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Inventing Logic: The Löwenheim-Skolem Theorem and First- and Second-Order Logic

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INTRODUCTION

On its own, the Löwenheim-Skolem theorem – as well as the ensuing Skolem Paradox – is of scant mathematical, logical, and even philosophical significance. It is of no consequence to practicing mathematicians, admits of extant resolutions at the logical level and is sufficiently vague to support all sorts of philosophical arguments.[*] Nevertheless, much enquiry and ink has flowed on the matter, and we intend here to contribute to this muddy river, all the while trying to avoid waxing poetic or waning technical. In this essay, we purport to use the paradox as a paradigm by which we mean to evaluate the role of first- and second-order predicate logic in a post-foundational context. As there is no reason to prefer one kind of logic over another unless we specify to what purpose we intend to use it, we shall argue that if we want mathematical logic to axiomatize, describe and couch the language of informal mathematical practice, then second-order logic yields more *intuitively appropriate* models. As well, since the Skolem paradox is a problem purely for standard model-theoretic semantics, our secondary purpose will simply be to show through this example how modern model-theoretic results can illuminate our current understanding of what logic is, as well as of what we conceive the role of logic to be in the greater scheme of human understanding. Indeed it is our over-arching intention to ponder the inherent limitations and, paradoxically, the intrinsic openness of human thought.

[*] Most notably, within Skolem's own expoundings on the subject. For a succinct rundown of this topic, we refer the reader to George (1985).

1.0 WHAT LOGIC?

Philosophic logic is generally understood as the formalization of vernacular languages. There are in fact many types of such formalizations: syllogistic, propositional and modal logics all attempt to capture the syntactic structure of speech patterns -- that is, of thought itself.* Much of modern logic, however, arose in a mathematical context, more precisely in the context of the foundational crisis of the early twentieth century. As a preamble to our discussion on the (epistemo)logical implications of the Löwenheim-Skolem theorem on the axiomatization of mathematics, we shall first briefly review the nature of mathematical logic as a peculiar branch of logic. As well, we shall find it worthwhile to touch upon the distinction and relation between model-theoretic semantics and proof-theoretic syntax which is crucial to our argument. Finally, we shall review in this section the characteristics of first- and second-order logic (and by extension, higher-order logic) – which is ultimately the object of our paper.

1.1 POST-FOUNDATIONAL MATHEMATICAL LOGIC

Nowadays, mathematical logic can be defined as the attempt to achieve an adequate formalization of mathematical language; alternatively, it can be considered as the study of the deductive and expressive power of formal theories. Historically, mathematical logic owes its inception to the search for the foundations of mathematics. The crisis in the foundations of mathematics was wrought with vigorous debate about purely abstract entities such as the uncountably infinite sets evoked by Cantor's Theorem. During the foundational crisis, logicism aimed to reduce all of mathematics to fundamental logical 'laws' of thought, to be expressed through an ideal and closed

* We must here note that there is a strong positive feedback effect between the thought expressed through language and the language that underlies the thought.

formal logical system with explicit axioms sufficient to characterize all abstract mathematics -- initially considered to be part and parcel with set theory. Set theory, however, was rife with contradictions and paradoxes; the first axiomatic wave in the field of logic focused quite exclusively on smoothing out and formalizing Cantor's paradise – work which was taken up notably by Hilbert, Russell, Whitehead, Zermelo, Skolem, and Fraenkel (Crossley et al. 1972, p. 5; Ferroiros 2001, p. 471).

Despite the obsolescence of foundational studies proper, this work remains a driving force behind much research in mathematical logic and many logicians and mathematicians still explicitly or implicitly harbour hope that some kind of non-absolute foundation can still be achieved. Most notably, the ghost of Hilbert's programmes can still be glimpsed in debates into the 'nature' of mathematical logic. As it had once been hoped that mathematics could be reduced to logic, it seems that the ideal that logic should be devoid of any mathematical content or presuppositions is still circulating: logic should formalize mathematics, but if mathematics impregnates logic, we arrive at a vicious circle. On the other hand, it is not clear that there is a non-ideological reason why mathematics must rest on a foundation that is itself non-mathematical in nature.[*] Hence, while we shall not discuss foundational studies in this paper, we do note that many prejudices still seem to hold: 'logic' logic is equated with first-order logic, and 'mathematical' logic is equated with second-order logic – which

[*] In fact, the reverse statement sounds rather absurd barring convincing non-ideological reasons. Evidently, much has been made of the presence or lack thereof of a border between logic and mathematics; generally, however, mathematical logic is considered a subdiscipline of both mathematics and logic. While some maintain that even a fuzzy border is a border – a border that restricts each discipline to its own sphere – we are not of that opinion. If anything, the haziness of this border renders it *a priori* 'undecidable' to which domain a 'foundation' (or, even a simple axiomatization) of mathematics must be sought. Regardless, as Boolos has remarked, if all the other sciences presuppose themselves, why shouldn't mathematics? See Boolos (1975, p. 517), Gauthier (1976, pp. 294-6), and Shapiro (1999, pp. 51-54).

is itself conflated with set-theory; Many important philosophical issues concerning the nature of logic itself surround the debate between first- and second-order logic – to which we shall soon turn our attention, after having first delved into the semantics/syntax dichotomy that pervades modern logic (Shapiro 1985, p. 742).

1.2 MODEL THEORETIC SEMANTICS

Although modern logic arose during the *Grundlagenkrise der Mathematik*, its crux occurred after its falter. We thus find it impossible to delineate the scope of what logic is, what it can do, and what we would like it to do, without wording it in proof- and model-theoretic terms. All mathematical logic consists of two parts: a formal language governed by recursive proof-theoretic syntax and an informal language governed by descriptive model-theoretic semantics. The formal language is ideally uncontaminated by the content or meaning of the linguistic sentences it codifies: the archetypal formal system is a sturdy logical skeleton that wavers not in the winds of philosophical opinions. The informal semantics is the interpretation of the theory that reflects unto the language its truth conditions.

As the Löwenheim-Skolem theorem is a model-theoretic problem, it is this aspect of the logical conundrum that retains our attention. Models accomplish the linguistic reference-fixing of the consequence relations delineated by the deductive system. At times, these models may reveal the defects in our logical systems that are in need of adjustments, though at others they may expose flaws in our broader epistemological systems instead. Indeed, since Tarski, model-theory has often been understood as a theory *about* truth. This is perhaps why model-theory has come to dominate and define mathematical logic.* But if "formal language is to model-theory what language is

* Harold Hodes even goes so far as to claim "truth in the model is a model of truth" (Shapiro 1999, p. 44).

to the world" (Shapiro 1999, p. 44), and if the world 'contains' the truth that the language seeks to reconstruct, then an analysis of the model brings to light not only what we know, but also what we *can* know, what we *cannot* know, as well as *how* we know (Gauthier 1976, pp. 121-2 and 218; Shapiro 1999, pp. 43-4, 56).

1.3 FIRST- AND SECOND-ORDER LOGIC

In this paper, we are concerned solely with first- and second-order predicate calculus with standard model-theoretic semantics. First-order predicate logic (henceforth, 'FOL') consists of a given non-empty domain *d* within which a countable infinity of quantified variables range over the individual elements. Standard first-order model-theoretic semantics are fundamentally characterized by completeness, compactness and the Löwenheim-Skolem theorem (henceforth, 'LST'). After the metalogical conclusions of Gödel's enquiries, FOL became highly valued for its ability to generate a full deductive system. This appraisal reflects the strong proof-theoretic tradition that suffuses all of mathematics: there can be no results without proof. For these reasons, first-order calculus is still the *de facto* fundamental logical language (Ferreiros 2001, p. 470; Gauthier 1976, pp. 64, 123; Manzano 1996, p. 112; Tharp 1975, pp. 4, 7).

Second-order predicate logic (abbreviated 'SOL') with standard model-theoretic semantics have interpretations wherein additional 'second-order' quantified variables range over all of the subsets of elements of a given non-empty domain *d*, along with the standard 'first-order' variables that range over its objects alone. Most early logicians, like Zermelo, initially worked purely with second-order logic. First-order logic was an offshoot of the original predicate logic and was later championed by logicians like Skolem for mostly ideological reasons – but practical reasons as well: first-order logic is more easily and crisply manipulated (Crossley et al. 1972, p. 5; Ferreiros 2001, p. 471). Since Gödel's incompleteness theorem shows that compactness and LST inherently fail in second-order axiom systems, no standard model-theory of SOL can produce a complete deductive system (Gauthier 1976, p. 64; Shapiro 1985, p. 714; Shapiro 1999, p. 42).*

Although it is generally considered that FOL trades expressive power in the name of securing clear epistemic gains, there is a damper: a complete deductive system alone neither provides knowledge nor generates new knowledge (Gauthier 1976, p. 218; Jané 1993, p. 67; Shapiro 1999, pp. 44-5). Nonetheless, the other side of the coin is that SOL is overly expressive: it has been claimed that, by saying everything, it in fact says nothing. Indeed, though its incompleteness lends itself to dizzying heights of expressive potential, its deductive mechanisms are cumbersome and often bewildering. For this reason, it is SOL that must bear the burden of proof (Gauthier 1976, p. 123; Jané 1993, p. 71; Manzano 1996, pp. 5, 60-2, 112).†

2.0 WHAT PARADOX?

One of the earliest meta-mathematical results to arise from model-theoretic research into FOL was the Löwenheim-Skolem theorem. In this section, we shall first explore the Löwenheim-Skolem theorem, including the upward and downward expansions of the theorem which we owe to Tarski. Then, we shall touch upon the ensuing Skolem paradox that arises in the semantic interpretations of first-order models,‡ as well as its basic philosophic import. The paradox

* On the other hand, some non-standard second-order models *do* have all three of these properties – most notably, Henkin semantics. However, all the same objections (including the ones that derive from the presence of LST) that we will address later hold for these second-order models (Gauthier 1976, p. 123; Shapiro 1985, p. 715).

† The problem with FOL is that its strong syntax cannot generate results that constitute new knowledge, whilst the problem with SOL is that its strong semantics do not allow logicians to show which of its disparate results constitute knowledge.

‡ Given that LST does not hold in second-order theories, Skolem's paradox cannot arise in second-order model-theoretic semantics.

not being a true antinomy, and it being but of minor significance to mathematicians, we shall describe a few solutions of which it admits within the framework of metamathematical logic. The impact of LST on our understanding of first- and second-order predicate logic, as well as on logic and epistemological possibilities themselves, will be addressed in the next chapter.

2.1 THE LÖWENHEIM-SKOLEM THEOREM

As we have seen, the Löwenheim-Skolem property – along with completeness and compactness – is a hallmark of first-order models. Quite simply, the Löwenheim-Skolem theorem states that any theory that is consistent (i.e., that has a model) has a countable model. Whilst prior to Gödel's completeness and compactness theorems,[*] the Löwenheim-Skolem theorem is nevertheless its immediate consequence, as it states that if a formula is satisfiable (i.e., has a model), then it is satisfiable within a countable domain. However, given Gödel's accompanying compactness theorem, LST follows from the completeness theorem in such a way that it is possible to derive the upwards and downwards Löwenheim-Skolem-Tarski theorems (or 'LSTT'). The original theorem, as refined by Tarski, can thus be formulated in two versions: a) the upward Lowenheim-Skolem-Tarski theorem and b) the downward Löwenheim-Skolem-Tarski theorem. The upwards LST states that if a theory has any model of infinite size, then that theory also has a model whose domain is the same size as an infinite set *A*; in other words, a satisfiable set of sentences always has a model of a greater infinite cardinal. As a corollary, the downward LST proves that if M is a model of cardinality K and if λ is a cardinality smaller than K, than M has a submodel of cardinality

[*] Gödel's completeness theorem states that FOL mechanically produces all the valid logical formulas that follow from its axioms, such that any logical expression is either satisfiable or refutable in a model. His compactness theorem states that a set of sentences in FOL has a model if and only if all of its finite subsets also have a model (Crossley 1972, p. 7).

λ which satisfies the same theory as M itself; in other words, every satisfiable sentence has at most a countable model (Shapiro 1985, p. 714; Schoenfield 2001, p. 79).

By Lindström's theorem – one of the pioneering technical results of model-theory – LST is required if we want a strong first-order theory.[*] However, the presence of the Löwenheim-Skolem property means that first-order theories cannot manage the cardinalities of its infinite models, in such a way that its models are not categorical – that is, they are not isomorphic. As a consequence, while LST plays an important role in proving the strength and completeness of FOL, it nevertheless has the serious unintended consequence of paving the way for Gödel's 1931 incompleteness theorem wherein (through a process now known as the 'arithmetization' of logical syntax) Gödel proved that FOL was not strong enough to either prove or disprove the formulas of classic mathematics, such as arithmetic. FOL is incomplete in regards to the theory of natural numbers based on Peano's well established axioms,[†] and this is a highly problem-

[*] Lindström's theorem establishes that if a model of any given logic *L* has either Löwenheim-Skolem-Tarski property and is also compact (and, therefore, complete), then *L* is equivalent to first-order logic. FOL is then equated with the maximal (strongest) logic possible – but only given these provisions. Indeed, the characterization of the 'strength' of FOL as defined by this theorem is dependant upon the presumption that LST (and compactness, and completeness) are *essential* characteristics of sound logic: if they are not, then invoking Lindström's theorem amounts to begging the question (Tharp 1975, pp. 4-9). The question of elucidating whether or not LST is a necessary feature of logic (which would then imply that FOL is the strongest logic period) is by no means one we are prepared to answer, but it is a question we will confront as we weigh the undesirable consequences of the Löwenheim-Skolem property against those of a logical system that does not possess it.

[†] The incompleteness theorem also had considerable ramifications on the burgeoning discipline of proof-theory – which has led, amongst other things, to the universally accepted (yet unprovable) Church-Turing thesis which states that the absolute undecidability of a formula can be decided via algorithms. This theorem was crisply trailed by a whole slew of undecidability results concluding that the whole of mathematics as well as a host of basic problems – such as the Continuum Hypothesis itself (when combining Gödel's and Cohen's consistency results) – were essentially undecidable. With these results arose for the first time a fundamental problem of consistency within elementary number theory. Of

atic situation. Indeed, LST has been deemed the first of the modern incompleteness theorems, casting shadows on the assumption that FOL can be the kind of strong logic that mathematical theory and practice can rely on (Kleene 1952, p. 427).

A final note: because of the lack of completeness and compactness of second-order theories, some theoreticians will admit only FOL as 'valid' logic. However, the lack of a Löwenheim-Skolem property in second-order models of these theories is not actually considered a problem: rather, it is thought of as a favourable characteristic – for reasons we shall see shortly. As such, while the Löwenheim-Skolem theorem and Skolem's paradox cannot quite be considered a logical deal-breaker, it is a tipping-point – a genuine model-theoretic problem. Of course, to understand how model-theoretic semantics can influence our understanding of the formal theories themselves and further delineate the scope of logical enquiries, it shall be necessary to describe the ensuing Skolem paradox.

2.2 SKOLEM'S PARADOX

The Löwenheim-Skolem theorem leads to a simple paradox: the appearance of a seeming contradiction between the Löwenheim-Skolem theorem (which proves that if a formula or list of formulas is satisfiable, then it is $\aleph_0$-satisfiable) and Cantor's theorem (which proves the existence of non-denumerable sets of cardinality $2^{\aleph_0}$). Overtly, since we have a countable theory that proves the undenumerability of some sets, how can the ensuing model account for the existence of uncountable objects within its countable domain? For we then have countable models for axiom systems that are intended to structure uncountable domains – which is unseemly. The

course, all of these considerations are also of consequence within the mathematical and philosophical discourse on the nature of truth (Boolos 1975, p. 523; Crossley 1972, pp. 7-10; Kleene 1952, pp. 300-1, 317-8, 436). As seminal as Gödel's incompleteness results are, we regret that we can provide no more space to the subject in this present paper.

paradox confronts us with the reality that any axiomatization of set theory employing a denumerable amount of formal axioms will fail to render and characterize the absolute concepts of set, power-set, bijection, non-denumerability, etc., which are the very fundamental set-theoretic notions that we intend mathematical logic to explain (Kleene 1952, p. 427; Kleene 1971, pp. 326-9).

Furthermore, as already alluded to, the upward and downward LST reveal the basic non-categoricity of first-order models, further reinforcing the non-absoluteness of set-theoretical notions in FOL. This has led to a (relatively) marginal meta-mathematical and philosophical conclusion called Skolemism. Skolemism is the idea that set-theoretical notions are inherently relativistic: what is non-denumerable in one interpretation of a formal system may be denumerable in another, as there is no prior absolute definition of non-denumerability; instead, uncountability is a property relative to a given model, not of the formal system.* For the Skolemite, the paradox proves the model-relative nature of all cardinality results. Skolem's paradox is thus a problem specifically for model-theory: it paradoxically reveals that a complete and compact formal axiom system produces many interpretations – including unintended ones that rub against the grain of the formal theory we thought we were building up. LST thus inherently provides the impetus for non-standard models (Crossley et al. 1972, pp. 6, 29; Ferreiros 2001, p. 472; Kleene 1952, p. 427).† Of course, to a certain extent, this is natural: the scientific method has a sneaky way of crushing our hunches, hypotheses and our desires;

* Though, to be fair, it is hotly contested whether we even have a prior solid reason to believe that these are notions that can be rendered at all, given set theory itself is in need of a formal 'foundation' (Jané 1993, pp. 78-83; Shapiro 1999, p. 58). However, it is not our aim to engage in foundational debates, and so this will remain here a moot point.

† The first of these models was constructed by Skolem himself. It is through the work of Henkin however that such non-standard models are revealed to be an inherent consequence of the model-theoretic relativity of complete and compact FOL with Löwenheim-Skolem properties (Crossley et al. 1972, p. 29).

but applied to model-theory, the consequences of LST seem to reveal that this is a problem *within* even the most rigid application of the scientific method itself.

2.3 SOLUTIONS TO SKOLEM'S PARADOX

Skolem's paradox has definite solutions available, whether we like these particular resolutions or not. As it is not an antinomy (but is, rather, a mere incongruity), it may be brushed aside by informal mathematic practice and, indeed, by much of formal mathematics. And therein lies a distinction between the logician and the mathematician, for while the mathematician is content to work with logical concepts and models for purely investigative and constructive reasons, it is the logician who must concern himself with the ontic starting point of the concepts and models invoked (Klenk 1976, pp. 476, 479). However, logic itself cannot produce any ontic knowledge without subscribing to an underlying ideological position postulating its prior existence. Therefore, we shall explore here only the solutions pertinent to the model-theoretic semantics of first- and second-order theories.

2.3.1 WITHIN FIRST ORDER LOGIC

The upward Löwenheim-Skolem-Tarski theorem fix

The aforementioned upward and downward versions of LST comprise what may be considered a reflection schema, wherein the upward version correlates to the ascending reflection of all acceptable models of higher infinite cardinality and the downward theorem represents a descending reflection of all admissible models of lower (in)finite cardinality. Viewed in this light, a dialectical interpretation through the upward and downward variants saves the Lowenheim-Skolem theorem from itself, with no need to step outside a first-order theory. Thus, within FOL, the upwards Löwenheim-Skolem-

Tarski theorem may be invoked to dispel the paradox and show that the continuum can be built up through an ascending reflection of the countable models which underlie it. Indeed, this may well account for the hypothesis that denumerable models are sufficient to describe set-theory (Gauthier 1976, pp. 297-8; Klenk 1976, pp. 475, 479, 485). However, the upward LSTT has its own host of drawbacks, which shall be succinctly addressed in the next section.

The misinterpretation interpretation

This solution can also be dubbed the 'much ado about nothing solution'. Indeed it may be claimed that while the model-theoretic interpretation understands the existential and universal quantifiers to range only over the domain of M, the observing logician 'intuitively' understands instead that '$\exists x$' and '$\forall x$' to range over the *entire* set-theoretic universe. In other words, while only $\aleph_0$ elements can be 'observed' within the model's domain, the full spectrum $2^{\aleph_0}$ can be 'observed' from without. However, within the countably infinite model, the infinite sets actually are countable – they may really be placed in bijection with the natural numbers. The apparent paradox arises simply because this bijection does not actually occur from within the perspective of the model. The denumerable model thus internally satisfies the notion of 'non-denumerable set', though the model is in fact countable to the external observer of the model. This is how a countable model can be said to sufficiently describe the continuum. Indeed, in retrospect, it seems rather trite to say that if M is countable – and the paradox is thus rendered somewhat banal. Of course, this is still an unexpected and unsettling explanation to the logician as it leaves him or her with no compass by which he may explain his own notion of (non-)denumerability (Crossley et al. 1972, pp. 6, 29; Kleene 1952, p. 426).

The non-standard model-theoretic semantics way out

Of course, it can also simply be accepted as fact that there are standard and non-standard models. It could very well be that we commonly use a particular arithmetic, but that there might be a manifold of consistent arithmetic that we could also work with in a sound manner. In fact, mathematicians do frequently build and explore non-standard models of theories, some mathematicians become specialists of these models, and some believe that one or more non-standard models are in some shape or form 'better' than non-standard ones.* Skolem's paradox isn't an antinomy, it is then just an unexpected consequence: one we must learn from. In actuality, there is no Platonist or intuitive conception of 'set-theory' as either an ideal fact or a fixed intuitive notion to which we ought to fit our model-theoretic semantics. Rather, we develop and construct what set-theory really is or is not based on the results achieved through axiomatization, model-theory and, especially, proof-theory. Nothing *beyond* what is proved is of any scientific substance; if a consistent, complete and compact logic necessarily entails LST and the Skolem 'paradox', then *that* is logic, *that* is reality. There is simply no other way to proceed than to attempt to solve 'known' problems – by any means necessary.† The costs of this resolution will be elucidated in the fourth section.

2.3.2 WITHIN SECOND-ORDER LOGIC

* 'Better' need not mean absolutely better. A non-standard model might simply be better at capturing a certain use or a certain possible application of a theory. As an analogy, we can truthfully say that the duodecimal notation is 'better' for counting, that a non-simple continued fraction is 'better' at expressing π, or that the binary system is 'better' for building Turing machines. Again, 'better' always refers to a certain context.

† The fix just mentioned smoothly recoups the constructivist position. In fact, according to McCarty and Tennant, Skolem's paradox does not actually arise at all in constructivist practice. Indeed, it is a basic underlying assumption among constructivists and some intuitionists that Cantor's theorem and the notion of 'non-denumerability' have no place in logic, as they cannot be constructed in the rigorous sense of the word. At best, it is possible to suspend belief as to the matter, but it is still widely believed within constructivist circles that even countable infinity is *de juris* unconstructible. Evidently, such a position entails that LST – and therefore Skolem's paradox itself – is unconstructible (McCarty & Tennant 1987).

As we know, LST only applies to first-order formal theories. It does not apply either to second-order structures, or to the informal mathematics that first-order model-theoretic semantics intends to capture. LST does not apply to SOL because SOL does not itself produce LST, and thus it does not produce the infamous paradox. Furthermore, second-order models of ZFC were shown in 1930 by Zermelo (himself an early champion of SOL) to accurately interpret the informal set-theoretic notions of cardinality and power-set, something with which first-order axiomatizations struggle. Furthermore, SOL characterizes up to isomorphism the classic theory of natural numbers as established by Peano's axioms (Shapiro, Second-Order Lang.). The details of this (and other benefits) incurred by SOL due to its lack of LST shall be dealt with later on in the next section.

3.0 WHAT SEMANTICS?

As we have seen, LST and Skolem's paradox are intrinsically tied to FOL. But as we have also seen, FOL is intrinsically tied to LST by Lindström's theorem.* Though the paradox is not a strict contradiction of the sort that poses a strong problem for mathematics itself, it does highlight some of the inescapable limitations of any first-order axiomatization of mathematics. Most immediately, the question arises as to whether there is an inherent contradiction in first-order model theory, and what this contradiction tells the respective logical and mathematical communities. Indeed, it seems to us that the paradox gives us a clearer idea of what logic can or cannot do, but most importantly, whether the interest lies in a secure but restrictive FOL, or in a treacherous but expressive SOL. This section shall thus review what the Löwenheim-Skolem theorem can tell us about first- and second-order logic.

* See footnote on p. 34.

3.1 THE LÖWENHEIM-SKOLEM THEOREM IN FIRST-ORDER LOGIC

A limitation imposed on FOL by the upward and downward LST is that it is in fact impossible from a model-theoretic perspective to say what the cardinality of the underlying model is: though we can say that it is infinite, we cannot say in an absolute way whether the model is actually imbedded in an uncountable model. Of course, the upwards Löwenheim-Skolem-Tarski states that if the string of formulas has a model, it admits also of models of every infinite cardinality K – but this does not allow us to deduce whether the countable model is characterizing an uncountable domain which is itself 'really' uncountable (Crossley et al. 1972, p. 29; Shapiro 1985, pp. 716-9).*

Given the ubiquity of the concept of non-denumerable sets in mathematics, it is expected of mathematical logic that its language allows for an expression of this property, as well as many others that are essential to mathematical theory. Yet applied to first-order logic, LST is tantamount to stating that FOL is not powerful enough to generate an enumerating function capable of expressing the concept of non-denumerability: as such, what appears to be non-denumerable within M, is in fact denumerable from without (Kleene 1952, p. 426; Klenk 1976, p. 475). In fact, the presence of LST renders FOL's semantics inadequate even to allow for a satisfactory expression of the basic arithmetical principle: Peano's axiom of induction. Also, proof-theory being inexorably intertwined with model-theoretic semantics, LST thus has indelible consequences on the strength of FOL as a deductive system: despite completeness, the range of what it can prove is so limited as to render it rather an absurd choice for the description of mathematical practice (Boolos 1975, p. 521; Jané 1993, pp. 68, 70; Shapiro 1985, pp. 716, 722, 727).

* This idea can be expressed something like stating that there exists an uncountable 'universe' outside the domain of the countable model at hand, a universe of which the model captures only a piece (Crossley et al. 1972, p. 70). But then that raises the question: can we provide a model for this universe?

3.2 THE LÖWENHEIM-SKOLEM THEOREM IN SECOND-ORDER LOGIC

Because LST fails in SOL, its model-theoretic semantics allow for the convenient and sound expression of countable and uncountable sets, since non-denumerability *is* characterizable in second-order model-theoretic semantics: the formula expressing the uncountable domain simply admits of no countable model. As such, SOL interprets 'correctly' our intuitive set-theoretic ideas. Furthermore, the crucial axiom of induction in ZFC, as well as Peano's axioms and the axioms of separation and replacements, are also best expressed at the second-order.[*] Arithmetic, real and complex number analysis, Euclidean spaces, real vector spaces... the list of theories uniquely and exactly expressible through second-order model-theoretic semantics is far too extensive to be dismissed out of hand (Boolos 1975, pp. 521-2; Jané 1993, pp. 68-71, 79-81; Manzano 1996, pp. 4-5; Shapiro 1985, pp. 722, 727, 729-30).

Another benefit conferred unto SOL by its lack of LST is its categoricity results. Because SOL is not compact and is not imbued with the Löwenheim-Skolem property, its models are categorical; for example, it can characterize the sets of natural and real numbers – and many other infinite structures – up to isomorphism (Shapiro 1985, p. 714; Shapiro 1999, p. 43). This may very well prove to be a greater strength than the syntax of FOL; after all, to correctly describe a structure is to describe it categorically, and second-order model-theoretic semantics does this for denumerable and non-denumerable structures, as well as most of the structures of classical mathematic theory. Categoricity also seems more essential than completeness when considering the interdisciplinarity of the mathematical subfields, as exemplified by the common practice of embedding specific

[*] Of course, this and other axioms may be 'replaced' with first-order axiom schemes. However, this solution requires that we postulate an infinity of axioms (to the unique second-order axioms), a move which is quite preposterous in light of a formal system that cannot even express what it is that is meant by 'infinite' within its own model- and proof-theory (Boolos 522; Jané 79-80; Shapiro, Second-Order Lang. 725-6)!

theories within another, most commonly set-theory – for which second-order axiomatization has thus far proven indispensable (Boolos 1975, pp. 523-5; Jané 1993, p. 70; Shapiro 1985, pp. 716-7, 722, 728, 739). After all, even if a theory were to be explainable through FOL, if its models are not categorical, not much is gained.

So then what is the advantage of limiting the scope of logic solely to FOL when it is well known that its expressive power is inherently poor and inadequate to express even the basic givens of classical mathematics? Well, SOL's semantic power are themselves impugned: it has been claimed that all the answers it claims to hold are merely hypothetical for lack of a complete deductive system to draw them out (Jané 1993, pp. 81-4; Klenk 1976). As both FOL and SOL have their advantages and disadvantages, it is impossible to say which one is the 'better', unless we specify what we are using the logic for. The answer seems deceptively simple, but is in reality quite treacherous: we want mathematical logic to reproduce our intended models. However, since the Löwenheim-Skolem property prohibits the isomorphic categorization of our intended infinite models, FOL inherently leads us to consider unintended models – and this is a serious problem for most logicians.

4.0 WHAT INTENDED MODELS?

Amongst other things, Skolem's paradox highlights a visceral metamathematical impasse: what constitutes an intuitively acceptable model? After all, we are wont to think that mathematicians have some grasp of the theory they are in the midst of elaborating; however, a naive Platonist vision of the mathematical objects that populate the reality 'behind the model' is anathema to the modern, level-headed theoretician. But are intended models 'real' models? Are unintended models aberrations of truth? Must we reject a formally true theory if it yields unintended models? Are intended models philosophically anachronistic? None of these questions can be acceptably answered lest we first ponder the question: what is model theory (and mathematical logic itself) a theory of?

4.1 MODEL-THEORY AND INFORMAL MATHEMATICS

We here mean to deepen our previous discussion: why should mathematical logic prefer semantic expressiveness over completeness? What exactly are we trying to express? And why are not proof-theory and formal axiom systems enough to express it? Of course, mathematical logic is engaged in an inexhaustible dialogue with standard informal mathematical practice.* Intended models are just the models that are intended by practicing mathematicians; it is their informal theories and models that mathematical logic intends to formalize. For while it may turn out that the informal theories are found wanting in some aspect or another, it is to informal mathematics that we must turn to if we are to give any meaning to the term 'intended model'. We want the models of our formal systems to coincide with the intended objects and relations we mean to describe – ideally categorically. These *are* the original models, the ones we hold all others up to (Shapiro 1999, pp. 45, 48).

After all, informal mathematics is both the beginning and end point of mathematical logic: it not only provides the immediate impetus and *raison-d'être* of logic, but it is also hoped that logic has applications – namely, that it is useful and illuminating to proof-seeking practitioners. It is through informal mathematics that we can conceive, however vaguely, of what we mean by the standard intended model that unites the majority of mathematicians and logicians (Jané 1993, pp. 68-9; Shapiro 1985, pp. 725-6). As such, what this practice says and to a certain extent how it does it constitutes our standard. We shall now contrast the correlations between first- and second-order models with standard semantics with the informal mathemati-

* Perhaps sometimes aptly referred to as pre-formal mathematics.

cal practice that grounds it.

4.2 THE LÖWENHEIM-SKOLEM THEOREM AND INFORMAL MATHEMATICS

As noted, LST is not generally considered to be a problem for practicing mathematicians. But despite the fact that LST admits of resolutions both at the first- and second-order level, it shall nevertheless prove instructive to enquire more into the preferences of the mathematicians themselves. As has been stated, FOL is generally considered incapable of axiomatizing some of the most elementary branches of mathematics – including set-theory, which is crucial to nearly all branches of the discipline. Not surprisingly, what FOL cannot axiomatize is what it can't model: infinite structures. This is where LST can be considered the tipping-point, especially if one considers the fact that the language of infinity is crucial to our modern understanding of mathematics and physics (Shapiro 1985, pp. 714-5, 719, 739). LST is an issue for model-theoretic semantics, and where FOL fails the test is precisely in the semantic section. The semantics involved in informal mathematical practice -- even such well-understood notions as finitude, mathematical induction, minimal closure and well-founded relations like the predecessor and 'less than' relations (which may all be easily constructed with second-order formulas) – simply cannot be expressed in first-order axiomatizations. FOL neither resembles our mathematical structures nor seems useful to the working mathematician looking for guidance in far-off places like philosophy, logic and meta-mathematics (Shapiro 1985, pp. 722-4, 727).

Of course, there are reasons to prefer FOL, but in a post-foundational epoch one must re-evaluate the traditional authority of first-order theories, especially if one considers that foundational studies are ultimately about founding pre-formal mathematics. While SOL can no more provide a foundation for all of mathematics, it does shed more light on mathematical theory and practice than its first-order counterpart. Second-order model-theoretical semantics simply provide better models of the infinite structures that are vital to modern mathematics. Also, they capture better the meanderings and sinews of the semantics of ordinary mathematic discourse;* indeed, it seems that informal mathematics, which dispenses totally of the LST, functions semantically at the second-order level of logic. Indeed, the universal preference for second-order languages by practicing mathematicians can be interpreted as their universal rejection of Skolemism (Jané 1993, p. 67; Shapiro 1985, pp. 720, 727, 739; Shapiro 1999, pp. 44, 62).

5.0 INVENTING LOGIC?

As we have seen, intended models refer to the *use* of particular terms and sentences in informal theory, but that such use is subject to some kind of evolution (especially when reflected to itself by a formal theory's model) is quite banally evident and inevitable. While the model then means to capture the use of informal notions, the model itself still must be semantically interpreted to gauge whether this use is in fact adequately expressed. Here, there seems to be a curious interplay: while we want our formal models to capture something of our intended models, the formal model itself does not ever point to the intended model and, in fact, provides no real means by which we may gauge the veracity of our intended models, or even what we ourselves mean by our vague idea of the intended model. Furthermore, without semantic interpretation, it is difficult to assess what we are building formal theories of, and perhaps most importantly, why we are building them in the first place. The intent of this section is to offer up a few further musings on what model-theoretic semantics can tell us about our broad meta-mathematical, logical and epistemological aims.

* Indeed, of ordinary vernacular discourse itself (Klenk 1976, p. 483).

5.1 MODEL-THEORETIC SEMANTICS AND INTENDED MODELS

The Löwenheim-Skolem theorem confronts us with the facile way in which we rely upon unspoken agreements as to our intended models. Indeed, the relativity results inherent to LST can lend credence to the idea that logic might just be a purely formal science. Indeed, given the LSTT, Skolem's paradox proves *neither* the existence nor the non-existence of non-denumerable sets as the first-order model actually allows sentences to be interpreted either in the denumerable or the non-denumerable domain (Klenk 1976, pp. 476, 479). The interpreter's choice is then guided by other than purely formal considerations – such as one's prior commitment to the ontological existence or characteristics of non-denumerable sets. All that the LST actually states is that the models generated by FOL cannot characterize, cannot *say* anything about whatever the hypothetical structure of non-denumerable sets might be (Klenk 1976, pp. 479, 484). Under this light, the results of the application of the LST to FOL are not criteria by which one can measure whether the intuitively intended model of sets has been achieved – rather, it begs such questions as: what 'intended model'? Why not unintended models? Why not the model that we completely, if counter-intuitively, built?

While the answer to such questions lie outside the scope of our essay, we find it sufficient for the moment to postulate that if we mean to describe mathematics – but especially if we mean to 'found' mathematics – then our models had better do this; if this is the case, it simply will not do to have an axiom system incapable of reproducing the basic conceptual underpinnings of modern mathematics. Of course, our conceptual understanding might need adjusting but it will have to be adjusted in light of a theory within which it can discuss itself. Besides describing mathematical structures, it would also be nice if we could derive as well a model of mathematical activity itself. However, since FOL needs to replace simple second-order axioms (like the axiom of induction) with infinite axiom schemes, it seems that even the syntax of FOL is insufficiently strong to deduce the most basic of the intuitions and principles that mathematicians work with. It thus seems that the informal logic of proof-theory itself is second-order – though we may with more or less success transcribe it to the first-order.

5.2 MODEL-THEORETIC SEMANTICS AND FORMAL SYSTEMS

Meta-mathematical speculation as to what *is* an intended model put aside, in actuality, formal logic *does* rely on informal mathematical theories and practice to guide its investigations: it is those theories that we want to generate in our models. However, modern axiomatics are not merely meant to codify and crystallize static states of knowledge or disembodied mathematical notions floating in the sky above. Or, rather, if this is what some intend it to do, then modern axiomatics also entails a host of unintended consequences. First and foremost, the formalization of informal theories often reveals unexpected epiphenomena – of which Skolem's paradox has struck us as a worthy paradigm. Because of such results, the formal theory itself morphs and adapts to these results through repeated intrusions and manipulations of its holdings. Sometimes, however, it is informal practice that is subtly or not so subtly modified by the observations and clarifications of model-theoretic results (Klenk 1976, pp. 480, 482).

As such, even though our mathematical hunches and techniques may need fine-tuning, a model in which classical mathematics cannot recognize itself will be very hard-pressed to provide answers, solutions and guidance to informal mathematics. The structural and formal approach is not sufficient to account for both the state and the needs of informal mathematics. Mathematics is a rigorous but inherently creative and ultimately intuitive discipline. Formalization is what happens *after* the edification of sufficient reasons warranting such a formalization, as well as sufficient guiding ideas. We cannot build a vacuum. In fact, we are tempted to say that 'incompleteness' is part and parcel with the way humans effectively think – even about mathematics.

As such, logicians must take into account the symbiotic relation between informal practice and formal system if logic is to reflect mathematics, and if logic is to be a productive science – especially as proof-theory becomes more and more entangled with model-theory (Gauthier 1976, p. 293; Shapiro 1985, pp. 716-7; Shapiro 1999, pp. 46, 50, 56-7). The dichotomy between informal mathematics and formal logic has indeed become a tenuous premise to uphold (Klenk 1976, p. 482), one that not only sheds little light on meta-logical aims, but cuts off its supply. Strict logic has not reified itself, it has become more and more acutely aware of its limitations. If anything, modern logic has not become static, it has exploded into a convoluted host of standard and non-standard models, first-, second- and higher-order logics and set theories, etc.

5.3 MODEL-THEORETIC SEMANTICS AND EPISTEMOLOGY

We want logic sometimes for logic's sake but also for the sake of interdisciplinary research. We wish to apply logic in order to discover inalienable truths about the world. Logic *is* a branch of epistemology; it is a branch of philosophy that perhaps more than any other is the science of knowledge, or of measuring knowledge (Gauthier 1976, 296-7). While we do not state that either the Skolem paradox or any particular interpretation of a model has any ontic import,* we do state that we intend to capture at least some sort of ontological reality, whether or not we ourselves (can) capture this reality.† Although an ideal logic will be devoid of any content (i.e., will be

* No existence claims stem either from first- or second-order logic (Jané 1993, p. 68).

† The extant literature spawned by Putnam's "Models and Reality" (1980) attests to the desirability and general plausibility of such a thesis. However, Putnam's hypotheses are of no weight to our present discussion, which limits itself to the meta-mathematical implications of model-theory.

purely syntactical) (Boolos 1975, p. 517; Jané 1993, pp. 67, 72-4), this strikes us as a Platonic ideal of sorts for many reasons. We ought to apply logic to elucidate a domain, a structure, a world, a universe that is itself inherently populated – by us.

While logic does not reveal meaning, and even though it is a highly contested idea that anything *has* meaning, we nevertheless intend it to be a tool by which we can explore, purify and understand the meanings that we do ascribe to the constructed universes within and without ourselves. It is to this extent that we do not wish to reduce logical results to simple epistemological and ontological epiphenomena. If logic means nothing, why do we build it? For what would we want models, other than for gleaning, gauging and separating some sorts of truth? The model is a tool: it can tell us what does not work, what is contradictory, what is consistent, what is satisfactory, and what is merely interesting. It confronts with a certain externalization, a map of particular mental labyrinths through which we daily cut our paths. It is a creative tool for investigation, for exploration. It characterizes equally the true, the false, the hypothetical, the fantastical, the interesting. What we do with these representations, how we semantically interpret the given formal logic is how we link what could be but merely interesting observations with an even richer and more complex reality within and without ourselves.

CONCLUSION

In light of its inherent limitations, FOL is inapt to characterize and structure informal mathematical theory and practice. Seeing as how a vast portion of mathematical theories (as well as physical theories) require that they be imbedded in set theory, and seeing as how formal logic itself has a set-theoretic pedigree, the role of second- and higher-order logic is currently of greater import. This is not to say that the soundness of set-theory itself does not need to be further explored and expounded, but it is to say that a development and consolidation of SOL is to be recommended if one wants to found mathematics albeit in a non-absolute manner (*pace* Jané 1993, pp. 68, 74-5, 78, 85) and understand what mathematics and logic is about and what they can *say* about human understanding and cognitive functions. However much we might want and even need to formalize theories, it would be foolish and unproductive to ignore the inherent creativity involved both in mathematical and logical practice. And if what we want logic to do is formalize that reality as much as it is possible for it to do, SOL (much like non-standard models, quantum logic and multivalent propositional logic) is unavoidably logic (Gauthier 1976, pp. 294-5). It is certainly not all that logic is, but to disregard SOL and its substantial weight is to do a great disservice to logic and the development of a fundamental branch of knowledge in regards to the human experience.

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