The (greatest) fragment of Classical Logic that respects the Variable-Sharing Principle (in the FMLA-FMLA framework)

Abstract

We examine the set of formula-to-formula valid inferences of Classical Logic, where the premise and the conclusion share at least a propositional variable in common. We review the fact, already proved in the literature, that such a system is identical to the first-degree entailment fragment of R. Epstein's Relatedness Logic, and that it is a non-transitive logic of the sort investigated by S. Frankowski and others. Furthermore, we provide a semantics and a calculus for this logic. The semantics is defined in terms of a p-matrix built on top of a 5-valued extension of the 3-element weak Kleene algebra, whereas the calculus is defined in terms of a Gentzen-style sequent system where the left and right negation rules are subject to linguistic constraints.

Keywords: relevant logics, non-transitive logics, p-matrix, weak Kleene algebra, infectious logics

1. Background and aim

In the wake of the so-called paradoxes of strict implication, characteristic of the systems presented by C. I. Lewis in the early decades of the last century, many logics were proposed whose featured notions of implication did not suffer such inconveniences. In contemporary terminology, systems of this sort are referred to as *relevant* or *relevance logics*—see, e.g., [28]. Work around these logics was usually done in a rather idiosyncratic way, having in mind a particular understanding of the characteristic *relevance* of an implication free of the paradoxes. For example, in [29] E. J. Nelson proposed a relevant implication, defined as the impossibility of the truth of the antecedent and the falsity of the consequent, the relevance of which lied in the requirement that both components be accessory for said impossibility to obtain—contrary to Lewis' implication, where the impossibility of either of these conditions above was sufficient for the impossibility of their conjunction. Alternatively, in [33] W. T. Parry proposed a relevant implication, called analytic implication, the relevance of which lied in the requirement that the content of the consequent is included or contained in the content of the antecedent—according to the exegesis of some scholars, which was nevertheless disputed by W. T. Parry himself.

Despite the debates that took place in the decades following Lewis' work, it is nowadays widely assumed that when working with propositional languages an implication is *relevant* only if its *antecedent and consequent share some propositional variable in common.* This seems to reflect the fact that these terms should not totally diverge with regard to their subject-matters, meaning by this that there should be some common subject-matter connecting the former and the latter—with systems satisfying this condition only sometimes being called "weakly" relevant. Granting a few idealized but relatively standard assumptions about the formalization of subject-matters in the context of propositional languages, this constraint is usually formalized by the so-called Variable-Sharing Principle (VSP, for short).[1] This principle requires the following of a theorem that is an implication of the form $\varphi \to \psi$, where $Var(\chi)$ refers to the set of propositional variables appearing in a formula χ:

$$Var(\varphi) \cap Var(\psi) \neq \emptyset$$

As it is known, there are many logics that respect the Variable-Sharing Principle—a paradigmatic example being A. Anderson and N. Belnap's logic R, for which see [1, p. 252-254]. In this vein, although it has been pointed out that the satisfaction of this criterion is only *necessary but not sufficient* to establish the relevance of a target notion of implication, it could be interesting to consider its satisfaction as an appropriate filter on a previously given independent notion of implication—thus rendering an (at least weakly) relevant subsystem thereof.[2] In this vein we could conceive, for example, filtering Classical Logic (CL, hereafter). Then, although truth-preservation in CL is an unacceptable guide to implication (due to its permeability to irrelevancies in the form of the paradoxes of material and strict implication), it might well be the case that the simultaneous satisfaction of truth-preservation and the Variable-Sharing Principle is an acceptable criterion. In fact, this is exactly the path followed by R.

[1] In particular, granting that the subject-matter of a complex proposition is to be identified with the sum or collection of the subject matter of all the propositional letters appearing in it—an idealized but relatively standard assumption, as discussed, e.g., in [5, p. 563]. Furthermore, in this respect it should be said that we are not considering languages with propositional constants—like *Verum* or the Ackermann constant—for different considerations need to be taken into account in such cases. We would like to thank Shawn Standefer for urging us to clarify this issue. As suggested by an anonymous reviewer, for references on this issue see [22] and [2].

[2] That the Variable-Sharing Principle can be seen as necessary but not sufficient is salient by noticing that there have been many relevant logicians (Anderson and Belnap among them) who rejected implications that are valid in Classical Logic and comply with the Variable-Sharing Principle—e.g. those going from $\neg\varphi \wedge (\varphi \vee \psi)$ to ψ, or from $\varphi \wedge \neg\varphi$ to $\varphi \wedge \psi$. We would like to thank Francesco Paoli for urging us to highlight this fact. Also, as pointed out by an anoynmous reviewer, it should be noticed that the VSP should be mainly predicated of systems and not of formulae. It may only be used metaphorically in the latter cases—and even then, not without some risk, since for example the schema $\varphi \to (\psi \to \varphi)$ "satisfies" the VSP but proves $\psi \to (\varphi \to \varphi)$ in any system with $\varphi \to \varphi$ as an axiom and and Modus Ponens as a rule.

Epstein in [16] where his propositional Relatedness Logic (REL, henceforth) is introduced.

Now, when working with relevant logics, it is standard to denote as "first-degree entailments" those implications of the form $\varphi \to \psi$ where φ and ψ contain no occurrence of the implication connective. As noted in [16], [30], and [31], it can be observed that whenever a first-degree entailment is valid in REL, the consequent preserves the truth of the antecedent and, moreover, the implication respects the Variable-Sharing Principle. More formally, when $\varphi \to \psi$ is a first-degree entailment:

$$\vdash_{\mathsf{REL}} \varphi \to \psi \quad \Longleftrightarrow \quad \begin{cases} \varphi \vdash_{\mathsf{CL}} \psi, \text{ and} \\ Var(\varphi) \cap Var(\psi) \neq \emptyset \end{cases}$$

It is also sometimes customary to think about such a set of valid first-degree entailments as a logical system on its own right. This can be easily done by considering the "first-degree fragment" $\mathsf{L}_{\mathsf{fde}}$ of a logic L formulated in a language with an implication connective, where $\varphi \to \psi$ is a first-degree entailment:

$$\vdash_{\mathsf{L}} \varphi \to \psi \quad \Longleftrightarrow \quad \varphi \vdash_{\mathsf{L}_{\mathsf{fde}}} \psi$$

In this respect, it is instructive to notice that valid first-degree entailments in REL encode certain validities in CL. Indeed, it can be easily seen that $\mathsf{REL}_{\mathtt{fde}}$ is the FMLA-FMLA fragment of $\mathsf{CL}_{\mathsf{VSP}}$—that is to say, the fragment of CL that respects the Variable-Sharing Principle.[3] That we choose to denote this fragment by $\mathsf{CL}_{\mathsf{VSP}}$ can be explained by noting that, in general, we may denote with the FMLA-FMLA fragment of $\mathsf{L}_{\mathsf{VSP}}$ the subsystem of a given logic L whose valid inferences are only those valid inferences of L that satisfy the Variable-Sharing Principle.[4] That is to say:

$$\varphi \vdash_{\mathsf{L}_{\mathsf{VSP}}} \psi \quad \Longleftrightarrow \quad \begin{cases} \varphi \vdash_{\mathsf{L}} \psi, \text{ and} \\ Var(\varphi) \cap Var(\psi) \neq \emptyset \end{cases}$$

With these clarifications in mind, let us state what our goals are with regard to $\mathsf{REL}_{\mathtt{fde}}$—i.e., the FMLA-FMLA fragment of $\mathsf{CL}_{\mathsf{VSP}}$. We aim at providing, first, an extensional semantics and, second, a simple Gentzen-style sequent calculus for it. Before detailing how the paper is structured in order to achieve our goals, let us discuss two aspects of the title of this article which are connected to said objectives. As an anonymous reviewer pointed out, the term "fragment" is often used to signal language restriction, instead of delimitation of a certain concrete and precisely delineated subsystem. However, in absence of a better and widespread term for this purpose, we prefer to stick to it and hope that the

[3] The FMLA-FMLA fragment of a logic L is the restriction of L to what is called, e.g., in [25, p. 108] the FMLA-FMLA framework. That is to say, set of inferences that are valid in such a logic which have exactly one formula as a premise and exactly one formula as a conclusion.

[4] As an anonymous reviewer points out, this constitutes a slightly different variant of the VSP—a deductive version of the VSP, one may claim. Here we are not concerned with logics and their theorems involving an implication connective, but in logics and their valid inferences, regardless of whether the system in question has a certified implication connective or not.

reader does not fall prey of any ambiguity—thus, in what follows fragments will not be linguistic but deductive restrictions of logical systems. Then, as another anonymous reviewer points out, as there are many such fragments of Classical Logic that respect the VSP, one may doubt the definite description element of the title—i.e., calling $\mathsf{CL}_{\mathsf{VSP}}$ *the* fragment of said logic that respects the VSP. However, we think it is clear enough that singling out the system that has *all* the deductive validities of Classical Logic that also comply with the VSP makes it an unequivocal qualification for this denomination. Furthermore, taking into account that this is the greatest collection of such valid inferences of Classical Logic that respect the VSP, also explains why our target subsystem of Classical Logic is denoted by this definite description.

Thus, for the purpose of achieving our goals, our work is structured as follows. In Section 2, we analyze with a certain degree of generality the fragment of any Tarskian logic that respects the Variable-Sharing Principle, establishing that in some important cases the resulting systems belong to a peculiar family—that of the non-transitive p-logics. In Section 3, we provide appropriate semantics for $\mathsf{REL}_{\mathtt{fde}}$ with the help of certain structures called p-matrices that generalize the so-called regular logical matrices. In Section 4, we present a sound and complete Gentzen-style sequent calculus for $\mathsf{REL}_{\mathtt{fde}}$ whose rules are bound to certain linguistic restrictions, guaranteeing the satisfaction of the Variable-Sharing Principle. Finally, in Section 5 we wrap up some concluding remarks and point towards directions of future work.

This being said, before delving into the proper contents of the article, let us briefly make explicit that we will be working with a propositional language $\mathcal{L}$ counting with a denumerable set Var of propositional variables $p, q, r, \dots$ and with logical connectives $\neg, \wedge, \vee$—intended to represent negation, conjunction, and disjunction, respectively. Thus, $\mathbf{FOR}(\mathcal{L})$ will be the algebra of well-formed formulae, standardly defined, whose carrier set is the set of well-formed formulae $FOR(\mathcal{L})$. In this respect, lower case Roman letters $\varphi, \psi, \chi, \dots$ will be considered as schematic formulae, whereas upper case Greek letters $\Gamma, \Delta, \Theta, \dots$ will be considered as schematic sets of formulae.

2. The fragment of a Tarskian logic that respects the Variable-Sharing Principle

In this section we analyze the fragment of any Tarskian logic that respects the Variable-Sharing Principle, paying special attention to the kind of systems that results from applying such a sieve, and to the semantic structures usually associated with said fragments. Thus, we notice that sometimes constraining Tarskian logics in this way results in a peculiar kind of systems called non-transitive p-logics. In this vein, we discuss logical matrices and related structures generalizing them, called p-matrices, furthermore focusing on some sufficient conditions that guarantee the satisfaction of the Variable-Sharing Principle in the systems induced by such matrices.

To begin with, let us recall what the literature usually understands by a Tarskian logic. By this it is usually meant a logical system formulated in the Set-Fmla framework, whose underlying consequence relation $\vdash$ has the following properties, where $\Gamma, \Delta \subseteq FOR(\mathcal{L})$ and $\varphi, \psi \in FOR(\mathcal{L})$:[5]

- $\Gamma \vdash \varphi$ if $\varphi \in \Gamma$ (Reflexivity)
- If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \varphi$ (Monotonicity)
- If $\Delta \vdash \varphi$ and $\Gamma \vdash \psi$ for every $\psi \in \Delta$, then $\Gamma \vdash \varphi$ (Transitivity)

In our case though—since we are interested in discussing semantics and calculi for $\mathsf{REL}_{\mathtt{fde}}$ which is the Fmla-Fmla fragment of $\mathsf{CL}_{\mathsf{VSP}}$—we are interested in the definition of Tarskian logics in the Fmla-Fmla framework. Whence, if a logic counts with connectives $\wedge$ and $\vee$ (to be interpreted, respectively, as conjunction and disjunction) we may say that a Tarskian logic is a logical system whose consequence relation $\vdash$ enjoys the following features, where $\varphi, \psi, \gamma, \delta \in FOR(\mathcal{L})$:

- $\varphi \vdash \varphi$ (Reflexivity)
- If $\varphi \vdash \psi$, then $\varphi \wedge \gamma \vdash \psi$ and $\varphi \vdash \psi \vee \delta$ (Monotonicity)
- If $\varphi \vdash \psi$ and $\psi \vdash \gamma$, then $\varphi \vdash \gamma$ (Transitivity)

Now, regarding the semantic interpretation of Tarskian logics, it is interesting to notice that all such systems can be semantically characterized by *logical matrices*. For a given propositional language $\mathcal{L}$ a logical matrix $\mathcal{M}$ is a pair $\langle \mathbf{A}, D \rangle$, where $\mathbf{A}$ is an algebra of the same similarity type than $\mathcal{L}$, and D is a subset of A, the universe or carrier set of $\mathbf{A}$. Letting an $\mathcal{M}$-valuation v be an homomorphism from $\mathbf{FOR}(\mathcal{L})$ to $\mathbf{A}$, a logical matrix $\mathcal{M}$ induces a Tarskian consequence relation $\vDash_{\mathcal{M}}$ in the following standard manner, where $\Gamma \cup \{\varphi\} \subseteq FOR(\mathcal{L})$:

$$\Gamma \vDash_{\mathcal{M}} \varphi \Longleftrightarrow \text{for every } \mathcal{M}\text{-valuation } v\text{: if } v(\Gamma) \subseteq D, \text{ then } v(\varphi) \in D$$

In this vein, it is a well-known result in Abstract Algebraic Logic—proved by R. Wójcicki in [42]—that for any Tarskian logic whose underlying consequence relation is $\vdash_{\mathsf{L}}$, there is a class $\mathbb{M}$ of logical matrices such that $\vdash_{\mathsf{L}} = \bigcap\{\vDash_{\mathcal{M}} \mid \mathcal{M} \in \mathbb{M}\}$. Whenever such a class is a singleton $\{\mathcal{M}\}$, we may say that $\vdash_{\mathsf{L}} = \vDash_{\mathcal{M}}$. In such a case, we will take the liberty of referring to $\vDash_{\mathcal{M}}$ as $\vDash_{\mathsf{L}}$. Thus, logical matrices allow understanding logical consequence in the context of Tarskian logics as preservation of designated values. Whence, *if all the premises are assigned a designated value, so must the conclusion.* This generalizes the idea, dear to Classical Logic, that valid arguments are such that if the premises are

[5] As pointed out by an anonymous reviewer, if we take into account sets of formulae—as opposed to sequences, lists, or multisets thereof—Reflexivity and Transitivity below imply Monotonicity. These properties are expressed here as standardly defined, e.g., in [19, p. 12].

all true, so must be the conclusion. Of course, all the previous remarks apply equally to a Tarskian logic formulated in the Fmla-Fmla framework—just that, instead of talking of a plurality of premises, we just need to consider a single premise.

Having clarified what Tarskian logics are, we may now move on to consider the main question of this section, namely, what kind of system results from focusing on the fragment of a Tarskian logic that respects the Variable-Sharing Principle. We hope that answering this question will provide us some clarity with regard to the semantic and proof-theoretic characterization of our target logic, $\mathsf{REL}_{\mathtt{fde}}$. But, to answer this question we must consider two scenarios. In the first, the Tarskian logic in question already satisfies the Variable-Sharing Principle. In the second, it does not. It is obvious then, that applying such a constraint to a logic in the first scenario does not change anything. Thus, we obtain the same system we started with.[6] It is the second scenario that is more interesting, because if the Tarskian logic we start with does not respect the Variable-Sharing Principle, then the system resulting from filtering out all its irrelevant impurities can be quite non-standard.

To observe why this may be the case, consider the following. For a logic whose underlying consequence relation is $\vdash$ let us a say that a *theorem* is a formula φ such that $\psi \vdash \varphi$, for all $\psi \in FOR(\mathcal{L})$, whereas an *anti-theorem* is a formula φ such that $\varphi \vdash \psi$, for all $\psi \in FOR(\mathcal{L})$. It should be clearly noticeable that a logic L cannot satisfy the Variable-Sharing Principle if it has either theorems or anti-theorems. Furthermore, as we will show below, if L has either theorems or anti-theorems, its fragment satisfying the Variable-Sharing Principle results in a logic that is *not Tarskian*—for it is non-transitive.

Interestingly enough, although non-transitive systems are not Tarskian logics, some of them belong to a special kind that generalizes Tarskian logics. These are the so-called *p-logics*, developed firstly by S. Frankowksi in [20]. When formulated either in the Set-Fmla or the Fmla-Fmla framework, p-logics should be considered as systems whose underlying consequence relation *respects both Reflexivity and Monotonicity, although it does not necessarily respect Transitivity.* By this, we mean that p-logics that are transitive are Tarskian logics, whereas p-logics that are non-transitive are not—and can be, thus, regarded as "proper" p-logics in some sense. In this spirit, consequence relations underlying proper p-logics can be rightfully referred to as proper p-consequence relations.[7]

Along these lines, it can be easily shown that whenever we start with a Tarskian logic L and later focus on its fragment satisfying the Variable-Sharing Principle—that is on $\mathsf{L}_{\mathsf{VSP}}$—there are some conditions that L may have which guarantee that $\mathsf{L}_{\mathsf{VSP}}$ be a non-transitive p-logic. These can be summarized as follows.

Observation 2.1. *If* L *is a Tarskian logic and has either theorems or anti-*

[6] One case of this sort is the logic $\mathsf{E}_{\mathtt{fde}}$, induced by a logical matrix built on top of the 4-element Belnap-Dunn algebra—discussed, e.g., in [15] by J. M. Dunn and in [4] by N. Belnap.

[7] The remark that some non-transitive systems are p-logics has substance to it as the latter comprises, e.g., reflexive systems. Thus, non-transitive systems that are also non-reflexive cannot be regarded as p-logics and therefore not all non-transitive systems are of this kind.

theorems, then the system $\mathsf{L_{VSP}}$ *is a non-transitive p-logic.*

Proof. We first establish the Reflexivity and Monotonicity of $\mathsf{L_{VSP}}$, for which it is important to remember the meaning that these properties have in the context of Tarskian logics formulated in the FMLA-FMLA framework. To prove the former, suppose $\varphi \vdash_{\mathsf{L}} \varphi$. Trivially, $Var(\varphi) \cap Var(\varphi) \neq \emptyset$. Whence, $\varphi \vdash_{\mathsf{L_{VSP}}} \varphi$. To prove the latter, suppose $\varphi \vdash_{\mathsf{L}} \psi$ and $Var(\varphi) \cap Var(\psi) \neq \emptyset$, whence $\varphi \vdash_{\mathsf{L_{VSP}}} \psi$. Since L is assumed to be Tarskian, in the FMLA-FMLA framework its Monotonicity amounts to the following inference being valid, for all $\gamma \in FOR(\mathcal{L})$: $\varphi \wedge \gamma \vdash_{\mathsf{L}} \psi$. Simple set-theoretic reasoning allows to establish that $Var(\varphi \wedge \gamma) \cap Var(\psi) \neq \emptyset$. Whence, $\varphi \wedge \gamma \vdash_{\mathsf{L_{VSP}}} \psi$. Similar reasoning establishes that $\varphi \vdash_{\mathsf{L_{VSP}}} \psi \vee \delta$.

We now prove that if L has either theorems or anti-theorems, then $\mathsf{L_{VSP}}$ is non-transitive—and, thus, a "proper" p-logic. For this purpose, consider first that L has theorems, letting ψ be a theorem, and φ and γ be arbitrary formulae, such that $Var(\varphi) \cap Var(\psi) = \emptyset$, but $Var(\varphi) \cap Var(\gamma) \neq \emptyset$ and $Var(\gamma) \cap Var(\psi) \neq \emptyset$. Since L is assumed to be Tarskian, in the FMLA-FMLA framework its Monotonicity implies the validity of $\varphi \vdash_{\mathsf{L}} \varphi \vee \gamma$, for all $\gamma \in FOR(\mathcal{L})$. Because of ψ being a theorem, we know that $\varphi \vee \gamma \vdash_{\mathsf{L}} \psi$. Given $Var(\varphi) \cap Var(\varphi \vee \gamma) \neq \emptyset$, and $Var(\varphi \vee \gamma) \cap Var(\psi) \neq \emptyset$—the latter by hypothesis—the previous remarks guarantee that $\varphi \vdash_{\mathsf{L_{VSP}}} \varphi \vee \gamma$ and $\varphi \vee \gamma \vdash_{\mathsf{L_{VSP}}} \psi$, although $\varphi \nvdash_{\mathsf{L_{VSP}}} \psi$. Thus, if L is a Tarskian logic that has theorems, $\mathsf{L_{VSP}}$ is a non-transitive p-logic.

The case for anti-theorems is analogous, and thus we leave it to the reader as an exercise. $\square$

Now, let us recall that the aim of this article is to provide a simple semantics and calculus for $\mathsf{REL_{fde}}$—that is to say, the FMLA-FMLA fragment of $\mathsf{CL_{VSP}}$. With the information of the previous result in hand, we may safely claim that $\mathsf{REL_{fde}}$ is a non-transitive p-logic.[8] But, if this is the case, it would be interesting to know whether p-logics in general (and non-transitive p-logics as a special case) can be associated with certain semantic structures, just like Tarskian logics can be identified with logical matrices.

Happily, the answer is affirmative in this respect. Indeed, there is a correspondence between p-logics and a family of structures that generalizes logical matrices—opportunely called *logical p-matrices*. Thus, for a given propositional language $\mathcal{L}$ a logical p-matrix $\mathcal{M}$ is a triple $\langle \mathbf{A}, D_p, D_c \rangle$, where $\mathbf{A}$ is an algebra of the same similarity type than $\mathcal{L}$, and D_p, D_c are subsets of A, the universe or carrier set of $\mathbf{A}$, such that $D_p \subseteq D_c$. These sets should be understood as a set of *designated* values for formulae conceived *as premises*, and a set of *designated* values for formulae conceived *as conclusions*. Hence, by the restrictions imposed above, if a formula is designated as a premise it must be designated as a conclusion—although if it is not designated as a premise, it may well be

[8]It should be duly noted that the non-transitive nature of $\mathsf{REL_{fde}}$ *as a deductive system* stems from the non-transitivity of the implication involved in the first-degree entailments that are valid in Epstein's logic REL, which was already discussed in [16] and [30]. We would like to thank an anonymous reviewer for urging us to clarify this.

designated as a conclusion.[9] Letting an $\mathcal{M}$-valuation v be an homomorphism from $\mathbf{FOR}(\mathcal{L})$ to $\mathbf{A}$, a logical p-matrix $\mathcal{M}$ induces a p-consequence relation $\vDash_{\mathcal{M}}$ in the following, standard manner, where $\Gamma \cup \{\varphi\} \subseteq FOR(\mathcal{L})$:

$$\Gamma \vDash_{\mathcal{M}} \varphi \iff \text{for every } \mathcal{M}\text{-valuation } v\text{: if } v(\Gamma) \subseteq D_p, \text{ then } v(\varphi) \in D_c$$

In this vein, S. Frankowski's shows in [20, p. 47] that for any p-logic whose underlying consequence relation is $\vdash_{\mathsf{L}}$, there is a class $\mathbb{M}$ of logical p-matrices such that $\vdash_{\mathsf{L}} = \bigcap\{\vDash_{\mathcal{M}} | \mathcal{M} \in \mathbb{M}\}$. Whenever such a class is a singleton $\{\mathcal{M}\}$, we may say that $\vdash_{\mathsf{L}} = \vDash_{\mathcal{M}}$. In such a case, we will take the liberty of referring to $\vDash_{\mathcal{M}}$ as $\vDash_{\mathsf{L}}$. It should be noticed that whenever $D_p = D_c$, the corresponding p-matrix is actually a regular logical matrix—justifying the claim that the former kind of structures generalizes the latter. In a nutshell, if being designated as a premise is the same as being designated as a conclusion, then we are in the presence of a regular logical matrix. When this is not the case and the p-matrix in question is not a regular logical matrix, it is interesting to observe that logical consequence cannot be understood as preservation of designated values, in the traditional sense. It is perhaps better to say that it can be understood in terms of preservation in a more liberal or generalized reading. Whence, *if all the premises are assigned a designated value for premises, then the conclusion must be assigned a designated value for conclusions.* Once again, the previous remarks apply equally to a p-logic formulated in the Fmla-Fmla framework.

Information of this sort is useful, as it suggests to us that $\mathsf{REL}_{\mathtt{fde}}$ being a proper p-logic, its semantics should be given in terms of a proper p-matrix. This, of course, does not suggest in itself the features of the semantics in question.[10] For this purpose, let us review a number of remarks that will make our approximation below more intelligible. These observations concern some sufficient—although not necessary—features that a logical matrix, and a p-matrix, in turn, may have that will make the system thereby induced to comply with the Variable-Sharing Principle. In this regard, adapting some of the terminology used in their article, we may paraphrase G. Robles and J. Mendez in [37] (see also [36] and [38]) by stating the following result.

Lemma 2.2 ([37])**.** *Let* L *be a Tarskian logic induced by the logical matrix* $\langle \mathbf{A}, D\rangle$*, formulated in the propositional language counting with connectives* $\neg, \wedge, \vee$*. If there are* $a_1, a_2 \in A$ *such that:*

- $a_1 \in D$ *and* $\neg^{\mathbf{A}}(a_1) = \wedge^{\mathbf{A}}(a_1, a_1,) = \vee^{\mathbf{A}}(a_1, a_1,) = a_1$
- $a_2 \notin D$ *and* $\neg^{\mathbf{A}}(a_2) = \wedge^{\mathbf{A}}(a_2, a_2,) = \vee^{\mathbf{A}}(a_2, a_2,) = a_2$.

[9] In S. Frankowski's words, this formalizes the idea that p-consequence relations represent the transition from premises which may be held to a stricter standard (of acceptance, or belief, or truth) to conclusions which may be held to a more tolerant standard—constituting plausible (whence the "p") conclusions rather than strictly certain conclusions thereof.

[10] Notice that a proper p-logic cannot receive other than proper p-matrix semantics. Were someone to offer regular matrix semantics for it, then the resulting system will be transitive, and thus not a proper p-logic. Therefore, it will not be a semantics *for it.* We would like to thank an anonymous reviewer for urging us to clarify this.

Then, L *satisfies the Variable-Sharing Principle.*

We can easily see that these remarks can be straightforwardly generalized so as to provide an analogous result concerning p-matrices, instead of regular logical matrices. To discuss such a generalization we now turn.

Lemma 2.3. *Let* L *be a p-logic induced by the p-logical matrix* $\langle \mathbf{A}, D_p, D_c \rangle$*, formulated in the propositional language counting with connectives* $\neg, \wedge, \vee$*. If there are* $a_1, a_2 \in A$ *such that:*

- $a_1 \in D_p$ *and* $\neg^{\mathbf{A}}(a_1) = \wedge^{\mathbf{A}}(a_1, a_1,) = \vee^{\mathbf{A}}(a_1, a_1,) = a_1$
- $a_2 \notin D_c$ *and* $\neg^{\mathbf{A}}(a_2) = \wedge^{\mathbf{A}}(a_2, a_2,) = \vee^{\mathbf{A}}(a_2, a_2,) = a_2$.

Then, L *satisfies the Variable-Sharing Principle.*

Proof. Assume L is a p-logic induced by the p-logical matrix $\langle \mathbf{A}, D_p, D_c \rangle$, where all the operations and the truth-values involved have the conditions outlined above. Suppose, then that there is a valid inference $\varphi \vDash_{\mathsf{L}} \psi$ such that $Var(\varphi) \cap Var(\psi) = \emptyset$. Then, consider an L-valuation v such that:

$$v(p) = \begin{cases} a_1 & \text{if } p \in \varphi \\ a_2 & \text{otherwise} \end{cases}$$

By the conditions assumed above, we know that $v(\varphi) = a_1$, whereas $v(\psi) = a_2$. Thus, $v(\varphi) \in D_p$ while $v(\psi) \notin D_c$, whence v is a valuation witnessing $\varphi \nvDash_{\mathsf{L}} \psi$. This contradicts our initial assumption, which then implies that if the aforementioned conditions are met, then every valid inference satisfies the Variable-Sharing Principle. □

In what follows, we will use these remarks in the investigation of semantic structures that will induce the fragment of Classical Logic that respects the Variable-Sharing Principle—i.e., in introducing semantics for $\mathsf{REL}_{\mathtt{fde}}$. By this, we mean that we will build a p-matrix that will induce the logic in question, where such a p-matrix will have two truth-values behaving in the way described by Lemma 2.3.

3. Semantics

The aim of this section is to present a simple extensional semantics for $\mathsf{REL}_{\mathtt{fde}}$. In this regard, it should be noted that algebraic semantics—particularly, semantics where logical consequence is defined in terms of certain order-theoretic relations holding between the elements of the carrier set of a given algebra as, e.g., in L. Humberstone's [25, p. 246]—have been introduced both for the full system REL by R. Epstein in [16] and for the restricted fragment $\mathsf{REL}_{\mathtt{fde}}$ that concerns us, by F. Paoli in [31]. Additionally, F. Paoli presents a more traditional algebraic semantics for it in [30], in the form of a class of products of Boolean algebras and τ-semilattices.

However, no extensional semantics where logical consequence is understood in terms of the assignment of designated values of some kind to premises and conclusions has been discussed so far, whence the material below constitutes a novel development in this respect.[11] Of course, since $\mathsf{REL}_{\mathtt{fde}}$ is a non-transitive p-logic and therefore a non-Tarskian logic, if it happens to be possible for it to be induced by a logical matrix of sorts, such a structure will not be a regular logical matrix, but rather a proper p-matrix. Thus, in what follows we present a route to arrive at such a p-matrix, highlighting that there might be other equally interesting manners of landing the same results.

In particular, we will go through a two-step process in order to define our target p-matrix. This process will consist, on the one hand, in finding a proper p-matrix that induces CL and, on the other hand, in extending said p-matrix with additional truth-values so as to guarantee the satisfaction of the Variable-Sharing Principle—without causing any other logical side-effects, as invalidating classically valid inferences that satisfy this principle.

Our *first step in the way to arriving at a p-matrix semantics for* $\mathsf{REL}_{\mathtt{fde}}$ is the presentation of a proper p-matrix that will induce CL. This already suggests a few discussions in itself. To wit, if the matrix in question is a proper p-matrix but not a regular matrix, one may wonder whether the resulting logic will be identical to CL, or if it will differ with this system in some respect. Lengthy debates have been had in the past few years in this regard, mostly revolving around the logic ST defended by Cobreros, Égré, Ripley and van Rooij in many works—some of which include [8], [9], [10], [11], [34] and [35]. For future reference, the logic ST is induced by the p-matrix $\langle \mathbf{SK}, \{\mathbf{t}\}, \{\mathbf{t}, \mathbf{n}\} \rangle$ built on top of the 3-element strong Kleene algebra $\mathbf{SK}$ from S. Kleene's [26], whose carrier set is $\{\mathbf{t}, \mathbf{n}, \mathbf{f}\}$ and whose characteristic operations can be presented in the form of the "truth-tables" appearing in Figure 3. These authors championed the view that Classical Logic can be legitimately seen as induced by a structure of this sort, whereas other scholars contested that although the resulting system called by them ST coincided with CL with regard to its set of valid inferences it did not coincide in what regards to its valid *metainferences*—which, roughly speaking, refers to inferences between inferences themselves. The jury is still out in this trial, as it is in a related meta-discussion, that of trying to determine whether the question itself is substantial or terminological.[12]

For the purpose of this article, however, we will admit that certain p-matrices can characterize CL, at least in what concerns to its set of valid inferences. This is instrumental for us, given the task we set for ourselves of trying to find a simple semantics for those inferences that not only are valid in Classical Logic, but that also respect the Variable-Sharing Principle. As a consequence of adopting this point of view, we will entertain these p-matrices as inducing CL, although we

[11] As an anonymous reviewer points out, the semantics by Paoli in [30] are extensional, although logical consequence there is not understood in terms of the usual notion of preservation of designated values for logical matrices, but instead in terms of the satisfaction of a binary relation *Imp*.

[12] Some of the crucial works on this debate are B. Dicher and F. Paoli's [14], E. Barrio, F. Pailos and D. Szmuc's [3], and C. Scambler's [39].

	$\neg$
t	**f**
n	**n**
f	**t**

$\wedge$	**t**	**n**	**f**
t	**t**	**n**	**f**
n	**n**	**n**	**f**
f	**f**	**f**	**f**

$\vee$	**t**	**n**	**f**
t	**t**	**t**	**t**
n	**t**	**n**	**n**
f	**t**	**n**	**f**

Figure 1: The strong Kleene truth-tables

will sometimes refer to the systems thereby induced with other names—because this will be useful for matters of clarity below, when we extend these structures to arrive at semantics for $\mathsf{REL}_{\mathtt{fde}}$.

Interestingly enough, recently p-matrix semantics for CL, different from those discussed by Cobreros, Égré, Ripley and van Rooij, have been presented. This alternative option is built on top of the 3-element weak Kleene algebra **WK**—instead of the aforementioned strong Kleene algebra. Thus, either implicitly or explicitly it is possible to find semantics along these lines in F. Correia's [13], F. Paoli and M. Pra Baldi's [32] and in [Author]. In order for things to be clear in what follows, let us state here that the **WK** algebra is the structure whose carrier set is $\{\mathbf{t}, \mathbf{e}, \mathbf{f}\}$ and whose characteristic operations can be presented in the form of the "truth-tables" appearing in Figure 2. In this respect, it was either shown or mentioned in the previously referred works that an interesting logic that we call wST can be shown to have the same valid inferences that CL, thereby offering a proper p-matrix semantics for it.

	$\neg$
t	**f**
e	**e**
f	**t**

$\wedge$	**t**	**e**	**f**
t	**t**	**e**	**f**
e	**e**	**e**	**e**
f	**f**	**e**	**f**

$\vee$	**t**	**e**	**f**
t	**t**	**e**	**t**
e	**e**	**e**	**e**
f	**t**	**e**	**f**

Figure 2: The weak Kleene truth-tables

Definition 3.1. wST is the p-logic induced by the following p-matrix:

$$\langle \mathbf{WK}, \{\mathbf{t}\}, \{\mathbf{t}, \mathbf{e}\}\rangle$$

Lemma 3.2 ([32], [Author])**.** *For all $\varphi, \psi \in FOR(\mathcal{L})$:*

$$\varphi \vDash_{\mathsf{CL}} \psi \quad \Longleftrightarrow \quad \varphi \vDash_{\mathsf{wST}} \psi$$

Before moving on to the further extension of this p-matrix in order to arrive at a structure inducing $\mathsf{REL}_{\mathtt{fde}}$, let us take a moment to understand why the introduction of the third value $\mathbf{e}$ is not disruptive, i.e., why the resulting logic has the same valid inferences than CL. The explanation appearing next is a straightforward adaptation of the one used to explain why ST has the same valid inferences than CL, in many places of the literature.

Let us first observe the *exclusion of* $\mathbf{e}$ *from the set of designated values for premises* guarantees that no inferences will be rendered invalid because the premises were assigned this new value. In other words, that only classically-satisfiable premises can be the premises of an inference having a counterexample. Secondly, the *inclusion of* $\mathbf{e}$ *in the set of designated values for conclusions* guarantees that no inference will be invalid because the conclusion was assigned this new value. Again, this means that only classically-falsifiable conclusions can be the conclusion of an inference having a counterexample. In a nutshell, with the help of the linguistic resources available, the introduction of the non-classical value $\mathbf{e}$ is ineffective for the generation of new counterexamples to classically valid inferences. Furthermore, whenever a wST-valuation constitutes a counterexample to some inference, the fact that the operations in **WK** are monotonic with regard to the partial order $\mathbf{i} \leq \mathbf{t}, \mathbf{i} \leq \mathbf{f}, \mathbf{i} \leq \mathbf{i}, \mathbf{t} \leq \mathbf{t}, \mathbf{f} \leq \mathbf{f}$ guarantees that these valuations can be transformed into Boolean valuations without altering the values of complex formulae assigned $\mathbf{t}$ and $\mathbf{f}$.[13]

Our *second step in the way to arriving at a proper p-matrix for* $\mathsf{REL}_{\mathtt{fde}}$ will be, then, to appropriately extend the previously discussed p-matrix in the spirit of the remarks made in Lemma 2.3. That is to say, we will have a p-matrix whose underlying algebra has two additional values with respect to the **WK** algebra—one of such values will be designated for premises and conclusions, while the other will be undesignated for premises and conclusions. In addition, these two elements will behave in the way described by Lemma 2.3, that is to say, whenever they are negated, conjoined with themselves, or disjoined with themselves, they will respectively return the same value. For reasons that will be clear below, let us refer to these truth-values as $\mathbf{o}^{\mathbf{e}}_1$ and $\mathbf{o}^{\mathbf{e}}_2$, respectively.

However, on top of securing this behavior, we need to make sure that the inclusion of such values is as effective and as innocuous as desired. In other words, that their inclusion renders invalid all inferences that are valid in CL which do not comply with the Variable-Sharing Principle, without invalidating some inferences that do comply with said principle. For this purpose, one way to extend the **WK** algebra to satisfy this demands is to allow for two additional elements working exactly like the non-classical value $\mathbf{e}$ whenever premises and conclusions share a propositional variable. Thus, it should be understood that, whenever premises and conclusions share a propositional variable, it should be impossible to generate counterexamples to the validity of the inference in question by assigning the formulae involved the newly introduced truth-values in a convenient way.

This can be done by letting the result of every operation in which the elements $\mathbf{o}^{\mathbf{e}}_1$ and $\mathbf{o}^{\mathbf{e}}_2$ are some, but not all of the inputs, be calculated as if these truth-values were replaced by $\mathbf{e}$—additionally, letting the result be $\mathbf{o}^{\mathbf{e}}_1$ if all inputs were $\mathbf{o}^{\mathbf{e}}_1$, and $\mathbf{o}^{\mathbf{e}}_2$ if all inputs were $\mathbf{o}^{\mathbf{e}}_2$, respectively. This guarantees that *new*

[13] From these remarks one may take away the fact that for any 3-element algebra $\mathbf{A}$ with carrier set $\{\mathbf{t}, \mathbf{i}, \mathbf{f}\}$ having the 2-element Boolean algebra as a subalgebra, the p-matrix $\langle \mathbf{A}, \{\mathbf{t}\}, \{\mathbf{t}, \mathbf{i}\} \rangle$ induces a logic that has the same valid inferences that CL, as long as all the operations in $\mathbf{A}$ are monotonic with regard to the aforementioned partial order—an observation already present *in nuce* in K. Schütte's [40], as reviewed by J.-Y. Girard in [23, p. 162]. As mentioned in several places, among them in S. Kripke's [27], both the operations in the **SK** algebra and the operations in the **WK** algebra are monotonic in this way.

counterexamples to classically valid inferences will only emerge when premises can be assigned the truth-value $\mathbf{o}_1^{\mathbf{e}}$ and conclusions can be assigned the truth-value $\mathbf{o}_2^{\mathbf{e}}$. A situation only possible if premises and conclusions do not share any propositional variable.

Finally, before moving on to defining the ingredients of the p-matrix inducing $\mathsf{REL}_{\mathtt{fde}}$, let us observe that the requirements above can be translated into general algebraic terminology, as follows.

Definition 3.3. An algebra $\mathbf{A}$ has distinct elements $k, \mathbf{o}^k \in A$ such that $\mathbf{o}^k$ "mimics" k if and only if for all n-ary operations ¶ and all $\{a_1, \ldots, a_n\} \subseteq A$:

$$\text{if } \{\mathbf{o}^k\} \subsetneq \{a_1, \ldots, a_n\}, \text{ then } \P^{\mathbf{A}}(a_1, \ldots, a_n) = \P^{\mathbf{A}}((a_1, \ldots, a_n)[\mathbf{o}^k/k])$$

where $(a_1, \ldots, a_n)[\mathbf{o}^k/k]$ is the result of replacing each occurrence of $\mathbf{o}^k$ for an occurrence of k in $a_1, \ldots, a_n$.

Naturally, this can be generalized to algebras counting with a set $\{\mathbf{o}_1^k, \ldots, \mathbf{o}_n^k\}$ of elements that "mimic" an element k.

Definition 3.4. An algebra $\mathbf{A}$ has a universally idempotent element k if and only if for all n-ary operations ¶ and all $\{a_1, \ldots, a_n\} \subseteq A$:

$$\text{if } \{k\} = \{a_1, \ldots, a_n\}, \text{ then } \P^{\mathbf{A}}(a_1, \ldots, a_n) = k$$

Definition 3.5. Given an algebra $\mathbf{A}$, the algebra $\mathbf{A}[\mathbf{o}^k]$ is its extension with a universally idempotent element $\mathbf{o}^k \notin A$ that "mimics" an element $k \in A$, such that for all n-ary operations ¶ and all $\{a_1, \ldots, a_n\} \subseteq A \cup \{\mathbf{o}^k\}$:

$$\P^{\mathbf{A}[\mathbf{o}^k]}(a_1, \ldots, a_n) = \begin{cases} \mathbf{o}^k & \text{if } \{\mathbf{o}^k\} = \{a_1, \ldots, a_n\} \\ \P^{\mathbf{A}}((a_1, \ldots, a_n)[\mathbf{o}^k/k]) & \text{if } \{\mathbf{o}^k\} \subsetneq \{a_1, \ldots, a_n\} \\ \P^{\mathbf{A}}(a_1, \ldots, a_n) & \text{otherwise} \end{cases}$$

where $(a_1, \ldots, a_n)[\mathbf{o}^k/k]$ is the result of replacing each occurrence of $\mathbf{o}^k$ for an occurrence of k in $a_1, \ldots, a_n$.

Once more, this can be generalized to extended algebras $\mathbf{A}[\mathbf{o}_1^k, \ldots, \mathbf{o}_n^k]$ with a set of universally idempotent elements $\{\mathbf{o}_1^k, \ldots, \mathbf{o}_n^k\}$ that "mimic" an element k previously available in the starting algebra $\mathbf{A}$—which can be redescribed as $\mathbf{A}[\mathbf{o}_1^k, \ldots, \mathbf{o}_{n-1}^k][\mathbf{o}_n^k] = \ldots = \mathbf{A}[\mathbf{o}_1^k] \ldots [\mathbf{o}_n^k]$.

Having clarified this, our requirements above concerning a semantic structure for $\mathsf{REL}_{\mathtt{fde}}$ can otherwise be phrased as saying that we need to extend the **WK** algebra with two universally idempotent elements that mimic $\mathbf{e}$, one of which should be designated for premises and conclusions in the context of the extended p-matrix, whereas the other should be undesignated for premises and conclusions in the context of the extended p-matrix. This algebra we call, correspondingly, the 5-element algebra $\mathbf{WK}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}]$, whose carrier set can be conspicuously described as $\{\mathbf{t}, \mathbf{o}_1^{\mathbf{e}}, \mathbf{e}, \mathbf{o}_2^{\mathbf{e}}, \mathbf{f}\}$ and whose operations can be described by the "truth-tables" in Figure 3.[14]

	$\neg$
$\mathbf{t}$	$\mathbf{f}$
$\mathbf{o}_1^{\mathbf{e}}$	$\mathbf{o}_1^{\mathbf{e}}$
$\mathbf{e}$	$\mathbf{e}$
$\mathbf{o}_2^{\mathbf{e}}$	$\mathbf{o}_2^{\mathbf{e}}$
$\mathbf{f}$	$\mathbf{t}$

$\wedge$	$\mathbf{t}$	$\mathbf{o}_1^{\mathbf{e}}$	$\mathbf{e}$	$\mathbf{o}_2^{\mathbf{e}}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{f}$
$\mathbf{o}_1^{\mathbf{e}}$	$\mathbf{e}$	$\mathbf{o}_1^{\mathbf{e}}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$
$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$
$\mathbf{o}_2^{\mathbf{e}}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{o}_2^{\mathbf{e}}$	$\mathbf{e}$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{f}$

$\vee$	$\mathbf{t}$	$\mathbf{o}_1^{\mathbf{e}}$	$\mathbf{e}$	$\mathbf{o}_2^{\mathbf{e}}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{t}$
$\mathbf{o}_1^{\mathbf{e}}$	$\mathbf{e}$	$\mathbf{o}_1^{\mathbf{e}}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$
$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$
$\mathbf{o}_2^{\mathbf{e}}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{o}_2^{\mathbf{e}}$	$\mathbf{e}$
$\mathbf{f}$	$\mathbf{t}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{e}$	$\mathbf{f}$

Figure 3: The five-valued $\mathsf{wST}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}]$ truth-tables

With these tools in hand, we turn to defining our target non-transitive p-logic and to proving that its Fmla-Fmla fragment is equal to $\mathsf{REL}_{\mathtt{fde}}$, that is to say, to the Fmla-Fmla fragment of $\mathsf{CL}_{\mathsf{VSP}}$.

Definition 3.6. $\mathsf{wST}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}]$ is the logic induced by the following p-matrix:

$$\langle \mathbf{WK}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}], \{\mathbf{t}, \mathbf{o}_1^{\mathbf{e}}\}, \{\mathbf{t}, \mathbf{o}_1^{\mathbf{e}}, \mathbf{e}\}\rangle$$

Lemma 3.7. *For every $\varphi, \psi \in FOR(\mathcal{L})$ if there is a $\mathsf{wST}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}]$-valuation v such that either $v(\varphi) = \mathbf{t}$ and $v(\psi) = \mathbf{o}_2^{\mathbf{e}}$, or $v(\varphi) = \mathbf{o}_1^{\mathbf{e}}$ and $v(\psi) = \mathbf{f}$, then $Var(\varphi) \cap Var(\psi) = \emptyset$.*

Proof. Firstly, that $v(\varphi) = \mathbf{t}$ implies that for all $p \in Var(\varphi)$, $v(p) \in \{\mathbf{t}, \mathbf{f}\}$. Simultaneously, that $v(\psi) = \mathbf{o}_2^{\mathbf{e}}$ implies that for all $q \in Var(\psi)$, $v(q) = \mathbf{o}_2^{\mathbf{e}}$. Whence, $Var(\varphi) \cap Var(\psi) = \emptyset$. Secondly, that $v(\varphi) = \mathbf{o}_1^{\mathbf{e}}$ implies that for all $p \in Var(\varphi)$, $v(p) = \mathbf{o}_1^{\mathbf{e}}$. Simultaneously, that $v(\psi) = \mathbf{f}$ implies that for all $q \in Var(\psi)$, $v(q) \in \{\mathbf{t}, \mathbf{f}\}$. Whence, $Var(\varphi) \cap Var(\psi) = \emptyset$. □

Theorem 3.8. *The* Fmla-Fmla *fragment of* $\mathsf{wST}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}] = \mathsf{REL}_{\mathtt{fde}}$

Proof. On the one hand, assume $\varphi \nvDash_{\mathsf{wST}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}]} \psi$. There are four ways in which this can happen. Either there is a $\mathsf{wST}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}]$-valuation v such that (i) $v(\varphi) = \mathbf{t}$ and $v(\psi) = \mathbf{f}$, or (ii) $v(\varphi) = \mathbf{t}$ and $v(\psi) = \mathbf{o}_2^{\mathbf{e}}$, or (iii) $v(\varphi) = \mathbf{o}_1^{\mathbf{e}}$ and $v(\psi) = \mathbf{f}$, or (iv) $v(\varphi) = \mathbf{o}_1^{\mathbf{e}}$ and $v(\psi) = \mathbf{o}_2^{\mathbf{e}}$. In case (i), we are guaranteed that v is a Boolean valuation, whence we know that $\varphi \nvDash_{\mathsf{CL}} \psi$. In cases (ii) and (iii), we know by Lemma 3.7 that $Var(\varphi) \cap Var(\psi) = \emptyset$. In case (iv), we know by Lemma 2.3 that $Var(\varphi) \cap Var(\psi) = \emptyset$. From all these considerations, it follows that $\varphi \nvDash_{\mathsf{REL}_{\mathtt{fde}}} \psi$.

On the other hand, assume that $\varphi \nvDash_{\mathsf{REL}_{\mathtt{fde}}} \psi$. That is to say, that either $\varphi \nvDash_{\mathsf{CL}} \psi$, or $Var(\psi) \cap Var(\varphi) = \emptyset$. If the former is the case, then there is a CL-valuation v such that $v(\varphi) = \mathbf{t}$ and $v(\psi) = \mathbf{f}$. However, given CL-valuations

[14] Notice that this algebra can be seen as the extension with a universally idempotent element that mimics the infectious element of a 4-element algebra appearing in the article [31] by F. Paoli, referred to as the **FP** algebra in [Author]. Whence, in turn, this last structure can be equally described as the extension of the **WK** algebra with a universally idempotent element that mimics the infectious element $\mathbf{e}$—that is to say, as $\mathbf{WK}[\mathbf{o}^{\mathbf{e}}]$.

are a subset of $\mathsf{wST}[\mathbf{o^e_1o^e_2}]$-valuations, this establishes that there is a $\mathsf{wST}[\mathbf{o^e_1o^e_2}]$-valuation v' such that $v'(\varphi) = \mathbf{t}$ and $v'(\psi) = \mathbf{f}$. From this it follows that $\varphi \nvDash_{\mathsf{wST}[\mathbf{o^e_1o^e_2}]} \psi$. If the latter is the case, it is possible to construct a $\mathsf{wST}[\mathbf{o^e_1o^e_2}]$-valuation v such that:

$$v(p) = \begin{cases} \mathbf{o^e_1} & \text{if } p \in Var(\varphi) \\ \mathbf{o^e_2} & \text{otherwise} \end{cases}$$

For such a valuation it is possible to show, as in Lemma 2.3, that $v(\varphi) = \mathbf{o^e_1}$ while $v(\psi) = \mathbf{o^e_2}$. From this it follows that $\varphi \nvDash_{\mathsf{wST}[\mathbf{o^e_1o^e_2}]} \psi$. □

Matters of interpretation of the truth-values involved are rather difficult. To wit, whereas usually the characteristic value of the weak Kleene algebra is understood as representing meaninglessness or nonsense of some sort, as in D. Bochvar's [6] and S. Halldén's [24], it is saliently complicated to explain how this reading spills into the interpretation of the mimicking values $\mathbf{o^e_1}$ and $\mathbf{o^e_2}$. Our intention here is not, however, to provide a cogent philosophical reading of the truth-values involved in a semantic presentation of $\mathsf{REL}_{\mathtt{fde}}$—the Fmla-Fmla fragment of $\mathsf{CL}_{\mathsf{VSP}}$—but simply to offer a semantic structure that will induce this target non-transitive p-logic. In this respect, an in-depth discussion of these matters, hoping to determine if there is a p-matrix with a cogent and perspicuous philosophical reading for $\mathsf{REL}_{\mathtt{fde}}$, will have to wait for another time.

A further question regarding this semantic rendering of $\mathsf{REL}_{\mathtt{fde}}$ lies in its being an extension of a proper p-matrix inducing a system with the same valid inferences that CL. Our semantics for this fragment of $\mathsf{CL}_{\mathsf{VSP}}$ consisted of extending a p-matrix built on top of the **WK** algebra with mimicking values appropriately taken to be designated or undesignated for premises and conclusions. One may, then, ask whether it is possible to build another different p-matrix for $\mathsf{REL}_{\mathtt{fde}}$ by means of extending a proper p-matrix for CL built on top of another structure. We leave this question for future research, although we provide some preliminary conjectures in Section 5.

Having provided a semantics for our target logic, in the following section, we devote ourselves to defining an appropriate calculus for this system.

4. Sequent Calculus

In this section we provide a sound and complete sequent calculus for $\mathsf{REL}_{\mathtt{fde}}$, that is to say, for the set of first-degree entailments valid in Epstein's Relatedness Logic, which incidentally coincides with the Fmla-Fmla fragment of $\mathsf{CL}_{\mathsf{VSP}}$.[15] Proof-systems for Epstein's logic as a whole have been given by R. Epstein himself in [16] in the form of an axiom system, by W. Carnielli in [7] in the form of a tableaux system, and by L. Fariñas del Cerro and V. Lugardon in [17] in the form of a Gentzen-style sequent calculus.

[15] I would like to thank Bruno Da Ré, Francesco Paoli and Shawn Standefer for very illuminating and insightful discussions on the content this section.

As regards $\mathsf{REL}_{\mathtt{fde}}$, a Hilbert-style axiomatization has been presented by F. Paoli in [30] , whereas a tableaux system is presented by him in [31]. Here, with the purpose of endowing our target logic with a Gentzen-style sequent calculus we will follow the ideas and techniques discussed by M. I. Corbalán and M. Coniglio in [12], and by R. French in [21], where calculi with linguistic restrictions are presented for 3-valued systems based on the **WK** algebra, as well as for subsystems thereof like the first-degree entailments of R. Angell's logic of Analytic Containment.

For this task, we will work with sequents of the form $\Gamma \succ \Delta$ defined as pairs $\langle \Gamma, \Delta \rangle$ where Γ and Δ are finite sets of formulae. In this context, sequents will receive a concrete interpretation, as we will establish that $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ is provable in the target calculus if and only if the first-degree entailment $\varphi_1 \wedge \cdots \wedge \varphi_n \to \psi_1 \vee \cdots \vee \psi_m$ is valid in REL—in other words, if and only if $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \psi_1 \vee \cdots \vee \psi_m$. It will be important to bear this in mind when conducting the soundness and completeness proofs.

The main idea behind the calculus that we introduce below is to have sequent rules (be it initial sequents, operational or structural rules), that are bound to linguistic restrictions. That is to say, rules that can be applied only if certain constraints regarding the parametric or active formulae are met. These restrictions guarantee that the rules preserve the satisfaction of the Variable-Sharing Principle or, put differently, that the rules guarantee that there is subject-matter overlap between premises and conclusions.

We will now proceed to present the set of rules that define our calculus $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$, later showing the adequacy of the formalism. Let us note, in passing, that for $\Theta \subseteq FOR(\mathcal{L})$, $Var(\Theta) = \bigcup\limits_{\theta \in \Theta} Var(\theta)$.

Definition 4.1. The calculus $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$ is constituted by the following rules:

Initial Sequents:

$$[Initial] \;\; \Gamma, p \succ p, \Delta$$

Structural Rules:

$$\frac{\Gamma, \varphi \succ \Delta \quad \Gamma \succ \varphi, \Delta}{\Gamma \succ \Delta} \; [Cut]^{\ddagger}$$

$$\ddagger: \text{ where } Var(\Gamma) \cap Var(\Delta) \neq \emptyset$$

Operational Rules:

$$\frac{\Gamma \succ \varphi, \Delta}{\Gamma, \neg\varphi \succ \Delta} \; [\neg L]^{\dagger} \qquad\qquad \frac{\Gamma, \varphi \succ \Delta}{\Gamma \succ \neg\varphi, \Delta} \; [\neg R]^{\ddagger}$$

$\dagger:$ where $Var(\Gamma, \varphi) \cap Var(\Delta) \neq \emptyset$ $\qquad$ $\ddagger:$ where $Var(\Delta, \varphi) \cap Var(\Gamma) \neq \emptyset$

$$\frac{\Gamma, \varphi \succ \Delta \quad \Gamma, \psi \succ \Delta}{\Gamma, \varphi \vee \psi \succ \Delta} \ [\vee L] \qquad \frac{\Gamma \succ \varphi, \psi, \Delta}{\Gamma \succ \varphi \vee \psi, \Delta} \ [\vee R]$$

$$\frac{\Gamma, \varphi, \psi \succ \Delta}{\Gamma, \varphi \wedge \psi \succ \Delta} \ [\wedge L] \qquad \frac{\Gamma \succ \varphi, \Delta \quad \Gamma \succ \psi, \Delta}{\Gamma \succ \varphi \wedge \psi, \Delta} \ [\wedge R]$$

Regarding the structural rules, it shall be noted that $[Initial]$ is a form of the structural rule of Identity or Reflexivity, with Left and Right Weakening "absorbed"—to some extent. Indeed, as we remark below, adopting these initial sequents allows for the left and right Weakening rules to be admissible in their unrestricted forms.

Lemma 4.2. *The following form of the Weakening rules are admissible in* $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$*:*

$$\frac{\Gamma \succ \Delta}{\Gamma, \varphi \succ \Delta} \ [KL] \qquad \frac{\Gamma \succ \Delta}{\Gamma \succ \varphi, \Delta} \ [KR]$$

Proof. Regarding $[KL]$, suppose we have a derivation of $\Gamma \succ \Delta$. We can turn this into a derivation of $\Gamma, \varphi \succ \Delta$ by adding φ to the left-hand side of each of the nodes of the derivation, as the uppermost node will still constitute a rightful instance of $[Initial]$. Similarly, regarding $[KR]$, suppose we have a derivation of $\Gamma \succ \Delta$. We can turn this into a derivation of $\Gamma \succ \varphi, \Delta$ by adding φ to the right-hand side of each of the nodes of the derivation, as the uppermost node will still constitute a rightful instance of $[Initial]$. □

The next result we discuss shows that every provable sequent of $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$ encodes a corresponding first-degree entailment that is valid in REL—or, what is the same, a valid inference of $\mathsf{REL}_{\mathtt{fde}}$. For the purpose of proving this, we will appeal to the characterization of said set of valid entailments in the paragraphs above.

Lemma 4.3. *All the rules of* $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$ *preserve* $\mathsf{REL}_{\mathtt{fde}}$*-validity. In other words, for each of the rules of the calculus, if the premise sequents are valid in* $\mathsf{REL}_{\mathtt{fde}}$*, so is the conclusion sequent of that rule.*

Proof. We show this by cases—focusing on the restricted rules and leaving the rest as exercises to the reader—assuming the premise sequents of a rule are valid in $\mathsf{REL}_{\mathtt{fde}}$, and later proving that its conclusion sequent is also valid in said logic. In all cases below, we will assume that Γ can be redescribed as $\gamma_1, \dots, \gamma_n$, and that Δ can be redescribed as $\delta_1, \dots, \delta_m$.

> $[\neg L]^{\dagger}$: Assume $\gamma_1 \wedge \cdots \wedge \gamma_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \varphi \vee \delta_1 \vee \cdots \vee \delta_m$, and $Var(\gamma_1 \wedge \cdots \wedge \gamma_n, \varphi) \cap Var(\delta_1 \vee \cdots \vee \delta_m) \neq \emptyset$. By simple reasoning this allows to establish that $\gamma_1 \wedge \cdots \wedge \gamma_n \vDash_{\mathsf{CL}} \varphi \vee \delta_1 \vee \cdots \vee \delta_m$ and, concomitantly, that $\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \neg\varphi \vDash_{\mathsf{CL}} \delta_1 \vee \cdots \vee \delta_m$. Furthermore, that $Var(\gamma_1 \wedge \cdots \wedge \gamma_n, \varphi) \cap Var(\delta_1 \vee \cdots \vee \delta_m) \neq \emptyset$ guarantees that $Var(\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \neg\varphi) \cap Var(\delta_1 \vee \cdots \vee \delta_m) \neq \emptyset$. Whence, $\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \neg\varphi \vDash_{\mathsf{REL}_{\mathtt{fde}}} \delta_1 \vee \cdots \vee \delta_m$.

$[\neg R]^{\ddagger}$: Assume $\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \varphi \vDash_{\mathsf{REL}_{\mathtt{fde}}} \delta_1 \vee \cdots \vee \delta_m$, and $Var(\delta_1 \vee \cdots \vee \delta_m, \varphi) \cap Var(\gamma_1 \wedge \cdots \wedge \gamma_n) \neq \emptyset$. By simple reasoning this allows to establish that $\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \varphi \vDash_{\mathsf{CL}} \delta_1 \vee \cdots \vee \delta_m$ and, concomitantly, that $\gamma_1 \wedge \cdots \wedge \gamma_n \vDash_{\mathsf{CL}} \neg\varphi \vee \delta_1 \vee \cdots \vee \delta_m$. Furthermore, that $Var(\delta_1 \vee \cdots \vee \delta_m, \varphi) \cap Var(\gamma_1 \wedge \cdots \wedge \gamma_n) \neq \emptyset$ guarantees that $Var(\gamma_1 \wedge \cdots \wedge \gamma_n) \cap Var(\neg\varphi \vee \delta_1 \vee \cdots \vee \delta_m) \neq \emptyset$. Whence, $\gamma_1 \wedge \cdots \wedge \gamma_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \neg\varphi \vee \delta_1 \vee \cdots \vee \delta_m$.

$[Cut]^{\ddagger}_{\ddagger}$: Assume that $\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \varphi \vDash_{\mathsf{REL}_{\mathtt{fde}}} \delta_1 \vee \cdots \vee \delta_m$, that $\gamma_1 \wedge \cdots \wedge \gamma_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \varphi \vee \delta_1 \vee \cdots \vee \delta_m$, and that $Var(\gamma_1 \wedge \cdots \wedge \gamma_n) \cap Var(\delta_1 \vee \cdots \vee \delta_m) \neq \emptyset$. By simple reasoning this allows to establish that $\gamma_1 \wedge \cdots \wedge \gamma_n \wedge \varphi \vDash_{\mathsf{CL}} \delta_1 \vee \cdots \vee \delta_m$ and $\gamma_1 \wedge \cdots \wedge \gamma_n \vDash_{\mathsf{CL}} \varphi \vee \delta_1 \vee \cdots \vee \delta_m$. In CL these two facts imply that $\gamma_1 \wedge \cdots \wedge \gamma_n \vDash_{\mathsf{CL}} \delta_1 \vee \cdots \vee \delta_m$, which together with the assumption that $Var(\gamma_1 \wedge \cdots \wedge \gamma_n) \cap Var(\delta_1 \vee \cdots \vee \delta_m) \neq \emptyset$ implies that $\gamma_1 \wedge \cdots \wedge \gamma_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \delta_1 \vee \cdots \vee \delta_m$.

The case of $[Initial]$, the $[\wedge]$ and $[\vee]$ rules are straightforward and thus are left to the reader as an exercise. □

Theorem 4.4 (Soundness)**.** *If the sequent $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ is provable in $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$, then $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \psi_1 \vee \cdots \vee \psi_m$.*

Proof. We know that the initial sequents are valid in $\mathsf{REL}_{\mathtt{fde}}$ and that all rules preserve $\mathsf{REL}_{\mathtt{fde}}$ validity. A straightforward induction on the height of the derivation shows (using Lemma 4.3 in the inductive step) that all provable sequents encode inferences that are valid in $\mathsf{REL}_{\mathtt{fde}}$. Thus, if $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ is provable in $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$, then the corresponding inference is valid in $\mathsf{REL}_{\mathtt{fde}}$—in other words, $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \psi_1 \vee \cdots \vee \psi_m$. □

Now, having discussed the soundness of our calculus, we will now turn to the more tiresome task of providing a completeness proof for $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$. For this purpose, we will show that whenever $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \psi_1 \vee \cdots \vee \psi_m$, there is a respective sequent that is provable in our Gentzen-style sequent calculus $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$.

Theorem 4.5 (Completeness)**.** *If $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \psi_1 \vee \cdots \vee \psi_m$, then the sequent $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ is provable in $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$.*

Proof. In the Appendix. □

Corollary 4.6 (Cut-elimination)**.** *The restricted version of the Cut rule is eliminable from $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$*

Proof. In the Appendix. □

5. Conclusions

In this article, we discussed $\mathsf{REL}_{\mathtt{fde}}$, the first-degree fragment of R. Epstein's Relatedness Logic—which is identical to the Fmla-Fmla fragment of $\mathsf{CL}_{\mathsf{VSP}}$. In this respect, we presented a p-matrix semantics and a Gentzen-style sequent calculus for this logic.

A couple of venues for further research are left open in this regard. First, our p-matrix semantics are based on the extension of the **WK** algebra with two universally idempotent elements that "mimic" the characteristic infectious element $\mathbf{e}$. It would be important to know whether it is possible to offer different semantics for $\mathsf{REL}_{\mathtt{fde}}$, which are not built on top of the **WK** algebra, but on top of a different algebraic structure. For example, extending a p-matrix for CL built on top of the **SK** algebra. This may as well be possible, but we should notice that an extension thereof like the one discussed above, with two mimicking values will not work. In fact, it is easy to check that the logic $\mathsf{ST}[\mathbf{o}_1^{\mathbf{n}}\mathbf{o}_2^{\mathbf{n}}]$ induced by the p-matrix $\langle \mathbf{SK}[\mathbf{o}_1^{\mathbf{n}}\mathbf{o}_2^{\mathbf{n}}], \{\mathbf{t}, \mathbf{o}_1^{\mathbf{n}}\}, \{\mathbf{t}, \mathbf{o}_1^{\mathbf{n}}, \mathbf{n}\}\rangle$ will invalidate the inference schema $\varphi \vee \psi \vDash_{\mathsf{ST}[\mathbf{o}_1^{\mathbf{n}}\mathbf{o}_2^{\mathbf{n}}]} \psi \vee \neg\psi$, which nevertheless satisfies the Variable-Sharing Principle.[16] Other routes may be available that make no appeal to mimicking values, starting from the **SK** algebra and obtaining a structure on top of which a proper p-matrix for $\mathsf{REL}_{\mathtt{fde}}$ can be built—these will definitely be interesting to explore.

Furthermore, it would be illuminating to learn, where L is a subclassical logic characterizable by a single finite matrix (like, e.g., S. Kleene's K_3 or G. Priest's LP) whether p-matrix semantics for the Fmla-Fmla fragment of $\mathsf{L}_{\mathsf{VSP}}$ can be obtained, in the spirit of the semantics for $\mathsf{REL}_{\mathtt{fde}}$. In other words, by expanding their characteristic regular matrix semantics to proper p-matrix semantics inducing systems having the same valid inferences, and later extending said conforming p-matrix semantics with two mimicking values of the appropriate kind. In this vein, a systematic and general way of obtaining proper p-matrix semantics for subclassical systems may be useful—and can be found in some recent developments by M. Fitting's works, like [18]. We hope to investigate these and other questions in the near future.

Appendix: Completeness and Cut-Elimination

Completeness Proof

We start by assuming that the sequent $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ is unprovable in $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$. We, then, consider two cases:

(i) $Var(\psi_1, \ldots, \psi_m) \cap Var(\varphi_1, \ldots, \varphi_n) = \emptyset$

(ii) $Var(\psi_1, \ldots, \psi_m) \cap Var(\varphi_1, \ldots, \varphi_n) \neq \emptyset$

[16] As a means of an example, let us assume that φ is p and ψ is q, and consider a $\mathsf{ST}[\mathbf{o}_1^{\mathbf{n}}\mathbf{o}_2^{\mathbf{n}}]$-valuation v such that $v(p) = \mathbf{t}$ and $v(q) = \mathbf{o}_2^{\mathbf{n}}$. This valuation is such that $v(p \vee q) = \mathbf{t}$, whereas $v(q \vee \neg q) = \mathbf{o}_2^{\mathbf{n}}$, witnessing $p \vee q \nvDash_{\mathsf{ST}[\mathbf{o}_1^{\mathbf{n}}\mathbf{o}_2^{\mathbf{n}}]} q \vee \neg q$, whence the invalidity of the aforementioned schema.

showing that in both cases we can design valuations that witness $\varphi_1 \wedge \cdots \wedge \varphi_n \nvDash_{\mathsf{REL_{fde}}} \psi_1 \vee \cdots \vee \psi_m$.

Case (i): if $Var(\psi_1, \ldots, \psi_m) \cap Var(\varphi_1, \ldots, \varphi_n) = \emptyset$ is the case, consider a $\mathsf{REL_{fde}}$-valuation v such that:

$$v(p) = \begin{cases} \mathbf{o}_1^{\mathbf{e}} & \text{if } p \in Var(\varphi_1, \ldots, \varphi_n) \\ \mathbf{o}_2^{\mathbf{e}} & \text{if } p \in Var(\psi_1, \ldots, \psi_m) \\ \mathbf{e} & \text{otherwise} \end{cases}$$

It is then straightforward to notice, as in Lemma 2.3, that all $\varphi_j \in \{\varphi_1, \ldots, \varphi_n\}$ will be such that $v(\varphi_j) = \mathbf{o}_1^{\mathbf{e}}$, whereas all $\psi_i \in \{\psi_1, \ldots, \psi_m\}$ will be such that $v(\psi_i) = \mathbf{o}_2^{\mathbf{e}}$. A quick inspections of the $\mathbf{WK}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}]$ algebra allows to notice that this renders $v(\varphi_1 \wedge \cdots \wedge \varphi_n) = \mathbf{o}_1^{\mathbf{e}}$, while at the same time giving $v(\psi_1 \vee \cdots \vee \psi_m) = \mathbf{o}_2^{\mathbf{e}}$. Whence, $\varphi_1 \wedge \cdots \wedge \varphi_n \nvDash_{\mathsf{REL_{fde}}} \psi_1 \vee \cdots \vee \psi_m$.

Case (ii): if $Var(\psi_1, \ldots, \psi_m) \cap Var(\varphi_1, \ldots, \varphi_n) \neq \emptyset$ is the case, in order to show that $\varphi_1 \wedge \cdots \wedge \varphi_n \nvDash_{\mathsf{REL_{fde}}} \psi_1 \vee \cdots \vee \psi_m$ we will apply a slight modification of the method of reduction trees as explored, e.g., in [41] by G. Takeuti and in [35] by D. Ripley.

The idea is to start with a sequent that we assume to be unprovable later extending it in a finite series of steps with the help of *reduction rules* that will finally render a *reduction tree*. Thus, we start with an unprovable sequent and build a tree above it, with each node consisting of a sequent that results from an application of the reduction rules to the sequent below it. As we extend the tree, we will sometimes find that the tip of a branch is an instance of one of [*Initial*]—in such a case we will consider this branch *closed* and will stop performing reductions on it. Contrary to that, if a branch is not closed after applying all the possible reduction rules, we will consider this branch *open*.

Below, we detail the rules that we apply to the sequents at the top of each branch of the tree, at each stage of the reduction process. Let us note, in passing, that this technique requires an enumeration of the formulae of our language, and that when the same sequent appears at the tip of some branch of more than one tree, they are simultaneously reduced.

- To reduce a sequent of the form $\Gamma, \varphi \wedge \psi \succ \Delta$, extend the branch with the sequent $\Gamma, \varphi, \psi \succ \Delta$.
- To reduce a sequent of the form $\Gamma \succ \varphi \wedge \psi, \Delta$, extend the branch by splitting in two. To one new branch, add the sequent $\Gamma \succ \varphi, \Delta$; to the other, add the sequent $\Gamma \succ \psi, \Delta$
- To reduce a sequent of the form $\Gamma \succ \varphi \vee \psi, \Delta$, extend the branch with the sequent $\Gamma \succ \varphi, \psi, \Delta$.
- To reduce a sequent of the form $\Gamma, \varphi \vee \psi \succ \Delta$, extend the branch by splitting in two. To one new branch, add the sequent $\Gamma, \varphi \succ \Delta$; to the other, add the sequent $\Gamma, \psi \succ \Delta$.
- To reduce a sequent of the form $\Gamma, \neg\varphi \succ \Delta$, consider whether it is the case that $Var(\Gamma, \varphi) \cap Var(\Delta) \neq \emptyset$. If this is the case, extend the branch with the sequent

$\Gamma, \neg\varphi \succ \varphi, \Delta$; otherwise, do nothing and proceed to reduce the next sequent, if there is one.

- To reduce a sequent of the form $\Gamma \succ \neg\varphi, \Delta$, consider whether it is the case that $Var(\Delta, \varphi) \cap Var(\Gamma) \neq \emptyset$. If this is the case, extend the branch with the sequent $\Gamma, \varphi \succ \neg\varphi, \Delta$; otherwise, do nothing and proceed to reduce the next sequent, if there is one.

Suppose we start with a sequent of the form $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ where $Var(\psi_1, \ldots, \psi_m) \cap Var(\varphi_1, \ldots, \varphi_n) \neq \emptyset$ and follow this process as many times as necessary for there to be no more legal applications of the reduction rules. Then, either all branches of the tree will be closed (whence, we have a proof of the sequent that was assumed to be unprovable, contradicting our initial hypothesis), or some branch will be open. Suppose the latter is the case.

The next step in our proof is to show that it is possible to find a $\mathsf{REL}_{\mathtt{fde}}$-valuation that witnesses $\varphi_1 \wedge \cdots \wedge \varphi_n \nvDash_{\mathsf{REL}_{\mathtt{fde}}} \psi_1 \vee \cdots \vee \psi_m$. For this purpose, let us temporarily relabel the sequents in the open branch as $\Gamma_1 \succ \Delta_1, \ldots, \Gamma_k \succ \Delta_k$, letting $\Gamma_1 \succ \Delta_1$ be $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ and letting $\Gamma_k \succ \Delta_k$ be the sequent at the tip of the open branch. Furthermore, let the sequent $\Gamma \succ \Delta$—where $\Gamma = \cup\{\Gamma_i \mid 1 \leq i \leq k\}$ and $\Delta = \cup\{\Delta_i \mid 1 \leq i \leq k\}$—be the sequent that "collects" all the sequents appearing in the nodes of the open branch.

Before going into the final stage of this proof, lets us highlight a number of facts regarding our newly defined Γ and Δ. These are: (i) for all propositional variables p, $p \notin \Gamma \cap \Delta$; (ii) there are $\Gamma', \Delta' \subseteq Var$ such that $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$; (iii) for all formulae φ, if $\neg\varphi \in \Gamma$ then $\varphi \in \Delta$; (iv) for all formulae φ, if $\neg\varphi \in \Delta$ then $\varphi \in \Gamma$. All these can be derived from the definition of Γ and Δ, the fact that none of the $\Gamma_i \succ \Delta_i$ ($1 \leq i \leq k$) is an instance of [*Initial*], and the fact that the reduction rules preserve the satisfaction of the Variable-Sharing Principle.

We prove here remark (iii), noting that the proof for remark (iv) is perfectly analogous.[17] Thus, suppose $\neg\varphi \in \Gamma$. By construction of Γ, either $\neg\varphi \in \Gamma_1$ or $\neg\varphi \in \Gamma_j$, for $j > 1$. We now reason focusing on when an appearance of $\neg\varphi$ is being reduced.

- Suppose it is being reduced in the sequent $\Gamma_1 \succ \Delta_1$, and that $\neg\varphi \in \Gamma_1$. Then, in this case, the restriction to reduce $\neg\varphi$ amounts to $Var(\Gamma_1 \setminus \{\neg\varphi\}, \varphi) \cap Var(\Delta_1) \neq \emptyset$. However, $Var(\Gamma_1 \setminus \{\neg\varphi\}, \varphi)$ is just $Var(\Gamma_1)$. Whence, given we know by hypothesis that $Var(\Gamma_1) \cap Var(\Delta_1) \neq \emptyset$, this restriction is satisfied and it is guaranteed by the reduction rules that $\varphi \in \Delta_{1+1}$. Therefore, by the construction process above $\varphi \in \Delta$. (Notice that this would not be guaranteed if it were not the case that, by hypothesis, $Var(\Gamma_1) \cap Var(\Delta_1) \neq \emptyset$)
- Suppose, alternatively, that it is being reduced in the sequent $\Gamma_j \succ \Delta_j$, for $j > 1$, and that $\neg\varphi \in \Gamma_j$. Then, in this case, the restriction to reduce $\neg\varphi$ amounts to $Var(\Gamma_j \setminus \{\neg\varphi\}, \varphi) \cap Var(\Delta_j) \neq \emptyset$. Recall that, by construction, $\Gamma_1 \subseteq \Gamma_j$ and $\Delta_1 \subseteq \Delta_j$. There are, now, two cases: either $\varphi \notin \Gamma_1$, or $\varphi \in \Gamma_1$. If the former, then by the above $Var(\Gamma_j \setminus \{\neg\varphi\}, \varphi) \cap Var(\Delta_j) \neq \emptyset$. If the latter, then once again $Var(\Gamma_j \setminus \{\neg\varphi\}, \varphi)$ is just $Var(\Gamma_j)$, and then by the above

[17] In turn, we stress here that remark (i) is true because otherwise $\Gamma \succ \Delta$ would be an instance of [*Initial*], and remark (ii) is true because otherwise $Var(\Gamma) \cap Var(\Delta) = \emptyset$, which would contradict the assumptions holding at this point of the proof.

$Var(\Gamma_j \setminus \{\neg\varphi\}, \varphi) \cap Var(\Delta_j) \neq \emptyset$. Therefore, the restriction is satisfied and it is guaranteed by the reduction rules that $\varphi \in \Delta_{j+1}$. Finally, by the construction process above $\varphi \in \Delta$. (Notice that this would not be guaranteed if it were not the case that, by hypothesis, $Var(\Gamma_1) \cap Var(\Delta_1) \neq \emptyset$)

Now, for the final stage of the proof, take the aforementioned sequent $\Gamma \succ \Delta$ and consider the $\mathsf{REL}_{\mathtt{fde}}$-valuation v such that:

$$v(p) = \begin{cases} \mathbf{t} & \text{if } p \in \Gamma \text{ or } \neg p \in \Delta \\ \mathbf{f} & \text{otherwise} \end{cases}$$

We now prove by induction on the complexity of φ that v is a $\mathsf{REL}_{\mathtt{fde}}$-valuation such that $v(\varphi) = \mathbf{t}$ if and only if $\varphi \in \Gamma$ and $v(\varphi) = \mathbf{f}$ if and only if $\varphi \in \Delta$.

Base case:

- $\varphi = p$. If $p \in \Gamma$, $v(p) = \mathbf{t}$ by definition of v. Otherwise, if $p \in \Delta$, for example, $v(p) = \mathbf{f}$ by definition. Notice that, by the remarks above, we know that either $p \notin \Gamma$, or $p \notin \Delta$—granting the well-definedness of v.

Inductive step: we assume that for all formulae of lesser complexity than φ, the hypothesis holds and show that it also holds for φ.

- $\varphi = \neg\psi$. If $\neg\psi \in \Gamma$, we know that $\psi \in \Delta$ by the remarks above. By the IH we know that $v(\psi) = \mathbf{f}$, whence $v(\neg\psi) = \mathbf{t}$. Otherwise, if $\neg\psi \in \Delta$, we know that $\psi \in \Gamma$ by the remarks above. By the IH we know that $v(\psi) = \mathbf{t}$, whence $v(\neg\psi) = \mathbf{f}$.
- $\varphi = \psi \wedge \chi$. $\psi \wedge \chi \in \Gamma$ we know that $\psi, \chi \in \Gamma$. By the IH we know that $v(\psi) = v(\chi) = \mathbf{t}$. Thus, $v(\psi \wedge \chi) = \mathbf{t}$. Otherwise, if $\psi \wedge \chi \in \Delta$, then either $\psi \in \Delta$ or $\chi \in \Delta$. By the IH we know that either $v(\psi) = \mathbf{f}$ or $v(\chi) = \mathbf{f}$. Whence, $v(\psi \wedge \chi) = \mathbf{f}$.
- $\varphi = \psi \vee \chi$. If $\psi \vee \chi \in \Gamma$ we know that either $\psi \in \Gamma$ or $\chi \in \Gamma$. By the IH we know that either $v(\psi) = \mathbf{t}$ or $v(\chi) = \mathbf{t}$. Whence, $v(\psi \vee \chi) = \mathbf{t}$. Otherwise, if $\psi \vee \chi \in \Delta$, we know that $\psi, \chi \in \Delta$. By the IH this implies that $v(\psi) = v(\chi) = \mathbf{f}$. Whence, $v(\psi \vee \chi) = \mathbf{f}$.

Given this, and since $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Gamma$ and $\{\psi_1, \ldots, \psi_m\} \subseteq \Delta$, we know that for all i such that $1 \leq i \leq n$, $v(\varphi_i) = \mathbf{t}$, and for all j such that $1 \leq j \leq m$, $v(\psi_j) = \mathbf{f}$. Whence, by looking at the Boolean reduct of the $\mathbf{WK}[\mathbf{o}_1^{\mathbf{e}}\mathbf{o}_2^{\mathbf{e}}]$ algebra it is easy to notice that $v(\varphi_1 \wedge \cdots \wedge \varphi_n) = \mathbf{t}$ and $v(\psi_1 \vee \cdots \vee \psi_m) = \mathbf{f}$. Therefore, v is a $\mathsf{REL}_{\mathtt{fde}}$-valuation witnessing the fact that $\varphi_1 \wedge \cdots \wedge \varphi_n \nvDash_{\mathsf{REL}_{\mathtt{fde}}} \psi_1 \vee \cdots \vee \psi_m$.

This establishes that if $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \psi_1 \vee \cdots \vee \psi_m$, then a sequent of the form $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ is provable in $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$.[18]

[18] Notice that this proof does not give full CL, because when considering whether we are in Case(i) or Case (ii), some classically valid inferences are discarded right away—namely, those inferences which do not satisfy the Variable-Sharing Principle.

Cut-Elimination Proof

By Theorem 4.4, if there is a proof of the sequent $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ in $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$, then $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \psi_1 \vee \cdots \vee \psi_m$. Furthermore, by Theorem 4.5, if $\varphi_1 \wedge \cdots \wedge \varphi_n \vDash_{\mathsf{REL}_{\mathtt{fde}}} \psi_1 \vee \cdots \vee \psi_m$, then applying the method of reduction trees gives a proof of the sequent $\varphi_1, \ldots, \varphi_n \succ \psi_1, \ldots, \psi_m$ in $\mathcal{G}_{\mathsf{REL}_{\mathtt{fde}}}$. However, notice that this proof does not feature any instance of the restricted version of the Cut rule and is, thus, a Cut-free proof. Whence, for any sequent provable in $\mathcal{G}_{\mathsf{PAI}_{\mathtt{fde}}}$, there is a proof of it that does not use the restricted version of the Cut rule.

References

[1] A. Anderson and N. Belnap. *Entailment: The Logic of Relevance and Neccessity*, volume 1. Princeton University Press, Princeton, 1975.

[2] G. Badia. Variable sharing in substructural logics: an algebraic characterization. *Bulletin of the Section of Logic*, 47(2):107–115, 2018.

[3] E. Barrio, F. Pailos, and D. Szmuc. A Hierarchy of Classical and Paraconsistent Logics. *Journal of Philosophical Logic*, 1(49):93–120, 2020.

[4] N. Belnap. How a Computer Should Think. In G. Ryle, editor, *Contemporary Aspects of Philosophy*, pages 30–55. Oriel Press, Stocksfield, 1977.

[5] F. Berto. Simple hyperintensional belief revision. *Erkenntnis*, 84(3):559–575, 2019.

[6] D. Bochvar. On a Three-Valued Calculus and its Application in the Analysis of the Paradoxes of the Extended Functional Calculus. *Matematicheskii Sbornik*, 4:287–308, 1938.

[7] W. Carnielli. Methods of proof for relatedness logic and dependence logics. *Reports on Mathematical Logic*, 21:35–46, 1987.

[8] P. Cobreros, P. Égré, D. Ripley, and R. van Rooij. Tolerant, classical, strict. *Journal of Philosophical Logic*, 41(2):347–385, 2012.

[9] P. Cobreros, P. Égré, D. Ripley, and R. van Rooij. Tolerance and mixed consequence in the s'valuationist setting. *Studia Logica*, 100(4):855–877, 2012.

[10] P. Cobreros, P. Égré, D. Ripley, and R. van Rooij. Priest's motorbike and tolerant identity. In R. Ciuni, H. Wansing, and C. Wilkommen, editors, *Recent Trends in Philosophical Logic*, pages 75–85. Springer, Cham, 2014.

[11] P. Cobreros, P. Égré, D. Ripley, and R. van Rooij. Reaching transparent truth. *Mind*, 122(488):841–866, 2014.

[12] M. Coniglio and M. I. Corbalán. Sequent calculi for the classical fragment of Bochvar and Halldén's Nonsense Logics. In *Proceedings of the 7th Workshop on Logical and Semantic Frameworks with Applications (LSFA)*, pages 125–136, 2012.

[13] F. Correia. Weak necessity on weak Kleene matrices. In F. Wolter, H. Wansing, M. de Rijke, and M. Zakharyashev, editors, *Advances in Modal Logic*, pages 73–90. World Scientific, River Edge, NJ, 2002.

[14] B. Dicher and F. Paoli. ST, LP, and tolerant metainferences. In C. Başkent and T. M. Ferguson, editors, *Graham Priest on Dialetheism and Paraconsistency*, pages 383–407. Springer, Dordrecht, 2020.

[15] M. Dunn. Intuitive semantics for first-degree entailments and 'coupled trees'. *Philosophical Studies*, 29(3):149–168, 1976.

[16] R. Epstein. *The Semantic Foundations of Logic*, volume I: Propositional Logics. Oxford University Press, New York, 2nd edition, 1995.

[17] L. Fariñas del Cerro and V. Lugardon. Sequents for dependence logics. *Logique et Analyse*, 133-134:57–71, 1991.

[18] M. Fitting. The Strict/Tolerant Idea and Bilattices. In A. Zamansky and O. Arieli, editors, *Outstanding Contributions to Logic volume dedicated to Arnon Avron*. Spriger. Forthcoming.

[19] J. M. Font. *Abstract Algebraic Logic*. College Publications, London, 2016.

[20] S. Frankowski. Formalization of a plausible inference. *Bulletin of the Section of Logic*, 33(1):41–52, 2004.

[21] R. French. A Simple Sequent Calculus for Angell's Logic of Analytic Containment. *Studia Logica*, 105(5):971–994, 2017.

[22] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*. Elsevier, San Diego, CA, USA, 2007.

[23] J.-Y. Girard. *Proof theory and logical complexity*. Bibliopolis, Napoli, 1987.

[24] S. Halldén. *The Logic of Nonsense*. Uppsala Universitets Arsskrift, Uppsala, 1949.

[25] L. Humberstone. *The Connectives*. MIT Press, Cambridge, MA, 2011.

[26] S. C. Kleene. *Introduction to metamathematics*. North-Holland, Amsterdam, 1952.

[27] S. Kripke. Outline of a theory of truth. *Journal of Philosophy*, 72(19): 690–716, 1975.

[28] E. Mares. Relevance Logic. In E. N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, spring 2014 edition, 2014.

[29] E. J. Nelson. Intensional relations. *Mind*, 39(156):440–453, 1930.

[30] F. Paoli. Semantics for first-degree relatedness logic. *Reports on Mathematical Logic*, 27:81–94, 1993.

[31] F. Paoli. Tautological entailments and their rivals. In J. Béziau, W. Carnielli, and D. Gabbay, editors, *Handbook of Paraconsistency*, pages 153–175. College Publications, London, 2007.

[32] F. Paoli and M. P. Baldi. Proof Theory of Paraconsistent Weak Kleene Logic. *Studia Logica*, 108(4):779–802, 2020.

[33] W. T. Parry. Ein axiomensystem für eine neue art von implikation (analytische implikation). *Ergebnisse eines mathematischen Kolloquiums*, 4: 5–6, 1933.

[34] D. Ripley. Conservatively extending classical logic with transparent truth. *Review of Symbolic Logic*, 5(02):354–378, 2012.

[35] D. Ripley. Paradoxes and failures of cut. *Australasian Journal of Philosophy*, 91(1):139–164, 2013.

[36] G. Robles and J. M. Méndez. A Class of Simpler Logical Matrices for the Variable-Sharing Property. *Logic and Logical Philosophy*, 20(3):241–249, 2011.

[37] G. Robles and J. M. Méndez. A general characterization of the variable-sharing property by means of logical matrices. *Notre Dame Journal of Formal Logic*, 53(2):223–244, 2012.

[38] G. Robles, J. M. Méndez, and F. Salto. A Modal Restriction of R-Mingle with the Variable-Sharing Property. *Logic and Logical Philosophy*, 19(4): 341–351, 2010.

[39] C. Scambler. Classical logic and the strict tolerant hierarchy. *Journal of Philosophical Logic*, 2(49):351–370, 2020.

[40] K. Schütte. *Proof Theory*. Springer, Berlin, 1977.

[41] G. Takeuti. *Proof Theory*. Elsevier, Amsterdam, 2nd edition, 1987.

[42] R. Wójcicki. Some Remarks on the Consequence Operation in Sentential Logics. *Fundamenta Mathematicae*, 68(1):269–279, 1970.