cdot P)},$$

which can be represented in the Weyl algebra as the change

$$W(a,b) \mapsto W(\sqrt{\hbar/\hbar'}\cdot a, \sqrt{\hbar/\hbar'}\cdot b).$$

On the interpretation of different values of Planck's constant as corresponding to different systems of units, then, for each $a, b \in \mathbb{R}^{2n}$, we should understand the one-parameter family generated from $\hbar' = 1$ and varying $\hbar \in (0,1]$ to be a continuous section of a corresponding bundle of Weyl algebras. In other words, this interpretation motivates our choice for an internal task—namely, our specification of a particular continuity structure as a physically significant one. Explicitly, one should stipulate that the maps

$$\hbar \in (0,1] \mapsto W(\sqrt{\hbar}\cdot a, \sqrt{\hbar}\cdot b)$$

are continuous sections and use them to generate a C*-algebra of continuous sections. In fact, Feintzeig (2020) proves that this gives rise to the same continuity structure for a bundle of Weyl algebras as the standard Weyl or Berezin quantization maps, which employ a classical, commutative algebra (Binz et al., 2004). However, since the motivation and construction just outlined do not employ the classical Weyl algebra or a quantization map, this shows that thinking about rescalings of values of quantities with changes of units is enough to motivate and determine the continuity structure of a bundle over $(0,1]$ from only the physical information encoded in a quantum theory and its unitful scaling behavior. Of course, the scaling behavior encoded in the continuity structure of the bundle is extra information not contained in the Weyl algebra itself; however, we take the scaling behavior to be well-motivated by the physical interpretation of the Weyl algebra as it represents quantities in a quantum theory alone, without any consideration of its corresponding classical limit.[15]

As flagged in §2.2, what we have done here is take what one might view as an external aspect of the interpretation of quantum physics—the units of physical quantities—and encoded at least a part of this interpretation—namely, how quantities with related units rescale—internally in the mathematical structure of bundles of C*-algebras, which we use to represent a quantum theory. We think that this interpretation and mathematical structure are appropriately treated as part of quantum theory alone, with no reference to classical physics. We hope this small digression also serves to clarify just what physical interpretation we are presupposing when we begin with a uniformly continuous bundle of C*-algebras.

Thus, we believe that one can understand bundles of C*-algebras over $(0,1]$ to represent the structure of quantum theories alone. And so, the question of the existence and uniqueness of extensions to the larger base space $[0,1]$ takes on immediate philosophical relevance. The existence and uniqueness results that we now summarize for such bundle extensions obtain in full generality for the case where the base space I is an arbitrary locally compact metric space. Such general results then apply immediately to the case where the base space is either $I = (0,1]$ or $I = \{1/N \mid N \in \mathbb{N}\}$, which are the most typical base spaces used in analyzing limits of quantum theories. As such, we define a general notion of extension.

[15] We have only here dealt with one example of a bundle of C*-algebras using the Weyl algebra. We believe this reasoning about rescaling units of physical quantities should generalize, not to arbitrary bundles of C*-algebras, but to those bundles that can be used to represent quantum systems.

Definition 3. Let $\mathcal{A}_I = ((\mathfrak{A}_\hbar, \phi^I_\hbar)_{\hbar \in I}, \mathfrak{A})$ and $\mathcal{B}_J = ((\mathfrak{B}_\hbar, \phi^J_\hbar)_{\hbar \in J}, \mathfrak{B})$ be uniformly continuous bundles of C*-algebras over a locally compact metric spaces I and J, respectively.

- $\mathcal{B}_J$ is an *extension* of $\mathcal{A}_I$ if there is a monomorphism of uniformly continuous bundles of C*-algebras $\sigma : \mathcal{A}_I \to \mathcal{B}_J$, i.e. a homomorphism $\sigma = (\alpha, \beta)$, where α and β are both injective.
- $\mathcal{B}_J$ is a *minimal extension* of $\mathcal{A}_I$ if, moreover, α and β are both dense embeddings.[16]

In either case, we say that the extension $\mathcal{B}_J$ is *associated with α via σ*.

We are especially interested in minimal extensions, since these encompass the extension from the base space $(0, 1]$ to $[0, 1]$, thus containing $\hbar = 0$. Thankfully, one can show that a minimal extension is guaranteed to exist for any accumulation point of interest.

Theorem 1 (Steeger and Feintzeig (2021))**.** *Let $\mathcal{A}_I = ((\mathfrak{A}_\hbar, \phi^I_\hbar)_{\hbar \in I}, \mathfrak{A})$ be a uniformly continuous bundle of C*-algebras over a locally compact metric space I. Suppose $\alpha : I \to J$ is a dense, isometric embedding. Then there exists a minimal extension $\tilde{A}_J$ of $\mathcal{A}_I$ associated with α.*

We provide a summary of the construction here for use in our conceptual discussion. The idea of the construction of $\tilde{A}_J$ is to define a C*-algebra $\mathfrak{A}_j$ for each $j \in (J \setminus \alpha[I])$ and then glue these algebras to the original fibers, as follows.

- *Definition of fibers.* For each $j \in (J \setminus \alpha[I])$, consider a Cauchy net (i_δ) in I such that $\alpha(i_\delta)$ converges to j in J. Define $\mathcal{K}_j$ as the collection of continuous sections $a \in \mathfrak{A}$ such that $\lim_{i_\delta \to j} \|\phi^I_{i_\delta}(a)\|_{i_\delta} = 0$. It turns out that $\mathcal{K}_j$ is a closed two-sided ideal in $\mathfrak{A}$, which allows us to define the fiber C*-algebra by $\mathfrak{A}_j := \mathfrak{A}/\mathcal{K}_j$.
- *Gluing new fibers to the bundle.* One can keep the same collection of sections $\mathfrak{A}$ (and so β is dense, as it is the identity map). Define new maps ϕ^J_j, which are the same as ϕ^I_j when $j \in \alpha[I]$ and give the appropriate limits when $j \in (J \setminus \alpha[I])$. The appropriate limit turns out to be $\phi^J_j(a) = [a]_{\mathcal{K}_j}$, where $[a]_{\mathcal{K}_j} \in \mathfrak{A}/\mathcal{K}_j$ is the equivalence class corresponding to $a \in \mathfrak{A}$.

The result is a uniformly continuous bundle of C*-algebras $\tilde{A}_J$ as desired. Hence, a minimal extension exists for any given "completion" of the parameter space with an accumulation point of interest.

However, in order to use this minimal extension to talk of *the* classical theory defined by quantum theory, we require something further: a clear sense in which this extension is unique. It turns out that all minimal extensions associated with a dense embedding α are isomorphic to each other—and so it makes good sense to talk of both *the* minimal extension of a bundle and *the* classical theory that it defines.

Theorem 2 (Steeger and Feintzeig (2021))**.** *Let $\mathcal{A}_I$ be a uniformly continuous bundle of C*-algebras over a locally compact metric space I. Suppose that $\mathcal{B}_J$ and $\mathcal{C}_J$ are two minimal extensions of $\mathcal{A}_I$ associated with a given dense, isometric embedding $\alpha : I \to J$. Then $\mathcal{B}_J$ and $\mathcal{C}_J$ are isomorphic as uniformly continuous bundles of C*-algebras.*

[16]We use the term "dense embedding" above to mean something slightly different for α and for β. For α to be a dense embedding, it must be an isometric isomorphism between its domain and its image (equipped with the subspace topology) and its image must be dense in J (equipped with the metric topology). Sometimes we will call α a "dense, isometric embedding" for emphasis; although, note that we do not require α to be bijective. For β to be a dense embedding, nothing more is required other than that its image must be dense in $\mathfrak{B}$ (according to the algebra's norm)—this condition, in conjunction with injectivity, makes β a *-isomorphism (Kadison and Ringrose 1997, p. 243).

Theorems 1 and 2 together allow us to refer to $\tilde{\mathcal{A}}_J$ as *the* minimal extension of a uniformly continuous bundle of C*-algebras $\mathcal{A}_I$ associated with a given dense embedding $\alpha : I \to J$. Moreover, Theorem 2 allows us to refer to the algebra $\mathfrak{A}_j$ as *the* algebra at the accumulation point $j \in (J \backslash \alpha[I])$. We will do so for the remainder of the paper. In particular, to represent the classical limit of a quantum theory we can set $I = (0,1]$, $J = [0,1]$, and look at the minimal extension associated with the inclusion map $\alpha : (0,1] \to [0,1]$.

5 Explicit Definitions

5.1 Kinematics

The construction underlying the existence and uniqueness of bundle extensions leads immediately to explicit definitions of classical kinematic concepts in terms of quantum kinematics through the classical limit. The recipe for building these definitions is present in our sketch of the proof of Theorem 1, but it is instructive to fill it in. To keep things general, suppose that $\tilde{\mathcal{A}}_J$ is the minimal extension of $\mathcal{A}_I$ for α with the privileged point $0 \in J \backslash \alpha[I]$. Given quantities in the quantum theory whose scaling behavior is described by sections $a, b \in \mathfrak{A}$, we explicitly define the classical limits of a and b as elements A, B of $\mathfrak{A}_0$ with definitions and algebraic operations given by

$$\begin{aligned} A &:= [a]_{\mathcal{K}_0} \qquad\qquad B := [b]_{\mathcal{K}_0} \\ A + B &:= [a+b]_{\mathcal{K}_0} \\ A \cdot B &:= [a \cdot b]_{\mathcal{K}_0} \\ c \cdot A &:= [c \cdot a]_{\mathcal{K}_0} \\ A^* &:= [a^*]_{\mathcal{K}_0} \end{aligned} \tag{D_1}$$

for $c \in \mathbb{C}$. The limit construction shows that physical information at $\hbar > 0$ contains enough information to uniquely fix kinematical quantities and their algebraic relations at $\hbar = 0$—and such a construction is precisely what one needs to give explicit definitions like the ones above.

In addition to defining the algebraic structure of observables at $\hbar = 0$, our setup can also be used to define the classical Poisson bracket. Consider any pair of sections $a, b \in \mathfrak{A}$ and suppose that there exists a section $\{a, b\} \in \mathfrak{A}$ such that

$$\lim_{\hbar \to 0} \left\| \frac{i}{|\hbar|} [\phi_\hbar(a), \phi_\hbar(b)] - \phi_\hbar(\{a, b\}) \right\|_\hbar = 0, \tag{3}$$

where we define $|\hbar|$ as the distance between $\hbar$ and 0 using the metric of J. Note, in particular, that if the minimal extension is generated by a strict quantization, then $\{a, b\}$ is guaranteed to exist for each pair A, B in the domain $\mathcal{P} \subseteq \mathfrak{A}_0$ of the quantization maps. We may now define a partial binary operation on $\mathfrak{A}_0$ using just the information in $\mathcal{A}_I$ by

$$\{A, B\} := \begin{cases} [\{a, b\}]_{\mathcal{K}_0} & \text{if an } \{a, b\} \in \mathfrak{A} \text{ satisfying (3) exists,} \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{D_2}$$

Note that the operation $\{\cdot, \cdot\}$ is a Poisson bracket where it is defined—it is anticommutative and bilinear, and it satisfies both Leibniz's rule and the Jacobi identity (all properties that it inherits from the commutator on the algebra of uniformly continuous sections). Clearly, in the case that the minimal extension is generated by a strict quantization, $\{A, B\}$ recovers the usual Poisson bracket structure of the Poisson algebra $\mathcal{P}$. Thus, the information encoded in the commutator of the quantum theory allows us to explicitly define the Poisson bracket on the algebra of classical observables.

So far, we have shown that one can use the information encoded in a quantum theory along with its scaling behavior for $\hbar$—formalized using uniformly continuous bundles of C*-algebras—to explicitly define the classical kinematics—formalized in the algebraic structure of a commutative C*-algebra and a Poisson bracket. Next, we show that under suitable conditions, one can furthermore explicitly define the classical dynamics from a given quantum dynamics without prior reference to the classical theory.

5.2 Dynamics

Our general strategy for constructing explicit definitions of classical dynamics mirrors that for kinematics. We start with dynamics on all quantum systems in a uniformly continuous bundle satisfying the minimal requirement that these dynamics scale continuously with $\hbar$. Then we repeat our two-step extension-and-restriction procedure from before. That is, we first *extend* the dynamics on the entire bundle to a dynamics on the unique extended bundle with a privileged accumulation point guaranteed by Theorems 1 and 2. Then we *restrict* the dynamics on the extended bundle to the fiber algebra at that privileged point. Although this construction is abstract and general, we will show that for a relatively broad class of systems, it reproduces the standard classical Hamiltonian dynamics.

To start, we treat dynamics as maps taking observables at an initial time to observables at some later time t while preserving their algebraic relations—that is, we view dynamics as (at the very least) encoded in a one-parameter automorphism group $(\tau_t : \mathfrak{C} \to \mathfrak{C})_{t\in\mathbb{R}}$ for an algebra of interest $\mathfrak{C}$. These automorphisms may be generated by some classical or quantum Hamiltonian, but we will not demand that they be so for the moment. Suppose again that $\mathcal{A}_I = ((\mathfrak{A}_\hbar, \phi^I_\hbar)_{\hbar\in I}, \mathfrak{A})$ is a uniformly continuous bundle of C*-algebras and allow $(\tau_{t;\hbar})_{t\in\mathbb{R}}$ to denote the dynamics on each fiber algebra $\mathfrak{A}_\hbar$. To enforce that these dynamics scale continuously with $\hbar$, we require that they lift to an automorphism group on the algebra of sections. That is, we require the existence of a one-parameter automorphism group $(\tau_t)_{t\in\mathbb{R}}$ on $\mathfrak{A}$ satisfying

$$\tau_{t;\hbar} \circ \phi^I_\hbar = \phi^I_\hbar \circ \tau_t \qquad \text{for all } \hbar \in I,\ t \in \mathbb{R}. \tag{4}$$

This is just the requirement that we are considering (in some sense) the "same" quantum dynamics at different scales.

Now we turn to the extension-and-restriction procedure. Let J be a locally compact metric space with dense metric embedding $\alpha : I \to J$, and let $\tilde{\mathcal{A}}_J$ again be the minimal extension of $\mathcal{A}_I$ associated with α. Recall that we can choose the minimal extension to have the same algebra of sections $\mathfrak{A}$, without loss of generality (by Theorem 2). The lifting to automorphisms $(\tau_t)_{t\in\mathbb{R}}$ on $\mathfrak{A}$ by (4) then immediately defines dynamical automorphisms on the algebra of sections of the extended bundle $\tilde{\mathcal{A}}_J$. Moreover, this naturally defines an extended dynamics on any other minimal extension of $\mathcal{A}_I$ because Theorem 2 guarantees all such extensions are bundle isomorphic. All that remains is to restrict these dynamics to the algebra at the privileged point, which we again denote by $0 \in (J \setminus \alpha[I])$. Given dynamics τ_t on the sections, we define a one-parameter automorphism group $(\tau_{t;0})_{t\in\mathbb{R}}$ representing the dynamics for the observables in $\mathfrak{A}_0$ by

$$\tau_{t;0}(A) := [\tau_t(a)]_{\mathcal{K}_0} \qquad \text{for all } t \in \mathbb{R}, \tag{D_3}$$

where $a \in \mathfrak{A}$ and $A \in \mathfrak{A}_0$ are associated via (D_1). In other words, we define the classical dynamics by enforcing the analog of (4), given by the condition

$$\tau_{t;\hbar} \circ \phi^J_\hbar = \phi^J_\hbar \circ \tau_t \qquad \text{for all } \hbar \in J,\ t \in \mathbb{R} \tag{5}$$

now on the extended bundle over J. This completes our two-step explicit definition. Moreover, the extended dynamics and its restriction to any given fiber are *unique* up to dynamics-preserving bundle isomorphism, which justifies us in referring to *the* dynamics defined through the bundle extension. Notice that in the particular case where $I = (0, 1]$ and $J = [0, 1]$, this defines a classical dynamics at $\hbar = 0$ from only the information encoded in the quantum theory, now including dynamics, at $\hbar > 0$.

One might still worry about whether our abstract definition of the classical dynamics agrees with the usual notion of the dynamics of a classical mechanical system. We now proceed to show that for a class of classical Hamiltonian dynamics, our definition does indeed recover the standard form.

To show this, we now restrict attention to the case of interest where $\hbar$ takes values in $I = (0, 1]$ or its extension $J = [0, 1]$. First, we use the extra structure of a quantization map to prove a convergence result for arbitrary automorphism groups that do not need to be generated by a particular Hamiltonian. Our conclusion then follows by a similar convergence result due by Landsman, which we state in the Appendix. Landsman's result holds when a Hamiltonian vanishing at infinity generates both the quantum and the classical dynamics (and we discuss generalizations to broader classes of Hamiltonians below). So if such a Hamiltonian generates the quantum dynamics $\tau_{t;\hbar}$, then the classical dynamics τ_t defined by (D_3) agrees with the standard classical Hamiltonian dynamics.[17]

Proposition 1. *Suppose the maps* $(\mathcal{Q}_\hbar : \mathcal{P} \to \mathfrak{A}_\hbar)_{\hbar\in[0,1]}$ *form a strict deformation quantization generating the bundle* $\mathcal{A}_{[0,1]} = ((\mathfrak{A}_\hbar, \phi_\hbar^{[0,1]})_{\hbar\in[0,1]}, \mathfrak{A})$*. Suppose further we have a one-parameter automorphism group* $(\tau_t)_{t\in\mathbb{R}}$ *on* $\mathfrak{A}$*, and that for each* $\hbar \in [0, 1]$*, we have one-parameter automorphism groups* $(\tau_{t;\hbar})$ *on* $\mathfrak{A}_\hbar$ *satisfying (5). Then for any* $A \in \mathcal{P}$*,*

$$\lim_{\hbar\to 0} \|\mathcal{Q}_\hbar(\tau_{t;0}(A)) - \tau_{t;\hbar}(\mathcal{Q}_\hbar(A))\|_\hbar = 0. \tag{6}$$

Proposition 1 implies that our definition of classical dynamics in (8) recovers standard Hamiltonian dynamics for the class of quantum theories on a Hilbert space $L^2(\mathbb{R}^n)$, with corresponding classical limit formulated on the phase space $T^*\mathbb{R}^n$, as follows.

Theorem 3. *Suppose the maps* $(\mathcal{Q}_\hbar : \mathcal{P} \to \mathfrak{A}_\hbar)_{\hbar\in[0,1]}$ *form a strict deformation quantization with* $\mathcal{P} \subseteq C_0(T^*\mathbb{R}^n)$ *and* $\mathfrak{A}_\hbar \subseteq \mathcal{B}(L^2(\mathbb{R}^n))$ *for* $\hbar > 0$*, generating the bundle* $\tilde{\mathcal{A}}_J = ((\mathfrak{A}_\hbar, \phi_\hbar^J)_{\hbar\in J}, \mathfrak{A})$*. Suppose further we have a one-parameter automorphism group* $(\tau_t)_{t\in\mathbb{R}}$ *on* $\mathfrak{A}$*, and that for each* $\hbar \in [0, 1]$*, we have one-parameter automorphism groups* $(\tau_{t;\hbar})$ *on* $\mathfrak{A}_\hbar$ *satisfying (5) with J=[0,1]. If there is an* $H \in \mathcal{P}$ *such that for each* $\hbar > 0$ *and* $B \in \mathfrak{A}_\hbar$

$$\tau_{t;\hbar}(B) = e^{-i\mathcal{Q}_\hbar(H)t} B e^{i\mathcal{Q}_\hbar(H)t}, \tag{7}$$

then for any $A \in \mathcal{P}$*, if* $\tau_{t;0}(A) \in \mathcal{P}$ *for all* $t \in \mathbb{R}$*, then*

$$\frac{d}{dt}\tau_{t;0}(A) = \{H, A\}. \tag{8}$$

We emphasize that our general convergence result (6) in Proposition 1 holds for any dynamics, regardless of whether it is generated by a Hamiltonian. When, in addition, the quantum dynamics are generated by the quantization of a Hamiltonian vanishing at infinity, Landsman's convergence result implies that the $\hbar \to 0$ limit recovers the classical dynamics defined by that Hamiltonian.

[17] We provide proofs in the Appendix for all propositions that are not proved in references.

Landsman (1998a, 2007) proves a number of further propositions providing variations on the convergence result. Notably, one result (Landsman, 1998a, Cor. 2.5.2, p. 141) implies that the conclusion of Theorem 3 holds as well for any quantization equivalent to Weyl quantization when H is an unbounded polynomial at most quadratic in momentum p and configuration q. This means that the dynamics of free particles and harmonic oscillators are both in the purview of the result. He also proves slightly weaker convergence results for other Hamiltonians (Landsman, 1998a, §II.2.7). However, it is still unclear whether any convergence result of this sort is general enough to cover the kinds of dynamics that are Batterman's focus, i.e., those of chaotic systems. Thus, a staunch opponent to eliminativism may justly demand more. Nonetheless, our result already shows how a large class of classical dynamics is definable from quantum dynamics without prior reference to the classical theory, making great headway on filling in the eliminativist's schema. We leave further investigation of unbounded Hamiltonians and chaotic systems for future work.

6 Interlude

We now pause to summarize the philosophical significance of our first contribution concerning explicit definability before moving on to the issue of the understanding such definitions engender. Recall that our goal is to argue against the claims of Batterman, Berry, and Bokulich that the classical $\hbar \to 0$ limit of quantum mechanics is "singular" in a way that prevents an eliminativist reduction of classical to quantum mechanics. By paying close attention to mechanisms of Landsman's $\hbar \to 0$ limit, we have shown that the eliminativist's case is more robust than previously understood. On one pass, this case relies on the construction of explicit definitions to justify the elimination of reference to the classical theory—and we have shown how to extract such definitions from a modification of Landsman's methods.

As flagged in §2.1, this modification enables us to align two schemas of reduction: the eliminativist's schema, on the one hand, and Nickles's schema, on the other. With uniformly continuous bundles of C*-algebras, we have shown how our top-level theory T_t (classical mechanics) is, at least partially, both explicitly defined by and recovered in the strict $\hbar \to 0$ limit of our bottom-level theory T_b (quantum mechanics). The sense in which our limit satisfies Nickles's schema (E') is clear enough from the above work and Landsman's work. However, it is worth taking a moment to spell out how our limit satisfies the eliminativist's schema (E), as characterized by Butterfield.

Recall that the eliminativist's schema (E) takes the bottom-level theory T_b in conjunction with a set of explicit definitions D to derive the top-level theory T_t. In our case, we take the bottom-level theory to be quantum mechanics with instructions on how quantities scale with $\hbar$—that is, a uniformly continuous bundle of C*-algebras $\mathcal{A}_I$ defined over the base space $I = (0, 1]$. Our theory includes two further bits of structure: the commutator $[\,\cdot\,,\,\cdot\,]_\hbar$ on each fiber (itself definable from the C*-algebraic structure) and the continuously-scaling dynamics τ_t on the algebra of sections $\mathfrak{A}$. All in all, we can define

$$T_b := (\mathcal{A}_I, [\,\cdot\,,\,\cdot\,]_I, \tau_t)\,, \tag{9}$$

where $[\,\cdot\,,\,\cdot\,]_I$ denotes the family of all the commutators. Our top-level theory is classical mechanics on algebraic approach. So we have a commutative C*-algebra $\mathfrak{A}_0$ with a densely defined[18] Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ and dynamics $\tau_{t;0}$ generated by the bracket and some Hamiltonian H via (8), yielding

$$T_t := (\mathfrak{A}_0, \{\,\cdot\,,\,\cdot\,\}, \tau_{t;0})\,. \tag{10}$$

[18]Note that we do not include a particular dense Poisson subalgebra of $\mathfrak{A}_0$ in the structure of the classical theory, to reflect that the choice of such a Poisson subalgebra (e.g., the compactly supported smooth functions, or the Paley-Wiener functions) does not make a substantive difference to the physics.

Finally, we let D denote the conjunction of our three explicit definitions,

$$D := D_1 \wedge D_2 \wedge D_3. \tag{11}$$

Now we can construct a partial explicit definition of a classical theory in terms of the quantum theory. To ensure that this definition recovers the desired classical theory in the Hamiltonian framework, the eliminativist needs to check a number of conditions. First, suppose we explicitly define the algebra $\mathfrak{A}_0$ via (D_1) and equip it with the Poisson bracket defined by (D_2). Our eliminativist can check whether there exists a dense Poisson subalgebra $\mathcal{P} \subseteq \mathfrak{A}_0$, closed under the operations defined in (D_1) and (D_2) satisfying the following. The first condition is that the minimal extension of our bundle to $J = [0, 1]$ is generated by a strict quantization. That is to say, the eliminativist can confirm:

There is a family of quantization maps $(\mathcal{Q}_\hbar : \mathcal{P} \to \mathfrak{A}_\hbar)_{\hbar \in J}$ satisfying Von Neumann's, Dirac's, and Rieffel's conditions such that $\tilde{\mathcal{A}}_J$ is generated by $\mathcal{Q}_\hbar$. (C_1)

The second condition is that the dynamics defined by (D_3) on the $\mathcal{P}$ in (C_1) preserves that algebra, i.e., the eliminativist can check whether:

$$\text{For all } A \in \mathcal{P}, \tau_{t;0}(A) \in \mathcal{P} \text{ for all } t \in \mathbb{R}. \tag{C_2}$$

Finally, the third condition needed is that the quantum dynamics are generated by an element of the $\mathcal{P}$ in (C_1) of the appropriately vanishing sort, i.e., the eliminativist can assess the following claim:

There exist faithful representations $(\pi_\hbar)_{\hbar \in J}$ of the Poisson algebra at $\hbar = 0$ and of the C*-algebras at $\hbar > 0$ such that

$$\pi_0[\mathcal{P}] \subseteq C_0(T^*\mathbb{R}^n), \qquad \pi_\hbar[\mathfrak{A}_\hbar] \subseteq \mathcal{B}(L^2(\mathbb{R}^n)) \text{ for all } \hbar > 0, \tag{C_3}$$

and there is some $H \in \mathcal{P}$ for which (7) holds for each $\hbar > 0$ and $B \in \mathfrak{A}_\hbar$.

We denote the conjunction of these "explicit definition conditions" by C, i.e.,[19]

$$C := C_1 \wedge C_2 \wedge C_3. \tag{C}$$

This yields the following partial instantiation of the eliminativist's schema:

$$\text{If } C, \quad \text{then } T_b \wedge D \Rightarrow T_t. \tag{12}$$

But what about the cases where C fails to hold? In this case, our construction still defines a theory at $\hbar = 0$, but it is unclear precisely how it links up with classical concepts. Specifically, we obtain in general

$$T_b \wedge D \Rightarrow T_t^*, \quad \text{where } T_t^* := \left(\mathfrak{A}_0^*, \{\,\cdot\,,\,\cdot\,\}^*, \tau_{t;0}^*\right), \tag{13}$$

where the C*-algebra $\mathfrak{A}_0^*$ is defined by (D_1), the binary operation $\{\,\cdot\,,\,\cdot\,\}^*$ is some (possibly partial) Poisson bracket defined by (D_2), and $\tau_{t;0}^*$ is the family of automorphisms defined by (D_3). At this juncture, the eliminativist is free to view T_t^* as the proper description of classical physics, given their commitment to T_b as the proper description of the quantum theory and its scaling structure. But it may or may not be the case that this theory relates in an interesting way to the usual

[19]Note that we are somewhat loose in our notation here: the Poisson algebra referenced in (C_1)-(C_3) must be the same, i.e., an existential quantifier over $\mathcal{P}$ must enclose the entire conjunction.

classical concepts in some cases where C fails to hold. We wager that T_t^* recovers standard classical Hamiltonian mechanics with more generality,[20] but we leave the evaluation of this conjecture for future work.

So the eliminativist's work is far from finished! We are not yet able to provide explicit definitions of *all* classical terms on the algebraic approach. Nonetheless, we take our work to dismantle a possible objection to the eliminativist's project that draws inspiration from Batterman's criticisms. Namely, as discussed in §2.1, Batterman's comments might be taken to imply that any attempt to fulfill the eliminativist's schema is doomed to failure. But as shown above, there is at least a partial fulfillment of the schema, and the eliminativist is free to reject using the instances of classical terms not covered by it.

Now we move on to the further issue introduced in §2.2 concerning whether the explicit definitions we have provided of classical concepts are enough to provide understanding of classical physics. Equation (12) already provides an explanation of a sort—namely, a deductive-nomological one, where the law-like explananda are the quantum kinematic and dynamical structures encoded in T_b and the explanans is the classical kinematic and dynamical structure encoded in T_t. But as discussed in §2.2, one might reasonably demand a more telling explanation, or at least a clarification of which bits of quantum structure are relevant for fixing which bits of classical structure. We will address this question with category-theoretic tools. The end result will show how the physically interpreted mathematical structure of quantum mechanics determines the corresponding structure of classical physics—and we take this determination to impart one useful way of understanding classical behavior.

7 Category-Theoretic Preliminaries

Recall from §2.2 that our strategy is to mathematically encode at least parts of the physical interpretations of the structure of classical and quantum physics, and show that the latter structurally determines the former. Following Dewar (2017), we believe that certain aspects of interpretation, which we are calling *internal*, can be encoded mathematically using the tools of category theory. Specifically, category theory provides a setting for reasoning about structure-preserving maps, and the internal task of saying which mathematical features of a theory are physically significant (separate from the external task of saying what in the world they represent) is often accomplished by restricting attention to those mathematical features invariant under the structure-preserving maps. In other words, one can specify physically significant structures by specifying the collection of maps between objects that preserve them. We will use these tools to show that internal aspects of the interpretation of quantum physics, encoded in a category of mathematical objects representing quantum theories, structurally determines internal aspects of the interpretation of classical physics, encoded in a category of mathematical objects representing classical theories. In this section, we introduce the necessary category-theoretic tools[21] as well as the categories encompassing quantum and classical theories that we work with.

A *category* $\mathbf{C}$ consists in a collection of *objects* and a collection of *arrows* or *morphisms*, each with a source and target. We denote a morphism by $f : A \to B$ for source A and target B. We will sometimes denote the collection of morphisms with source A and target B by $\mathrm{Hom}_{\mathbf{C}}(A, B)$. Moreover, a category contains an operation of morphism composition, denoted $\circ$, which is total in the sense that for each $f : A \to B$ and $g : B \to C$, there is an $h : A \to C$ such that $h = g \circ f$. The composition operation is required to be associative, and moreover, each object A has a unique

[20] In particular, we conjecture that one can loosen the requirements on the Hamiltonian H in (C_3).

[21] For more background on category theory, see, e.g., Awodey (2010).

identity morphism 1_A whose composition with other morphisms leaves them unchanged. We call a morphism $f : A \to B$ an *isomorphism* if there exists a morphism $f^{-1} : B \to A$ such that $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$. It is instructive to think of morphisms as maps preserving relevant structure for the objects in the category and of isomorphisms as maps identifying when two objects have the "same" structure, relative to the kind of structure encoded in the category.

The primary example of a category we are concerned with is the category of uniformly continuous bundles of C*-algebras, which we define now.

Definition 4. The category $\mathbf{UBunC^*Alg}$ consists in:

- *objects*: uniformly continuous bundles of C*-algebras whose base space is a locally compact metric space,
$$\mathcal{A}_I = \left(\left(\mathfrak{A}_\hbar, \phi_\hbar^I \right)_{\hbar \in I}, \mathfrak{A} \right);$$
- *morphisms*: bundle homomorphisms $\sigma : \mathcal{A}_I \to \mathcal{B}_J$.

The category $\mathbf{UBunC^*Alg}$ encodes the structure picked out by the definition of uniformly continuous bundles of C*-algebras. One can use objects in this category to represent quantum theories with information about the scaling behavior of physical quantities.

One other example of a category that we will use is the category of C*-algebras, defined as follows.

Definition 5. The category $\mathbf{C^*Alg}$ consists in:

- *objects*: C*-algebras;
- *morphisms*: *-homomorphisms.

The category $\mathbf{C^*Alg}$ encodes the structures picked out by the axioms for C*-algebras. We will think of certain objects in this category—in particular, the commutative C*-algebras, which form a subcategory—as representing classical theories.

We are now concerned with whether the explicit definitions (D_1)-(D_3), which amount to the construction of an object in $\mathbf{C^*Alg}$ from an object in $\mathbf{UBunC^*Alg}$, preserve the structure encoded in these different categories. In other words, we are interested in structure-preserving maps between categories. Such maps are given by the notion of a functor. A *functor* F between two categories $\mathbf{C}$ and $\mathbf{D}$ consists in two maps: one map between the objects of $\mathbf{C}$ and the objects of $\mathbf{D}$, and another map between the morphisms of $\mathbf{C}$ and the morphisms of $\mathbf{D}$. We denote both maps by F and require that if $f : A \to B$ is a morphism in $\mathbf{C}$, then $F(f) : F(A) \to F(B)$ in $\mathbf{D}$ (i.e., if $f \in \mathrm{Hom}_{\mathbf{C}}(A, B)$), then $F(f) \in \mathrm{Hom}_{\mathbf{D}}(F(A), F(B))$). Moreover, we require that F preserves the composition of arrows in the sense that if $f : A \to B$ and $g : B \to C$ are morphisms in $\mathbf{C}$, then $F(f \circ g) = F(f) \circ F(g)$.

We will use functors to compare the structure represented in different categories encoding partial interpretations of classical and quantum physics—in general, we can do this by analyzing the information that a functor "forgets." A useful schema for identifying precisely *what* a functor forgets is to consider each category as a collection of stuff equipped with structure* that has some properties[22]—for example, the category of groups consists of sets of elements (stuff) equipped with

[22] In what follows, we employ the terminology of Nguyen et al. (2020) by reserving the term "structure" for the pre-theoretical notion meaning (roughly) "information encoded in a mathematical tool" and using the term "structure*" to refer to the technical notion due to Baez et al. (2004). Note that stuff, structure*, and properties are all special sorts of structure, as we use the term.

a group operation (structure*) satisfying the usual axioms (properties). Intuitively, the natural embedding of the category of abelian groups in the category of all groups ought to preserve the stuff and the structure* (the set and group operation, respectively) but forget a property (abelianness). Following Baez et al. (2004), we make this intuition precise with the following scheme.

Suppose we have a functor $F : \mathbf{C} \to \mathbf{D}$. We call F *full* if for each pair of objects A, B in $\mathbf{C}$, the map $F : \mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{D}}(F(A), F(B))$ is surjective. We call F *faithful* if for each pair of objects A, B in $\mathbf{C}$, the map $F : \mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{D}}(F(A), F(B))$ is injective. We call F *essentially surjective* if for each object A' in $\mathbf{D}$, there is some object A in $\mathbf{C}$ such that $F(A)$ is isomorphic to A' in $\mathbf{D}$. We employ the following rough interpretations of these technical features of functors. We say that a functor *forgets only stuff* when it is not faithful, but it is essentially surjective and full. We say that a functor *forgets only structure** if it is not full, but it is essentially surjective and faithful. We say that a functor *forgets only properties* if it is not essentially surjective, but it is faithful and full. Note that the natural embedding of the category of abelian groups in the category of all groups is faithful and full, but not essentially surjective—so this schema characterizes it as preserving stuff and structure* and forgetting just a property, as desired. If a functor is full, faithful, and essentially surjective, then we say that it *forgets nothing* and call it a *categorical equivalence*. Categorical equivalence provides a relevant standard for when two categories share the same structure.[23] Later on, we will establish that the classical limit can be interpreted as a functor that forgets stuff and structure*. We will provide additional discussion of these notions of forgetfulness in §9, but we note here that interpretations of these notions of forgetfulness have been discussed extensively already in the literature (Weatherall, 2017; Bradley and Weatherall, 2020). We encourage the reader to review those discussions for further introduction.

We now proceed to summarize how, when our bundles are understood to form a category, the $\hbar \to 0$ limit can be represented as a functor from (a subcategory of) $\mathbf{UBunC^*Alg}$ to $\mathbf{C^*Alg}$. We analyze what this functor forgets and argue for a sense in which the structure of quantum physics determines the structure of classical physics.

8 Functoriality of the Classical Limit

In §4, we analyzed the classical limit in terms of extensions of uniformly continuous bundles to larger base spaces. Recall that we supposed our quantum theories to come with at least enough interpretation to make sense of the scaling behavior of quantities in different systems of units. That information in the quantum theory can be modeled as a bundle over the base space $(0, 1]$—representing values of $\hbar > 0$. The extension of this bundle to the larger base space $[0, 1]$ would encompass the corresponding classical theory—including the limit at $\hbar = 0$. Recall that Theorems 1 and 2 established the existence and uniqueness of exactly this kind of bundle extension under quite general conditions. The result was the explicit definition (D_1) of an algebra of observable quantities at $\hbar = 0$, i.e., the fiber algebra representing the classical limit.

In this section, we argue that the construction yielding (D_1) moreover *determines the structure* of the classical theory at the limit. In service of this goal, we will understand the classical limit as the composition of two functors. First, there is a functor F implementing the extension of a uniformly continuous bundle of C*-algebras as in Theorems 1 and 2. Second, there is a functor G representing the restriction from a bundle to the C*-algebra of the classical theory in the fiber over $\hbar = 0$. We understand the classical limit to be the functor L obtained from the composition $G \circ F$. We show in the Appendix that F forgets nothing, but G forgets structure* and stuff; it then follows immediately that L forgets structure* and stuff. We interpret this result to establish

[23]For more on applications of this classification scheme in philosophy of physics, see Weatherall (2016b, 2017).

a sense in which the structure of a classical theory is determined by that of its corresponding quantum theory. Due to constraints of space, this section gives only an informal summary of the construction of these functors provided in Steeger and Feintzeig (2021). We focus on discussing the philosophical significance of these functors and what they forget.

8.1 Extension Functor

Recall that an extension of a bundle of C*-algebras from some base space I to a larger base space J by the construction in §4 always proceeds relative to a dense isometric embedding $\alpha : I \to J$. The extension from the base space $I = (0, 1]$ for values of $\hbar > 0$ to the space $J = [0, 1]$ with $\hbar = 0$ included can be naturally understood to proceed with respect to the standard inclusion mapping $(0, 1] \to [0, 1]$. But to represent bundle extensions functorially, we need a common way of extending all bundles in a category, or a corresponding common way of providing a dense isometric embedding of every base space in a larger metric space. To that end, we think of the extension from $(0, 1]$ to $[0, 1]$ as an instance of the more general construction of the *one-point compactification* (see, e.g., Engelking, 1989, p. 169, Theorem 3.5.11), which can be applied to a broad class of base spaces. We summarize the construction of a functor representing bundle extensions associated with the one-point compactification, following the more detailed presentation in Steeger and Feintzeig (2021).

To use the one-point compactification for our purposes, we must impose some conditions on the base spaces of our bundles and the morphisms between them we consider. First, we consider only morphisms between bundles that act as proper maps between base spaces because only these maps are guaranteed to have continuous extensions to the one-point compactification. Second, we consider only those base spaces that are locally compact, non-compact metric spaces with metrizable one-point compactifications, and whose embedding in their one-point compactification is an isometric map. This class of base spaces includes $I = (0, 1]$, but rules out $I = \mathbb{R}$ since the embedding of the latter in its one-point compactification (the circle S^1) is not an isometric map. We denote the collection of such base spaces by $\mathcal{I}$. We will work with the subcategory of uniformly continuous bundles of C*-algebras whose base spaces belong to $\mathcal{I}$, and where morphisms act as proper maps between base spaces, which we denote by $\mathbf{UBunC^*Alg}_{\mathcal{I}}$. This category forms the domain of our extension functor.

Next, we need to specify the codomain category for our extension functor. We again impose some conditions on the base spaces of our bundles and the morphisms between them. First, we focus only on those base spaces that are obtained as the one-point compactification of some base space in $\mathcal{I}$, a collection we denote $C(\mathcal{I})$. Second, we consider only morphisms between bundles whose actions on base spaces preserve the way a base space is embedded in its one-point compactification. We call the category of uniformly continuous bundles satisfying these conditions $\mathbf{UBunC^*Alg}_{C(\mathcal{I})}$, which will form the codomain of our extension functor. The additional condition on morphisms in this category means only that such bundles have enough structure to distinguish the original base space from the added limit point obtained in the one-point compactification. For example, we understand such bundles to come with enough structure to pick out $(0, 1]$ as a privileged subspace of $[0, 1]$. The physical motivation for this condition is that there is a physically significant difference between the quantum theories at $\hbar > 0$ and the corresponding classical theory at $\hbar = 0$. We have thus endowed our bundles with enough structure to reflect this physical distinction.

The extension functor $F : \mathbf{UBunC^*Alg}_{\mathcal{I}} \to \mathbf{UBunC^*Alg}_{C(\mathcal{I})}$ is defined as follows. First, F acts on objects by taking any bundle to its unique extension associated with the one-point compactification guaranteed by Theorems 1 and 2. Second, F acts on arrows by taking the unique continuous extension of any proper isometric map between base spaces to a map between their one-point compactifications and leaving the *-homomorphisms between algebras of continuous sections unchanged. Notice that the condition on *-homomorphisms between algebras of sections makes sense because in the construction of the extended bundle, we employ the *same* algebra of sections and only define new evaluation maps to extend those sections continuously to new points in the base space. Steeger and Feintzeig (2021) establish that this assignment F of objects and morphisms is indeed a functor. Here, we note that the functor F representing bundle extensions forgets nothing.

Proposition 2. *The extension functor F is a categorical equivalence.*

Thus, relative to the conditions specified above, or in other words relative to the structure encoded in the categories $\mathbf{UBunC^*Alg}_{\mathcal{I}}$ and $\mathbf{UBunC^*Alg}_{C(\mathcal{I})}$, any uniformly continuous bundle of C*-algebras over a base space I completely determines the structure of its extension associated with the one-point compactification of I. Since the functor realizing this extension is an equivalence, there is a sense in which these objects have precisely *the same structure.*

8.2 Restriction Functor

We now aim to encode the remainder of the classical limit in the restriction from the base space $[0,1]$ to the structure of classical physics at $\hbar = 0$. Since we are representing the extension from $(0,1]$ to $[0,1]$ as an instance of an extension associated with the one-point compactification, we will treat one-point compactifications more generally and generically denote the added point to a base space I by 0_I. The restriction functor G we now describe has the category of extended bundles $\mathbf{UBunC^*Alg}_{C(\mathcal{I})}$ as its domain, which is the codomain of the extension functor F. The codomain category of the restriction functor G is the category $\mathbf{C^*Alg}$ of C*-algebras.

The restriction functor $G : \mathbf{UBunC^*Alg}_{C(\mathcal{I})} \to \mathbf{C^*Alg}$ is defined as follows. First, G acts on objects by taking any extended bundle over $I \cup \{0_I\}$ to the fiber C*-algebra over 0_I. Second, G acts on morphisms by constructing from any bundle morphism between bundles over base spaces I and J a *-homomorphism between the fibers at 0_I and 0_J by factoring through the corresponding evaluation maps. As remarked already, a bundle morphism (α, β) always determines morphisms between the fiber algebras over $\hbar \in I$ and $\alpha(\hbar) \in J$. To see that this gives rise to a morphism between the fibers over 0_I and 0_J, one only needs to note that if α is the extension of a proper map between base spaces $I, J \in \mathcal{I}$, then $\alpha(0_I) = 0_J$. Hence, it follows that there is a unique *-homomorphism between the fiber algebras at 0_I and 0_J determined by a bundle morphism. Steeger and Feintzeig (2021) establish that this assignment G of objects and morphisms is indeed a functor. Here, we note that the functor G representing bundle restrictions forgets structure* and stuff.

Proposition 3. *The restriction functor G is essentially surjective but neither full nor faithful.*

We note briefly that the proof that G is essentially surjective relies only on constructing a trivial bundle whose fibers all are isomorphic to whatever limit algebra one wants to reproduce (see Appendix). This somewhat artificial construction does not suffice to show that every classical theory is the classical limit of some quantum theory as one might hope.

But given Proposition 3, we can interpret G as forgetting stuff and structure*. It should be unsurprising that G forgets stuff. After all, a uniformly continuous bundle is much larger than the algebra over 0_I; as such, we can understand the stuff that G forgets to be the fibers $\mathfrak{A}_\hbar$ above all other points $\hbar \neq 0_I$. We return in §9 to the physical interpretation of the stuff that G forgets and how it bears on our central philosophical questions concerning structural determination and intertheoretic reduction.

It is a bit more difficult to characterize the structure* that G forgets. To understand this structure*, consider our initial motivation for employing continuous bundles of C*-algebras—namely, that they arise in a natural way from strict deformation quantizations. Recall that a strict deformation quantization relies for its definition on the further structure of a Poisson bracket in the classical theory. However, objects in $\mathbf{C^*Alg}$ do not contain the structure of a Poisson bracket and morphisms in this category need not preserve a Poisson bracket when one exists. As such, if we are given two continuous bundles of C*-algebras $\mathcal{A}_I$ and $\mathcal{B}_I$ over the base space $I = [0, 1]$, and $\mathcal{A}_I$ is determined from a strict deformation quantization, then there will, in general, be morphisms in $\mathbf{C^*Alg}$ between the fibers $\mathfrak{A}_0$ and $\mathfrak{B}_0$ at $\hbar = 0$ that do *not* preserve the Poisson bracket on $\mathfrak{A}_0$. However, morphisms between continuous bundles determined by a strict deformation quantization are generally constrained to preserve the Poisson bracket in a classical theory. The proof that G is not full in Proposition 3 proceeds, roughly, by showing that there are *-homorphisms σ_0 between the fibers $\mathfrak{A}_0$ and $\mathfrak{B}_0$ that cannot be mapped to by G because they fail to preserve the induced Poisson bracket in the classical limit. One way to understand this fact is that morphisms in $\mathbf{UBunC^*Alg}_{C(\mathcal{I})}$ encode not only the structure of the symmetric Jordan product on the fibers, but also the structure of the anti-symmetric Lie bracket determined by the commutator on the fibers, which corresponds in the classical limit with Poisson structure. Morphisms in $\mathbf{C^*Alg}$ that do not preserve the Poisson structures of classical theories (of which there are many) will not in general lie in the range of the functor G. So we can understand G to forget the structure* of the Lie brackets determined by the commutators on fibers in $\mathbf{UBunC^*Alg}_{C(\mathcal{I})}$, and thus to forget the structure* that encodes the classical Poisson bracket in the classical limit.

8.3 Limit Functor

We are now in a position to describe a single functor from quantum theories to classical theories. Recall that so far, we have an extension functor associated with the one point compactification $F : \mathbf{UBunC^*Alg}_{\mathcal{I}} \to \mathbf{UBunC^*Alg}_{C(\mathcal{I})}$, which forgets nothing, and a restriction functor $G : \mathbf{UBunC^*Alg}_{C(\mathcal{I})} \to \mathbf{C^*Alg}$, which forgets stuff and structure*. We now define the classical limit $L := G \circ F$ as the composition of the two. It follows immediately that L is a functor, which is characterized by the following proposition.

Proposition 4. *The functor L is essentially surjective but neither full nor faithful.*

Now we have characterized the classical limit with the functor L. This functor takes any quantum theory defined only for the values $\hbar > 0$ by a bundle of C*-algebras, extends this structure to a bundle whose base space includes the value $\hbar = 0$, and then restricts attention to the unique fiber C*-algebra glued on at $\hbar = 0$. The functor L forgets stuff and structure*. We take this fact to show that classical theories have *less* structure than quantum theories, demonstrating a sense in which the structure of the latter determines that of the former.

However, one should be careful in explicating the sense in which this claim is true. First, quantum mechanics determines its classical limit *only when* quantum mechanics is given a physical interpretation strong enough to motivate the use of the category $\mathbf{UBunC^*Alg}_{\mathcal{I}}$. It is not the mathematical structure of quantum mechanics that determines the structure of classical physics on its own, but rather the structure we understand quantum theories to possess when they are interpreted in a way that can be represented by the category $\mathbf{UBunC^*Alg}_{\mathcal{I}}$. Since we represent quantum theories not by the fiber C*-algebras at a fixed value of $\hbar$, but rather by an entire bundle of C*-algebras, we are understanding quantum theories to come with enough of an interpretation that we understand their scaling behavior as $\hbar$ varies. The choice of morphisms in $\mathbf{UBunC^*Alg}_{\mathcal{I}}$ is motivated by our focus on maps that preserve the scaling behavior of quantum theories as we vary $\hbar$.

Further, the sense in which classical physics has less structure than quantum physics holds only relative to the functor L and so *only when* classical physics is given a physical interpretation weak enough to motivate the use of the category $\mathbf{C^*Alg}$. This assumption is actually somewhat unnatural. The use of C*-algebras in classical physics provides only enough information to recover the topological structure of a phase space—it does not provide enough information to recover the Poisson or even the differentiable structure of a phase space as a manifold. If one thinks that this structure is essential to classical kinematics, then our results so far will not be satisfying. Further, none of these results so far relate classical dynamics to quantum physics.

However, we now summarize briefly some further technical results that show a sense in which both Poisson and dynamical structure in classical theories is determined from that in quantum theories. Steeger and Feintzeig (2021) show that the assignment defined by L sends (under mild conditions) any morphism of bundles defined by strict quantizations to a classical morphism that preserves the Poisson structure. And further, the assignment defined by L sends any morphism that preserves quantum dynamics to a classical morphism that preserves dynamics. This means that one can define adapted functors L_P and L_D capturing the classical limit on adapted categories of bundles that encode Poisson and dynamical structure, respectively. One can then analyze what these functors forget.

First, we discuss the functor L_P preserving Poisson structure.

- L_P is not faithful for precisely the same reason L is not faithful. One can interpret this to just show L_P forgets the same stuff that L does.

- As far as we can tell, the question of whether L_P is essentially surjective is open. Recall that this is the question of whether every classical theory is (isomorphic to) the classical limit under L_P of some quantum theory. In other words, this is just the question of how broad a collection of classical theories we can quantize with strict deformation quantization. It is known that one can construct Weyl-type quantizations for the almost periodic functions on a phase space that is the dual space to a symplectic topological vector space (Binz et al., 2004). It is also known that when one does not have a phase space with the linear structure of a vector space, one can still construct a Weyl-type quantization for a C*-algebra with an isometric action of $\mathbb{R}^d$ (Rieffel, 1993); for some generalizations, see Landsman (1998b) and Bieliavsky and Gayral (2015). Each of these constructions induces a bundle whose classical limit is the associated structure. These examples may provide enough generality to encompass many of the models of physical systems used in classical physics. But we do not know whether there is a construction general enough to quantize arbitrary classical theories.

- Similarly, it is an open question (as far as we can tell) whether L_P is full. We know only for particular kinds of morphisms σ_0 in the category of classical theories that there is some morphism σ in the category of quantum theories with $L_P(\sigma) = \sigma_0$. For example, if σ_0 is an automorphism of a classical Weyl algebra given by some combination of Bogoliubov transformations and gauge transformations (i.e., determined by an element of the affine symplectic group of the underlying symplectic vector space), then this holds (Binz et al., 2004, §5.3). But these are only very special cases of morphisms in the category of classical theories. The results of Proposition 5.15 (p. 74) and Proposition 8.40 (p. 142) of Bieliavsky and Gayral (2015) are closely related and provide significantly more generality, but further scrutiny is needed to determine whether they settle the question about the fullness of L_P.

Finally, we briefly discuss the functor L_D preserving dynamical structure.

- L_D is not faithful for precisely the same reason L is not faithful. One can again interpret this to just show L_D forgets the same stuff that L does.

- L_D is essentially surjective for precisely the same reason L is: one can always define a trivial bundle whose fibers all look like the limit theory one wants to reproduce, which gets mapped by L or L_D to the desired object.
- L_D also fails to be full for precisely the same reason as L. There are morphisms of classical theories that preserve dynamics, but not Poisson structure, that cannot be recovered as the classical limit of any morphism of quantum theories.

It is also worth noting that L_D factors into an extension and restriction functor in just the same way as L. In this case, the extension of the dynamics also gives rise to a categorical equivalence while the restriction of the dynamics to $\hbar = 0$ is represented by a functor that forgets stuff and structure*.

This discussion of Poisson structure and dynamics only briefly summarizes the constructions in Steeger and Feintzeig (2021), where more details can be found. The next and concluding section of this paper returns to the interpretation of these functors and argues that they provide a sense in which the structure of quantum physics determines the structure of classical physics. This, we claim, suffices to show that internal aspects of the interpretation of quantum physics determine internal aspects of the interpretation of classical physics.[24]

9 Discussion

In this paper, we have argued against the claims of Batterman, Berry, and Bokulich that the classical $\hbar \to 0$ limit of quantum mechanics is "singular" in a way that frustrates intertheoretic reduction. We took on two tasks. First, §1-6 provided (partial) explicit definitions of classical structure in terms of quantum structure. We summarized the results of those sections in §6. But one may still rightly worry about whether this definability entails understanding. We took on this task in §7-8, which we now discuss.

We have argued that if the issue is whether the structure of quantum physics determines that of classical physics, then the issue can be resolved: structural determination does, indeed, hold, in a precise sense, for kinematics, Poisson structure, and dynamical structure. We have argued for this claim by formulating the classical limit as a functor from a relevant category capturing the (partially) physically interpreted mathematical structure of quantum mechanics to a category capturing the (partially) physically interpreted mathematical structure of classical mechanics. In the remainder of the discussion, we focus on the functor L determining the kinematics, and ignore the extra technical and interpretive complications that arise in the consideration of Poisson and dynamical structure. We showed in §8 that this functor L can be interpreted as forgetting stuff and structure*.

It is worth returning once again to the question of the interpretation of our results, since here stuff and structure* are themselves technical notions. What is the philosophical significance of L forgetting stuff and structure*? Specifically, what is the sense in which this forgetfulness demonstrates structural determination?

First, we remark that the existence of the classical limit as a functor already provides a sense in which quantum kinematics determines classical kinematics in a natural way. The naturalness of

[24] We have taken the internal interpretation as given, although it is often motivated by external factors, as in the discussion of changes of units in the Weyl algebra in §4. We leave open whether the internal structure of bundles of C*-algebras can be justified in the same way for all cases. But what we have now shown is that *if* bundles of C*-algebras encode the internal interpretation of quantum physics, then they determine the internal interpretation of classical physics, as long as *that* interpretation is encoded by commutative C*-algebras (possibly with extra structure).

the construction is captured by the functor acting in what we interpret as "the same way" on all models of quantum kinematics and all morphisms between them. The functor provides a way of matching the structure-preserving maps in the category capturing quantum kinematics to structure-preserving maps in the category capturing classical kinematics. Moreover, the morphisms in the category of quantum theories naturally determine *a subset* of the classical ones; this is just what it means for L to fail to be full. As a general heuristic, fewer structure-preserving maps suggests more structure* (see, e.g., Barrett, 2015, p. 814). In at least this sense, quantum kinematics determines possibly *even more* structure* than that captured by our category representing classical kinematics.

The fact that L forgets structure* shows that its domain, the category of quantum theories, contains *strictly more* relevant physical information than its codomain, the category of classical theories. This approach to forgetting structure* has been offered already in the literature (e.g., Weatherall, 2016b); philosophers often describe a theory represented by one category as possessing "surplus structure" relative to another. We can justify the applicability of this talk to our case by comparing with the alternative functor L_P. The classical limit determines at least the Poisson structure of classical kinematics in addition to C*-algebraic structure, and so the functor L can be thought of as forgetting Poisson structure in virtue of the choice of category representing classical kinematics. In sum, the stuff (fiber algebras away from $\hbar = 0$) and structure* (Poisson bracket) forgotten by L each have a natural physical interpretation—and these interpretations provide the sense in which L witnesses structural determination.

There is a subtlety worth discussing regarding the significance of L forgetting stuff: although this forgetfulness witnesses a clear case of structural determination in our case, it need not do so in general. As Bradley and Weatherall (2020) note in a response to Nguyen et al. (2020), there are cases of functors forgetting stuff where the codomain seems to have *more* structure than the domain. Their paradigm example is a functor from a category of two-dimensional vector spaces to a category of two-dimensional vector spaces with a fixed ordered basis. The functor they provide is essentially surjective and full, but not faithful—so it forgets only stuff. What stuff? The freedom to choose a basis—a freedom that, Bradley and Weatherall argue, intuitively denotes more structure in vector spaces with ordered bases than in vector spaces simpliciter. Thus, there are some examples where a functor forgetting stuff signifies that the codomain category has more structure than the domain category.

Bradley and Weatherall's diagnosis is of interest, but we believe it is not applicable to the cases studied in this paper. Bradley and Weatherall aim to establish a sense in which surplus structure and representational redundancy pull in opposite directions: the more redundancy in representational tools, the more freedom in choosing which tools to use—and hence the more structure. But in our case, *neither* the stuff nor the structure* that L forgets is representationally redundant: the codomain *loses its capacity to represent* both the observables in algebras at $\hbar > 0$ (quantum stuff) and their Poisson or commutator structures (quantum structure*). So our L does not fit with Bradley and Weatherall's discussion.

There is another way of arguing for this point from our construction of the functor L. We showed that L is the composition of two functors, i.e. $L = G \circ F$. The functor F extends a quantum theory from values of $\hbar$ in $(0, 1]$ to values of $\hbar$ in $[0, 1]$, and the functor G then restricts to the value $\hbar = 0$. F is an equivalence, so all of L's forgetfulness comes from G's forgetfulness. But surely the extension of values of $\hbar$ to $[0, 1]$ already provides *enough* structure to represent the classical theory at $\hbar = 0$. In other words, the fact that F is an equivalence shows that the structure encoded in the "frame theory" containing values of $\hbar$ in $[0, 1]$—of which the classical kinematics at $\hbar = 0$ is a part—is equivalent to the structure of the quantum theory for values of $\hbar$ in $(0, 1]$. Since the structure of classical kinematics at $\hbar = 0$ is *part* of the structure of the "frame theory" and since the latter is equivalent to the structure of the quantum theory, this shows the classical kinematics can be captured as *part* of the structure of the quantum theory. We interpret the fact that the extension functor F forgets nothing, but the restriction functor G forgets structure* and stuff, as showing a sense in which classical kinematics has *strictly* less structure than quantum kinematics—structure determined from quantum kinematics through the classical limit.

Lastly, we make one caveat about the essential surjectivity of the functor L. As it is stated in this paper, the essential surjectivity of L does not have the philosophical significance one would like: it does not show that all desired classical theories can be obtained as the $\hbar \to 0$ limit of desired quantum theories via a strict quantization. The proof given in the Appendix constructs, for any C*-algebra, a trivial uniformly continuous bundle of C*-algebras containing the initial algebra as its limit. But the bundle constructed will not contain anything that looks like a familiar quantum algebra of observables in its fibers. The specific question of how broad a class of classical theories can be subject to strict deformation quantization, thus leading to their own recovery in the $\hbar \to 0$ limit, is an open question, as the domain of applicability of strict quantization continues to be extended in the literature (Rieffel, 1993; Landsman, 1998a; Bieliavsky and Gayral, 2015). We believe this open question clearly has philosophical significance for the extent to which quantum mechanics determines the structure of classical physics. We take our comments in §8 to make a small contribution to this literature by framing the philosophical significance of this question. But note well that the functor L simply does not address this question.

In sum, we have shown a precise sense in which the structure of quantum kinematics determines the structure of classical kinematics though the classical $\hbar \to 0$ limit. This sense of structural determination should be amenable to the view, shared by Batterman and Belot, that appropriate intertheoretic reductions should explain the physically-interpreted structures—rather than just the mathematical formalisms alone—of less fundamental theories from more fundamental theories. We have mathematically encoded internal aspects of the interpretation of classical and quantum physics in a category-theoretic framework. Along the way, we have shown that applying category theoretic tools to questions concerning structural determination imparts a fruitful way of understanding classical behavior, at least in part by producing interesting results bearing on philosophical questions about intertheoretic reduction. There is much further work to be done in this direction.

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Appendix

This appendix includes proofs of all new propositions stated in the body of the paper.

Proof of Proposition 1. For ease of notation, we define the global quantization map $\mathcal{Q} : \mathcal{P} \to \mathfrak{A}$ by

$$\mathcal{Q}(A) := [\hbar \mapsto \mathcal{Q}_\hbar(A)]$$

for each $A \in \mathcal{P}$ (cf. continuous quantization in Landsman, 1998a, p. 112). Notice that $\mathcal{Q}_\hbar = \phi_\hbar^J \circ \mathcal{Q}$, and also recall that (5) yields

$$\tau_{t;0} \circ \phi_0^J = \phi_0^J \circ \tau_t.$$

Moreover, since $\phi_0^J \circ \mathcal{Q} = \mathcal{Q}_0 = \mathrm{id}_{\mathcal{P}}$, the above immediately implies

$$\phi_0^J \circ \mathcal{Q}(\tau_{t;0}(A)) = \tau_{t;0}(\phi_0^J(\mathcal{Q}(A))) = \phi_0^J(\tau_t(\mathcal{Q}(A)))$$

for all $A \in \mathcal{P}$. Then we have for any $A \in \mathcal{P}$ that

$$\begin{aligned}
\lim_{\hbar \to 0} \|\mathcal{Q}_\hbar(\tau_{t;0}(A)) - \tau_{t;\hbar}(\mathcal{Q}_\hbar(A))\|_\hbar &= \lim_{\hbar \to 0} \left\|\phi_\hbar^J \circ \mathcal{Q}(\tau_{t;0}(A)) - \tau_{t;\hbar}(\phi_\hbar^J \circ \mathcal{Q}(A))\right\|_\hbar \\
&= \lim_{\hbar \to 0} \left\|\phi_\hbar^J \circ \mathcal{Q}(\tau_{t;0}(A)) - \phi_\hbar^J(\tau_t \circ \mathcal{Q}(A))\right\|_\hbar \\
&= 0
\end{aligned}$$

by the previous equation, since both $\mathcal{Q}(\tau_{t;0}(A))$ and $\tau_t \circ \mathcal{Q}(A)$ are uniformly continuous sections (belonging to $\mathfrak{A}$) that agree at $\hbar = 0$. □

Theorem 3 then follows immediately from a convergence result of Landsman (1998a), which we now state. The upshot is that if we take (7) and (8) to *define* the dynamics, rather than using the dynamics defined by (D_3), then one gets convergence in the limit $\hbar \to 0$ in precisely the same sense as in Proposition 1.

Lemma 1 (Landsman 1998a, Proposition 2.7.1, p. 148)**.** *Suppose the maps* $\{\mathcal{Q}_\hbar : \mathcal{P} \to \mathfrak{A}_\hbar\}_{\hbar \in [0,1]}$ *form a strict deformation quantization with* $\mathcal{P} \subseteq C_0(T^*\mathbb{R}^n)$ *and* $\mathfrak{A}_\hbar \subseteq \mathcal{B}(L^2(\mathbb{R}^n))$ *for* $\hbar > 0$*. Suppose further that dynamics* $\tau_{t;\hbar}$ *are defined from a given Hamiltonian* $H \in \mathcal{P}$ *by (7) for each* $\hbar > 0$ *and (8) for* $\hbar = 0$*. Then for each* $A \in \mathcal{P}$*, if* $\tau_{t;0}(A) \in \mathcal{P}$ *for all* $t \in \mathbb{R}$*, then (6) holds.*

Proof of Theorem 3. Every continuous section has a unique limit; thus, the theorem follows from the fact that (6) holds for both the dynamics defined for Lemma 1 and the dynamics defined by (D_3) for Proposition 1. □

We now characterize what the functors F, G, and L of §8 forget. For more detailed definitions and preliminaries, see Steeger and Feintzeig (2021). Note that we have two changes of notation from Steeger and Feintzeig (2021). First, we denote the one-point compactification by C rather than C_1 since it is the only extension of base spaces we consider. Second, we here denote by $\mathbf{UBunC^*Alg}_{C(\mathcal{I})}$ a subcategory of what Steeger and Feintzeig (2021) call $\mathbf{UBunC^*Alg}_{C(\mathcal{I})}$ in which all morphisms preserve the embeddings of base spaces in their one-point compactifications (i.e., for a morphism $\alpha : C(I) \to C(J)$ between base spaces $C(I)$ and $C(J)$, we require $\alpha \circ \alpha_I[I] \subseteq \alpha_J[J]$). All other notation is the same as in that reference.

Proof of Proposition 2. We consider faithfulness, essential surjectivity, and fullness in turn.

1. *Faithfulness.* Suppose that for two morphisms (α_1, β_1) and (α_2, β_2) between $F(\mathcal{A}_I) = \tilde{\mathcal{A}}_{C(I)}$ and $F(\mathcal{B}_J) = \tilde{\mathcal{B}}_{C(J)}$, we have $F(\alpha_1, \beta_1) = F(\alpha_2, \beta_2)$. Then it immediately follows that $\beta_1 = \beta_2$ from the definition of F. Moreover, we have $C(\alpha_1) = C(\alpha_2)$, which implies $\alpha_J \circ \alpha_1 = C(\alpha_1) \circ \alpha_I = C(\alpha_2) \circ \alpha_I = \alpha_J \circ \alpha_2$, which implies $\alpha_1 = \alpha_2$ because α_J is injective.

2. *Essential surjectivity.* Consider an arbitrary bundle $\mathcal{A}_{C(I)}$ in $\mathbf{UBunC^*Alg}_{C(\mathcal{I})}$. The canonical restriction $\mathcal{A}_{C(I)|I}$ along α_I is a bundle in $\mathbf{UBunC^*Alg}_{\mathcal{I}}$, and since $\mathcal{A}_{C(I)}$ is an extension of $\mathcal{A}_{C(I)|I}$ associated with α_I, it follows from Theorem 2 of Part I that $\mathcal{A}_{C(I)}$ is isomorphic in $\mathbf{UBunC^*Alg}_{C(\mathcal{I})}$ to $F_0(\mathcal{A}_{C(I)|I}) = \tilde{A}_{C(I)}$.

3. *Fullness* If $\sigma = (\alpha, \beta)$ is a morphism in $\mathbf{UBunC^*Alg}_{C(\mathcal{I})}$, then $\sigma = F(\sigma')$ for the morphism $\sigma' = (\alpha \circ \alpha_I, \beta)$ in $\mathbf{UBunC^*Alg}_{\mathcal{I}}$.

$\square$

Proof of Proposition 3. We consider faithfulness, fullness, and essential surjectivity in turn.

1. *Faithfulness.* Let $I = (0,1]$ so that $C(I) = [0,1]$. Suppose $\mathcal{A}_J$ and $\mathcal{B}_I$ are objects in $\mathbf{UBunC^*Alg}'_{C(\mathcal{I})}$ and $\sigma = (\alpha, \beta) : \mathcal{A}_J \to \mathcal{B}_I$ is a morphism in $\mathbf{UBunC^*Alg}'_{C(\mathcal{I})}$. Then so is the morphism $\sigma' = (\frac{1}{2}\alpha, \beta) : \mathcal{A}_J \to \mathcal{B}_I$. But $G(\sigma) = G(\sigma')$. Hence, G fails to be faithful.

2. *Fullness.* We consider two particular objects $\mathcal{A}_I$ and $\mathcal{B}_J$ in $\mathbf{UBunC^*Alg}'_{C(\mathcal{I})}$. First, fix $I = J = [0,1]$, and consider some Riemannian manifold M. $\mathcal{A}_I$ will be the trivial bundle, each of whose fibers is isomorphic to the abelian C*-algebra $C_0(T^*M)$. $\mathcal{B}_J$ will be the nontrivial bundle determined by the Weyl quantization $\mathcal{Q}_\hbar$ of T^*M. Explicitly,

$$\mathcal{A}_I := \left(\left(\mathfrak{A}_\hbar, \phi^I_\hbar\right)_{\hbar \in I}, \mathfrak{A} \right); \qquad \mathcal{B}_J := \left(\left(\mathfrak{B}_\hbar, \psi^J_\hbar\right)_{\hbar \in J}, \mathfrak{B} \right);$$

$$\mathfrak{A}_0 := C_0(T^*M); \qquad \mathfrak{B}_0 := C_0(T^*M);$$

$$\mathfrak{A}_\hbar := C_0(T^*M) \text{ for all } \hbar \in (0,1]; \qquad \mathfrak{B}_\hbar := \mathcal{K}(L^2(M)) \text{ for all } \hbar \in (0,1];$$

$$\mathfrak{A} := \{ fa \mid f \in C([0,1]), a \in C_0(T^*M) \}; \qquad \mathfrak{B} := \left\{ a \in \prod_{\hbar \in J} \mathfrak{B}_\hbar \;\middle|\; a \in C^*(\mathcal{Q}) \right\};$$

$$\phi^I_\hbar(fa) := f(\hbar)a \text{ for each } fa \in \mathfrak{A}; \qquad \psi^J_\hbar(a) := a(\hbar) \text{ for each } a \in \mathfrak{B}.$$

Now consider the identity map $\mathrm{id}_{C_0(T^*M)}$ on the fiber $C_0(T^*M)$ at $\hbar = 0$ in $\mathbf{C^*Alg}$. We will show that there is no morphism $\sigma : \mathcal{B}_J \to \mathcal{A}_I$ in $\mathbf{UBunC^*Alg}'_{C(\mathcal{I})}$ with $G(\sigma) = \mathrm{id}_{C_0(T^*M)}$.

Suppose $\sigma = (\alpha, \beta) : \mathcal{B}_J \to \mathcal{A}_I$ is a morphism in $\mathbf{UBunC^*Alg}'_{C(\mathcal{I})}$. Fix any $A, B \in C_c^\infty(T^*M)$ with $\{A, B\} \neq 0$. We will show that $G(\sigma)(\{A, B\}) = 0$, thus establishing that $G(\sigma)$ is not injective. It then follows that $G(\sigma) \neq \mathrm{id}_{C_0(T^*M)}$.

Denote the corresponding continuous sections by $a = [\hbar \mapsto \mathcal{Q}_\hbar(A)]$ and $b = [\hbar \mapsto \mathcal{Q}_\hbar(B)]$ in $\mathfrak{B}$. Further, let c be the continuous section given by $c = [\hbar \mapsto \mathcal{Q}_\hbar(\{A, B\})]$ in $\mathfrak{B}$.

We know from Dirac's condition in a strict deformation quantization that

$$\lim_{\hbar \to 0} \left\| \psi^J_\hbar(c) - \frac{i}{\hbar} \left[\psi^J_\hbar(a), \psi^J_\hbar(b) \right] \right\|_\hbar = 0.$$

Moreover, let $c' \in \mathfrak{B}$ be the continuous section defined by

$$\psi^J_\hbar(c') = \frac{i}{\hbar} \left[\psi^J_\hbar(a), \psi^J_\hbar(b) \right]$$

for each $\hbar \in [0,1]$. Now we have $\lim_{\hbar \to 0} \|\psi^J_\hbar(c - c')\|_\hbar = 0$. Further, by Lemma 3 in the Appendix to Part I, the map $\psi^J_\hbar(d) \mapsto \phi^I_\hbar(\beta(d))$ for $d \in \mathfrak{B}$ is a *-homomorphism, so it follows that

$$\phi^I_\hbar(\beta(c')) = \frac{i}{\hbar} \left[\phi^I_\hbar(\beta(a)), \phi^I_\hbar(\beta(b)) \right] = 0$$

for each $\hbar \in [0,1]$ because $\mathfrak{A}$ is abelian. Now, we have

$$\lim_{\hbar \to 0} \left\| \phi^I_\hbar(\beta(c)) \right\|_\hbar = \lim_{\hbar \to 0} \left\| \phi^I_\hbar(\beta(c - c')) \right\|_\hbar \leq \lim_{\hbar \to 0} \left\| \psi^J_\hbar(c - c') \right\|_\hbar = 0,$$

which implies $\phi_0^I(\beta(c)) = 0$.

It follows that $G(\sigma)(\{A, B\}) = \phi_0^I(\beta(c)) = 0$, so $G(\sigma)$ is not injective and hence $G(\sigma) \neq \mathrm{id}_{C_0(T^*M)}$. Therefore, G fails to be full.

3. *Essential surjectivity.* Consider an arbitrary C*-algebra $\mathfrak{A}$ and $I \in \mathcal{I}$. Define a trivial bundle by $\mathcal{A}_{C(I)} = ((\mathfrak{A}_\hbar, \phi_\hbar^{C(I)})_{\hbar \in C(I)}, \hat{\mathfrak{A}})$, where

$$\mathfrak{A}_\hbar := \mathfrak{A} \text{ for all } \hbar \in C(I);$$
$$\hat{\mathfrak{A}} := \{fa \mid f \in C_0(C(I)), a \in \mathfrak{A}\};$$
$$\phi_\hbar^{C(I)}(fa) := f(\hbar)a \text{ for each } \hbar \in C(I).$$

Then $\mathcal{A}_{C(I)}$ is an object in $\mathbf{UBunC^*Alg}'_{C(\mathcal{I})}$ and $G(\mathcal{A}_{C(I)}) \cong \mathfrak{A}$. Hence, G is essentially surjective.

□

Proof of Proposition 4. Immediate from Propositions 2 and 3. □

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