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Article

Symmetry, Invariance and Ontology in Physics and Statistics

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Abstract: This paper has three main objectives: (a) Discuss the formal analogy between some important symmetry-invariance arguments used in physics, probability and statistics. Specifically, we will focus on Noether's theorem in physics, the maximum entropy principle in probability theory, and de Finetti-type theorems in Bayesian statistics; (b) Discuss the epistemological and ontological implications of these theorems, as they are interpreted in physics and statistics. Specifically, we will focus on the positivist (in physics) or subjective (in statistics) interpretations *vs.* objective interpretations that are suggested by symmetry and invariance arguments; (c) Introduce the cognitive constructivism epistemological framework as a solution that overcomes the realism-subjectivism dilemma and its pitfalls. The work of the physicist and philosopher Max Born will be particularly important in our discussion.

Keywords: cognitive constructivism; de Finetti's and Noether's theorems; information geometry; MaxEnt formalism; objective ontologies; reference priors; subjectivism

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Objectivity means invariance with respect to the group of automorphisms.
Hermann Weyl. Symmetry, (1989, p.132).

Let us turn now to the relation of symmetry or invariance principles to the laws of nature... It is good to emphasize at this point the fact that the laws of nature, that is, the correlations between events, are the entities to which the symmetry laws apply, not the events themselves.
Eugene Wigner. Symmetry and Conservation Laws, (1967, p.16).

We cannot understand what is happening until we learn to think of probability distributions in terms of their demonstrable information content.
Edwin Jaynes. Probability Theory: The Logic of Science, (2003, p.198).

1. Introduction

In order to make this article accessible for readers in the physics and the statistics communities, we give a succinct review of several concepts, terminology and results involved, and also include simple derivations of a few classical results. Section 2 presents an overview of calculus of variations and Noether's theorem. Section 3 contrasts positivist and objective-ontological views of the invariant objects discussed in Section 2. Section 4 gives a very basic overview of Bayesian statistics and some results concerning the construction of invariant priors and posterior convergence theorems. Section 5 presents de Finetti's theorem and his subjectivist interpretation of Bayesian statistics. Sections 6 and 7 extend the discussion of symmetry-invariance in statistics to the MaxEnt framework in statistics and Noether's theorem in physics. These sections also include the discussion of objective ontologies for statistics, and a critique to de Finetti's position. Section 8 presents the cognitive constructivism epistemological framework as solution to the realist-subjectivist dilemma. This third alternative offers a synthesis of the classical realist and subjectivist positions that, we believe, is also able to overcome some of their shortcomings. Section 9 presents our final conclusions and directions for further research.

2. Minimum Action Principle and Physical Invariants

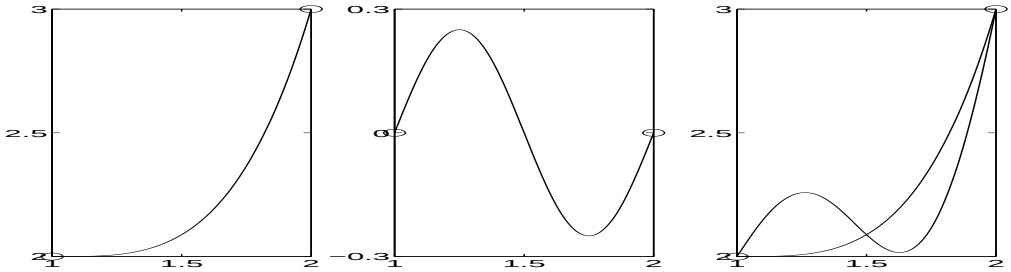
This section is a brief overview of symmetry and invariance in classical physics. Variational calculus is briefly reviewed in the following subsection. This section also presents the minimum action principle and a basic interpretation of Noether's theorem, linking symmetry and invariance in classical physics, see [1,2]. Many readers with a good backgrounds in physics will be well-acquainted with this material and may rather skip the following examples and derivations. These are kept as simple as possible, in order to make them easily accessible to readers from different backgrounds. We also hope that this approach will help them to understand and recognize the similarities between the techniques presented in this section and analogous techniques presented in Section 7 concerning probability and statistics. The presentation in the next subsections follows Section 2.7 of [3], see also [4,5], For an historical perspective see [6]. For thorough reviews of Noether's theorem and its consequences, see [2].

2.1. Variational Principles

Variational problem asks for the function $q(t)$ that minimizes a global functional (function of a function), $J(q)$, with fixed boundary conditions, $q(a)$ and $q(b)$, as shown in Figure 1. Its general form is given by a local functional, $F(t, q, q')$, and an integral or global functional, where the prime indicates, as usual, the simple derivative with respect to t, that is, $q' = dq/dt$,

$$J(q) = \int_a^b F(t, q, q')dt$$

Figure 1. Variational problem, $q(x)$, $\eta(x)$, $q(x) + \epsilon\eta(x)$.



2.2. Euler–Lagrange Equation

Consider a "variation" of $q(t)$ given by another curve, $\eta(t)$, satisfying the fixed boundary conditions, $\eta(a) = \eta(b) = 0$,

$$q = q(\epsilon, t) = q(t) + \epsilon\eta(t) \text{ and}$$

$$J(\epsilon) = \int_a^b F\left(t, q(\epsilon, t), q'(\epsilon, t)\right) dt$$

A minimizing $q(t)$ must be stationary, that is,

$$\frac{\partial J}{\partial \epsilon} = \frac{\partial}{\partial \epsilon} \int_a^b F\left(t, q(\epsilon, t), q'(\epsilon, t)\right) dt = 0$$

Since the boundary conditions are fixed, the differential operator affects only the integrand, hence

$$\frac{\partial J}{\partial \epsilon} = \int_a^b \left(\frac{\partial F}{\partial q}\frac{\partial q}{\partial \epsilon} + \frac{\partial F}{\partial q'}\frac{\partial q'}{\partial \epsilon} \right) dt$$

From the definition of $q(\epsilon, t)$ we have

$$\frac{\partial q}{\partial \epsilon} = \eta(t)\ , \quad \frac{\partial q'}{\partial \epsilon} = \eta'(t)\ , \text{ hence}$$

$$\frac{\partial J}{\partial \epsilon} = \int_a^b \left(\frac{\partial F}{\partial q}\eta(t) + \frac{\partial F}{\partial q'}\eta'(t) \right) dt$$

Integrating the second term by parts, we get

$$\int_a^b \frac{\partial F}{\partial q'}\eta'(t)dt = \frac{\partial F}{\partial q'}\eta(t)\Big|_a^b - \int_a^b \frac{d}{dt}\left(\frac{\partial F}{\partial q'}\right)\eta(t)dt$$

where the first term vanishes, since the extreme points, $\eta(a) = \eta(b) = 0$, are fixed. Hence

$$\frac{\partial J}{\partial \epsilon} = \int_a^b \left(\frac{\partial F}{\partial q} - \frac{d}{dt}\frac{\partial F}{\partial q'} \right) \eta(t) dt$$

Since $\eta(t)$ is arbitrary and the integral must be zero, the parenthesis in the integrand must be zero. This is the Euler–Lagrange equation:

$$\frac{\partial F}{\partial q} - \frac{d}{dt}\frac{\partial F}{\partial q'} = 0$$

If F does not depend explicitly on t, only implicitly through q and q', Euler–Lagrange equation implies

$$\frac{d}{dt}\left(q'\frac{\partial F}{\partial q'} - F \right) = q''\frac{\partial F}{\partial q'} + q'\frac{d}{dt}\frac{\partial F}{\partial q'} - \frac{\partial F}{\partial t} - \frac{\partial F}{\partial q}q' - \frac{\partial F}{\partial q'}q'' =$$

$$-q'\left(\frac{\partial F}{\partial q} - \frac{d}{dt}\frac{\partial F}{\partial q'} \right) - \frac{\partial F}{\partial t} = 0 \;\Rightarrow\; \left(q'\frac{\partial F}{\partial q'} - F \right) = \text{const}$$

2.3. Hamilton's Principle and Classical Mechanics

The Lagrangian of a classical mechanics system specified by generalized vector coordinates $q(t)$, kinetic energy $T(q, q')$, and potential energy $V(q, t)$, is defined as $L(t, q, q') = T(q, q') - V(q, t)$.

Hamilton's Principle states that, if a classical mechanics system is anchored at initial and final fixed positions $q(a)$ and $q(b)$, $a < b$, then the system follows the trajectory that minimizes its physical action,

$$J(q) = \int_a^b L(t, q, q') dt$$

defined as the integral of the Lagrangian over time along the system's trajectory. Noether's theorem establishes very general conditions under which the existence of a symmetry in the system, described by the action of a continuous group in the system, implies the existence of a quantity that remains constant in the system's evolution, that is, a conservation law, see for example Section 2.7 of [3].

As a first example, consider a Lagrangian $L(t, q, q')$ that does not depend explicitly of q. This situation reveals a symmetry: The system is invariant by a translation on the coordinate q. From Euler–Lagrange equation, it follows that the quantity $p = \partial F/\partial q'$ is conserved. In the language of classical mechanics, q would be called a *cyclic coordinate*, while p would be called a *generalized moment.*

As a second example, consider a Lagrangian that does not depend explicitly on t. In this case, Euler–Lagrange equation implies $q'\frac{\partial L}{\partial q'} - L = E$, where the constant E is interpreted as the system's conserved energy. Indeed, if the potential, $V(q)$, is only function of the system position, and the kinetic energy is a quadratic form on the velocities, $T = \sum_{i,j} c_{i,j}\, q'_i\, q'_j$, the last equation assumes the familiar form $E = 2T - L = T + V$.

As a particular instance of the examples above, consider a particle in a plane orbit around a radially symmetric and time independent potential, $L = \frac{m}{2}(r'^2 + r^2\theta') - V(r)$. Since the Lagrangian does not depend explicitly on θ, the system conserves its angular momentum, A, displayed in the next equation. Since the Lagrangian also does not depend explicitly on t, the system conserves its energy, E,

$$\frac{\partial L}{\partial q'} = mr^2q' = A\,, \quad T + V = \frac{m}{2}(r'^2 + r^2\theta') + V(r) = E$$

These two constants of motion, A and E, are in fact a complete set of parameters for the particle's orbit, that is, these constants fully specify the possible orbits, as becomes clear by combining the last equations into the following system of ordinary differential equations,

$$r'^2 = \frac{2}{m}\left(E - V(r)\right) - \frac{m^2 r^2}{A}\ , \quad \theta' = \frac{A}{mr^2}$$

3. Positivism *vs.* Objective Ontologies

In this section we discuss the ontological status of the constants or conserved quantities that emerge from Noether's theorem, such as the energy or angular momentum. In Sections 6 and 7 we are going to discuss similar formalisms in the realm of mathematical statistics, and will raise similar questions concerning the ontological status of parameters in statistical models.

Ontology is the study of $o\nu\tau o\varsigma$, "of that which is". Hence, we pose the question: Are these parameters real, or do they represent or correspond to something real? The commonsense or naïve realism answer is in the affirmative: These parameters correspond to real entities, things that actually exist out there, in the external world or in the environment. These entities can be pointed at and even be precisely measured; they are as real as the kitchen sink. The positivist philosophical school strongly dissents from the naïve realism position, claiming that these parameters are nothing but artificial constructs, things that only exist inside the system, model or theory. They are in reality only ghosts. These two conflicting philosophical positions reappear in a great variety of forms, with many subtle differences and interesting nuances. We will not venture into the analysis of these fine distinctions, for our main goal is to overcome this dichotomy with the introduction, in Section 8, of the cognitive constructivism epistemological framework. However, in order to reach this goal, we must analyze several connections relating ontology and the concepts of symmetry and invariance, for these are the keys that can unlock our ontological enigmas.

3.1. Positivism, Subjectivism and Solipsism

According to the positivist school, closely related to the skeptical or solipsist traditions, the primary physical quantities must be directly measurable, like (non-relativistic) time, position, *etc*. In the example at hand, the variables of direct interest, corresponding to the primary physical quantities, describe the system's trajectory, giving the time-evolution of the particle's position, $[r(t), \theta(t)]$.

For the positivist school, quantities like A and E have a *metaphysical* character. The literal meaning of the word meta-physical is just beyond-physical, designating entities that are not accessible to our senses or, in our case, quantities that are not directly measurable. Therefore, from a positivist perspective, metaphysical entities have a very low ontological status, that is, they do not correspond to objects having a real existence in the world. Rather, they are ghost that only exist inside our formalism.

In the example at hand, E and A constitute a very convenient *index system* for the collection of all possible orbits, but that does not grant these quantities any real existence. On the contrary, these quantities must be considered just as integration constants, useful variables in an intermediate step for the calculation of the system's orbits. They can be used as convenient auxiliary variables, generating secondary or derived concepts, but that is all that they can ever aspire to be. The physicist Max Born, in pp.49,144,152 of [7], summarizes this position:

Positivism assumes that the only primary statements which are immediately evident are those describing direct sensual impressions. All other statements are indirect, theoretical constructions to describe in short terms the connections and relations of the primary experiences. Only these have the character of reality. The secondary statements do not correspond to anything real, and have nothing to do with an existing external world; they are conventions invented artificially to arrange and simplify "economically" the flood of sensual impressions.

[The] strictly positivistic standpoint: The only reality is the sense impressions. All the rest are "constructs" of the mind. We are able to predict, with the help of the mathematical apparatus ... what the experimentalist will observe under definite experimental conditions ... But it is meaningless to ask what there is behind the phenomena, waves or particles or what else.

We have before us a standpoint of extreme subjectivism, which may rightly be called "physical solipsism". It is well-known that obstinately held solipsism cannot be refuted by logical argument. This much, however, can be said, that solipsism such as this does not solve but evades the problem. Logical coherence is a purely negative criterion; no system can be accepted without it, but no system is acceptable just because it is logically tenable. The only positive argument in support of this abstract type of ultra-subjectivism is [that] the belief in the existence of an external world is irrelevant and indeed detrimental to the progress of science.

3.2. Born's Criticism of Positivism

The *gnosiological* sense of the word metaphysics has its roots in Aristotelic philosophy: In a nutshell, it designates concepts that enable possible answers to why-questions, that is, metaphysical concepts are the basis for any possible answer explaining "why" things are the way they do. Hence, metaphysical concepts are at the core of metaphorical figures and explanation mechanisms, providing an articulation point between the empirical and the hypothetical, a bridge from the rational and the intuitive, a link between calculation and insight. In this role, metaphysical concepts give us access to a "much deeper" level of reality. Without metaphysical concepts, physical reasoning would be downgraded to merely cranking the formalism, either by algebraic manipulation of the symbolic machinery or by sheer number crunching. Furthermore, metaphysical concepts are used to express metaphorical figures that constitute the basis of our communication. These points are addressed by Max Born in pp.144,152 of [7]:

Many physicists have adopted [the positivist] standpoint. I dislike it thoroughly ... The actual situation is very different. All great discoveries in experimental physics have been due to the intuition of men who made free use of models, which were for them not products of the imagination, but representatives of real things. How could an experimentalist work and communicate with his collaborators and his contemporaries without using models composed of particles, electrons, nucleons, photons, neutrinos, fields and waves, the concepts of which are condemned as irrelevant and futile?

However, there is of course some reason for this extreme standpoint. We have learned that a certain caution is necessary in using these concepts ... Modern theories demand a reformulation. This new formulation is slowly evolving, but has probably not reached a final expression.

The final lines of the last quotation address a distinct characteristic of modern science, namely, its dynamism, the rapid pace of its growth, the creation and reformulation of theories. However, formulating new theories implies creating new concepts and, sometimes, abandoning an old one. The frustration of finding out that some of the traditional good-old "things" do not "exist", may be followed by the post-traumatic reaction of denying the existence of anything that is not unequivocally perceived by direct sensorial experience. In this setting, the popularity of positivism or subjectivism may be understood as a first (although clumsy) reaction against naïve realism. We believe that the two extreme epistemological positions of naïve realism and subjective solipsism can be seen as two tentatives of building an ontology based on a correspondence principle, as opposed to the approach taken in epistemological framework of cognitive constructivism. We will return to this theme in Section 8.

3.3. Ontology by Invariance and Autonomy

The idea of founding an ontology on invariance properties has its origin in the Erlangen Program of Felix Klein, [8]. His approach to geometry provided a unified treatment for Euclidean, non-Euclidean and Riemannian geometry, see [9] and [10]. In Klein's approach, the intrinsic properties of a geometric space is specified by a set of privileged actions, effected by a discrete or continuous group of possible transformations. Well-known examples are the rigid translations and rotations of Euclidean geometry, the parallel transport of Riemannian geometry, and so on. Moreover, the essential geometric objects or properties are characterized as invariants under the specified actions.

Furthermore, invariance suggests self-determination or autonomy: Invariance properties can be interpreted as stating the irrelevance of peripheral peculiarities. For example, the irrelevance of the particular coordinate system or reference frame being used (in a well-defined class of equivalence), *etc*. Hence, invariant objects are perceived to be detached from any idiosyncratic condition, to stand by themselves, to be autonomous. Max Born, in pp.158,163,104,51 of [7], summarizes this position as applied to building an ontology for physical sciences:

> I think the idea of invariant is the clue to a rational concept of reality, not only in physics but in every aspect of the world. The theory of transformation groups and their invariants is a well established part of mathematics. Already in 1872 the great mathematician Felix Klein discussed in his famous 'Erlanger Program' the classification of geometry according to this point of view; the theory of relativity can be regarded as an extension of this program to the four-dimensional geometry of space-time. The question of reality in regard to gross matter has from this standpoint a clear and simple answer.
>
> Thus we apply analysis to construct what is permanent in the flux of phenomena, the invariants. Invariants are the concepts of which science speaks in the same way as ordinary language speaks of "things", and which it provides with names as if they were ordinary things.
>
> The words denoting things are applied to permanent features of observation or observational invariants.
>
> The invariants of [a] theory have the right to be considered as representations of objects in the real world. The only difference between them and the objects of everyday life is that the

latter are constructed by the unconscious mind, whereas the objects of science are constructed by conscious thinking.

In the next sections we will analyze some formalisms for dealing with symmetry and invariance in Bayesian statistics, their traditional subjectivist interpretations, and alternative ontological or objective interpretations based on the concepts of symmetry and invariance.

4. Bayesian Statistics

A standard model of (parametric) Bayesian statistics concerns an observed (vector) random variable, x, in the sampling space, $\mathcal{X}$, that has a *sampling* distribution with a specified functional form, $p(x \mid \theta)$, indexed by the (vector) parameter, θ, in the parameter space, Θ. This same functional form, regarded as a function of the free variable θ with a fixed argument x, is the model's *likelihood* function. In *frequentist* or classical statistics, one is allowed to use probability calculus in the sample space, but strictly forbidden to do so in the parameter space, that is, x is to be considered as a random variable, while θ is not to be regarded as random in any way. In frequentist statistics, θ should be taken as a "fixed but unknown quantity".

In the Bayesian context, the parameter θ is regarded as a latent (non-observed) random variable. Hence, the same formalism used to express credibility or (un)certainty, namely, probability theory, is used in both the sample and the parameter space. Accordingly, the joint probability distribution, $p(x, \theta)$ should summarize all the information available in a statistical model. Following the rules of probability calculus, the model's joint distribution of x and θ can be factorized either as the likelihood function of the parameter given the observation times the *prior* distribution on θ, or as the *posterior* density of the parameter times the observation's marginal density,

$$p(x, \theta) = p(x \mid \theta)p(\theta) = p(\theta \mid x)p(x)$$

The *prior* probability distribution $p_0(\theta)$ represents the initial information available about the parameter. In this setting, a *predictive* distribution for the observed random variable, x, is represented by a mixture (or superposition) of stochastic processes, all of them with the functional form of the sampling distribution, according to the prior mixing (or weights) distribution,

$$p(x) = \int_\theta p(x \mid \theta)p_0(\theta)d\theta$$

If we now observe a single event, x, it follows from the factorizations of the joint distribution above that the *posterior* probability distribution of θ, representing the available information about the parameter after the observation, is given by

$$p_1(\theta) \propto p(x \mid \theta)p_0(\theta)$$

In order to replace the "proportional to" symbol, $\propto$, by an equality, it is necessary to divide the right hand side by the normalization constant, $c_1 = \int_\theta p(x \mid \theta)p_0(\theta)d\theta$. This is the *Bayes rule*, giving the (inverse) probability of the parameter given the data. That is the basic learning mechanism of Bayesian statistics. Computing normalization constants is often difficult or cumbersome. Hence, especially in large models, it is customary to work with unnormalized densities or *potentials* as long as possible

in the intermediate calculations, computing only the final normalization constants. It is interesting to observe that the joint distribution function, taken with fixed x and free argument θ, is a potential for the posterior distribution.

Bayesian learning is a potentially recursive process, where the posterior distribution after a learning step becomes the prior distribution for the next step. Assuming that the observations are i.i.d. (independent and identically distributed) the posterior distribution after n observations, $x^{(1)}, \ldots x^{(n)}$, becomes,

$$p_n(\theta) \;\propto\; p(x^{(n)} \,|\, \theta) p_{n-1}(\theta) \;\propto\; \prod_{i=i}^{n} p(x^{(i)} \,|\, \theta) p_0(\theta)$$

If possible, it is very convenient to use a *conjugate prior*, that is, a mixing distribution whose functional form is invariant by the Bayes operation in the statistical model at hand. For example, the conjugate priors for the Normal and Multivariate models are, respectively, Wishart and the Dirichlet distributions. The explicit form of these distributions is given in the next sections.

In the next two subsections we further discuss the dynamics of the Bayesian learning process, that is, we present some rationale for choosing the prior distribution used to start the learning process, and some convergence theorems for the posterior as the number observations increases.

4.1. Fisher's Metric and Jeffreys' Prior

The Fisher information matrix, $J(\theta)$, is defined as minus the expected Hessian of the log-likelihood. The Fisher information matrix can also be written as the covariance matrix of for the gradient of the same likelihood, *i.e.*,

$$J(\theta) \equiv -\,\mathrm{E}\frac{\partial^2 \log p(x \,|\, \theta)}{\partial\, \theta^2} \;=\; \mathrm{E}\left(\frac{\partial\, \log p(x \,|\, \theta)}{\partial\, \theta} \; \frac{\partial\, \log p(x \,|\, \theta)}{\partial\, \theta} \right)$$

The physicist Harold Jeffreys [11] used the Fisher's metric to define a class of prior distributions proportional to the determinant of the information matrix,

$$p(\theta) \propto |J\,(\theta)|^{1/2}$$

Jeffreys' priors are geometric objects in the sense of being invariant by a continuous and differentiable change of coordinates, $\eta = f(\theta)$, in the parameter space, see pp. 41–54 of [12]. That is:

$$J(\theta) = \left[\frac{\partial\, \eta}{\partial\, \theta} \right] J(\eta) \left[\frac{\partial\, \eta}{\partial\, \theta} \right]' \;, \;\; \text{hence}$$

$$|J\,(\theta)|^{1/2} \; d\theta \;=\; \left| \frac{\partial\, \eta}{\partial\, \theta} \right| \; |J\,(\eta)|^{1/2} \;=\; |J\,(\eta)|^{1/2} \; d\eta$$

It is important to realize that the argument of transformation-invariance used in the derivation of Jeffrey's priors is quite distinct from the argument of symmetry-invariance relative to the action of a given group in the system used in Noether's theorem. Nevertheless, in Sections 5, 6 and 7 this later type of argument will be used to derive parametric families of sampling distributions for which, in turn, Jeffrey's priors offer a coherent starting point or ground state in the Bayesian inferential process. Moreover, transformation-invariance is, in its own right, an important property intimately related to our leitmotif of ontology by invariance and autonomy, as further discussed in Sections 8 and 9.

Example (multinomial distribution):

$$p(y \mid \theta) = n! \prod_{i=1}^{m} \theta_i^{x_i} \Big/ \prod_{i=1}^{m} x_i! \ , \ \theta_m = 1 - \sum_{i=1}^{m-1} \theta_i \ , \ x_m = n - \sum_{i=1}^{m-1} x_i$$

$$L = \log p(x \mid \theta) = \sum_{i=1}^{m} x_i \log \theta_i$$

$$\frac{\partial^2 L}{(\partial \theta_i)^2} = -\frac{x_i}{\theta_i^2} + \frac{x_m}{\theta_m^2} \ , \ \ \frac{\partial^2 L}{\partial \theta_i \partial \theta_j} = -\frac{x_m}{\theta_m^2} \ , \ \ i, j = 1 \ldots m-1$$

$$-\mathrm{E}\frac{\partial^2 L}{(\partial \theta_i)^2} = \frac{n}{\theta_i} + \frac{n}{\theta_m} \ , \ \ -\mathrm{E}\frac{\partial^2 L}{\partial \theta_i \partial \theta_j} = \frac{n}{\theta_m}$$

$$|J(\theta)| = (\theta_1 \theta_2 \ldots \theta_m)^{-1} \ , \ \ p(\theta) \propto (\theta_1 \theta_2 \ldots \theta_m)^{-1/2}$$

$$p(\theta \mid x) \propto \theta_1^{x_1 - 1/2} \theta_2^{x_2 - 1/2} \ldots \theta_m^{x_m - 1/2}$$

Hence, in the multinomial example, Jeffreys' prior "discounts" half an observation of each kind, while the flat prior discounts none.

4.2. Posterior Convergence

In this section we briefly state two asymptotic results that are fundamental to Bayesian statistics. For a detailed, intuitive and very understandable discussion, see Appendix B of [13].

Theorem 4.1 (Posterior Consistency for Discrete Parameters). Consider a model where $f(\theta)$ is the prior in a discrete (vector) parameter space, $\Theta = \{\theta^{(1)}, \theta^{(2)}, \ldots\}$, $X = [x^{(1)}, \ldots x^{(n)}]$ is a series of observations, and the posterior is given by

$$f(\theta^{(k)} \mid X) \propto f(\theta^{(k)})\, p(X \mid \theta^{(k)}) = f(\theta^{(k)}) \prod_{i=1}^{n} p(x^{(i)} \mid \theta^{(k)})$$

Further, assume that in this model there is a single value for the vector parameter, $\theta^{(0)}$, that gives the best approximation for the "true" predictive distribution $g(x)$ (in the sense that it minimizes the information divergence, as defined in Section 7). Then,

$$\lim_{n \to \infty} f(\theta^{(k)} \mid X) = \delta(\theta^{(k)}, \theta^{(0)})$$

We can extend this result to continuous parameter spaces, assuming several regularity conditions, like continuity, differentiability, and having the argument $\theta^{(0)}$ as a interior point of Θ with the appropriate topology. In such a context, we can state that, given a pre-established small neighborhood around $\theta^{(0)}$, like $C(\theta^{(0)}, \epsilon)$ the cube of side size ϵ centered at $\theta^{(0)}$, this neighborhood concentrates almost all mass of $f(\theta \mid X)$, as the number of observations grows to infinity. Under the same regularity conditions, we also have that Maximum a Posteriori estimator is a consistent estimator, *i.e.*, $\widehat{\theta} \to \theta^{(0)}$.

The next result states the convergence in distribution of the posterior to a Normal distribution, using the Fisher information matrix as defined in the last subsection, see Appendix B of [13].

Theorem 4.2 (Posterior Normal Approximation). The posterior distribution converges to a Normal distribution with mean $\theta^{(0)}$ and precision $nJ(\theta^{(0)})$.

4.3. Subjective Coherence vs. *Objective Priors*

The invariance properties of Jeffreys' priors entail objective interpretations that may be suggested in analogy with Noether's theorem. In pp. 132–133 of [14], we find some further explanations for the meaning of Weyl's opening quote in this paper:

> We found that objectivity means invariance with respect to the group of automorphisms ... This is what Felix Klein called a 'geometry' in the abstract sense. A geometry, Klein said, is defined by a group of transformations, and investigates everything that is invariant under the transformations of this given group.

Under appropriate regularity conditions, the *information geometry* is defined by the metric in the parameter space given by the Fisher information matrix, that is, the geometric length of a curve is computed integrating the differential form $dl^2 = d\theta' J(\theta) d\theta$, see [15–17]. Hence, in analogy with Klein's Erlangen Program, Jeffreys' priors can be interpreted as giving the underlying geometry of the parameter space. In this role, Jeffreys' priors provide reference metrics for the parameter space of statistical models, metrics that we so often take for granted and use for a great variety of purposes without further thought or consideration. Under the objective perspective, the posterior convergence theorem characterizes "true" parameter values as fixed points of the learning process, that is, as (stochastic) invariants. Naturally, all these interpretations have a strong objective character.

However, subjective interpretations of the results presented in this section are also possible. Under the subjectivist perspective, Jeffreys' prior can be understood as coherence conditions for the subject's opinion. For a subjectivist, invariance by reparameterization is only a consistency condition on rational but personal prior beliefs. Even convergence results can be re-interpreted as leading not to to any kind of objectivity, but to inter-subjectivity, that is, as leading, in due time, to the mutual agreement of several solipsist agents. However, perhaps the best justification or the most powerful defense for the subjectivist position was presented by Bruno de Finetti, the undisputed champion of the subjectivist position. In the next section we review de Finetti's theorem and its original interpretation. These considerations, not surprisingly, are based on yet another symmetry relation, namely, exchangeability.

5. Exchangeability and De Finetti Theorems

In Section 4 we derived several results of Bayesian statistics assuming that observations are i.i.d. random variables, Nevertheless, the i.i.d. hypothesis has a strong "objectivist" flavor. De Finetti's theorem, [18,19], allows us to replace the i.i.d. hypothesis by a symmetry condition, namely, exchangeability.

A finite sequence of n random (vector) variables, $y^{(1)}, y^{(2)}, \dots y^{(n)}$ is (finitely) exchangeable if the joint distribution of these random variables is permutation-symmetric, that is, if, for any permutation π in $\{1, 2, \dots n\}$,

$$p(y^{(1)}, y^{(2)}, \dots y^{(n)}) = p(y^{\pi(1)}, y^{\pi(2)}, \dots y^{\pi(n)})$$

An infinite sequence of random variables is infinitely exchangeable if any of its finite subsequences is exchangeable.

Bruno de Finetti's original exchangeability theorem states that, for an infinite exchangeable sequence of (scalar) Boolean random variables, partial joint probability distributions can always be represented as a mixture on the parameter space of a product of Bernoulli distributions,

$$p(y_1, \ldots y_n) \;=\; \int_0^1 \prod_{i=1}^n \theta^{y_i}(1-\theta)^{1-y_i}\; q(\theta)d\theta \;=\; \int_0^1 \theta^{x_n}(1-\theta)^{1-x_n}\; q(\theta)d\theta$$

the probability distributions of the partial sums, $x_n = \sum_{i=1}^n y_i$, can always be represented as the corresponding mixture on the parameter of binomial distributions,

$$f(x_n) \;=\; \left(\begin{array}{c} n \\ x_n \end{array} \right) \int_0^1 \theta^{x_n}(1-\theta)^{1-x_n}\; g(\theta)d\theta$$

and the mixing cumulative distribution, $G(\theta)$, is the limit cumulative distribution for the partial frequencies,

$$G_n(\theta) = \Pr\left(x_n/n \leq \theta\right)$$

The statement above assumes that there is a limit density, $g(\theta)$; otherwise one has to employ the abstract limit probability measure, $dG(\theta)$. For a simple and intuitive proof, see Section VII.1.4 of [20]. Feller's proof follows some of the same arguments used by de Finetti, including some interesting demonstration techniques based on the theory of difference equations, see [18,19,21]. For a short and elegant proof that relies on measure theoretic analysis, see [22]. For several extensions of de Finetti's theorem, see [23,24].

From simple characterization theorems, see [25], it is immediate to extend de Finetti's theorem from the Binomial to the Multinomial case. Arguments of analysis, see Chapter 4 of [26] or Section VII.12 of [27], allow us to further extend de Finetti's theorem to obtain a general representation theorem for finite subsequences of an infinitely exchangeable sequence of real-valued random variables. Assuming that ordinary densities can replace the more abstract probability measures, the general form of the joint probability distribution for a finite set of (observed) random variables, $x_1, \ldots x_n$, can be represented by the product of a parameterized density for individual observations mixed by a density on the (latent) parameter,

$$p(x^{(1)}, \ldots x^{(n)}) = \int_\Theta \prod_{i=1}^n p(x^{(i)} \mid \theta) q(\theta) d\theta$$

If these probability models are subject to additional constraints, like the symmetry conditions considered in next sections, the general representation takes the form of a mixture in the corresponding family of invariant distributions.

5.1. De Finetti's Subjectivism

Bruno de Finetti is the founder of modern Bayesian statistics, and an outspoken advocate of extreme subjectivism. In his lecture—Exchangeability, Induction and "Unknown" Probabilities—Chapter 8 of [28], de Finetti offers a synthesis of his interpretation of probability as subjective uncertainty in the context of the theorem presented in the last section:

> Every time that probability is given by a mixture of hypotheses, with independence holding for each of them respectively, it is possible to characterize the inductive relevance that results on the

basis of [the mixture] equation in terms of exchangeability ... [This] shows how the concept of exchangeability is necessary in order to express in a meaningful way what is usually referred to as "independence with constant but unknown probability". This expression is not correct because in fact (a) there is no independence; (b) the probability is not constant because the composition of the urn being unknown, after learning the outcome of the draws, the probability of the various compositions is subject to variation.

Hence, according to de Finetti, the "objective" i.i.d. condition is a mirage, a displaced reflection "outside" of a more fundamental symmetry condition "inside", namely, exchangeability. Naturally, for de Finetti, exchangeability is on the eyes of the beholder, expressing his or her own explicit or implicit opinions on how observational data should be correctly handled.

Irving John Good was another leading figure of the early days of the Bayesian revival movement. Contrary to de Finetti, Good has always been aware of the dangers of an extreme subjectivist position. Some of his conclusions seem to echo Born's arguments presented in Section 1, see for example, Random Thoughts about Randomness, p. 93 in Chapter 8 of [29].

Some of you might have expected me, as a confirmed Bayesian, to restrict the meaning of the word 'probability' to subjective (personal) probability. That I have not done so is because I tend to believe that physical [objective] probability exists and is in any case a useful concept... The philosophical impact of de Finetti's theorem is that it supports the view that solipsism cannot be logically disproved. Perhaps it is the mathematical theorem with most potential philosophical impact.

In Sections 6 and 7, we shall widen the scope of our discussion, presenting probability distributions derived from symmetry conditions far more general than exchangeability. Some of these examples are of great historical importance. However, in contrast to de Finetti's theorem, some of these results seem to favor objective interpretations. As planned in the introduction, we will continue to explore the importance of symmetry arguments in the realism-subjectivism debate. Nevertheless, the increasing power and sophistication of the available symmetry arguments seem to have the effect of generating ever more violent oscillations or inducing an even greater polarization between the two alternatives in the realism-subjectivism dilemma. Sections 6 and 7 are the last steps leading to the presentation, in Section 8, of the cognitive constructivism epistemological framework that, as promised in the introduction, will be able to solve (many aspects of) the ongoing ontological conundrum.

6. Simple Symmetric Probability Distributions

Sections 6 and 7 present important results linking symmetry constraints and the functional form of probability distributions. We follow the very simple approach to functional equations of probability distributions in [30]. For related but more general and sophisticated approaches, see [31–33].

6.1. Cauchy's Functional Equations

In this subsection we introduce the additive and multiplicative forms of Cauchy functional equation, since this functional equation will be used several times in the sequel.

Cauchy's additive functional equation has the form

$$f(x+y) = f(x) + f(y)$$

The following argument from Cauchy (1821) shows that a continuous solution of this functional equation must have the form

$$f(x) = cx$$

Repeating the sum of the same argument, x, n times, we must have $f(nx) = nf(x)$. If $x = (m/n)t$, then $nx = mt$ and

$$nf(x) = f(nx) = f(mt) = mf(t)\ , \quad \text{hence}$$
$$f\left(\frac{m}{n}t\right) = \frac{m}{n}f(t)$$

taking $c = f(1)$, and $x = m/n$, it follows that $f(x) = cx$, over the rationals, $x \in Q$. From the continuity condition for $f(x)$, the last result must also be valid over the reals, $x \in R$. Q.E.D.

Cauchy's multiplicative functional equation has the form

$$f(x+y) = f(x)f(y)\ , \quad \forall x, y > 0\ , f(x) \geq 0$$

The trivial solution of this equation is $f(x) \equiv 0$. Assuming $f(x) > 0$, we take the logarithm, reducing the multiplicative equation to the additive equation,

$$\ln f(x+y) = \ln f(x) + \ln f(y)\ , \quad \text{hence}$$

$$\ln f(x) = cx\ , \text{or}\ \ f(x) = \exp(cx)$$

6.2. Radial Symmetry and the Gaussian Distribution

John Herschel [34] gave the following derivation of the Gaussian or Normal distribution, concerning the distribution of errors in astronomical measurements, see also [35].

A few years later, James Clerk Maxwell used the same arguments to derive the functional form of the probability distribution of molecular velocities in statistical physics. The derivation follows three simple steps, based on arguments of (1) independence; (2) radial symmetry; and (3) normalization. For the sake of simplicity, we give the derivation using a two dimensional Cartesian coordinate system, $x = [x_1, x_2]$.

Axial independence (a.k.a. orthogonality condition): Measurements on orthogonal axes should be independent. Hence, the joint distribution should be decomposed in product form,

$$f(x) = f(x_1)f(x_2)$$

Radial symmetry (a.k.a. isotropy condition): Measurement errors should not depend on the current orientation of the coordinate system. This step uses the exponential solution to Cauchy multiplicative functional equation.

$$f(x) = f(x_1)f(x_2) = f(x_1^2 + x_2^2) \Rightarrow f(x) \propto \exp(x_1^2 + x_2^2) = \exp(||x||^2)$$

Normalization: The probability density must integrate to unity.

$$\int f(x)dx_1dx_2 = 1 \Rightarrow f(x) = \frac{a}{\pi}\exp(-a||x||^2)\ ,\ a > 0$$

In the next section, we give a second derivation of this functional form based on the MaxEnt formalism, given more general constraints on the first and second statistical moments, that is, the expectation vector and covariance matrix.

6.3. Homogeneous Discrete Markov Processes

We seek the general form of a homogeneous discrete Markov process. Let $w_k(t)$, for $t \geq 0$, be the probability of occurrence of exactly k events. Let us also assume the following hypotheses:

Time Locality: If $t_1 \leq t_2 \leq t_3 \leq t_4$, then the number of events in $[t_1, t_2]$ is independent of the number of events in $[t_3, t_4]$.

Time Homogeneity: The distribution for the number of events occurring in $[t_1, t_2]$ depends only on the interval length, $t = t_2 - t_1$.

From time locality and homogeneity, we can decompose the occurrence of no (zero) events in $[0, t + u]$ as,

$$w_0(t + u) = w_0(t)w_0(u)$$

Hence, $w_0(t)$ must obey Cauchy's functional equation, and

$$w_0(t) = \exp(ct) = \exp(-\lambda t)$$

Since $w_0(t)$ is a probability distribution, $w_0(t) \leq 1$, and $\lambda > 0$.

Hence, $v(t) = 1 - w_0(t) = 1 - \exp(-\lambda t)$, the probability of one or more events occurring before $t > 0$, must be the familiar exponential distribution.

For $k \geq 1$ occurrences before $t + u$, the general decomposition relation is

$$w_n(t + u) = \sum_{k=0}^{n} w_k(t)w_{n-k}(u)$$

Theorem 6.1 (Renyi–Aczél). The Poisson distribution,

$$w_k(t) = e^{-\lambda t}(\lambda t)^k/k!$$

is the (non-trivial) solution of this system of functional equations under the *rarity condition*: The probability that an event occurs in a short time at least once is approximately equal to the probability that it occurs exactly once, that is, the probability of simultaneous occurrences is zero. The general solution has the form of a mixture of Poisson processes, each one counting bursts of a fixed number of events. For the proof, by induction, see Section 2.1–2.3 in [30].

Pereira and Stern [25] present the characterization of the Multinomial by the Poisson distribution, and vice versa. These characterizations allow us to interpret the exchangeability symmetry characterizing a Multinomial processes in terms of the time homogeneity symmetry and superposition principle characterizing corresponding Poisson counting processes, and vice versa. From a strict predictivist standpoint, it can be observed, our analogy between de Finetti's and Renyi–Aczél theorems is not fully complete, because an exogenous (to the characterization) mixing distribution on the Poisson parameter has to be integrated into the Bayesian inferential model. In order to present our analogy argument in full strength, the straight i.i.d. hypothesis for non-overlapping same-length intervals in Renyi-Aczél

theorem, as stated in Section 2.3.6 of [30], should be replaced by a weaker conditional i.i.d. hypothesis, given the asymptotic frequency rate of the stochastic process. A direct characterization of the mixed Poisson process can be found in Proposition 1.28 at p.66 of [36].

7. Minimum Information or MaxEnt Principle

Section 2 presented Noether's theorem that, based on the minimum action principle and the system's invariance under a group action produces an invariant (physical) quantity. This section presents the Maximum Entropy (MaxEnt) argument to derive the functional form of an invariant distribution. It is possible to derive the functional form of many of the most probability distributions used in statistical models, see [37]. We also present Bregman's algorithm as simple and elegant way to compute finite MaxEnt distributions even when no closed analytical form is available. The formulation of Bregman's algorithm also provides a useful tool for quickly validating the analytic form of many MaxEnt solutions. Many readers with a good background in probability and statistics will be well-acquainted with this material and may rather skip the following examples and derivations. These are kept as simple as possible, in order to make them easily accessible to readers from different backgrounds. We also hope that this approach will help them to understand and recognize the similarities between the techniques presented in this section and analogous techniques presented in Sections 2.

The formal similarities between the minimum action and MaxEnt frameworks imply that all the positivist and anti-positivist positions analyzed in Section 3 can be carried over to the realm of statistics. In particular, all the heated debates concerning the reality of parameters, or what they represent, can be restated, almost *ipsis litteris*, in the context of statistical models. Furthermore, the same formal similarities will allow many epistemological arguments used to introduce the cognitive constructivism framework to be carried from the domain of physics to statistics, and vice versa.

7.1. Maximum Entropy under Constraints

The origins of the entropy concept lay in the fields of thermodynamics and statistical physics, but its applications have extended far and wide to many other phenomena, physical or not. The entropy of a probability distribution, $H(p(x))$, is a measure of uncertainty (or impurity, confusion) in a system whose states, $x \in \mathcal{X}$, have $p(x)$ as probability distribution. For detailed analysis and further references, see [38–40] and Appendix E of [41]. For the sake of simplicity, we present most of the following discussion in the context of a finite system, with states spanned by an indexed $i \in \{1, 2, \ldots n\}$.

The Boltzmann–Gibbs–Shannon measure of entropy is defined as

$$H_n(p) = -I_n(p) = -\sum_{i=1}^{n} p_i \log(p_i) = -\,\mathrm{E}_i \log(p_i) \quad , \quad 0 \log(0) \equiv 0$$

The opposite of the entropy, $I(p) = -H(p)$, the negentropy, is a measure of information available about the system.

Shannon's inequality, a theorem that follows directly from the definition of entropy, can be stated as follows: If p and q are two distributions over a system with n possible states, and $q_i \neq 0$, then the *information divergence* of p relative to q, $I_n(p, q)$, is positive, except if $p = q$, when it is null,

$$I_n(p, q) \equiv \sum_{i=1}^{n} p_i \log\left(\frac{p_i}{q_i}\right) \quad , \; I_n(p, q) \geq 0 \; , \; I_n(p, q) = 0 \Rightarrow p = q$$

Shannon's inequality motivates the use of the information divergence as a measure of (non-symmetric) "distance" between distributions. In statistical sciences, this measure is known as the Kullback–Leibler distance. The denominations directed divergence or cross information are used in Engineering.

Given a prior distribution, q, we would like to find a vector p that minimizes the information divergence $I_n(p, q)$, where p is under the constraint of being a probability distribution, and maybe also under additional constraints over the expectation of functions taking values on the system's states. In the specification of the constraints, A is an $(m-1) \times n$ real matrix, and the problem is formulated as

$$p^* \in \arg\min I_n(p, q) \ , \quad p \geq 0 \ | \, \mathbf{1}'p = 1 \text{ and } Ap = b$$

The solution p^* is the *minimum information divergence* distribution, relative to q, given the constraints $\{A, b\}$. We can write the probability normalization constraint as a generic linear constraint, including $\mathbf{1}$ and 1 as the m-th (or 0-th) rows of matrix A and vector b. By doing so, we do not need to keep any distinction between the normalization and the other constraints. The operators $\odot$ e $\oslash$ indicate the point (element) wise product and division between matrices of same dimension.

The Lagrangian function of this optimization problem and its derivatives are:

$$L(p, w) = p' \log(p \oslash q) + w'(b - Ap)$$

$$\frac{\partial L}{\partial p_i} = \log(p_i/q_i) + 1 - w' A_{\bullet,i} \ , \quad \frac{\partial L}{\partial w_k} = b_k - A_{k,\bullet} \, p$$

Equating the $n+m$ derivatives to zero, we have a system with $n+m$ unknowns and equations, giving viability and optimality conditions (VOCs) for the problem:

$$\begin{aligned} p_i &= q_i \exp\left(w' A_{\bullet,i} - 1\right) \text{ or } p = q \odot \exp\left((w' A)' - \mathbf{1}\right) \\ A_{k,\bullet} \, p &= b_k \ , \ p \geq 0 \end{aligned}$$

We can further replace the unknown probabilities, p_i, writing the VOCs only on w, the dual variables (Lagrange multipliers),

$$h_k(w) \equiv A_{k,\bullet} \left(q \odot \exp\left((w' A)' - \mathbf{1}\right)\right) - b_k = 0$$

In the case of finite systems, the last form of the VOCs motivates the use of iterative algorithms of Gauss–Seidel type, solving the problem by cyclic iteration. In this type of algorithm, one cyclically "fits" one equation of the system, for the current value of the other variables. For a detailed analysis of this type of algorithm, see [42–46].

Bregman's Algorithm:

Initialization: Take $t = 0$, $w^{(t)} \in R^m$, and

$$p_i^{(t)} = q_i \exp\left({w^{(t)}}' A_{\bullet,i} - 1\right)$$

Iteration step: for $t = 1, 2, \ldots$, Take

$$k = (t \bmod m) \quad \text{and} \quad \nu \,|\, \varphi(\nu) = 0 \ , \quad \text{where}$$

$$\begin{aligned} w^{(t+1)} &= \left[w_1^{(t)}, \ldots w_{k-1}^{(t)}, w_k^{(t)} + \nu, w_{k+1}^{(t)}, \ldots w_m^{(t)}\right]' \\ p_i^{(t+1)} &= q_i \exp(w^{(t+1)'} A_{\bullet,i} - 1) \;=\; p_i^{(t)} \exp(\nu A_{i,k}) \\ \varphi(\nu) &= A_{k,\bullet}\, p^{(t+1)} - b_k \end{aligned}$$

From our discussion of entropy optimization under linear constraints, it should be clear that the minimum information divergence distribution for a system under constraints on the expectation of functions taking values on the system's states,

$$E_{p(x)} a_k(x) = \int a_k(x) p(x) dx = b_k$$

(including the normalization constraint, $a_0 = \mathbf{1}, b_0 = 1$) has the form

$$p(x) = q(x) \exp\left(-\theta_0 - \theta_1\, a_1(x) - \theta_2\, a_2(x) \ldots\right)$$

Note that we took $\theta_0 = -(w_0 - 1)$, $\theta_k = -w_k$, and we have also indexed the state i by variable x, so to write the last equation in the standard form used in the statistical literature. Hence, this form is a convenient way to check that several distributions commonly used in statistics can be interpreted as minimum information (or MaxEnt) densities (relative to the uniform distribution, if not otherwise stated) given some constraints over the expected value of state functions. For example:

- The Binomial distribution is characterized as the distribution of maximum entropy for $i \in \{1 \ldots n\}$, given the expected value of the mean, relative to the combinatorial prior $C(n, i)$.
- The Normal distribution is characterized as the distribution of maximum entropy on R^n, given the expected values of its first and second moments, *i.e.*, mean vector and covariance matrix.
- The Wishart distribution:

$$f(S \,|\, \nu, V) \equiv c(\nu, V) \exp\left(\frac{\nu - d - 1}{2} \log(\det(S)) - \sum\nolimits_{i,j} V_{i,j} S_{i,j}\right)$$

is characterized as the distribution of maximum entropy in the support $S > 0$, given the expected value of the elements and log-determinant of matrix S. That is, writing Γ' for the digamma function,

$$\mathrm{E}(S_{i,j}) = V_{i,j} \;\;,\;\; \mathrm{E}(\log(\det(S))) = \sum\nolimits_{k=1}^{d} \Gamma'\left(\frac{\nu - k + 1}{2}\right)$$

- The Dirichlet distribution

$$f(x \,|\, \theta) = c(\theta) \exp\left(\sum\nolimits_{k=1}^{m} (\theta_k - 1) \log(x_k)\right)$$

is characterized as the distribution of maximum entropy in the simplex support, $x \geq 0 \,|\, \mathbf{1}'x = 1$, given the expected values of the log-coordinates, $\mathrm{E}(\log(x_k))$.

For the Multinomial distribution, the mean vector represents expected frequency rates, and the combinatorial prior represents the equiprobability condition. The Multinomial distribution is only a function of accumulated counts, it is not dependent on the particular order of the outcomes. Hence it represents exchangeable events, generalizing the Binomial case examined in Section 5.

For the Normal distribution, the mean represents a translation vector, and the coefficients of the correlation matrix represent rotation angles. Translations and rotations constitute the action group defining Euclidean geometry. In particular, Herschel's isotropy and orthogonality conditions, see Section 6.2, are expressed by the zero mean vector and (multiples of) the identity correlation matrix. For a general discussion for symmetry characterizations of the Normal distribution, see Section 8.2 of [32].

The Dirichlet and the Wishart distribution can be characterized as conjugate and transformation-invariant priors for the first two distributions, or as MaxEnt distributions in their on right. Dirichlet-Multinomial and Normal-Wishart models, and many of its variations, like discrete Bayesian networks, regression, factor analysis, generalized linear models, and combinations of the above, known as finite mixture models, encompass the most widely used classes of statistical models. This is due to the formal convenience for mathematical manipulation offered by these models, and also by the importance (in the target application) of the symmetries encoded by these models. The next section should help to clarify this last statement.

8. Cognitive Constructivism

In the preceding sections, we have discussed several forms of invariance found in statistical models. However, a fundamental epistemological inquiry concerns whether these invariances concern: (a) the observer; (b) the observed phenomena *an sich* (in itself); or (c) the observation process. These three possibilities are related to the epistemological frameworks of (a) subjectivism, skepticism, solipsism, *etc*.; (b) naïve or dogmatic forms of realism; (c) cognitive constructivism. Previous sections discussed some aspects of the first two epistemological frameworks. The last sections make additional considerations about these two frameworks and also introduce a third option, Cognitive Constructivism, or Cog-Con.

8.1. Objects as Eigen-Solutions

The Cog-Con framework rests upon Heinz von Foerster's metaphor of *Objects as tokens for eigen-solutions*, the key to Cog-Con ontology and metaphysics. The recursive nature of a learning system interacting with its environment produces recurrent states or stable solutions. Under appropriate conditions, such a (learned or known) solution, if presented to the system, will regenerate itself as a fixed point, an equilibrium or homeostatic state, *etc*. These are called eigen-values, eigen-vectors, eigen-functions, eigen-behaviors or, in general, eigen-solutions. The circular or cyclic characteristic of recursive processes and their eigen (auto, equilibrium, fixed, homeostatic, invariant, recurrent, recursive) states are investigated by von Foerster in [47,48]. The concept of eigen-solution is the key to distinguish specific objects in the cognitive domain of a learning system. Objects are *"tokens for eigen-solutions"*. (A soccer ball is something that interacts with a human in the exact way it is supposed to do for playing soccer.) Eigen-solutions can also be tagged or labeled by words, and these words can be articulated in language. Of course, the articulation rules defined for a given language, its grammar and semantics, only make the language useful if they somehow correspond to the composition rules for the objects the words stand for.

Moreover, von Foerster establishes four essential attributes of eigen-solutions: Eigen-values are ontologically discrete, stable, separable and composable. It is important to realize that, in the sequel, the term "discrete", used by von Foerster to qualify eigen-solutions in general, should be replaced, depending on the specific context, by terms such as lower-dimensional, precise, sharp, singular *etc*. In several well-known examples in exact sciences, these four essential properties lead to the concept of basis, basis of a finite vector space, like in linear algebra, basis of a Hilbert space, like in Fourier or Wavelet analysis, or more abstract settings, like basis for matroid structure, generators for an algebraic group, *etc*. Nevertheless, the concept of eigen-solution and its four essential properties is so important in the Cog-Con framework that it is used as a fundamental metaphor in far more general—and not necessarily formalized—contexts. For detailed interpretations of the of these four essential attributes on eigen-solutions in the context of Bayesian learning systems, see [41,49–52].

8.2. Semantics by Construction and by Correspondence

In this subsection we further investigate some consequences of the Cog-Con perspective on objects and their representation in language, contrasting it with more traditional approaches, based on dyadic cuts and subsequent correspondences. Correspondence approaches start by making a distinction that cuts the world in two, and then choose or decide if objects are correctly placed "in here" or "out there": Are they internal concepts in a given system or are they external entities in the environment? Do they belong to the "subjective" or "upper" world of the mind, spirit, intuitive thought *etc*., or do they belong to "reality" or "lower" world of the body, matter, *etc*.?

The predetermined cut splitting the world in two also suggests two natural alternative ways to travel the epistemological path: Either the correct ideas above are those corresponding to the "reality" below, or the correct things below are those corresponding to the "good" ideas above, *etc*. The existence of true and static correspondence principle is a necessary pre-requisite, but there are different ways to establish the connection, or to learn it (or to remember it). An empiricist diligently observes the world, expecting to develop convenient tools and technology that can, in turn, be used to climb the track of scientific discovery. A dogmatic idealist works hard at his metaphysical doctrine, in order to secure a good spot at the top, expecting to have an easy ride sliding down the trail of knowledge.

The dyadic correspondence approach is simple and robust. It can be readily adapted to many different situations and purposes. It also has attractive didactic qualities, being easy to understand and to teach. The dyadic correspondence approach has low entrance fees and low maintenance costs, as long as one understands that the assumption of a predetermined correspondence makes the whole system essentially static. Its major weakness relates to this rigidity. It is not easy to consider new hypothesis or original concepts, and even harder to handle the refutation or the dismissal of previously accepted ones. New world orders always need to be, at least conceptually, proven afresh or build up from scratch.

In Cognitive Constructivism, language can be seen as a third pole in the epistemological framework, a third element that can play the role of a buffer, moderating or mitigating the interaction of system and environment, the relation of theory and experiment, *etc*. After all, it is only in language that it is possible to enunciate statements, which can then be judged for truthfulness or falsehood. Moreover, language gives us a shelf to place our objects (representations of eigen-solutions), a cabinet to store these (symbolic) tokens. Even if the notion of object correspondence, to either purely internal concepts

to a given system or to strictly external entities in the environment, is inconsistent with the Cog-Con framework, this framework is perfectly compatible with having objects re-presented as symbols in one or several languages. This view is very convenient and can be very useful, as long as we are not carried away, and start attributing to language magic powers capable of creating *ex nihilo* the world we live in. As naïve as this may seem, this is a fatal mistake made by some philosophers in the radical constructivist movement, see [41].

The Cog-Con approach requires, from the start, a more sophisticated construction, but it should compensate this trouble with the advantage of being more resilient. Among our goals is escaping the dilemmas inherent to predetermined correspondence approaches, allowing more flexibility, providing dynamic strength and stability. In this way, finding better objects, representing sharper, stabler, easier to compose, or more generally valid eigen-solutions (or even better representations for these objects in the context of a given system) does not automatically imply the obliteration of the old ones. Old concepts or notations can be replaced by better ones, without the need of ever being categorically discredited. Hence, theories have more room to continuously grow and adapt, while a concept at one time abandoned may be recycled if its (re)use is convenient at a later opportunity. In this way, the Cog-Con epistemological framework naturally accommodates dynamic concepts, change of hypotheses and evolution of theories, all so characteristic of modern science.

9. Conclusions and Further Research

When Bayesian statistical models are examined in the Cog-Con framework, the role played by the model's parameters becomes clear: They are (converging to stochastic) eigen-solutions of the Bayesian learning process of information acquisition by the incorporation of observational data. These eigen-solutions are characterized by several important invariance properties including asymptotic, symmetry and transformational aspects. All these distinct invariance properties are interrelated by the structure of the statistical model and, in the spirit of Herman Weyl's opening quote in this paper, mutually support the notion of emerging eigen-solutions as autonomous ontological objects. The precision aspect of the quality of such an eigen-solution can be evaluated by its current posterior density and such an evaluation can, in turn, be used to access its *objectivity*. Similarly, we can access the objective truthfulness of more general hypotheses, stated as equations on the parameters. The Full Bayesian Significance Test was designed having this very purpose in mind, including the evaluation of other aspects of eigen-solutions represented by statistical hypotheses, like their stability and compositionality, see [49,53–55].

Frequentist statistics categorically forbids probabilistic statements in the parameter space. De Finettian subjectivist interpretation of Bayesian statistics "justifies" the use of prior and posterior distributions for the parameters, but their role is to be used as nothing more than integration variables in the process of computing predictive distributions. In a very positivist spirit, like any other non-directly observable or latent quantity, parameters are labeled as meta-physical entities, and given a very low ontological status. Finally, in the Cog-Con interpretation of Bayesian statistics, parameters can take their rightful place on the center stage of the inferential process. They are not just auxiliary variables, but legitimate objects of knowledge that can be properly used in true bearing statements. We claim

that the last situation corresponds to what a scientist naturally finds, needs and wants in the practice of empirical science.

Ontological frameworks based on correspondence semantics seem to be particularly inappropriate in quantum mechanics. We believe that many of the arguments and ideas presented in this paper can be re-presented, even more emphatically, for statistical models concerning experimental measurements in this realm. The formal analogies between the invariance arguments and their use in statistics and physics is even stronger in quantum mechanics than in the case of classical physics, see [56–60]. For example, the idea of eigen-solution, which already plays a very important role in the epistemology of classical physics (see for example [41]), plays an absolutely preponderant role in quantum mechanics. Furthermore, the quantum mechanics superposition principle makes the use of mixtures of basic symmetric solutions as natural (and necessary) in quantum mechanics as they are in statistics. Moreover, quantum mechanics theory naturally leads to interpretations that go beyond the classical understanding of probability as "uncertainty about a well-defined but unknown state of nature". As stated in p. 168 of [7], in quantum mechanics *"the true physical variable is the probability density"*.

Perhaps Born's statement ending the last paragraph is also pertinent in application areas (sometimes derogatorily) labeled "soft" science. In the context of physics and other "hard" sciences, we search for anchors that can hold to the very fabric of existence, which can grip rock-bottom reality, or provide irreducible elements that can be considered as basic constituents or components of a multitude of derived complex systems. In this context, this article argued that the objective perspective offered by the Cog-Con epistemological framework is far superior to the strict subjective (or at most inter-subjective) views of the decision theoretic epistemological framework of traditional Bayesian statistics. Meanwhile, in the context of "soft" sciences, like marketing, psychology or sociology, the subjective and inter-subjective views of traditional Bayesian statistics may not contradict but rather complement the objective perspective offered by Cog-Con. We intend to explore this possibility in future research.

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