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THE AGM THEORY AND INCONSISTENT BELIEF CHANGE

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Abstract
The problem of how to accommodate inconsistencies has attracted quite a number of researchers, in particular, in the area of database theory. The problem is also of concern in the study of belief change. For inconsistent beliefs are ubiquitous. However, comparatively little work has been devoted to discussing the problem in the literature of belief change. In this paper, I examine how adequate the AGM theory is as a logical framework for belief change involving inconsistencies. The technique is to apply to Grove's sphere system, a semantical representation of the AGM theory, logics that do not infer everything from contradictory premises, viz., paraconsistent logics. I use three paraconsistent logics and discuss three sphere systems that are based on them. I then examine the completeness of the postulates of the AGM theory with respect to the systems. At the end, I discuss some philosophical implications of the examination.

1. *Inconsistent Beliefs*

When Alchourrón, Gärdenfors and Makinson (1985) (hereafter AGM) proposed their theory of belief change, known as the AGM theory, they defined a doxastic state (state of beliefs) as a set and the dynamics of beliefs as

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changes in the set. For AGM, not all sets are belief sets. Belief sets are special kind of sets that satisfy the following two criteria (Gärdenfors (1988) p. 22):

1. The set must be consistent.
2. The set must be logically closed.

AGM held that a belief set containing inconsistent beliefs could not be held by a rational agent. When Gärdenfors (1988) re-presented the AGM theory in his canonical text (once upon a time at least), it became apparent that the AGM notion of rationality was mainly driven by consistency.

This consistency driven rationality of beliefs is mainly due to the underlying logic that the AGM theory employs, viz., classical logic. Classical logic seems to be useful as AGM show. Yet, since a belief set is closed under logical consequence, *any* belief is contained in the belief set once a contradictory belief such as $A \wedge \neg A$ is involved. For the logical principle *ex contradictione quodlibet* (ECQ), that anything follows from inconsistent premises ($A, \neg A \models B$ for any A and B) is valid in classical logic. Consequently, belief sets containing contradictions are trivial, i.e., containing every sentence of the language. Hence anyone who holds inconsistent beliefs is thought of as being irrational. For his/her beliefs are trivial.

This classical view of rationality has been criticised, for example, by Priest (1987).[1] The main objection, in the context of beliefs, is notoriously that beliefs are often inconsistent yet may be non-trivial. This does not mean that we often believe in 'true contradictions' or have beliefs which are both true and false. But we often catch ourselves having a belief which is inconsistent with another belief. (See the example given in Lewis (1982).) Moreover, inconsistent beliefs may even be held rationally. For example, consider the paradox of the preface. People, after thorough research, write a book in which they claim in the preface $A_1, ..., A_n$ with rational reasons to believe them. But they are aware that no books contain no falsehoods. So they believe $\neg(A_1 \wedge \cdots \wedge A_n)$ too. Clearly, the beliefs are inconsistent. Yet they believe them rationally and the beliefs are non-trivial.

Whether or not one can rationally hold inconsistent beliefs, the imposition of consistency criterion on beliefs is too strong a requirement for a logical framework of belief change. There *may* be some empirical reasons for thinking that our beliefs are consistent. Our beliefs *may* indeed turn out to be consistent. Yet this is a matter of empirical investigation. There are no *a priori* or logical reasons to think that our beliefs cannot be inconsistent. The only logical reason seems to be given once we collapse beliefs into knowledge. There may be logical reasons for the consistency of knowledge. Yet it is not clear how to motivate collapsing beliefs into knowledge. Consistency

[1] See also Priest (2001) and Tanaka (1998) in the context of belief change.

is thus a too strong requirement for beliefs in a logical framework of belief change, whether we subscribe to classical logic or not.

Logical closure, on the other hand, seems to be a necessary feature of a *logical* framework of belief change. Against this criterion, Fuhrmann (1991), Hansson (1989), Hansson (1992) and Nebel (1992), for example, have proposed the approach which is based on belief *bases* that are not closed under logical consequence. Their approach is tantamount to separating explicit beliefs from implicit beliefs which are derived from the explicit beliefs, or separating relevant beliefs from irrelevant beliefs. Based on this approach, several formal techniques have been developed in recent years to deal with inconsistent beliefs; for example, Chopra and Parikh (2000), Hansson and Wassermann (2002) and Wassermann (2003). These techniques allow implicit or irrelevant inconsistencies to arise in the belief system as a whole by 'localising' explicit or relevant consistent belief bases.

There are two things that should be said about the above approach. Firstly, even though the approach may have some advantages, if we give up logical closure, it is not clear that our attempt will provide a logical framework for belief change. It is true that the belief base approach does not give up logical closure all together; implicit beliefs, or all beliefs (whether relevant or irrelevant), are represented by logically closed sets. Yet the fundamental units of the framework are logic free: as Rott (2001) points out, the approach invokes extra-logical factors. Thus, the question arises, in what sense are we providing a logical framework?

Secondly, the belief base approach itself is not a way to handle inconsistent beliefs in a sensible manner. The above mentioned techniques to handle inconsistencies are based on the assumption that consistent belief bases can be isolated. However, this seems too strong an assumption for a logical framework that can be said to accommodate inconsistent beliefs in a sensible manner. To see this, consider the Liar sentence: This sentence is false. Though there is a long history of debates concerning the truth value of the sentence, if we admit that the sentence is both true and false, a belief about this sentence itself is inconsistent. Hence, if we decide to revise our belief about the Liar sentence, we have to deal with one inconsistent belief. It is not clear how to localise consistency in this case. Moreover, even if we assume that belief bases are consistent, at some stage we have to consider inferring beliefs from contradictory beliefs in order to sensibly accommodate inconsistent beliefs. In other words, in order to present a logical model of belief change that involves inconsistencies, we have to develop a technique to allow such beliefs in a logically closed set.

For these reasons, I develop in this paper an approach to accommodate inconsistent beliefs in a sensible manner that has been overlooked by the advocates of the belief base approach. Specifically, I represent beliefs as a logically closed set without separating two kinds of beliefs. However, I do not impose the criterion that the set must be consistent. Thus, I am concerned with a belief set defined as follows:

Definition 1: (Belief Set) *Let K be a set of sentences. Then K is a belief set iff K is logically closed, i.e., $K = Cn(K)$.*

In order to accommodate inconsistent beliefs, I employ *paraconsistent logics* that do not validate ECQ ($A, \neg A \models B$ for any A and B). As Rott (2001) notes, this paraconsistent approach is not the same as a belief base approach. The former involves only logical factors; the latter invokes extra-logical factors. Hence the paraconsistent approach to belief change developed in this paper provides a *logical* framework of belief change. And to provide a logical framework is the point of the logical study of belief change.

2. *The AGM Theory and Inconsistent Beliefs*

Against the consistency-driven notion of rationality adopted by AGM, Priest (2001) proposes an alternative framework for belief change. His approach is motivated by the thought that the notion of rationality involves more than consistency. Priest proposes a framework that considers multiple criteria for rationality such as a low degree of *ad-hoc*ness, fruitfulness, explanatory power, unifying power as well as consistency. As a result, accepting inconsistency may turn out to score a high degree of rationality.

Perhaps, proposing an alternative framework to the AGM theory is the right path to take in formalising a theory of belief change involving inconsistent beliefs. However, the question of how incompatible the AGM theory is with the paraconsistent approach remains to be answered. As a step towards answering this question, I examine the AGM theory in the presence of inconsistent beliefs. The technique is to appeal to a semantic representation of the AGM theory, viz., Grove's sphere system, which is known to be sound and complete with respect to the AGM postulates. Instead of using Grove's original system which is based on classical logic, I construct three sphere systems based on three paraconsistent logics. I present semantics for dealing with inconsistent belief change and examine which of the AGM postulates fail to be complete with respect to the semantics. That is, I examine how inadequate the formal AGM theory is if a belief set is allowed to be inconsistent.[2]

[2] Soundness does not concern us here since the adequacy of the paraconsistent sphere systems with respect to the AGM postulates is not what is examined in this paper.

3. *Grove's Systems of Spheres*

In presenting a semantic model of the AGM theory, Grove (1988) introduces *systems of spheres* that are similar to the sphere semantics for counterfactuals proposed by Lewis (1973). He shows that the systems of spheres are directly related to the belief change postulates for the three operations proposed by AGM:

Expansion

(K^+1) K^+_A is a belief set.
(K^+2) $A \in K^+_A$.
(K^+3) $K \subseteq K^+_A$.
(K^+4) If $A \in K$ then $K^+_A = K$.
(K^+5) If $K \subseteq H$ then $K^+_A \subseteq H^+_A$.
(K^+6) For all belief sets K and all sentences A, K^+_A is the smallest belief set that satisfies (K^+2)–(K^+5).

Contraction

(K^-1) K^-_A is a belief set.
(K^-2) $K^-_A \subseteq K$.
(K^-3) If $A \notin K$ then $K^-_A = K$.
(K^-4) If $\nvdash A$ then $A \notin K^-_A$.
(K^-5) If $A \in K$ then $K \subseteq (K^-_A)^+_A$.
(K^-6) If $\vdash A \leftrightarrow B$ then $K^-_A = K^-_B$.
(K^-7) $K^-_A \cap K^-_B \subseteq K^-_{A \wedge B}$.
(K^-8) If $A \notin K^-_{A\wedge B}$ then $K^-_{A\wedge B} \subseteq K^-_A$.

Revision

(K^*1) K^*_A is a belief set.
(K^*2) $A \in K^*_A$.
(K^*3) $K^*_A \subseteq K^+_A$.
(K^*4) If $\neg A \notin K$ then $K^+_A \subseteq K^*_A$.
(K^*5) $K^*_{\neg A}$ is trivial iff $\vdash A$.
(K^*6) If $\vdash A \leftrightarrow B$ then $K^*_A = K^*_B$.
(K^*7) $K^*_{A\wedge B} \subseteq (K^*_A)^+_B$.
(K^*8) If $\neg B \notin K^*_A$ then $(K^*_A)^+_B \subseteq K^*_{A\wedge B}$.

The first postulates of each operation, i.e., (K^+1), (K^-1) and (K^*1), ensure that $+$, $-$ and $*$ are functions that map a pair of a belief set and a sentence to a new belief set that satisfies the criteria for a belief set. Though they are important, these postulates are of a little interest to us in this paper. These first postulates will not be considered in the following sections.

To make this paper self-contained, I rehearse Grove's systems of spheres and their relationship to the AGM theory. Let M_L be a set of all maximal *consistent* sets of sentences of a language L. Then any belief set K can be represented by a subset $|K|$ of M_L. $|K|$ consists of all maximal consistent sets in which all the sentences in K are contained. Because of isomorphism between maximal consistent sets and models, $|K|$ can be thought of as a set of models of $|K|$, and formally defined as:

$$|K| = \{m \in M_L : m \models K\}.$$

The set of models of a sentence A, $|A|$, can be defined in a similar fashion.

A system of spheres, $\mathcal{S}$, centred on some subset X of M_L, is a collection of subsets, called *spheres*, of M_L. In our context, we only consider the cases where X is $|K|$. Then $\mathcal{S}$ has to satisfy the following conditions:

(S1) $\mathcal{S}$ is totally ordered by $\subseteq$; that is, if $S, S' \in \mathcal{S}$, then $S \subseteq S'$ or $S' \subseteq S$.

(S2) $|K|$ is the $\subseteq$-minimum of $\mathcal{S}$; that is, $|K| \in \mathcal{S}$ and if $S \in \mathcal{S}$ then $|K| \subseteq S$.

(S3) M_L is in $\mathcal{S}$ (and so the largest element of $\mathcal{S}$).

(S4) If A is a sentence and there is any sphere in $\mathcal{S}$ intersecting $|A|$, then there is a smallest sphere in $\mathcal{S}$ intersecting $|A|$.

Intuitively, a system of spheres centred on $|K|$ is a series of sets of possible worlds which, starting from $|K|$, covers every possible world or every way in which the belief set K could be. And each set, represented by a sphere, gives a measure of closeness to $|K|$ which is ordered by $\subseteq$.

Expansion K^+_A, contraction K^-_A and revision K^*_A of K by A can then be defined by the systems of spheres. Let t be a one place function. Then $t(S)$ defines the set of all formulas in all the elements in $S \subseteq M_L$. So $t(S)$ is the theory $\bigcap S$. Precisely, $t(S)$ is defined as follows:

$$t(S) = \{A : m \in S \Rightarrow m \models A\}.$$

Then $t(S)$ will be a belief set. Grove uses the function t to define the belief change functions that are sound and complete with respect to the AGM postulates.

Grove's proof for soundness contains an error, as is reported in Priest, Surendonk and Tanaka (1996) and Priest and Tanaka (1997). There are several ways of repairing it. The most general way is to require that any sphere, $S \subseteq M_L$, is an elementary class, which is defined as follows:

Definition 2: (Elementary Class) *S is an elementary class iff there is a sentence A such that $S = |A|$.*

This means that every sphere including $|K|$ for any K represents a finitely axiomatisable theory. Note that elementary classes have the following properties:

Lemma 1: *For any elementary classes S_1 and S_2,*

(EC1) *$S_1 \cap S_2$ is an elementary class.*
(EC2) *$S_1 \cup S_2$ is an elementary class.*[3]

Before presenting the relationship to the AGM theory, we require the following definitions. For any sentence A, if $|A|$ intersects any sphere (i.e., any elementary class) in $\mathcal{S}$, the condition (S4) ensures that there will be some spheres in $\mathcal{S}$ which intersects $|A|$, yet there is exactly one sphere $S(A)$ which is smaller than any other such sphere. If $|A|$ does not intersect any spheres, which by (S3) occurs only if $|A| = \phi$, then $S(A)$ is taken to be M_L.

By using $S(A)$, we can define the closest worlds in M_L to $|K|$ that play a crucial role in the systems of spheres. Such worlds are represented by the set $C(A)$, defined as:

$$C(A) = |A| \cap S(A).$$

It is worth noting that, since $|A|$ and $S(A)$ are elementary classes, $C(A)$ is also an elementary class, as can easily be shown.

3.1. ... *and Expansion*

We are now in a position to define the belief change functions in terms of the systems of spheres. For expansion, K^+_A is defined as:

Definition 3: $K^+_A = t(|K| \cap |A|)$.

[3] Proofs for all the theorems and the lemmas in this paper are included in the Appendix.

Then the following theorem shows that K^+_A given by the above definition satisfies the AGM expansion postulates.

Theorem 1: *Let $\mathcal{S}$ be any system of spheres in M_L centred on $|K|$. If an expansion function K^+_A is defined as in Definition 3 then the postulates (K^+2) – (K^+6) are satisfied.*

3.2. *... and Contraction*

For contraction, K^-_A is defined as follows:

Definition 4: $K^-_A = t(|K| \cup C(\neg A))$.

Theorem 2: *Let $\mathcal{S}$ be any system of spheres in M_L centred on $|K|$. If a contraction function K^-_A is defined as in Definition 4 then the postulates (K^-2) – (K^-8) are satisfied.*

3.3. *... and Revision*

Now for revision, K^*_A is defined as follows:

Definition 5: $K^*_A = t(C(A)) = t(|A| \cap S(A))$.

Theorem 3: *Let $\mathcal{S}$ be any system of spheres in M_L centred on $|K|$. If a revision function K^*_A is defined as in Definition 5 then the postulates (K^*2) – (K^*8) are satisfied.*

4. *Relevant Systems*

The development of relevant logics was pioneered by Anderson and Belnap (1975). They proposed a number of systems in proof theoretic form. There are several ways to introduce relevant logics. I choose one that seems to be simple to grasp due to the simplified semantics provided by Priest and Sylvan (1992) and Restall (1993). An interpretation for the language is a 5-tuple $\langle g, W, R, *, \nu\rangle$, where W is a set of worlds; $g \in W$ (the base world); R is a ternary relation on W; $*$ is a one place function from W to W; and ν is an evaluation function that assigns to each pair of world, w, and propositional parameter, p, a truth value $\nu_w(p) \in \{1, 0\}$. Truth values at worlds are then assigned to formulas by the following conditions:

$$\nu_w(\neg A) = 1 \text{ iff } \nu_{w^*}(A) \neq 1$$

$\nu_w(A \wedge B) = 1$ iff $\nu_w(A) = 1$ and $\nu_w(B) = 1$
$\nu_w(A \vee B) = 1$ iff $\nu_w(A)$ 1 or $\nu_w(B) = 1$
$\nu_g(A \rightarrow B) = 1$ iff for all $x \in W$, if $\nu_x(A) = 1$ then $\nu_x(B) = 1$

and for $x \neq g$;

$\nu_x(A \rightarrow B) = 1$ iff for all $y, z \in W$ such that $Rxyz$,
if $\nu_y(A) = 1$ then $\nu_z(B) = 1$.

Semantic consequence is then defined in the usual way in terms of truth preservation at g:

$\Sigma \models A$ iff for all interpretations $\langle g, W, R, \nu \rangle$, if $\nu_g(B) = 1$
for all $B \in \Sigma$ then $\nu_g(A) = 1$.

This system is called BM. To obtain semantics for B, $*$ is required to satisfy the condition that $w^{**} = w$ in each interpretation. Then it may be that $\nu_w(A) = \nu_w(\neg A) = 1$ and that $\nu_w(A) \neq 1$ and $\nu_w(\neg A) \neq 1$. Extensions of B, such as R and T, are obtained by placing conditions on the accessibility relation R.[4]

4.1. *Systems of Spheres and Relevant Logics*

We now construct the system of spheres that is based on relevant logics, in particular the relevant logic B. The replacement of classical logic by a relevant logic leads to two important changes to Grove's systems of spheres.[5] Firstly, instead of maximal *consistent* sets of sentences, the main focus is *prime theories*, where a prime theory is defined as follows:

[4] For the simplified semantics for the extensions of B, see Restall (1993).

[5] These changes are also discussed by Restall and Slaney (1995) who show, *inter alia*, that the system of spheres based on FDE, a fragment of relevant logic containing just $\neg$, $\wedge$ and $\vee$, is sound with respect to all of the AGM postulates. Note, however, that there are some differences between the FDE sphere system presented by Restall and Slaney and the relevant sphere system developed in this paper. Firstly, Restall and Slaney employ a four-valued semantics instead of a two-valued semantics. Secondly, their language contains two propositional constants, $\perp$ and $\top$, which denote the 'false only' truth value and the 'true only' truth value respectively. The introduction of $\perp$ and $\top$ gives rise to the possibility that a belief set be trivial. The approach adopted in this paper is to disallow that possibility.

Definition 6: (Prime Theory) *Let Σ be a set of sentences. Σ is a prime theory iff it satisfies the following conditions:*

1. *if $A, B \in \Sigma$ then $A \wedge B \in \Sigma$*
2. *if $\vdash A \rightarrow B$ then (if $A \in \Sigma$ then $B \in \Sigma$)*
3. *if $A \vee B \in \Sigma$ then $A \in \Sigma$ or $B \in \Sigma$*

for some sentences A and B.

The semantics of relevant logic ensures that the set of sentences made true by an evaluation function is a prime theory. Hence such sets replace maximal consistent sets in systems of spheres.[6] Among all prime theories of a language, there is a theory in which all sentences of the language are true (the trivial theory), denoted by $\perp$, and a theory in which no sentences are true (or every sentence is not true and not just false), (the empty theory), denoted by $\top$. Because of these theories, systems of spheres have some properties different from those that they have in classical case, as is shown below.

Secondly, the definition of the contraction function has to be changed. In the original definition, the set $|K| \cup C(\neg A)$ determines the belief set K^-_A. However, this definition is not suitable to the systems of spheres which are based on prime theories. The first reason is that a prime theory that contains A may also contain $\neg A$. As a result, K^-_A could still contain A. The second is that $A \vee \neg A$ is true in a prime theory where $\neg A$ is true. So if the original definition is used, the systems of spheres based on relevant logic gives that $A \vee \neg A \in K^-_A$. Yet this is not in accordance with the semantics of relevant logic which allows incompleteness (and inconsistencies). One way to overcome these problems is to define K^-_A as $|K| \cup \Pi$ where Π is a class of prime theories m in which A fails (or not true). Then if there is such an m, $A \notin K^-_A$. Also m could reject $A \vee \neg A$, and so it could well be the case that $A \vee \neg A \notin K^-_A$.

By applying the changes mentioned in the previous section, we construct the system of spheres, so that the sphere semantics is suitable for the theory of belief change based on relevant logics, B at least. Let M_R be a set of all prime theories of a language L. Then, as mentioned above, $\perp \in M_R$ and $\top \in M_R$. $\perp \in M_R$ ensures that every $S \subseteq M_R$ intersects every other sphere in M_R. For $\perp$ is an element of all spheres in M_R. In particular, $|K|$ intersects every $S \subseteq M_R$.

A system of spheres, $\mathcal{S}_R$, centred on $|K|$, is a collection of subsets of M_R. Let $\overline{|A|}$ be the complement of $|A|$, i.e., $\overline{|A|} = M_R - |A|$. Define $S(\overline{A})$ to

[6] Models and maximal consistent sets are interchangeable in classical logic because of the isomorphism between them. In relevant logics, however, prime theories and models are quite distinct. In this paper, I consider relevant sphere systems that are based on prime theories.

be the smallest sphere in $\mathcal{S}_R$, which intersects $\overline{|A|}$. Then $\mathcal{S}_R$ satisfies the conditions (S1) – (S3) and the following which is a stronger form of (S4):

(S4)′ If A is a sentence and there is any sphere in $\mathcal{S}_R$ intersecting $|A|$ or $\overline{|A|}$, then there is a smallest sphere in $\mathcal{S}_R$ intersecting $|A|$ or $\overline{|A|}$ respectively.

Remark: When relevant logics are applied to the theory of belief change, belief sets are not closed under logical consequences. They are closed under provable entailments (see the definition of prime theory). So strictly speaking, (K^+1), (K^-1) and (K^*1) do not hold. However, as is shown below, this difference between classical and relevant logics does not produce harmful consequences.

4.1.1. *... and Expansion*

As in Grove's original formulation, the AGM belief change operations can be defined in the systems of spheres based on relevant logics. For expansion, the classical definition serves in the relevant case. So an expansion function is defined as in Definition 3 ($K^+_A = t(|K| \cap |A|)$).

Theorem 4: *Let $\mathcal{S}_R$ be any system of spheres in M_R centred on $|K|$. If an expansion function K^+_A is defined as in Definition 3 then the postulates (K^+2) – (K^+6) are satisfied.*

4.1.2. *... and Contraction*

As is mentioned above, we cannot appeal to prime theories where $\neg A$ is true in defining contraction. The closest theories to $|K|$ to be considered in contraction must be the complement of the prime theories in which A is true. So we have to consider a class of prime theories in which A is not true and not just false. We define this class, $C\overline{(A)}$, as follows:

$$C\overline{(A)} = S\overline{(A)} \cap \overline{|A|}.$$

Classically the complement of an elementary class, $\overline{S}$, is also an elementary class. For if $S = |A|$ for some sentence A then $\overline{S} = |\neg A|$ and so $\overline{S}$ is an elementary class. Yet in a relevant logic even though $S = |A|$, $\overline{S} \neq |\neg A|$. In fact, there is no sentence B such that $\overline{S} = |B|$. To verify this claim, suppose that $S = |A|$ and $\overline{S} = |B|$ for some sentences A and B. Then for all $m \in S$, $m \models A$, and for all $n \in \overline{S}$, $n \models B$. But this is impossible, for $\top$ must be in either S or $\overline{S}$. So if $S = |A|$ then there is no sentence B such that $\overline{S} = |B|$. The preceding argument shows that $\overline{|A|}$ is not an elementary class. Hence there is no guarantee that $C\overline{(A)}$ is an elementary class.

We are now in a position to define contraction in the systems of spheres based on a relevant logic. For contraction, we define a contraction function as follows:

Definition 7: $K^-_A = t(|K| \cup C\overline{(A)})$.

Note that, as we saw above, $C\overline{(A)}$ is not an elementary class. So $|K| \cup C\overline{(A)}$ is not an elementary class in general.

Theorem 5: *Let $\mathcal{S}_R$ be any system of spheres in M_R centred on $|K|$. If a contraction function K^-_A is defined as in Definition 7 then the postulates (K^-2) – (K^-4) and (K^-6) – (K^-8) are satisfied.*

The classical proof for (K^-5) does not hold in the relevant case. Yet whether or not it is satisfied is an open question at time of writing. This is mainly because of the assumption that every $S \subseteq M_R$ is an elementary class. If this assumption is abandoned, a counter-example for (K^-5) can be established.[7] Yet the assumption is needed in order to show that Grove's (classical) systems of spheres is sound with respect to the AGM postulates.[8] Makinson (1987) calls contraction operations that do not hold (K^-5) *withdrawals*. Perhaps, a relevant sphere system satisfies only withdrawal rather than contraction. However, I leave a discussion of (K^-5) for another occasion.

4.1.3. *... and Revision*

An intuitive understanding of revision is somewhat related to the *Ramsey Test*. (See Gärdenfors (1988) for the relationship between the AGM theory and the Ramsey Test.) A belief set K is revised when the input sentence

[7] A counter-example: $K = t(|p|)$ and $A = p \vee q$, and $|p|$ and M_R are the only members of $\mathcal{S}_R$. Then it can be checked that $p \in K$. Suppose that we have the following easy to prove lemma: if $S = |A|$ then $\top \in \overline{S}$. By this lemma, $\top \in C\overline{(p \vee q)}$, and so $K^-_A = \phi$. But then $p \notin (K^-_A)^+_A$, since $\not\vdash p \vee q \rightarrow p$. Hence $K \not\subseteq (K^-_A)^+_A$.

[8] There is another solution to the problem of Grove's proof. This solution does not require introducing the elementary class assumption. See Priest, Surendonk and Tanaka (1996) and Priest and Tanaka (1997) for this solution. Yet I do not pursue this approach in this paper. For it works only in the classical case.

A would not commit one to believe all of the beliefs together with A; in particular, A contradicts the beliefs that are already in K. In such cases it is *classically* necessary to revise K by giving up some beliefs in order to maintain consistency. For an inconsistent belief set is trivial. The need for the revision operation is thus driven by consistency. Yet if the concept of inconsistency is separated from that of triviality, the need for revision seems to disappear. The new belief A may simply be added to K without concern for the consistency of the resulting belief set. Hence a relevant logic, which allows inconsistency without trivialising the belief set, does not seem to give rise to the need to distinguish revision from expansion.[9]

This is exactly what the relevant sphere semantics shows. The semantics of relevant logic ensures that $\bot \in M_R$ and $\top \in M_R$. In terms of systems of spheres, $\bot$ is an element of all spheres, i.e., all elementary classes, and $\top$ is an element of no spheres, i.e., no elementary classes. Thus in M_R, $|K| \cap |A| \neq \phi$ for all sentences A. So the smallest sphere which intersects $|A|$, $S(A)$, is always $|K|$ for all A. Hence by defining a revision function as in the classical case, i.e., Definition 5 ($K^*_A = t(C(A)) = t(|K| \cap S(A))$), we have the following lemma:

Lemma 2: $K^*_A = K^+_A$ *if the underlying logic is a relevant logic.*

Though the revision functions are shown to be the same as the expansion functions, the spirit of revision is satisfied as the following theorem shows.

Theorem 6: *Let* $\mathcal{S}_R$ *be any system of spheres in* M_R *centred on* $|K|$. *If a revision function* K^*_A *is defined as in Definition 5 then the postulates* (K^*2) *–* (K^*4) *and* (K^*6) *–* (K^*8) *are satisfied.*

As was expected, (K^*5), which allows the belief set to be trivial, fails in the relevant systems of spheres.

Theorem 7: *Let* $\mathcal{S}_R$ *be any system of spheres in* M_R *centred on* $|K|$. *If a revision function* K^*_A *is defined as in Definition 5 then the postulates* (K^*5) *is not satisfied.*[10]

[9] This does not mean that an inconsistent belief set should not be revised. For a discussion of how to motivate revision in the presence of paraconsistent logics, see Priest (2001). A discussion on revision also appears at the end of this paper.

[10] One may wonder what happens if (K^*5) is reformulated as: $K^*_{\neg A}$ is inconsistent iff $\vdash A$. Since the paraconsistent approach developed in this paper is to challenge the possibility of an inconsistent belief set being trivial, I leave such a discussion for another occasion.

5. *Positive-Plus Systems*

The study of positive-plus systems was initiated by da Costa who proposed a family of paraconsistent logics $\mathsf{C_i}$, where $1 \leq i \leq \omega$. (For example, da Costa (1974) and da Costa and Alves (1977).) The semantics of system C_ω is an illuminating path to da Costa's systems, so we discuss it first.

Let a *da Costa semivaluation* be a function ν which maps every proposition in a language to 1 or 0. Then in C_ω, ν satisfies the following conditions: (See Loparić (1977) and Priest and Routley (1989)).)

1. $\nu(A \wedge B) = 1$ iff $\nu(A) = 1$ and $\nu(B) = 1$.
2. $\nu(A \vee B) = 1$ iff $\nu(A) = 1$ or $\nu(B) = 1$.
3. If $\nu(A) = 0$ then $\nu(\neg A) = 1$.
4. If $\nu(\neg\neg A) = 1$ then $\nu(A) = 1$.
5. If $\nu(A \supset B) = 0$ then $\nu(B) = 0$.
6. If $\nu(A \supset B) = 1$ then $\nu(A) = 0$ or $\nu(B) = 1$.

A C_ω *valuation* is any semivaluation ν such that for any formula B of the form $A_1 \supset (A_2 \supset (A_3 \supset \cdots (A_{n-1} \supset A_n)))$ where A_n is not a conditional, if $\nu(B) = 0$ there is a semivaluation ν' such that $\nu'(A_i) = 1$, for each i such that $1 \leq i < n$, and $\nu'(A_n) = 0$. (See Priest and Routley (1989) p. 175.) It is worth noting that if $\nu(A) = 1$ then $\nu(\neg A)$ is under-determined, so it may be the case that $\nu(A) = 1$ and $\nu(\neg A) = 1$. The value of A is assigned independently of the value of $\neg A$ under the valuation. As a consequence of this, the classical rule of double negation: $\nu(A) = \nu(\neg\neg A)$, fails.

Semantic consequence is then defined in the usual way:

$$\Sigma \models A \text{ iff for all } \nu, \text{ if } \nu(B) = 1 \text{ for all } B \in \Sigma \text{ then } \nu(A) = 1.$$

As can be checked, neither $\{A, \neg A\} \models B$ nor $\{A \wedge \neg A\} \models B$.

Da Costa strengthens system C_ω to produce systems $\mathsf{C_n}$ $(1 \leq n < \omega)$. In introducing system $\mathsf{C_n}$, it is convenient to abbreviate $\neg(A \wedge \neg A)$ to A°. In other words, A° expresses the consistency of A, and $\circ$ may be construed as a *classicality operator*.

On a C_ω valuation, there are two kinds of sentences: those that are consistent, i.e., $\nu(A) \neq \nu(\neg A)$, and those that are dialetheic, i.e., $\nu(A) = \nu(\neg A) = 1$. The motivation for $\mathsf{C_n}$ is to make this point explicit and achieve the following: (See Priest and Routley (1989) p. 166.)

> If B is a compound of $A_1, ..., A_n$ and $\Sigma \models A_1^\circ \wedge \cdots \wedge A_n^\circ$ then $\Sigma \models B$ iff B is a classical consequence of Σ.

Now $\mathsf{C_1}$ adds to the above conditions 1–6 the following:

7. $\nu(A^\circ) = 1$ if $\nu(A) \neq \nu(\neg A)$

8. $\nu(A^\circ) = 0$ if $\nu(A) = \nu(\neg A) = 1$.

C_1 also adds the following to fulfill the motivation:

9. $\nu(A) = 0$ if $\nu(B^\circ) = \nu(A \supset B) = \nu(A \supset \neg B) = 1$
10. $\nu((A \wedge B)^\circ) = \nu((A \vee B)^\circ) = \nu((A \supset B)^\circ) = \nu((\neg A)^\circ) = 1$ if $\nu(A^\circ) = \nu(B^\circ) = 1$.

Let A^n be an abbreviation of $A^{\circ\circ\cdots\circ}$ where $\circ$ appears n (≥ 1) times, and $A^{(n)}$ of $A^1 \wedge A^2 \wedge \cdots \wedge A^n$. Then the extension of the semantics of C_1 to the systems C_n is immediate. The semantics of C_n gives the inductive truth conditions of A^n as:

7′. $\nu(A^n) = 1$ if $\nu(A^{n-1}) \neq \nu(\neg A^{n-1})$
8′. $\nu(A^n) = 0$ if $\nu(A^{n-1}) = \nu(\neg A^{n-1}) = 1$

and replaces 9 and 10 by the following

9′. $\nu(A) = 0$ if $\nu(B^{(n)}) = \nu(A \supset B) = \nu(A \supset \neg B) = 1$
10′. $\nu((A \wedge B)^{(n)}) = \nu((A \vee B)^{(n)}) = \nu((A \supset B)^{(n)}) = \nu((\neg A)^{(n)}) = 1$ if $\nu(A^{(n)}) = \nu(B^{(n)}) = 1$.

Adding the semantic conditions for the classicality operator, i.e., 7′ and 8′, to those for the semantics of C_ω, simplifies the semantic condition for $\supset$ to: (See Priest and Routley (1989) pp. 176–7.)

11. $\nu(A \supset B) = 1$ iff $\nu(A) = 0$ or $\nu(B) = 1$.

Then the difference between valuations and semivaluations vanishes. Semantic consequence in C_n is defined as in C_ω. The semantic conditions for C_n ($1 \leq n < \omega$) show that the positive fragment of C_n is exactly that of classical logic. Hence C_n is the positive fragment of classical logic plus da Costa negation. (See Priest and Routley (1989) and Priest and Tanaka (2004).)

5.1. *Systems of Spheres and C_ω*

The adoption of C_ω requires some changes to systems of spheres. In the context of relevant logic, maximal consistent sets of sentences were replaced by prime theories. In C_ω, they are replaced by maximal *non-trivial* sets of sentences, where a maximal non-trivial set of sentences is defined as follows:

Definition 8: (Maximal Non-Trivial Set) *Let Σ be a set of sentences in a language L. Σ is a maximal non-trivial set iff if $A \notin \Sigma$ then $\Sigma \cup \{A\}$ is trivial, i.e., the set of all sentences of L.*

In C_ω (and all C_n), this means that either A or $\neg A$ (and maybe both) is in a maximal non-trivial set.

Now, let M_{C_ω} be a set of all maximal non-trivial sets of sentences in a language L. Then there are no trivial set, $\perp$, and no empty set, $\top$, in M_{C_ω}.

However, while $|A| \cup |\neg A| = M_{C_\omega}$, as in the classical case, it may be the case that $|A| \cap |\neg A| \neq \phi$.

A system of spheres, $\mathcal{S}_{C_\omega}$, centred on $|K|$, is then a collection of subsets of M_{C_ω}. Although $\mathcal{S}_{C_\omega}$ is different from $\mathcal{S}_R$, $\mathcal{S}_{C_\omega}$ satisfies (S1) – (S4)$'$.

5.1.1. ... *and Expansion*

For expansion, the classical definition serves here. So an expansion function is defined as in Definition 3 ($K^+_A = t(|K| \cap |A|)$).

Theorem 8: *Let $\mathcal{S}_{C_\omega}$ be any system of spheres in M_{C_ω} centred on $|K|$. If an expansion function K^+_A is defined as in Definition 3 then the postulates (K^+2) – (K^+6) are satisfied.*

5.1.2. ... *and Contraction*

For contraction, the definition in the relevant case can be used. That is, a contraction function is defined as in Definition 7 ($K^-_A = t(|K| \cup C(\overline{A}))$). Note that $C(\overline{A})$ is not an elementary class in general. Thus $|K| \cup C(\overline{A})$ is not an elementary class.

Theorem 9: *Let $\mathcal{S}_{C_\omega}$ be any system of spheres in M_{C_ω} centred on $|K|$. If a contraction function K^-_A is defined as in Definition 7 then the postulates (K^-2) – (K^-4) and (K^-6) – (K^-8) are satisfied.*

For (K^-5), whether or not it is satisfied is an open question at time of writing.

5.1.3. ... *and Revision*

A revision function can also be defined as in the classical and relevant case. So it is defined as in Definition 5 ($K^*_A = t(C(A)) = t(|K| \cap S(A))$). Although $\perp \notin M_{C_\omega}$, we have

Lemma 3: $K^*_A = K^+_A$ *if the underlying logic is* C_ω.

This is because, for any sentences A and B, $|A| \cap |B| \neq \phi$. In particular, $|K| \cap |A| \neq \phi$. So the smallest sphere intersecting $|A|$ is $|K|$.

Theorem 10: *Let $\mathcal{S}_{C_\omega}$ be any system of spheres in M_{C_ω} centred on $|K|$. If a revision function K^*_A is defined as in Definition 5 then the postulates (K^*2) – (K^*4) and (K^*6) – (K^*8) are satisfied.*

Theorem 11: *Let $\mathcal{S}_{C_\omega}$ be any system of spheres in M_{C_ω} centred on $|K|$. If a revision function K^*_A is defined as in Definition 5 then the postulate (K^*5) is not satisfied.*

5.2. *Systems of Spheres and* $\mathsf{C_n}$

The sphere semantics based on $\mathsf{C_n}$ ($1 \leq n < \omega$) seems to be a combination of that of C_ω and that of classical logic. Let M_{C_n} be the set of all maximal non-trivial sets of sentences in a language L. M_{C_n} is somewhat different from all of M_L, M_R and M_{C_ω}. Firstly, unlike the case of M_R, the trivial set, $\bot$, and the empty set, $\top$, are not in M_{C_n}, i.e., $\bot \notin M_{C_n}$ and $\top \notin M_{C_n}$. Yet as in the case of M_{C_ω}, $|A| \cup |\neg A| = M_{C_n}$ and it may be the case that $|A| \cap |\neg A| \neq \phi$, though if A behaves consistently, then $|A| \cap |\neg A| = \phi$. Secondly, any sentence A divides M_{C_n} into two parts: one in which $\nu(A^\circ) = 1$, indicating the sets where A behaves consistently, and the other in which $\nu(A^\circ) = 0$, indicating the sets where A maybe inconsistent. The second part is in fact the complement of the first. We let $|A^\circ|$ denote the first and $\overline{|A^\circ|}$ ($= M_{C_n} - |A^\circ|$) the second.

A system of spheres, $\mathcal{S}_{Cn}$, centred on $|K|$, is a collection of subsets of M_{C_n}. Then $\mathcal{S}_{Cn}$ satisfies the conditions (S1) – (S4)$'$. Despite the fact that M_{C_n} is divided into two parts for any A, $\mathcal{S}_{Cn}$ does not introduce any complexity with respect to the belief change functions, as we see below.

5.2.1. *... and Expansion*

For expansion, as in the cases of relevant logic and C_ω, the function is defined as in the classical case. So an expansion function is defined as in Definition 3 ($K^+_A = t(|K| \cap |A|)$).

Theorem 12: *Let $\mathcal{S}_{Cn}$ be any system of spheres in M_{C_n} centred on $|K|$. If an expansion function K^+_A is defined as in Definition 3 then the postulates (K^+2) – (K^+6) are satisfied.*

5.2.2. *... and Contraction*

A contraction function is defined as in the cases of relevant logic and C_ω as well, i.e., Definition 7 ($K^-_A = t(|K| \cup C(\overline{A}))$). Note that in the case of $\mathsf{C_n}$, $\overline{|A|}$ is an elementary class. For $\overline{|A|} = |\neg A \wedge A^\circ|$. Hence $C(\overline{A})$ is an elementary class. Thus $|K| \cup C(\overline{A})$ is an elementary class.

Theorem 13: *Let $\mathcal{S}_{Cn}$ be any system of spheres in M_{C_n} centred on $|K|$. If a contraction function K_A^- is defined as in Definition 7 then the postulates (K^-2) – (K^-8) are satisfied.*

Note that (K^-5) is shown to be satisfied. The fact that $|K| \cup \overline{C(A)}$ is an elementary class allows the classical proof to be applicable.

5.2.3. *... and Revision*

A revision function is defined as in Definition 5 ($K_A^* = t(C(A)) = t(|K| \cap S(A))$). Unlike the cases of relevant logic and C_ω, $K_A^* \neq K_A^+$ in general if the underlying logic is $\mathsf{C_n}$. For if $|K| \subseteq |A^\circ|$ and $|K| \subseteq |\neg A|$ then $S(A) \neq |K|$. Yet we have the following lemma:

Lemma 4: *Let $|K| \cap \overline{|A^\circ|} \neq \phi$. Then $K_A^* = K_A^+$ if the underlying logic is $\mathsf{C_n}$ ($1 \leq n < \omega$).*

That is, if K allows A to be inconsistent, revision of K by A collapses into expansion.

Theorem 14: *Let $\mathcal{S}_{Cn}$ be any system of spheres in M_{C_n} centred on $|K|$. If a revision function K_A^* is defined as in Definition 5 then the postulates (K^*2) – (K^*4) and (K^*6) – (K^*8) are satisfied.*

Theorem 15: *Let $\mathcal{S}_{Cn}$ be any system of spheres in M_{C_n} centred on $|K|$. If a revision function K_A^* is defined as in Definition 5 then the postulate (K^*5) is not satisfied.*

6. *Non-Adjunctive Systems*

Non-adjunctive systems were perhaps the first formal systems that were developed for a paraconsistent purpose. The development of the systems was pioneered by Jaśkowski (1969) who introduced *discussive (discursive) logic* in an axiomatic form. The semantics of discussive logic was later investigated by da Costa and his co-workers. (For example, da Costa and Dubikajtis (1977). The development of the semantics started earlier than their paper however.) Non-adjunctive systems were further developed by a number of people such as Schotch and Jennings (1980), Rescher and Brandom (1980) and Rescher and Manor (1970).

The main idea of non-adjunctive systems, in particular of discussive logic, is as follows. A discourse may be advanced by a number of participants.

Each participant puts forward some information, beliefs or opinions that are assumed to be self-consistent. Conceivably, the opinions of one person may sometimes contradict that of others. So a discourse as a whole may be inconsistent. For what is true in a discourse is a sum of information put forward by participants.

In order to formalise this idea, imagine each participant's information set as the set of things true in a possible world of standard modal logic. An interpretation, I, is a Kripke interpretation of some modal logic, say S5, employing the usual truth conditions for $\wedge$, $\vee$ and $\neg$. Conditionals will be looked at later. Then I is a discussive model of sentence A iff A holds at some world. Hence, by appealing to S5, semantic consequence is defined as: (See Priest (2002).)

$$\Sigma \models A \text{ iff } \Diamond\Sigma \models_{S5} \Diamond A \text{ where } \Diamond\Sigma \text{ is } \{\Diamond A : A \in \Sigma\}.$$

It is easy to check that $A, \neg A \not\models B$. For it may be the case that $\Diamond A$ and $\Diamond\neg A$, but not $\Diamond B$, in an S5 interpretation. It should be noted, however, that $A \wedge \neg A \models B$. This means that a single contradictory premise yields an explosive inference in discussive logic, as in the case of standard modal logic. Hence adjunction has to fail, i.e., $A, B \not\models A \wedge B$. For otherwise $A, \neg A \models A \wedge \neg A \models B$. Thus conjunction has non-standard behaviour which makes discussive logic paraconsistent.

This leaves us to discuss the conditional. Let $\supset_{\mathsf{d}}$ be a discussive implication. Then $A \supset_{\mathsf{d}} B$ is defined as $\Diamond A \supset B$. It is easy to check that discussive implication satisfies *modus ponens*: $A, A \supset_{\mathsf{d}} B \models B$ (at least by appealing to S5 semantic consequence). For the definition of the discussive biconditional $\equiv_{\mathsf{d}}$, it is natural to define $A \equiv_{\mathsf{d}} B$ as $(A \supset_{\mathsf{d}} B) \wedge (B \supset_{\mathsf{d}} A)$. Yet this definition gives that $A \equiv_{\mathsf{d}} \neg A \models B$, as can easily be checked. Presumably to avoid the problem, Jaśkowski chose to define $A \equiv_{\mathsf{d}} B$ as $(A \supset_{\mathsf{d}} B) \wedge (B \supset_{\mathsf{d}} \Diamond A)$ (or $(\Diamond A \supset B) \wedge (\Diamond B \supset \Diamond A)$).

6.1. *Systems of Spheres and Discussive Logic*

We now construct the systems of spheres that are based on discussive logic. In doing so, we appeal to the modal logic S5. Let M_D be the set of all S5 interpretations. $|A|$ is defined to be the set of all interpretations I such that for *some* w in I, $w \models A$. Note that it may be the case that in an interpretation I, $w \models A$ and $w' \models \neg A$. Hence it may be that $|A| \cap |\neg A| \neq \phi$. Yet $\bot \notin M_D$. For there is a sentence B such that $w \not\models B$ for all w in I, for example, the negation of an S5 logical truth. Also $\top \notin M_D$ and $|A| \cup |\neg A| = M_D$, as every $I \in M_D$ is an S5 interpretation.

A system of spheres, $\mathcal{S}_D$, centred on $|K|$, is a collection of subsets of M_D. It satisfies the conditions (S1) – (S4)$'$.

6.1.1. ... *and Expansion*

For expansion, the classical definition serves in the case of discussive logic. So an expansion function is defined as in Definition 3 ($K^+_A = t(|K| \cap |A|)$).

Theorem 16: *Let $\mathcal{S}_D$ be any system of spheres in M_D centred on $|K|$. If an expansion function K^+_A is defined as in Definition 3 then the postulates (K^+2) – (K^+6) are satisfied.*

6.1.2. ... *and Contraction*

Contraction is defined as in the cases of relevant logic, C_ω and $\mathsf{C_n}$. That is, a contraction function is defined as in Definition 7 ($K^-_A = t(|K| \cup C(\overline{A}))$). Note that $C(\overline{A})$ is not an elementary class in general. Thus, $|K| \cup C(\overline{A})$ is not an elementary class. In fact, $C(A)$ may not be an elementary class either, since (EC1) of Lemma 1 fails.

Theorem 17: *Let $\mathcal{S}_D$ be any system of spheres in M_D centred on $|K|$. If a contraction function K^-_A is defined as in Definition 7 then the postulates (K^-2) – (K^-4) and (K^-6) – (K^-8) are satisfied.*

For (K^-5), whether or not it is satisfied is an open question at time of writing.

6.1.3. ... *and Revision*

A revision function is defined as in Definition 5 ($K^*_A = t(C(A)) = t(|K| \cap C(A))$). If the underlying logic is discussive logic, then $K^*_A \neq K^+_A$. For $S(A) \neq |K|$ in general, since $\perp \notin M_D$ and it is not the case that $|A| \cap |B| \neq \phi$ for any A and B.

Theorem 18: *Let $\mathcal{S}_D$ be any system of spheres in M_D centred on $|K|$. If a revision function K^*_A is defined as in Definition 5 then the postulates (K^*2) – (K^*4) and (K^*6) are satisfied.*

Theorem 19: *Let $\mathcal{S}_D$ be any system of spheres in M_D centred on $|K|$. If a revision function K^*_A is defined as in Definition 5 then the postulates (K^*5) and (K^*7) are not satisfied.*

The non-standard behaviour of conjunction in discussive logic seems to affect those revision postulates which involve conjunction, though it does not affect contraction postulates. For (K^*8) too, the classical proof does not hold. However, whether or not it is satisfied is an open question at time of writing.

7. *Some Implications*

We saw in the previous sections four types of systems of spheres, each of which is based on classical logic, relevant logics, positive-plus logics and non-adjunctive logic.[11] In general, two AGM postulates seem to require some attention in the context of paraconsistent logics. The first is $(K^{-}5)$ the so called *recovery postulate*. Most paraconsistent sphere systems do not validate nor invalidate the postulate. As has been argued by many, including Gärdenfors (1988) himself, the recovery postulate is problematic and open to many criticisms, as many intuitive contraction operations do not validate it. For example, suppose that $A \wedge B$ is in a belief set. Then if A is contracted from the belief set, $A \wedge B$ may go with it. So when A is added again, B is not recovered. The information B is lost during the process. This example indicates that the question about the recovery postulate is not specific to the context of paraconsistent logics.[12]

Secondly, paraconsistent sphere systems do not satisfy $(K^{*}5)$. Classically, the main objective of introducing this postulate is to maintain that $K^{*}_{\neg A}$ is consistent, unless A is a logical truth in which case $K^{*}_{\neg A}$ becomes inconsistent. For the negation of a logical truth is always false classically. And when a belief set, e.g., $K^{*}_{\neg A}$, is inconsistent, it becomes trivial. According to paraconsistent approaches, however, beliefs may be non-trivial even if they are inconsistent. Since beliefs should not be trivial simply because they are inconsistent, as was argued at the beginning of the paper, the classical motivation for $(K^{*}5)$ goes astray. In order to accommodate inconsistent beliefs sensibly, therefore, $(K^{*}5)$ is not adequate as part of the logical framework of belief change.

By and large, however, the accommodation of inconsistent beliefs does not require a wholesale rejection of the AGM theory. All we need to reject, as far as the AGM postulates go, is $(K^{*}5)$ which is problematic in the presence of inconsistent beliefs. Although the AGM theory as a whole is motivated by the consistency criterion, $(K^{*}5)$ is therefore independent of other AGM

[11] The main aim of this paper is to examine the adequacy of the AGM theory in the presence of inconsistent beliefs. For this reason, I leave aside for now the important question of which paraconsistent logic is best suited to model inconsistent belief change.

[12] For a discussion of recovery postulate in classical context, see for example Makinson (1987).

postulates. This seems to indicate that the AGM theory is not totally an inadequate theory of belief change in handling inconsistent beliefs in a sensible fashion.

Now putting aside the AGM postulates, one interesting result of our investigation is that revision collapses into expansion. This does not mean, however, that the revision operation cannot be defined in dealing with inconsistent beliefs. Indeed, it is more intuitive to think that expansion and contraction are primitive and revision is a derived operation.[13] There are several ways to define revision in terms of expansion and contraction. The 'standard' definition is the *Levi Identity*: $K^*_A = (K^-_{\neg A})^+_A$. If we define revision in this way, some AGM postulates no longer hold in the contexts of paraconsistent logics.[14] However, since inconsistent beliefs can be accommodated sensibly in the context of paraconsistent logics, revision can be, so it seems, defined alternatively by the *Reverse Levi Identity*: $K^*_A = (K^+_A)^-_{\neg A}$, or by the *Consolidated Expansion*: $K^*_A = (K^+_A)^-_{\mathrm{I}}$ where I is an inconsistency.[15] However, I leave a discussion of the issue of defining revision in terms of expansion and contraction for another occasion.[16]

One consequence of our result, that revision collapses into expansion, is that our framework of belief change in general satisfies the *Ramsey Test* which can be formalised as: (RM) $A > B \in K$ iff $B \in K^*_A$, where $A > B$ is a conditional statement.[17] For revision is *ipso facto* expansion and hence (RM) is tantamount to our assumption that a belief set is closed under logical consequence, or closed under provable entailments in the case of relevant logics. In fact, it was (RM) that was the motive behind the idea that revision needs not be distinguished from expansion.

This consequence, however, goes against the argument of Fuhrmann (1991) that (RM) should be rejected, even though the choice of underlying logic is largely in agreement with our approach in this paper. Whether a revision operation should satisfy (RM) or not, a framework of belief change based on a paraconsistent logic needs not reject (RM), as we have seen above. The issue

[13] Gärdenfors (1988) introduces revision before contraction. It is not clear what the reason behind it is, in particular whether or not he thinks that revision is a derived operation.

[14] Proof in the context of relevant logics is included in the Appendix.

[15] Thanks are due to Krister Segerberg for bringing my attention to the latter.

[16] See Hansson (1992) for a discussion on reversing the Levi Identity.

[17] See for example Gärdenfors (1988) p. 148.

seems to be a matter of further investigation into the definition of revision, just as in the context of classical logic.[18]

Appendix

This section contains proofs for the theorems and lemmas that appear in the paper. To make proofs self-contained, lemmas that are used to prove the theorems are also included below. In proving the theorems and lemmas, it is useful to note the completeness theorem which holds in all of the logics that we saw above:

Completeness Theorem: Let Σ be a collection of sentences. Then for every sentence A, if $\Sigma \not\vdash A$ then for some $m \in M_X$, $m \models \Sigma$ and $m \not\models A$, where M_X is either M_L, M_R, M_{C_ω}, M_{C_n} or M_D.

Grove's Systems of Spheres

Lemma 1: *For any elementary classes S_1 and S_2,*

(EC1) $S_1 \cap S_2$ *is an elementary class.*
(EC2) $S_1 \cup S_2$ *is an elementary class.*

Proof.

(EC1) Since S_1 and S_2 are elementary classes, there are sentences A and B such that $S_1 = |A|$ and $S_2 = |B|$. So for any $m \in S_1 \cap S_2$, $m \models A \wedge B$. Hence $m \in |A \wedge B|$. Conversely, for any $m \in |A \wedge B|$, $m \models A \wedge B$. So $m \in |A| \cap |B|$. Hence $m \in S_1 \cap S_2$.

(EC2) For any $m \in S_1 \cup S_2 = |A| \cup |B|$, $m \models A$ or $m \models B$. In either case $m \models A \vee B$. Hence $m \in |A \vee B|$. Conversely, for any $m \in |A \vee B|$, $m \models A \vee B$. So $m \models A$ or $m \models B$. Hence $m \in |A| \cup |B|$, and so $m \in S_1 \cup S_2$. □

Lemma 5: *The function t has the following properties:*

(t1) $t(|K|) = K$ *for all belief sets* K.
(t2) $S = |t(S)|$ *for* $S \subseteq M_X$ *if* S *is an elementary class.*
(t3) $t(S)$ *is non-trivial if* S *is nonempty, for* $S \subseteq M_X$.
(t4) *If* $S \subseteq S'$ *then* $t(S') \subseteq t(S)$, *for* $S, S' \subseteq M_X$.

[18] For a discussion of (RM) in the context of classical logic, see Gärdenfors (1988).

(t5) $t(S) \cap t(S') = t(S \cup S')$ *for* $S, S' \subseteq M_X$.
(M_X *is either* M_L*,* M_R*,* M_{C_ω}*,* M_{C_n} *or* M_D*.)*

Proof.

(t1) Take any $A \in t(|K|)$. Then
$A \in t(|K|)$ iff for all $m \in |K|$, $m \models A$
iff for all m such that $m \models K$, $m \models A$.
If $A \in K$ then it is clear that for all m such that $m \models K$, $m \models A$. If $A \notin K$, then by the completeness theorem, for some m, $m \models K$ and $m \not\models A$. Hence $A \in t(|K|)$ iff $A \in K$.

(t2) By the definition of elementary class, $S = |A|$ for some sentence A. Then
$m \in |t(|A|)|$ iff $m \models t(|A|)$
iff $m \models A$
iff $m \in |A|$.
Hence $|A| = |t(|A|)|$, and so $S = |t(S)|$.

(t3) If $t(S)$ is trivial for $S \subseteq M_X$, then S is empty.

(t4) Suppose $S \subseteq S'$. Take any $A \in t(S')$. Then for all $m \in S'$, $m \models A$. Since $S \subseteq S'$, for all $n \in S$, $n \models A$. Hence $A \in t(S)$.

(t5) By general set theory $\bigcap S \cap \bigcap S' = \bigcap (S \cup S')$. Since $t(S)$ is defined as $\bigcap S$, $t(S) \cap t(S') = t(S \cup S')$. □

Theorem 1: Let $\mathcal{S}$ *be any system of spheres in* M_L *centred on* $|K|$*. If an expansion function* K^+_A *is defined as in Definition 3 then the postulates (*K^+2*) – (*K^+6*) are satisfied.*

Proof.

(K^+2) For all $m \in |K| \cap |A|$, $m \models A$. So $A \in t(|K| \cap |A|)$. Hence by Definition 3, $A \in K^+_A$.

(K^+3) $|K| \cap |\alpha| \subseteq |K|$
iff $t(|K|) \subseteq t(|K| \cap |A|)$ by (t4)
iff $K \subseteq K^+_A$ by Definition 3 and (t1)

(K^+4) Suppose $A \in K$. Then for all $m \in |K|$, $m \models A$. Hence $|K| \cap |A| = |K|$. So $t(|K| \cap |A|) = t(|K|)$. Hence by Definition 3 and (t1), $K^+_A = K$.

(K^+5) Suppose $K \subseteq H$. Then

$$\begin{array}{llll} |H| \subseteq |K| & \text{iff} & |H| \cap |A| \subseteq |K| \cap |A| & \\ & \text{iff} & t(|K| \cap |A|) \subseteq t(|H| \cap |\alpha|) & \text{by (t4)} \\ & \text{iff} & K^+_A \subseteq H^+_A & \text{by Definition 3.} \end{array}$$

(K^+6) Suppose there is a function $K^\sharp_A$ for some belief set K and sentence A satisfying (K^+2) – (K^+5) and $K \subseteq K^\sharp_A \subseteq K^+_A$. By ($K^+5$), $K^+_A \subseteq (K^\sharp_A)^+_A \subseteq (K^+_A)^+_A$. But by ($K^+2$), $A \in K^\sharp_A$. So by (K^+4), $K^+_A \subseteq K^\sharp_A \subseteq K^+_A$. Hence $K^\sharp_A = K^+_A$. Hence K^+_A is the smallest belief set that satisfies (K^+2) – (K^+5). □

Lemma 6: *If $|A| \subseteq |B|$ for any sentences A and B then $S(B) \subseteq S(A)$.*

Proof. Suppose $|A| \subseteq |B|$. Then any sphere intersecting $|A|$ also intersects $|B|$. Hence $S(B) \subseteq S(A)$. □

Theorem 2: *Let $\mathcal{S}$ be any system of spheres in M_L centred on $|K|$. If a contraction function K^-_A is defined as in Definition 4 then the postulates (K^-2) – (K^-8) are satisfied.*

Proof.

(K^-2) $|K| \subseteq |K| \cup C(\neg A)$

$$\begin{array}{lll} \text{iff} & t(|K| \cup C(\neg A)) \subseteq t(|K|) & \text{by (t4)} \\ \text{iff} & K^-_A \subseteq K & \text{by Definition 4 and (t1)} \end{array}$$

(K^-3) Suppose $A \notin K$. By the completeness theorem, for some $m \in M_L$, $m \models K$ and $m \not\models A$. So $m \models \neg A$. Hence $|K| \cap |\neg A| \neq \phi$. So by (S2), $|K|$ is the smallest sphere intersecting $|\neg A|$. Hence $|K| \cup C(\neg A) = |K|$. Hence $|K^-_A| = |K| \cup C(\neg A) = |K|$. Then $t(|K^-_A|) = t(|K|)$. Hence by (t1) and Definition 4, $K^-_A = K$.

(K^-4) Suppose $\not\vdash A$. Then, since there is at least one sphere intersecting $|\neg A|$ (viz., M_L itself), by (S4) there is a smallest such sphere. So $C(\neg A) \neq \phi$. Hence $A \notin t(|K| \cup C(\neg A))$, i.e. $A \notin K^-_A$.

(K^-5) Suppose $A \in K$. Then $|K| \cap |A| = |K|$. Now

$$\begin{array}{rcll} (K_A^-)_A^+ & = & t(|K_A^-| \cap |A|) & \\ & = & t(|t(|K| \cup C(\neg A))| \cap |A|) & \text{by Definition 4} \\ & = & t((|K| \cup C(\neg A)) \cap |A|) & \text{by (EC2) and (t2)} \\ & = & t((|K| \cap |A|) \cup (C(\neg A) \cap |A|)) & \end{array}$$

But $C(\neg A) \cap |A| = \phi$. Hence $(K_A^-)_A^+ = t(|K| \cap |A|) = t(|K|)$. Hence $K = (K_A^-)_A^+$.

(K^-6) If $\vdash A \leftrightarrow B$ then $\vdash \neg A \leftrightarrow \neg B$. So for all m in M_L, $m \models \neg A$ iff $m \models \neg B$. So $C(\neg A) = C(\neg B)$, and so $K_A^- = K_B^-$.

(K^-7) Since $\vdash \neg A \rightarrow \neg A \vee \neg B$, $|\neg A| \subseteq |\neg A \vee \neg B|$. Hence by Lemma 6, $S(\neg A \vee \neg B) \subseteq S(\neg A)$ iff $|\neg A| \cap S(\neg A \vee \neg B) \subseteq S(\neg A) \cap |\neg A|$. Similarly, $\vdash \neg B \rightarrow \neg A \vee \neg B$. So $|\neg B| \cap S(\neg A \vee \neg B) \subseteq S(\neg B) \cap |\neg B|$. Hence

$$\begin{array}{ll} & (|\neg A| \cap S(\neg A \vee \neg B)) \cup (|\neg B| \cap S(\neg A \vee \neg B)) \\ & \qquad \subseteq (S(\neg A) \cap |\neg A|) \cup (S(\neg B) \cap |\neg B|) \\ \text{iff} & (|\neg A| \cup |\neg B|) \cap S(\neg A \vee \neg B) \\ & \qquad \subseteq (S(\neg A) \cap |\neg A|) \cup (S(\neg B) \cap |\neg B|) \\ \text{iff} & C(\neg(A \wedge B)) \subseteq C(\neg A) \cup C(\neg B). \end{array}$$

Now take any $D \in K_A^- \cap K_B^-$. Then by (t5), $t(|K| \cup C(\neg A)) \cap t(|K| \cup C(\neg B)) = t((|K| \cup C(\neg A)) \cup (|K| \cup C(\neg B)))$. So $m \models D$ for any $m \in (|K| \cup C(\neg A)) \cup (|K| \cup C(\neg B)) = |K| \cup C(\neg A) \cup C(\neg B)$. Since $C(\neg(A \wedge B)) \subseteq C(\neg A) \cup C(\neg B)$, for any $n \in |K| \cup C(\neg(A \wedge B))$, $n \in |K| \cup C(\neg A) \cup C(\neg B)$. So $n \models D$. Hence $D \in t(|K| \cup C(\neg(A \wedge B)))$, i.e. $D \in K_{A \wedge B}^-$. Hence $K_A^- \cap K_B^- \subseteq K_{A \wedge B}^-$.

(K^-8) Suppose $A \notin K_{A \wedge B}^-$. There are three cases.

i) $|K| \cap |A| \cap |B| \neq \phi$ but $|K| \not\subseteq |A|$ and $|K| \not\subseteq |B|$. $K \not\models A$. Then $S(\neg A) = |K| = S(\neg(A \wedge B))$. So

$$\begin{array}{rcll} K_A^- & = & t(|K| \cup (|K| \cap |\neg A|)) & \\ & = & t(|K|) & \\ & = & K & \text{by (t1).} \\ K_{A \wedge B}^- & = & t(|K| \cup (|K| \cap |\neg(A \wedge B)|)) & \\ & = & t(|K|) & \\ & = & K & \text{by (t1).} \end{array}$$

Hence $K^-_{A\wedge B} = K^-_A$, and so $K^-_{A\wedge B} \subseteq K^-_A$.

ii) $|K| \cap |A| \cap |B| \neq \phi$ and $|K| \subseteq |A|$ but $|K| \not\subseteq |B|$.
$K \models A$ and $K \not\models B$. Then $S(\neg(A \wedge B)) = |K|$. So $K^-_{A\wedge B} = K$ as in i). Since $K \models A$, $A \in K^-_{A\wedge B}$, which contradicts assumption.

iii) $|K| \cap |A| \cap |B| \neq \phi$ and $|K| \subseteq |A|$ and $|K| \subseteq |B|$.
$K \models A$ and $K \models B$. Since $\vdash A \wedge B \rightarrow A$, $|A \wedge B| \subseteq |A|$. So $|\neg A| \subseteq |\neg(A \wedge B)|$. Hence $S(\neg(A \wedge B)) \subseteq S(\neg A)$ by Lemma 6. Now by assumption, $A \notin K^-_{A\wedge B} = t(|K| \cup C(\neg(A \wedge B)))$. So for some $m \in C(\neg(A \wedge B))$, $m \not\models A$, since $K \models A$. Hence $S(\neg A) \subseteq S(\neg(A \wedge B))$. Thus $S(\neg A) = S(\neg(A \wedge B))$. Then

$$
\begin{aligned}
& |\neg A| \cap S(\neg A) \subseteq |\neg(A \wedge B)| \cap S(\neg(A \wedge B)) \\
\text{iff}\quad & |K| \cup (|\neg A| \cap S(\neg A)) \subseteq |K| \\
& \qquad \cup(|\neg(A \wedge B)| \cap S(\neg(A \wedge B))) \\
\text{iff}\quad & t(|K| \cup (|\neg(A \wedge B)| \cap S(\neg(A \wedge B))) \\
& \qquad \subseteq t(|K| \cup (|\neg A| \cap S(\neg A))) \text{ by (t4)} \\
\text{iff}\quad & t(|K| \cup C(\neg(A \wedge B))) \subseteq t(|K| \cup C(\neg A)) \\
\text{iff}\quad & K^-_{A\wedge B} \subseteq K^-_A \qquad \text{by Definition 4.}
\end{aligned}
$$
□

Theorem 3: *Let $\mathcal{S}$ be any system of spheres in M_L centred on $|K|$. If a revision function K^*_A is defined as in Definition 5 then the postulates (K^*2) – (K^*8) are satisfied.*

Proof.

(K^*2) For any $m \in |A| \cap S(A)$, $m \models A$. So $A \in t(|A| \cap S(A))$. Hence by Definition 5, $A \in K^*_A$.

(K^*3) By (S2), $|K| \subseteq S(A)$. So $|K| \cap |A| \subseteq S(A) \cap |A|$. By (t4), $t(S(A) \cap |A|) \subseteq t(|K| \cap |A|)$. Hence by Definition 3 and Definition 5, $K^*_A \subseteq K^+_A$.

(K^*4) Suppose that $\neg A \notin K$. By the completeness theorem, for some $m \in M_L$, $m \models K$ and $m \not\models \neg A$, i.e., $m \models A$. By (S2), $|K|$ is the smallest sphere intersecting $|A|$. So $S(A) = |K|$. Hence $K^*_A = t(|A| \cap S(A)) = t(|A| \cap |K|) = K^+_A$. Hence $K^+_A \subseteq K^*_A$.

(K^*5) If $\vdash A$ then $|\neg A| = \phi$, so $C(\neg A) = \phi$. Hence $K^*_{\neg A}$ is trivial. If $\nvdash A$ then $|\neg A| \neq \phi$. Since there is at least one sphere intersecting $|\neg A|$ (viz., M_L itself), by (S4) there is a smallest such sphere $S(\neg A)$. So $|\neg A| \cap S(\neg A) \neq \phi$. Hence by (t3) and Definition 5, $K^*_{\neg A}$ is non-trivial. Hence $K^*_{\neg A}$ is trivial iff $\vdash A$.

(K^*6) If $\vdash A \leftrightarrow B$ then for all m in M_L, $m \models A$ iff $m \models B$. Hence $C(A) = C(B)$, and so $K^*_A = K^*_B$.

(K^*7) Since $\vdash A \wedge B \rightarrow A$, $|A \wedge B| \subseteq |A|$. Hence by Lemma 6

$$
\begin{aligned}
& S(A) \subseteq S(A \wedge B) \\
\text{iff}\quad & |A| \cap |B| \cap S(A) \subseteq |A| \cap |B| \cap S(A \wedge B) \\
& \qquad = |A \wedge B| \cap S(A \wedge B) \\
\text{iff}\quad & |t(|A| \cap S(A))| \cap |B| \subseteq |t(|A \wedge B| \cap S(A \wedge B))| && \text{by (EC1) and (t2)} \\
\text{iff}\quad & |K^*_A| \cap |B| \subseteq |K^*_{A\wedge B}| && \text{by Definition 5} \\
\text{iff}\quad & t(|K^*_{A\wedge B}|) \subseteq t(|K^*_A| \cap |B|) && \text{by (t4)} \\
\text{iff}\quad & K^*_{A\wedge B} \subseteq (K^*_A)^+_B && \text{by Definition 5 and (t1)}
\end{aligned}
$$

(K^*8) Suppose that $\neg B \notin K^*_A$. By the completeness theorem, for some $m \in M_L$, $m \models K^*_A$ and $m \not\models \neg B$, i.e., $m \models B$. So $|K^*_A| \cap |B| \neq \phi$. Hence by Definition 5 and (t2), $|A| \cap S(A) \cap |B| \neq \phi$, i.e., $|A \wedge B| \cap S(A) \neq \phi$. Hence $S(A \wedge B) = S(A)$, and so $S(A \wedge B) \subseteq S(A)$. Then as in ($K^*7$), $(K^*_A)^+_B \subseteq K^*_{A\wedge B}$. □

Relevant Systems

Theorem 4: *Let $\mathcal{S}_R$ be any system of spheres in M_R centred on $|K|$. If an expansion function K^+_A is defined as in Definition 3 then the postulates (K^+2) – (K^+6) are satisfied.*

Proof. The proof is the same as in the classical case, except that $\mathcal{S}$ and M_L are replaced by $\mathcal{S}_R$ and M_R respectively. □

Theorem 5: *Let $\mathcal{S}_R$ be any system of spheres in M_R centred on $|K|$. If a contraction function K^-_A is defined as in Definition 7 then the postulates (K^-2) – (K^-4) and (K^-6) – (K^-8) are satisfied.*

Proof.

($K^{-}2$) Proof is the same as in the classical case, except that $C(\neg A)$ is replaced by $C(\overline{A})$.

($K^{-}3$) Suppose $A \notin K$. By the completeness theorem, for some $m \in M_R$, $m \models K$ and $m \not\models A$. So by (S2), $|K|$ is the smallest sphere intersecting $\overline{|A|}$. Hence $|K| \cup C(\overline{A}) = |K|$. Hence $K^{-}_{A} = t(|K| \cup C(\overline{A})) = t(|K|) = K$.

($K^{-}4$) Since there is at least one sphere intersecting $\overline{|A|}$ (viz., M_R itself) by (S4)$'$ there is a smallest such sphere. So $C(\overline{A}) \neq \phi$. Hence $A \notin t(|K| \cup C(\overline{A}))$, i.e., $A \notin K^{-}_{A}$.[19]

($K^{-}6$) If $\vdash A \leftrightarrow B$ then for all m in M_R, $m \not\models A$ iff $m \not\models B$. So $C(\overline{A}) = C(\overline{B})$, and so $K^{-}_{A} = K^{-}_{B}$.

($K^{-}7$) Since $\vdash A \wedge B \rightarrow A$, $|A \wedge B| \subseteq |A|$. So $\overline{|A|} \subseteq \overline{|A \wedge B|}$. Hence by Lemma 6, $S(\overline{A \wedge B}) \subseteq S(\overline{A})$. So $\overline{|A|} \cap S(\overline{A \wedge B}) \subseteq S(\overline{A}) \cap \overline{|A|}$. Similarly, $\overline{|B|} \cap S(\overline{A \wedge B}) \subseteq S(\overline{B}) \cap \overline{|B|}$, since $\vdash A \wedge B \rightarrow B$. Then as in the classical case, $K^{-}_{A} \cap K^{-}_{B} \subseteq K^{-}_{A \wedge B}$, except that everything of the form $S(\neg A)$, $|\neg A|$, and $C(\neg A)$ is replaced by $S(\overline{A})$, $\overline{|A|}$, and $C(\overline{A})$ respectively.

($K^{-}8$) Proof is the same as in the classical case, except that everything of the form $S(\neg A)$, $|\neg A|$, and $C(\neg A)$ is replaced by its appropriate form. □

Lemma 2: *$K^{*}_{A} = K^{+}_{A}$ if the underlying logic is a relevant logic.*

Proof. Since $S(A) = |K|$, $K^{*}_{A} = t(|A| \cap |K|) = K^{+}_{A}$. □

Theorem 6: *Let $\mathcal{S}_R$ be any system of spheres in M_R centred on $|K|$. If a revision function K^{*}_{A} is defined as in Definition 5 then the postulates ($K^{*}2$) – ($K^{*}4$) and ($K^{*}6$) – ($K^{*}8$) are satisfied.*

[19] Note that the condition of ($K^{-}4$), $\not\vdash A$, is not required in the proof, and so what is proved here is stronger than ($K^{-}4$).

Proof.

(K^*2) $K^*_A = t(|A| \cap S(A)) = t(|A| \cap |K|)$. For any $m \in |K| \cap |A|$, $m \models A$. So $A \in t(|K| \cap |A|)$. Hence $A \in K^*_A$.

(K^*3) By Lemma 2, $K^*_A = K^+_A$. Hence $K^*_A \subseteq K^+_A$.

(K^*4) As in (K^*3), $K^+_A = K^*_A$. Hence $K^+_A \subseteq K^*_A$.[20]

(K^*6) If $\vdash A \leftrightarrow B$ then for all $m \in M_R$, $m \models A$ iff $m \models B$. Hence $C(A) = C(B)$, and so $K^*_A = K^*_B$.

(K^*7) $K^*_{A\wedge B} = K^+_{A\wedge B} = t(|A \wedge B| \cap |K|)$. And

$$\begin{aligned} (K^*_A)^+_B &= t(|B| \cap |K^*_A|) \\ &= t(|B| \cap |K^+_A|) && \text{by Lemma 2} \\ &= t(|B| \cap (|A| \cap |K|)) && \text{by Definition 5 and (t2)} \\ &= t(|A \wedge B| \cap |K|) \end{aligned}$$

Hence $K^*_{A\wedge B} = (K^*_A)^+_B$, and so $K^*_{A\wedge B} \subseteq (K^*_A)^+_B$.

(K^*8) As in (K^*7), $(K^*_A)^+_B = K^*_{A\wedge B}$. Hence $(K^*_A)^+_B \subseteq K^*_{A\wedge B}$.[21] □

Theorem 7: *Let $\mathcal{S}_R$ be any system of spheres in M_R centred on $|K|$. If a revision function K^*_A is defined as in Definition 5 then the postulates (K^*5) is not satisfied.*

Proof.

(K^*5) A counter-example: Let $K = Cn(p)$ and $A = p \rightarrow p$. Then $\vdash A$ and $K^*_{\neg A} = t(|\neg(p \rightarrow p)| \cap |p|)$ by Lemma 2. But then $q \notin K^*_{\neg A}$ for some q such that p and q are different propositional parameters, since $p, \neg(p \rightarrow p) \nvdash q$ in any standard system of relevant logic. Hence $K^*_{\neg A}$ is non-trivial. □

Positive-Plus Systems

Theorem 8: *Let $\mathcal{S}_{C_\omega}$ be any system of spheres in M_{C_ω} centred on $|K|$. If an expansion function K^+_A is defined as in Definition 3 then the postulates (K^+2) – (K^+6) are satisfied.*

[20] Note that the condition, $\neg A \notin K$, is not required in the proof.

[21] Note that the condition, $\neg B \notin K^*_A$, is not required in the proof.

Proof. The proof is the same as in the classical case, except that $\mathcal{S}$ and M_L are replaced by $\mathcal{S}_{C_\omega}$ and M_{C_ω} respectively. □

Theorem 9: *Let $\mathcal{S}_{C_\omega}$ be any system of spheres in M_{C_ω} centred on $|K|$. If a contraction function K^-_A is defined as in Definition 7 then the postulates (K^-2) – (K^-4) and (K^-6) – (K^-8) are satisfied.*

Proof. Proofs for (K^-2) – (K^-3) and (K^-6) – (K^-8) are the same as in the case of relevant logic except that M_R is replaced by M_{C_ω}. The proof for (K^-4) is the same as in the classical case, except that M_L is replaced by M_{C_ω} and everything of the form $|\neg A|$, $S(\neg A)$, and $C(\neg A)$ is replaced by its appropriate form. Also (S4)$'$ is used instead of (S4). □

Lemma 3: $K^*_A = K^+_A$ *if the underlying logic is* C_ω.

Proof. In virtue of the semantics of C_ω, for any sentences A and B, $|A| \cap |B| \neq \phi$. In particular, $|K| \cap |A| \neq \phi$. So $S(A) = |K|$. Hence $K^*_A = t(|A| \cap |K|) = K^+_A$. □

Theorem 10: *Let $\mathcal{S}_{C_\omega}$ be any system of spheres in M_{C_ω} centred on $|K|$. If a revision function K^*_A is defined as in Definition 5 then the postulates (K^*2) – (K^*4) and (K^*6) – (K^*8) are satisfied.*

Proof. The proof is the same as in the relevant case, except that $\mathcal{S}_R$ and M_R are replaced by $\mathcal{S}_{C_\omega}$ and M_{C_ω} respectively, and Lemma 3 is used instead of Lemma 2. □

Theorem 11: *Let $\mathcal{S}_{C_\omega}$ be any system of spheres in M_{C_ω} centred on $|K|$. If a revision function K^*_A is defined as in Definition 5 then the postulate (K^*5) is not satisfied.*

Proof.

(K^*5) A counter-example: Let $K = Cn(p)$ and $A = p \supset p$. Then $\vdash A$ and $K^*_{\neg A} = t(|\neg(p \supset p)| \cap |p|)$ by Lemma 3. But then $q \notin K^*_{\neg A}$ for some q such that p and q are different propositional parameters, since $p, \neg(p \supset p) \nvdash q$ in C_ω. Hence $K^*_{\neg A}$ is non-trivial. □

Theorem 12: *Let $\mathcal{S}_{C_n}$ be any system of spheres in M_{C_n} centred on $|K|$. If an expansion function K^+_A is defined as in Definition 3 then the postulates (K^+2) – (K^+6) are satisfied.*

Proof. The proof is the same as in the classical case, except that $\mathcal{S}$ and M_L are replaced by $\mathcal{S}_{Cn}$ and M_{C_n} respectively. $\square$

Theorem 13: Let $\mathcal{S}_{Cn}$ be any system of spheres in M_{C_n} centred on $|K|$. If a contraction function K^-_A is defined as in Definition 7 then the postulates (K^-2) – (K^-8) are satisfied.

Proof. Proofs for (K^-2) – (K^-4) and (K^-6) – (K^-8) are the same as in the case of C_ω, except that M_{C_ω} is replaced by M_{C_n}. The proof for (K^-5) is the same as in the classical case, except that M_L is replaced by M_{C_n} and everything of the form $|\neg A|$, $S(\neg A)$, and $C(\neg A)$ is replaced by its appropriate form. $\square$

Lemma 4: Let $|K| \cap \overline{|A^\circ|} \neq \phi$. Then $K^*_A = K^+_A$ if the underlying logic is $\mathsf{C_n}$ *($1 \leq n < \omega$).*

Proof. If $|K| \cap \overline{|A^\circ|} \neq \phi$ then $S(A) = |K|$. Hence $K^*_A = t(|A| \cap |K|) = K^+_A$. $\square$

Lemma 7: If $m \in \overline{|A|}$ then $m \in |\neg A|$ for any sentence A and $m \in M_{C_n}$.

Proof. The proof is immediate in virtue of the semantics of $\mathsf{C_n}$. $\square$

Theorem 14: Let $\mathcal{S}_{Cn}$ be any system of spheres in M_{C_n} centred on $|K|$. If a revision function K^*_A is defined as in Definition 5 then the postulates (K^*2) – (K^*4) and (K^*6) – (K^*8) are satisfied.

Proof. The proof is the same as in the classical case, except:

(K^*4) Suppose that $\neg A \notin K$. By the completeness theorem, for some $m \in M_{C_n}$, $m \models K$ and $m \not\models \neg A$. Then $m \in |K|$ and $m \in \overline{|\neg A|}$. So $m \in |A|$ by Lemma 7. Thus by (S2), $|K|$ is the smallest sphere intersecting $|A|$. Then as in the classical case, $K^+_A \subseteq K^*_A$.

(K^*8) Suppose that $\neg B \notin K^*_A$. By the completeness theorem, for some $m \in M_{C_n}$, $m \models K^*_A$ and $m \not\models \neg B$. Then $m \in |K^*_A|$ and $m \in \overline{|\neg B|}$. So $m \in |B|$ by Lemma 7. Hence $|K^*_A| \cap |B| \neq \phi$. Then as in the classical case, $(K^*_A)^+_B \subseteq K^*_{A \wedge B}$. $\square$

Theorem 15: *Let $\mathcal{S}_{Cn}$ be any system of spheres in M_{C_n} centred on $|K|$. If a revision function K^*_A is defined as in Definition 5 then the postulate (K^*5) is not satisfied.*

Proof.

(K^*5) A counter-example: Let $K = Cn(p)$, $A = p \supset p$. Then $\vdash A$. Also $|K| \cap \overline{|A^\circ|} \neq \phi$. Hence $K^*_{\neg A} = t(|\neg(p \supset p)| \cap |p|)$ by Lemma 4. But then $q \notin K^*_{\neg A}$ for some q such that p and q are different propositional parameters, since $p, \neg(p \supset p) \not\vdash q$ in C_n. Hence $K^*_{\neg A}$ is non-trivial. □

Non-Adjunctive Systems

Theorem 16: *Let $\mathcal{S}_D$ be any system of spheres in M_D centred on $|K|$. If an expansion function K^+_A is defined as in Definition 3 then the postulates (K^+2) – (K^+6) are satisfied.*

Proof. The proof is the same as in the classical case, except that $\mathcal{S}$ and M_L are replaced by $\mathcal{S}_D$ and M_D respectively. □

Theorem 17: *Let $\mathcal{S}_D$ be any system of spheres in M_D centred on $|K|$. If a contraction function K^-_A is defined as in Definition 7 then the postulates (K^-2) – (K^-4) and (K^-6) – (K^-8) are satisfied.*

Proof. The proof is the same as in the case of C_ω, except:

(K^-7) There are two cases.

i) $K \not\models A \wedge B$. Then $S\overline{(A \wedge B)} = |K|$. So $|K| \cup C\overline{(A \wedge B)} = |K|$. Hence

$$
\begin{aligned}
& |K| \cup C\overline{(A \wedge B)} \subseteq |K| \cup C\overline{(A)} \cup C\overline{(B)} \\
\text{iff}\quad & t(|K| \cup C\overline{(A)} \cup C\overline{(B)}) \subseteq t(|K| \cup C\overline{(A \wedge B)}) && \text{by (t4)} \\
\text{iff}\quad & t((|K| \cup C\overline{(A)}) \cup (|K| \cup C\overline{(B)})) \subseteq t(|K| \cup C\overline{(A \wedge B)}) \\
\text{iff}\quad & t(|K| \cup C\overline{(A)}) \cap t(|K| \cup C\overline{(B)}) \subseteq t(|K| \cup C\overline{(A \wedge B)}) && \text{by (t5)} \\
\text{iff}\quad & K^-_A \cap K^-_B \subseteq K^-_{A \wedge B}.
\end{aligned}
$$

ii) $K \models A \wedge B$. This part of the proof requires epistemic entrenchment which is not provided in this paper. So the proof is not given here. None the less, it can be shown that (K^-7) is satisfied. □

Theorem 18: *Let $\mathcal{S}_D$ be any system of spheres in M_D centred on $|K|$. If a revision function K^*_A is defined as in Definition 5 then the postulates (K^*2) – (K^*4) and (K^*6) are satisfied.*

Proof. The proof is the same as in the case of C_n, except that M_{C_n} is replaced by M_D. □

Theorem 19: *Let $\mathcal{S}_D$ be any system of spheres in M_D centred on $|K|$. If a revision function K^*_A is defined as in Definition 5 then the postulates (K^*5) and (K^*7) are not satisfied.*

Proof.

(K^*5) A counter-example: Let $A = \Box\neg p \vee p$. Then $\vdash A$, since $\vdash_{S5} \Diamond(\Box\neg p \vee p)$.[22] But $|\neg A| \neq \phi$, since $\nvdash_{S5} \neg\Diamond\neg(\Box\neg p \vee p)$. Now since there is at least one sphere intersecting $|\neg A| = |\neg(\Box\neg p \vee p)|$ (viz., M_D itself), by (S4)′, there is a smallest such sphere $S(\neg(\Box\neg p \vee p))$. So $C(\neg(\Box\neg p \vee p)) \neq \phi$. Hence by (t3) and Definition 5, $K^*_{\neg A}$ is non-trivial.

(K^*7) A counter-example: Let $A = p$ and $B = \neg p$. Since $|p \wedge \neg p| = \phi$, $C(p \wedge \neg p) = \phi$. Thus $K^*_{p\wedge\neg p}$ is trivial. But for any $m \in |t(|p| \cap S(p))| \cap |\neg p|$, $m \not\models p \wedge \neg p$. Hence $p \wedge \neg p \notin t(|t(|p| \cap S(p))| \cap |\neg p|) = (K^*_p)^+_{\neg p}$. So $(K^*_p)^+_{\neg p}$ is non-trivial. Hence $K^*_{p\wedge\neg p} \not\subseteq (K^*_p)^+_{\neg p}$. Thus $K^*_{A\wedge B} \not\subseteq (K^*_A)^+_B$. □

Levi Identity and Relevant Logics

Lemma 8: $(K^+_\alpha)^+_\beta \subseteq K^+_{\alpha\wedge\beta}$.

Proof. By (K^+3), $K \subseteq K^+_{\alpha\wedge\beta}$. So $(K^+_\alpha)^+_\beta \subseteq ((K^+_{\alpha\wedge\beta})^+_\alpha)^+_\beta$ by (K^+5) twice. By (K^+2), $\alpha, \beta \in K^+_{\alpha\wedge\beta}$, since $\vdash \alpha \wedge \beta \rightarrow \alpha$ and $\vdash \alpha \wedge \beta \rightarrow \beta$. So $((K^+_{\alpha\wedge\beta})^+_\alpha)^+_\beta = K^+_{\alpha\wedge\beta}$ by (K^+4) twice. Hence $(K^+_\alpha)^+_\beta \subseteq K^+_{\alpha\wedge\beta}$. □

[22] This can be shown by the following:

	$\Box A \supset A$	Reflexivity axiom
iff	$\Box\Diamond p \supset \Diamond p$	
iff	$\neg\Box\Diamond p \vee \Diamond p$	
iff	$\Diamond\Box\neg p \vee \Diamond p$	
iff	$\Diamond(\Box\neg p \vee p)$.	

I owe this proof to Tim Surendonk.

Theorem 20: *If a revision function K^*_α is defined as $K^*_\alpha = (K^-_{\neg\alpha})^+_\alpha$ then the postulates (K^*2)–(K^*4), (K^*6) and (K^*8) are satisfied.*

Proof.

(K^*2) The result follows from (K^+2).

(K^*3) By (K^-2), $K^-_{\neg\alpha} \subseteq K$. Hence by (K^+5), $(K^-_{\neg\alpha})^+_\alpha \subseteq K^+_\alpha$.

(K^*4) Suppose $\neg\alpha \notin K$. Then $K^-_{\neg\alpha} = K$ by (K^-3). Hence $K^+_\alpha = (K^-_{\neg\alpha})^+_\alpha$. Then the result follows.

(K^*6) Suppose $\vdash \alpha \leftrightarrow \beta$. Then $\vdash \neg\alpha \leftrightarrow \neg\beta$. So $K^-_{\neg\alpha} = K^-_{\neg\beta}$ by (K^-6). Hence $(K^-_{\neg\alpha})^+_\alpha = (K^-_{\neg\beta})^+_\beta$.

(K^*8) Suppose that $\neg\beta \notin (K^-_{\neg\alpha})^+_\alpha$. Then $\neg\beta \notin K^-_{\neg\alpha}$. Also $\neg\alpha \notin K^-_{\neg\alpha}$. Hence by the primeness of a belief set, $\neg\alpha \vee \neg\beta \notin K^-_{\neg\alpha}$. Since $\vdash \neg\alpha \leftrightarrow (\neg\alpha \vee \neg\beta) \wedge \neg\alpha$, $K^-_{\neg\alpha} = K^-_{(\neg\alpha\vee\neg\beta)\wedge\neg\alpha}$ by (K^-6). So by (K^-8), $K^-_{\neg\alpha} = K^-_{(\neg\alpha\vee\neg\beta)\wedge\neg\alpha} \subseteq K^-_{\neg\alpha\vee\neg\beta} = K^-_{\neg(\alpha\wedge\beta)}$. Hence $(K^-_{\neg\alpha})^+_{\alpha\wedge\beta} \subseteq (K^-_{\neg(\alpha\wedge\beta)})^+_{\alpha\wedge\beta}$ by (K^+5). Hence $((K^-_{\neg\alpha})^+_\alpha)^+_\beta \subseteq (K^-_{\neg(\alpha\wedge\beta)})^+_{\alpha\wedge\beta}$ by Lemma 8.[23] □

Theorem 21: *If a revision function K^*_α is defined as $K^*_\alpha = (K^-_{\neg\alpha})^+_\alpha$ then the postulates (K^*5) and (K^*7) are not satisfied.*

Proof.

(K^*5) A counter-model: Let $K = Cn(p)$ and $\alpha = p \rightarrow p$. Then $\vdash \alpha$, and $p, \neg(p \rightarrow p) \in (K^-_\alpha)^+_{\neg\alpha}$ by (K^+2). But $q \notin (K^-_\alpha)^+_{\neg\alpha}$ for some q such that p and q are different propositional parameters, since $p, \neg(p \rightarrow p) \nvdash q$ in any standard system of relevant logic. Hence $(K^-_\alpha)^+_{\neg\alpha}$ is non-trivial.

[23] The proofs given here are the same as the classical proofs. See Gärdenfors (1988), p. 215.

(K^*7) The proof requires more machinery than we have got in this paper. So the proof is not provided here. None the less, it can be shown that (K^*7) is not satisfied.[24] □

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