# Multivector Amplitudes: A Superset of Complex Amplitudes Yielding Quantum Gravity and the Standard Model? 

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#### Abstract

A quantum theory utilizing multivector amplitudes instead of complex amplitudes has been developed within the framework of geometric algebra. This theory generalizes the Born rule to a multivector probability measure that is invariant under a wide range of geometric transformations. In this formalism, the gamma matrices become operators, enabling the construction of the metric tensor as a quantum observable. By requiring time invariance of the probability measure under all multi-vectorial amplitude transformations, specifically the gauge symmetries $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ retain conserved charge density, thus introducing them without the need for additional assumptions. Remarkably, the multivector amplitude formalism is found to be consistent only with 3+1-dimensional spacetime, encountering various obstructions in other dimensional configurations. This finding aligns with the observed dimensionality of the universe and suggests a possible explanation for the specific gauge symmetries of the Standard Model. Furthermore, the incorporation of the metric tensor as a quantum observable provides a natural pathway to integrate gravity with quantum mechanics.


## 1 Introduction

In this paper, we introduce a novel quantum theory that employs multivector amplitudes instead of complex amplitudes. The theory is entirely derived by solving an entropy maximization problem, yielding a probability measure and an associated vector space in which the multivector-valued wavefunction resides. The maximization problem also generates the complete set of requisite mathematical tools for a comprehensive quantum mechanical treatment, including a product form yielding non-negative probabilities, an evolution operator, transition amplitudes, superposition, interference, and observables, all generalized to

[^0]the geometric domain via multivectors. By formulating the theory as a solution to an entropy optimization problem, its consistency and well-definedness are mathematically assured.

Within this framework, we find that the gamma matrices are elevated to the status of operators, enabling the construction of the metric tensor as a quantum observable. Remarkably, the gauge symmetries of the standard model of particle physics, namely $\mathrm{U}(1), \mathrm{SU}(2)$, and $\mathrm{SU}(3)$, along with their associated conserved charge density, naturally emerge to preserve the time invariance of the probability measure under multi-vectorial amplitude transformations. Furthermore, multivector amplitudes are found to be free of obstructions exclusively in 3+1D spacetime, potentially offering insights into the dimensional specificity of the universe.

This innovative approach to quantum mechanics extends the 'Prescribed Observation Problem' (POP), a methodology we previously proposed [1], which applies entropy maximization techniques, well-established in statistical mechanics, to derive the axioms of quantum mechanics from first principles. The natural extension of this methodology to multivectors gives rise to the most geometrically rich quantum theory that can be formulated in terms of a wavefunction residing in a vector space and possessing a product form yielding non-negative probabilities.

In the results section, we will delve into the properties and implications of this multivector-based quantum mechanical theory. We commence with a concise overview of entropy maximization techniques as employed in statistical mechanics, followed by a summary of our previous work applying these techniques to quantum mechanics, and finally, their generalization to multivectors.

## Statistical Mechanics

Let us now begin with entropy maximization in the field of statistical mechanics (SM). The microcanonical ensemble of SM can be derived from an entropy maximization problem:
Definition 1 (Lagrange equation of SM).

$$
\begin{equation*}
\mathcal{L}(\rho, \lambda, \beta)=\underbrace{-k_{B} \sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q)}_{\text {Boltzmann entropy }}+\underbrace{\lambda\left(1-\sum_{q \in \mathbb{Q}} \rho(q)\right)}_{\text {Normalization Constraint }}+\underbrace{\beta\left(\bar{E}-\sum_{q \in \mathbb{Q}} \rho(q) E(q)\right)}_{\text {Average Energy Constraint }} \tag{1}
\end{equation*}
$$

Solving this optimization problem[2] yields the celebrated Gibbs' measure:

$$
\begin{equation*}
\frac{\partial \mathcal{L}(\rho, \lambda, \beta)}{\partial \rho}=0 \Longrightarrow \rho(q)=\underbrace{\frac{1}{\sum_{r \in \mathbb{Q}} \exp (-\beta E(r))}}_{\text {Microcanonical Ensemble }} \underbrace{\exp (-\beta E(q))}_{\text {Gibbs' Measure }} \tag{2}
\end{equation*}
$$

## Quantum Mechanics

Inspired by the result of Gibbs, in our previous work [1], we reformulated QM as a solution to an entropy maximization problem. The Lagrange equation defining the optimization problem is:

Definition 2 (Lagrange equation of QM).

$$
\mathcal{L}(\rho, \lambda, t)=\underbrace{-\sum_{q \in \mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)}}_{\substack{\text { Relative }  \tag{3}\\
\text { Shannon } \\
\text { Entropy }}}+\underbrace{\lambda\left(1-\sum_{q \in \mathbb{Q}} \rho(q)\right)}_{\substack{\text { Normalization } \\
\text { Constraint }}}+\underbrace{t / \hbar\left(\operatorname{tr} \sum_{q \in \mathbb{Q}} \rho(q)\left[\begin{array}{cc}
0 & -E(q) \\
E(q) & 0
\end{array}\right]\right)}_{\text {Phase Anti-Constraint }}
$$

The phase anti-constraint serves as a formal device to expand the solution space, allowing for the incorporation of complex phases into the probability measure. As it expands rather the constrict the solution space, the expression is the opposite of a constraint - hence we named it an anti-constraint.

Theorem 1. Solving this optimization problem yields the Born rule as the probability measure, $p(q)$ as the wavefunction initial state, and a partition function that is unitarily invariant:

$$
\begin{equation*}
\frac{\partial \mathcal{L}(\rho, \lambda, t)}{\partial \rho}=0 \Longrightarrow \rho(q)=\underbrace{\frac{1}{\sum_{r \in \mathbb{Q}} p(r)\|\exp (-i t E(r) / \hbar)\|} \underbrace{\|\exp (-i t E(q) / \hbar)\|}_{\text {Born Rule }} \underbrace{p(q)}_{\text {Initial State }}}_{\text {Unitarily Invariant Ensemble }} \tag{4}
\end{equation*}
$$

The solution resolves[1] into the five canonical axioms of QM [3, 4].
Proof. The optimization problem is solved as follows:

$$
\begin{align*}
\frac{\partial \mathcal{L}(\rho, \lambda, \tau)}{\partial \rho(q)} & =-\ln \frac{\rho(q)}{p(q)}-1-\lambda-\tau \operatorname{tr}\left[\begin{array}{cc}
0 & -E(q) \\
E(q) & 0
\end{array}\right]  \tag{5}\\
0 & =\ln \frac{\rho(q)}{p(q)}+1+\lambda+\tau \operatorname{tr}\left[\begin{array}{cc}
0 & -E(q) \\
E(q) & 0
\end{array}\right]  \tag{6}\\
\Longrightarrow \ln \frac{\rho(q)}{p(q)} & =-1-\lambda-\tau \operatorname{tr}\left[\begin{array}{cc}
0 & -E(q) \\
E(q) & 0
\end{array}\right]  \tag{7}\\
\Longrightarrow \rho(q) & =p(q) \exp (-1-\lambda) \exp \left(\begin{array}{cc}
\left.-\tau \operatorname{tr}\left[\begin{array}{cc}
0 & -E(q) \\
E(q) & 0
\end{array}\right]\right) \\
& =\frac{1}{Z(\tau)} p(q) \exp \left(-\tau \operatorname{tr}\left[\begin{array}{cc}
0 & -E(q) \\
E(q) & 0
\end{array}\right]\right)
\end{array} \$ . \begin{array}{l}
\text { (q) }
\end{array}\right) \tag{8}
\end{align*}
$$

The partition function is obtained as follows:

$$
\begin{align*}
1 & =\sum_{r \in \mathbb{Q}} p(r) \exp (-1-\lambda) \exp \left(-\tau \operatorname{tr}\left[\begin{array}{cc}
0 & -E(r) \\
E(r) & 0
\end{array}\right]\right) \\
\Longrightarrow(\exp (-1-\lambda))^{-1} & =\sum_{r \in \mathbb{Q}} p(r) \exp \left(-\tau \operatorname{tr}\left[\begin{array}{cc}
0 & -E(r) \\
E(r) & 0
\end{array}\right]\right)  \tag{10}\\
Z(\tau) & :=\sum_{r \in \mathbb{Q}} p(r) \exp \left(-\tau \operatorname{tr}\left[\begin{array}{cc}
0 & -E(r) \\
E(r) & 0
\end{array}\right]\right) \tag{12}
\end{align*}
$$

The probability measure is given by:

$$
\rho(q)=\frac{p(q) \exp \left(-\tau \operatorname{tr}\left[\begin{array}{cc}
0 & -E(q)  \tag{13}\\
E(q) & 0
\end{array}\right]\right)}{\sum_{r \in \mathbb{Q}} p(r) \exp \left(-\tau \operatorname{tr}\left[\begin{array}{cc}
0 & -E(r) \\
E(r) & 0
\end{array}\right]\right)}
$$

Transforming the representation of complex numbers from $\left[\begin{array}{cc}a & -b \\ b & a\end{array}\right]$ to $a+i b$ and associating the exponential trace with the complex norm using $\exp \operatorname{tr} \mathbf{M} \equiv$ det $\exp \mathbf{M}$, we obtain:

$$
\begin{align*}
\exp \operatorname{tr}\left[\begin{array}{cc}
a & -b \\
b & a
\end{array}\right]=\operatorname{det} \exp \left[\begin{array}{cc}
a & -b \\
b & a
\end{array}\right]= & r^{2} \operatorname{det}\left[\begin{array}{cc}
\cos (b)-\sin (b) \\
\sin (b) & \cos (b)
\end{array}\right], \text { where } r=\exp a  \tag{14}\\
& =r^{2}\left(\cos ^{2}(b)+\sin ^{2}(b)\right)  \tag{15}\\
& =\|r(\cos (b)+i \sin (b))\|  \tag{16}\\
& =\|r \exp (i b)\| \tag{17}
\end{align*}
$$

Substituting $\tau=t / \hbar$ and applying the complex-norm representation to both the numerator and denominator yields the following probability measure:

$$
\begin{equation*}
\rho(q)=\frac{1}{\sum_{r \in \mathbb{Q}} p(r)\|\exp (-i t E(r) / \hbar)\|}\|\exp (-i t E(q) / \hbar)\| p(q) \tag{18}
\end{equation*}
$$

Let us recall the five principal axioms of the canonical formalism of QM $[3,4]$ :

Axiom 1 State Space: Each physical system corresponds to a complex Hilbert space, with the system's state represented by a ray in this space.

Axiom 2 Observables: Physical observables correspond to Hermitian operators within the Hilbert space.

Axiom 3 Dynamics: The time evolution of a quantum system is dictated by the Schrödinger equation, where the Hamiltonian operator signifies the system's total energy.

Axiom 4 Measurement: The act of measuring an observable results in the system's transition to an eigenstate of the associated operator, with the measurement value being one of the eigenvalues.

Axiom 5 Probability Interpretation: The likelihood of a specific measurement outcome is determined by the squared magnitude of the state vector's projection onto the relevant eigenstate.

We now explore how these axioms are recovered from the expanded solution space engendered by the anti-constraint.

The wavefunction is delineated by decomposing the complex norm into a complex number and its conjugate, visualized as a vector within a complex n-dimensional Hilbert space, with the partition function acting as the inner product:

$$
\begin{equation*}
\sum_{r \in \mathbb{Q}} p(r)\|\exp (-i t E(r) / \hbar)\|=Z=\langle\psi \mid \psi\rangle \tag{19}
\end{equation*}
$$

where

$$
\left[\begin{array}{c}
\psi_{1}(t)  \tag{20}\\
\vdots \\
\psi_{n}(t)
\end{array}\right]=\left[\begin{array}{ccc}
\exp \left(-i t E\left(q_{1}\right) / \hbar\right) & & \\
& \ddots & \\
& & \exp \left(-i t E\left(q_{n}\right) / \hbar\right)
\end{array}\right]\left[\begin{array}{c}
\psi_{1}(0) \\
\vdots \\
\psi_{n}(0)
\end{array}\right]
$$

Here, $p(q)$ represents the probability associated with the initial preparation of the wavefunction, where $p\left(q_{i}\right)=\left\langle\psi_{i}(0) \mid \psi_{i}(0)\right\rangle$, and $Z$ is invariant under unitary transformations.

The axioms of quantum mechanics are recovered as follows:

1. The entropy maximization procedure inherently normalizes the vectors $|\psi\rangle$ with $1 / Z=1 / \sqrt{\langle\psi \mid \psi\rangle}$, linking $|\psi\rangle$ to a unit vector in Hilbert space. As the POP formulation of QM associates physical states with its probability measure, and the probability is defined up to a phase, physical states map to rays within Hilbert space, demonstrating Axiom 1.
2. In $Z$, an observable must satisfy:

$$
\begin{equation*}
\bar{O}=\sum_{r \in \mathbb{Q}} p(r) O(r)\|\exp (-i t E(r) / \hbar)\| \tag{21}
\end{equation*}
$$

Since $Z=\langle\psi \mid \psi\rangle$, any self-adjoint operator satisfying $\langle\mathbf{O} \psi \mid \phi\rangle=\langle\psi \mid \mathbf{O} \phi\rangle$ will equate the above equation, demonstrating Axiom 2.
3. Transforming Equation 20 out of its eigenbasis through unitary operations, the energy $E(q)$ typically transforms as a Hamiltonian operator:

$$
\begin{equation*}
|\psi(t)\rangle=\exp (-i t \mathbf{H} / \hbar)|\psi(0)\rangle \tag{22}
\end{equation*}
$$

The system's dynamics emerge from differentiating the solution with respect to the Lagrange multiplier:

$$
\begin{align*}
\frac{\mathrm{d}}{\mathrm{~d} t}|\psi(t)\rangle & =\frac{\mathrm{d}}{\mathrm{~d} t}(\exp (-i t \mathbf{H} / \hbar)|\psi(0)\rangle)  \tag{23}\\
& =-i \mathbf{H} / \hbar \exp (-i t \mathbf{H} / \hbar)|\psi(0)\rangle  \tag{24}\\
& =-i \mathbf{H} / \hbar|\psi(t)\rangle  \tag{25}\\
\Longrightarrow \mathbf{H}|\psi(t)\rangle & =i \hbar \frac{\mathrm{~d}}{\mathrm{~d} t}|\psi(t)\rangle \tag{26}
\end{align*}
$$

which is the Schrödinger equation, demonstrating Axiom 3.
4. From Equation 20, the possible microstates $E(q)$ of the system correspond to the eigenvalues of $\mathbf{H}$. An observation can be conceptualized as sampling from $\rho(q, t)$, with the post-measurement state being the occupied microstate $q$ of $\mathbb{Q}$. Consequently, when a measurement occurs, the system invariably emerges in one of these microstates, corresponding to an eigenstate of $\mathbf{H}$. Measured in the eigenbasis, the probability distribution is:

$$
\begin{equation*}
\rho(q, t)=\frac{1}{\langle\psi \mid \psi\rangle}(\psi(q, t))^{\dagger} \psi(q, t) \tag{27}
\end{equation*}
$$

In scenarios where the probability measure $\rho(q, \tau)$ is expressed in a basis other than its eigenbasis, the probability $P\left(\lambda_{i}\right)$ of obtaining the eigenvalue $\lambda_{i}$ is given as a projection on an eigenstate:

$$
\begin{equation*}
P\left(\lambda_{i}\right)=\left|\left\langle\lambda_{i} \mid \psi\right\rangle\right|^{2} \tag{28}
\end{equation*}
$$

Here, $\left|\left\langle\lambda_{i} \mid \psi\right\rangle\right|^{2}$ signifies the squared magnitude of the amplitude of the state $|\psi\rangle$ when projected onto the eigenstate $\left|\lambda_{i}\right\rangle$. As this argument holds for any observable, it demonstrates Axiom 4.
5. Since the probability measure (Equation 4) replicates the Born rule, Axiom 5 is also demonstrated.

Revisiting quantum mechanics from this perspective offers a coherent and unified narrative. Specifically, the phase anti-constraint is sufficient to entail the foundations of quantum mechanics (Axiom 1, 2, 3, 4, and 5) through the principle of entropy maximization. The phase anti-constraint becomes the formulation's sole axiom, and Axioms 1, 2, 3, 4, and 5 now emerge as theorems. For a more in-depth analysis of the POP in the context of QM, the reader is invited to consult our previous work [1].

Multivector Amplitudes
In this paper, we present a natural generalization of the reformulation of quantum mechanics based on the POP methodology. We extend the "phase anti-constraint" from our previous work to a more general "geometric anticonstraint," which is the geometrically richest anti-constraint that resolves into
a wavefunction living in a vector space and into a non-negative product form associated with probabilities. This generalization leads to a quantum theory based on multivector amplitudes. The Lagrange multiplier equation for this generalized formulation becomes:

Definition 3 (Lagrange equation of multivector-valued QM).

$$
\begin{equation*}
\mathcal{L}(\rho, \lambda, \tau)=\underbrace{-\sum_{q \in \mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)}}_{\substack{\text { Relative } \\ \text { Shannon } \\ \text { Entropy }}}+\underbrace{\lambda\left(1-\sum_{q \in \mathbb{Q}} \rho(q)\right)}_{\substack{\text { Normalization } \\ \text { Constraint }}}+\underbrace{\tau\left(\frac{1}{d} \operatorname{tr} \sum_{q \in \mathbb{Q}} \rho(q) \mathbf{M}(q)\right)}_{\text {Geometric Anti-Constraint }} \tag{29}
\end{equation*}
$$

where $d$ is the dimension of the space or spacetime, $\mathbf{M}$ is a traceless square matrix and $\tau$ is a Lagrange multiplier that will represent the proper time.

As we will see, the resolution of this Lagrange equation generates an extension of the five canonical axioms of QM that incorporates multivector amplitudes. This multivector-based quantum mechanical theory provides a unified framework that naturally includes the metric tensor of gravity as a quantum observable and the standard model gauge symmetries $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ including their associated conserved charge densities that leave the probability measure time-invariant. Solving the optimization problem also generates all the necessary tools for a consistent quantum mechanical treatment, from probability measure to observables, to self-adjointness, to superposition, to sum over geometries and interference extending them to the realm of multivector amplitudes, etc.

## 2 Results

Theorem 2. The solution to the Lagrange multiplier equation (Equation 29) resolves to the following probability measure:

$$
\begin{equation*}
\frac{\partial \mathcal{L}(\rho, \lambda, t)}{\partial \rho}=0 \Longrightarrow \rho(q)=\underbrace{\frac{1}{\sum_{r \in \mathbb{Q}} p(r) \exp \left(-\frac{1}{d} \tau \operatorname{tr} \mathbf{M}(r)\right)} \underbrace{\exp \left(-\frac{1}{d} \tau \operatorname{tr} \mathbf{M}(q)\right)}_{\text {Geometric Born Rule }} \underbrace{p(q)}_{\text {Initial State }}}_{\text {Geometrically Invariant Ensemble }} \tag{30}
\end{equation*}
$$

Proof.

$$
\begin{align*}
\frac{\partial \mathcal{L}(\rho, \lambda, \tau)}{\partial \rho(q)} & =-\ln \frac{\rho(q)}{p(q)}-1-\lambda-\tau \frac{1}{d} \operatorname{tr} \mathbf{M}(q)  \tag{31}\\
0 & =\ln \frac{\rho(q)}{p(q)}+1+\lambda+\tau \operatorname{tr} \frac{1}{d} \mathbf{M}(q)  \tag{32}\\
\Longrightarrow \ln \frac{\rho(q)}{p(q)} & =-1-\lambda-\tau \operatorname{tr} \frac{1}{d} \mathbf{M}(q)  \tag{33}\\
\Longrightarrow \rho(q) & =p(q) \exp (-1-\lambda) \exp \left(-\tau \operatorname{tr} \frac{1}{d} \mathbf{M}(q)\right)  \tag{34}\\
& =\frac{1}{Z(\tau)} p(q) \exp \left(-\tau \operatorname{tr} \frac{1}{d} \mathbf{M}(q)\right) \tag{35}
\end{align*}
$$

The partition function $Z(\tau)$, serving as a normalization constant, is determined as follows:

$$
\begin{align*}
1 & =\sum_{r \in \mathbb{Q}} p(r) \exp (-1-\lambda) \exp \left(-\tau \operatorname{tr} \frac{1}{d} \mathbf{M}(r)\right)  \tag{36}\\
\Longrightarrow(\exp (-1-\lambda))^{-1} & =\sum_{r \in \mathbb{Q}} p(r) \exp \left(-\tau \operatorname{tr} \frac{1}{d} \mathbf{M}(r)\right)  \tag{37}\\
Z(\tau) & :=\sum_{r \in \mathbb{Q}} p(r) \exp \left(-\tau \operatorname{tr} \frac{1}{d} \mathbf{M}(r)\right) \tag{38}
\end{align*}
$$

Consequently, the optimal probability distribution is given by:

$$
\begin{equation*}
\rho(q)=\frac{1}{\sum_{r \in \mathbb{Q}} p(r) \operatorname{det} \exp \left(-\frac{1}{d} \tau \mathbf{M}(r)\right)} \operatorname{det} \exp \left(-\frac{1}{d} \tau \mathbf{M}(q)\right) p(q) \tag{39}
\end{equation*}
$$

where $\operatorname{det} \exp M=\exp \operatorname{tr} M$.
This theorem generalizes the Born rule to a probability measure that is invariant under a wide range of geometric transformations. The geometrically invariant ensemble serves as a normalization factor, while the initial state $p(q)$ represents the probability associated with the initial preparation of the system.
Corollary 2.1. QM is a special solution of Theorem 2.
Proof.
$\left.\rho(q)\right|_{d \rightarrow 1, \mathbf{M}(q) \rightarrow\left[\begin{array}{cc}0 & -E(q) \\ E(q) & 0\end{array}\right]}=\underbrace{\frac{1}{\sum_{r \in \mathbb{Q}} p(r)\|\exp (-i t E(r) / \hbar)\|}}_{\text {Unitarily Invariant Ensemble }} \underbrace{\|\exp (-i t E(q) / \hbar)\|}_{\text {Born Rule }} \underbrace{p(q)}_{\text {Initial State }}$

This corollary demonstrates that quantum mechanics is a special case of the generalized probability measure derived in Theorem 2. By setting the dimension $d=1$ and choosing the traceless matrix $\mathbf{M}(q)$ to represent a complex phase within the energy of the system, we recover the familiar Born rule and the unitarily invariant ensemble of quantum mechanics from which the five canonical axioms of QM (Theorem 1) are provable.

Corollary 2.2. SM is a special solution of Theorem 2
Proof.

$$
\begin{equation*}
\left.\rho(q)\right|_{d \rightarrow 1, \mathbf{M}(q) \rightarrow[E(q)], p(q) \rightarrow 1}=\underbrace{\frac{1}{\sum_{r \in \mathbb{Q}} \exp (-\beta E(r))}}_{\text {Microcanonical Ensemble }} \underbrace{\exp (-\beta E(q))}_{\text {Gibbs Measure }} \tag{41}
\end{equation*}
$$

Similarly, this corollary shows that statistical mechanics is another special case of the generalized probability measure. By setting the dimension $d=1$, choosing the traceless matrix $\mathbf{M}(q)$ to represent the energy of the system, and assuming a uniform initial state $p(q)=1$, we recover the Gibbs measure and the microcanonical ensemble of statistical mechanics.

The theorem and associated corollaries provides a common framework for understanding the foundations of these theories (e.g. SM, QM and Multivectorvalued QM) and highlights the central role of entropy maximization in their construction.

### 2.1 Obstructions to Multivector amplitudes in 2D

In this section, we apply Theorem 2 to a two-dimensional (2D) space, where the dimension $d=2$ and the traceless matrix $\mathbf{M}$ is a $2 \times 2$ matrix. Although all dimensional configurations except $3+1 \mathrm{D}$ contain obstructions, which will be discussed later in this section, the 2 D case provides a valuable starting point before addressing the more complex $3+1 \mathrm{D}$ case. The probability measure in 2D takes the form:
$\rho(q)=\frac{1}{\sum_{r \in \mathbb{Q}} p(r) \operatorname{det} \exp \left(-\frac{1}{2} \tau\left[\begin{array}{cc}x(q) & y(q)-b(q) \\ y(q)+b(q) & -x(q)\end{array}\right]\right)} \operatorname{det} \exp \left(-\frac{1}{2} \tau\left[\begin{array}{c}x(q) \\ y(q)+b(q) \\ y(q)-b(q) \\ -x(q)\end{array}\right]\right) p(q)$

To represent this probability measure in terms of multivectors, we choose a matrix representation that is group isomorphic to the geometric algebra in 2D over the reals, denoted as $\mathrm{GA}(2) \cong \mathbb{M}(2, \mathbb{R})$ :

$$
\left[\begin{array}{ll}
a+x & y-b  \tag{43}\\
y+b & a-x
\end{array}\right] \cong a+x \hat{\mathbf{x}}+y \hat{\mathbf{y}}+b \hat{\mathbf{x}} \wedge \hat{\mathbf{y}}
$$

where the basis elements of this geometric algebra are defined as:

$$
\hat{\mathbf{x}}=\left[\begin{array}{cc}
1 & 0  \tag{44}\\
0 & -1
\end{array}\right], \hat{\mathbf{y}}=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right], \hat{\mathbf{x}} \wedge \hat{\mathbf{y}}=\left[\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right]
$$

A more compact notation for this multivector $\mathbf{u}$ is as follows:

$$
\begin{equation*}
\mathbf{u}=a+\mathbf{x}+\mathbf{b} \tag{45}
\end{equation*}
$$

where $a$ is a scalar, $\mathbf{x}$ is a vector, and $\mathbf{b}$ is a pseudo-scalar.
Using this notation, the evolution operator in the probability measure can be written as:

$$
\exp \left(-\frac{1}{2} \tau\left(\begin{array}{cc}
x(q) & y(q)-b(q)  \tag{46}\\
y(q)+b(q) & -x(q)
\end{array}\right)\right)=e^{-\frac{1}{2} \tau(\mathbf{x}(q)+\mathbf{b}(q))}
$$

We now introduce the multivector conjugate, also known as the Clifford conjugate, which generalizes the concept of complex conjugation to multivectors.
Definition 4 (Multivector conjugate (a.k.a Clifford conjugate)). Let $\mathbf{u}=a+$ $\mathbf{x}+\mathbf{b}$ be a multi-vector of the geometric algebra over the reals in two dimensions $\mathrm{GA}(2)$. The multivector conjugate is defined as:

$$
\begin{equation*}
\mathbf{u}^{\ddagger}=a-\mathbf{x}-\mathbf{b} \tag{47}
\end{equation*}
$$

The determinant of the matrix representation of a multivector can be expressed as a self-product:

Theorem 3 (Determinant as a Multivector Self-Product).

$$
\begin{equation*}
\mathbf{u}^{\ddagger} \mathbf{u}=\operatorname{det} \mathbf{M}_{\mathbf{u}} \tag{48}
\end{equation*}
$$

Proof. Let $\mathbf{u}=a+x \hat{\mathbf{x}}+y \hat{\mathbf{y}}+b \hat{\mathbf{x}} \wedge \hat{\mathbf{y}}$, and let $\mathbf{M}_{\mathbf{u}}$ be its matrix representation $\left[\begin{array}{ll}a+x & y-b \\ y+b & a-x\end{array}\right]$. Then:

$$
\begin{align*}
& 1: \quad \mathbf{u}^{\ddagger} \mathbf{u}  \tag{49}\\
& \quad=(a+x \hat{\mathbf{x}}+y \hat{\mathbf{y}}+b \hat{\mathbf{x}} \wedge \hat{\mathbf{y}})^{\ddagger}(a+x \hat{\mathbf{x}}+y \hat{\mathbf{y}}+b \hat{\mathbf{x}} \wedge \hat{\mathbf{y}})  \tag{50}\\
& \quad=(a-x \hat{\mathbf{x}}-y \hat{\mathbf{y}}-b \hat{\mathbf{x}} \wedge \hat{\mathbf{y}})(a+x \hat{\mathbf{x}}+y \hat{\mathbf{y}}+b \hat{\mathbf{x}} \wedge \hat{\mathbf{y}})  \tag{51}\\
& \quad= a^{2}-x^{2}-y^{2}+b^{2}  \tag{52}\\
& 2: \quad \operatorname{det} \mathbf{M}_{\mathbf{u}}  \tag{53}\\
& \quad=\operatorname{det}\left[\begin{array}{ll}
a+x & y-b \\
y+b & a-x
\end{array}\right]  \tag{54}\\
&=(a+x)(a-x)-(y-b)(y+b)  \tag{55}\\
&= a^{2}-x^{2}-y^{2}+b^{2} \tag{56}
\end{align*}
$$

Building upon the concept of the multivector conjugate, we introduce the multivector conjugate transpose, which serves as an extension of the Hermitian conjugate to the domain of multivectors.

Definition 5 (Multivector Conjugate Transpose). Let $|V\rangle\rangle \in(\operatorname{GA}(2))^{n}$ :

$$
|V\rangle\rangle=\left[\begin{array}{c}
a_{1}+\mathbf{x}_{1}+\mathbf{b}_{1}  \tag{57}\\
\vdots \\
a_{n}+\mathbf{x}_{n}+\mathbf{b}_{n}
\end{array}\right]
$$

The multivector conjugate transpose of $|V\rangle\rangle$ is defined as first taking the transpose and then the element-wise multivector conjugate:

$$
\left\langle\langle V|=\left[\begin{array}{lll}
a_{1}-\mathbf{x}_{1}-\mathbf{b}_{1} & \ldots & a_{n}-\mathbf{x}_{n}-\mathbf{b}_{n} \tag{58}
\end{array}\right]\right.
$$

Definition 6 (Bilinear Form). Let $|V\rangle$ and $|W\rangle$ be two vectors valued in GA(2). We introduce the following bilinear form:

$$
\begin{equation*}
\langle V V \mid W\rangle\rangle=\left(a_{1}-\mathbf{x}_{1}-\mathbf{b}_{1}\right)\left(a_{1}+\mathbf{x}_{1}+\mathbf{b}_{1}\right)+\ldots\left(a_{n}-\mathbf{x}_{n}-\mathbf{b}_{n}\right)\left(a_{n}+\mathbf{x}_{n}+\mathbf{b}_{n}\right) \tag{59}
\end{equation*}
$$

The partition function (Equation 42) can be expressed using the bilinear form:

Theorem 4 (Partition Function). $Z=\langle\langle V \mid V\rangle$
Proof.

$$
\begin{equation*}
\langle\langle V \mid V\rangle\rangle=\sum_{q \in \mathbb{Q}} V(q)^{\ddagger} V(q)=\sum_{q \in \mathbb{Q}} \operatorname{det} \mathbf{M}_{V(q)}=Z \tag{60}
\end{equation*}
$$

Theorem 5 (Inner Product). In the even sub-algebra of $\mathrm{GA}(2)$, the bilinear form is an inner product.

Proof.

$$
\begin{equation*}
\langle V V \mid W\rangle\rangle_{\mathbf{x} \rightarrow 0}=\left(a_{1}-\mathbf{b}_{1}\right)\left(a_{1}+\mathbf{b}_{1}\right)+\ldots\left(a_{n}-\mathbf{b}_{n}\right)\left(a_{n}+\mathbf{b}_{n}\right) \tag{61}
\end{equation*}
$$

This is isomorphic to the inner product of a complex Hilbert space, with the identification $i \cong \hat{\mathbf{x}} \wedge \hat{\mathbf{y}}$.

Since the even sub-algebra of GA(2) is closed under addition and multiplication, and the bilinear form constitutes an inner product, it follows that it can be employed to construct a Hilbert space. As this leads to a well-defined quantum theory, we will henceforth focus on the $\mathbf{x} \rightarrow 0$ case throughout the remainder of this section.

We now introduce the wavefunction, which is rotor-valued:

Definition 7 (Rotor-valued Wavefunction). The rotor-valued wavefunction is defined as follows:

$$
|\psi\rangle\rangle=\left[\begin{array}{c}
e^{\frac{1}{2}\left(a_{1}+\mathbf{b}_{1}\right)}  \tag{62}\\
\vdots \\
e^{\frac{1}{2}\left(a_{n}+\mathbf{b}_{n}\right)}
\end{array}\right]=\left[\begin{array}{c}
\sqrt{\rho_{1}} R_{1} \\
\vdots \\
\sqrt{\rho_{n}} R_{n}
\end{array}\right]
$$

The rotor wavefunction leads to the (2D) Dirac current:
Definition 8 (Dirac Current). Let $\psi(q)=\sqrt{\rho(q)} R(q)$. Then,

$$
\begin{equation*}
J \equiv \psi(q)^{\ddagger} \hat{\mathbf{x}}_{\mu} \psi(q)=\rho(q) \mathbf{e}_{\mu}(q) \tag{63}
\end{equation*}
$$

The Lagrange multiplier $\tau$ leads to a proper-time valued Schrödinger equation:

Definition 9 (Rotor Flow Generating Schrödinger equation).

$$
\begin{equation*}
\left.\frac{\mathrm{d}}{\mathrm{~d} \tau}|\psi(\tau)\rangle\right\rangle=-\frac{1}{2} \mathbf{b}|\psi(\tau)\rangle \tag{64}
\end{equation*}
$$

The resulting theory is very similar to David Hestenes' geometric algebra formulation of QM[5], but applied to the 2D case.

### 2.1.1 Obstructions

We identify three obstructions in the 2D case:

1. The Lagrange multiplier requires the proper time $\tau$, but the 2 D space considered contains 2 spatial dimensions and 0 time dimensions, leading to an inconsistency.
2. The $1+1 \mathrm{D}$ theory results in a split-complex quantum theory due to the bilinear form $(a-b \hat{\mathbf{t}} \wedge \hat{\mathbf{x}})(a+b \hat{\mathbf{t}} \wedge \hat{\mathbf{x}})$, which yields negative probabilities: $a^{2}-b^{2} \in \mathbb{R}$ for certain wavefunction states, in contrast to the non-negative probabilities $a^{2}+b^{2} \in \mathbb{R}^{\geq 0}$ obtained in the Euclidean 2D case.
3. In 2 D , the matrices $\hat{\mathbf{x}}_{\mu}$ are not operators because they are not self-adjoint. Although often used in the context defining the Dirac current, their nonstatus as observables prevent the construction of the metric tensor as a quantum observable. The benefits of having the basis matrices $\hat{\mathbf{x}}_{\mu}$ as operators will become obvious in the $3+1 \mathrm{D}$ case, where the gamma matrices will be self-adjoint operators. Indeed, in 2D:

$$
\begin{equation*}
\left(\hat{\mathbf{x}}_{\mu} \mathbf{u}\right)^{\ddagger} \mathbf{u}=\mathbf{u}^{\ddagger} \hat{\mathbf{x}}_{\mu}^{\ddagger} \mathbf{u}=\mathbf{u}^{\ddagger}\left(-\hat{\mathbf{x}}_{\mu}\right) \mathbf{u} \quad \neq \mathbf{u}^{\ddagger} \hat{\mathbf{x}}_{\mu} \mathbf{u} \tag{65}
\end{equation*}
$$

Since $\left(\hat{\mathbf{x}}_{\mu} \mathbf{u}\right)^{\ddagger} \mathbf{u} \neq \mathbf{u}^{\ddagger} \hat{\mathbf{x}}_{\mu} \mathbf{u}$, it follows that $\hat{\mathbf{x}}_{\mu}$ is not self-adjoint.
In the following section, we will explore the $3+1 \mathrm{D}$ case and subsequently investigate obstructions in higher-dimensional configurations. This analysis will demonstrate that the $3+1 \mathrm{D}$ multivector quantum theory is the only one that remains obstruction-free.

### 2.2 Multivector Amplitudes in 3+1D

In this section, we extend the concepts and techniques developed for multivector amplitudes in 2D to the more physically relevant case of $3+1 \mathrm{D}$ dimensions. We begin by defining a general multivector in the geometric algebra GA $(3,1)$.

Definition 10 (Multivector). Let $\mathbf{u}$ be a multivector of $\mathrm{GA}(3,1)$. Its general form is:

$$
\begin{align*}
\mathbf{u}= & a  \tag{66}\\
& +x \hat{\mathbf{x}}+y \hat{\mathbf{y}}+z \hat{\mathbf{z}}+t \hat{\mathbf{t}}  \tag{67}\\
& +f_{01} \hat{\mathbf{t}} \wedge \hat{\mathbf{x}}+f_{02} \hat{\mathbf{t}} \wedge \hat{\mathbf{y}}+f_{03} \hat{\mathbf{t}} \wedge \hat{\mathbf{z}}+f_{12} \hat{\mathbf{x}} \wedge \hat{\mathbf{y}}+f_{13} \hat{\mathbf{x}} \wedge \hat{\mathbf{z}}+f_{23} \hat{\mathbf{y}} \wedge \hat{\mathbf{z}}  \tag{68}\\
& +v_{0} \hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \wedge \hat{\mathbf{z}}+v_{1} \hat{\mathbf{t}} \wedge \hat{\mathbf{y}} \wedge \hat{\mathbf{z}}+v_{2} \hat{\mathbf{t}} \wedge \hat{\mathbf{x}} \wedge \hat{\mathbf{z}}+v_{3} \hat{\mathbf{t}} \wedge \hat{\mathbf{x}} \wedge \hat{\mathbf{y}}  \tag{69}\\
& +b \hat{\mathbf{t}} \wedge \hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \wedge \hat{\mathbf{z}} \tag{70}
\end{align*}
$$

A more compact notation for $\mathbf{u}$ is

$$
\begin{equation*}
\mathbf{u}=a+\mathbf{x}+\mathbf{f}+\mathbf{v}+\mathbf{b} \tag{71}
\end{equation*}
$$

where $a$ is a scalar, $\mathbf{x}$ a vector, $\mathbf{f}$ a bivector, $\mathbf{v}$ is pseudo-vector and $\mathbf{b}$ a pseudoscalar.

This general multivector can be represented by a $4 \times 4$ real matrix using the real Majorana representation, which establishes a connection between the geometric algebra and matrix algebra.

Definition 11 (Matrix Representation $\mathbf{M}_{\mathbf{u}}$ of $\mathbf{u}$ ). In a 3+1-dimensional context, a $4 \times 4$ real matrix, M, can be expressed using the real Majorana representation. Such a matrix has the general form:

$$
\mathbf{M}=\left[\begin{array}{cccc}
a+x-f_{02}+q & -z-f_{13}+w-b & f_{03}-f_{23}-p-v & t+y+f_{01}+f_{12}  \tag{72}\\
-z-f_{13}+w+b & a-x-f_{02}-q & -t+y+f_{01}+f_{12} & f_{03}-f_{23}-p-v \\
f_{03}+f_{23}-p+v & t+y-f_{01}+f_{12} & a+x+f_{02}-q & -z-f_{13}-w+b \\
-t+y+f_{01}-f_{12} & -f_{03}-f_{23}-p+v & -z+f_{13}-w-b & a-x+f_{02}+q
\end{array}\right]
$$

To manipulate and analyze multivectors in $\mathrm{GA}(3,1)$, we introduce several important operations, such as the multivector conjugate, the 3,4 blade conjugate, and the multivector self-product.

Definition 12 (Multivector Conjugate (in 4D)).

$$
\begin{equation*}
\mathbf{u}^{\ddagger}=a-\mathbf{x}-\mathbf{f}+\mathbf{v}+\mathbf{b} \tag{73}
\end{equation*}
$$

Definition 13 (3,4 Blade Conjugate). The 3,4 blade conjugate of $\mathbf{u}$ is

$$
\begin{equation*}
\lfloor\mathbf{u}\rfloor_{3,4}=a+\mathbf{x}+\mathbf{f}-\mathbf{v}-\mathbf{b} \tag{74}
\end{equation*}
$$

We can now express the determinant of the matrix representation of a multivector via a self-product[6]:

Theorem 6 (Determinant as a Multivector Self-Product).

$$
\begin{equation*}
\left\lfloor\mathbf{u}^{\ddagger} \mathbf{u}\right\rfloor_{3,4} \mathbf{u}^{\ddagger} \mathbf{u}=\operatorname{det} \mathbf{M}_{\mathbf{u}} \tag{75}
\end{equation*}
$$

Proof. Omitted due to space constraint. See [6] for a proof.
Definition 14 (GA(3, 1)-valued Vector).

$$
|V\rangle\rangle=\left[\begin{array}{c}
\mathbf{u}_{1}  \tag{76}\\
\vdots \\
\mathbf{u}_{n}
\end{array}\right]=\left[\begin{array}{c}
a_{1}+\mathbf{x}_{1}+\mathbf{f}_{1}+\mathbf{v}_{1}+\mathbf{b}_{1} \\
\vdots \\
a_{n}+\mathbf{x}_{n}+\mathbf{f}_{n}+\mathbf{v}_{n}+\mathbf{b}_{n}
\end{array}\right]
$$

These constructions allow us to express the probability measure in terms of the multivector self-product.

Definition 15 (Multilinear Form).

$$
\langle V| V|V| V\rangle\rangle=\left\lfloor\left[\begin{array}{lll}
\mathbf{u}_{1}^{\ddagger} & \ldots & \mathbf{u}_{n}
\end{array}\right]\left[\begin{array}{ccc}
\mathbf{u}_{1} & \ldots & 0  \tag{77}\\
\vdots & \ddots & \vdots \\
0 & \ldots & \mathbf{u}_{n}
\end{array}\right]\right\rfloor_{3,4}\left[\begin{array}{ccc}
\mathbf{u}_{1}^{\ddagger} & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & \mathbf{u}_{n}^{\ddagger}
\end{array}\right]\left[\begin{array}{c}
\mathbf{u}_{1} \\
\vdots \\
\mathbf{u}_{n}
\end{array}\right]
$$

Theorem 7 (Partition Function). $Z=\langle\langle V| V| V|V\rangle\rangle$
Proof.

$$
\begin{align*}
& \langle V| V|V| V\rangle  \tag{78}\\
& \quad=\left\lfloor\left[\begin{array}{lll}
\mathbf{u}_{1}^{\ddagger} & \ldots & \mathbf{u}_{n}
\end{array}\right]\left[\begin{array}{ccc}
\mathbf{u}_{1} & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & \mathbf{u}_{n}
\end{array}\right]\right\rfloor_{3,4}\left[\begin{array}{ccc}
\mathbf{u}_{1}^{\ddagger} & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & \mathbf{u}_{n}^{\ddagger}
\end{array}\right]\left[\begin{array}{c}
\mathbf{u}_{1} \\
\vdots \\
\mathbf{u}_{n}
\end{array}\right]  \tag{79}\\
& \quad=\left\lfloor\left[\begin{array}{lll}
\mathbf{u}_{1}^{\ddagger} \mathbf{u}_{1} & \ldots & \mathbf{u}_{n} \mathbf{u}_{n}
\end{array}\right]\right\rfloor_{3,4}\left[\begin{array}{c}
\mathbf{u}_{1}^{\ddagger} \mathbf{u}_{1} \\
\vdots \\
\mathbf{u}_{n}^{\ddagger} \mathbf{u}_{n}
\end{array}\right]  \tag{80}\\
& \quad=\left\lfloor\mathbf{u}_{1}^{\ddagger} \mathbf{u}_{1}\right\rfloor_{3,4} \mathbf{u}_{1}^{\ddagger} \mathbf{u}_{1}+\cdots+\left\lfloor\mathbf{u}_{n}^{\ddagger} \mathbf{u}_{n}\right\rfloor_{3,4} \mathbf{u}_{n}^{\ddagger} \mathbf{u}_{n}  \tag{81}\\
& \quad=\sum_{i=1}^{n} \operatorname{det} \mathbf{M}_{\mathbf{u}_{i}}  \tag{82}\\
& \quad=Z \tag{83}
\end{align*}
$$

Theorem 8 (Non-negative inner product). The multilinear form, applied to the even sub-algebra of $\mathrm{GA}(3,1)$ is awlays non-negative.

Proof. Let $|V\rangle\rangle=\left[\begin{array}{c}a_{1}+\mathbf{f}_{1}+\mathbf{b}_{1} \\ \vdots \\ a_{n}+\mathbf{f}_{n}+\mathbf{b}_{n}\end{array}\right]$. Then,
$\langle\langle V| V| V|V\rangle\rangle$

$$
=\left\lfloor\left[\begin{array}{ll}
\left(a_{1}+\mathbf{f}_{1}+\mathbf{b}_{1}\right)^{\ddagger}\left(a_{1}+\mathbf{f}_{1}+\mathbf{b}_{1}\right) & \ldots .]\rfloor_{3,4}\left[\begin{array}{c}
\left(a_{1}+\mathbf{f}_{1}+\mathbf{b}_{1}\right)^{\ddagger}\left(a_{1}+\mathbf{f}_{1}+\mathbf{b}_{1}\right) \\
\vdots
\end{array}\right], ~ \tag{84}
\end{array}\right.\right.
$$

$=\left\lfloor\left[\begin{array}{lll}\left(a_{1}-\mathbf{f}_{1}+\mathbf{b}_{1}\right)\left(a_{1}+\mathbf{f}_{1}+\mathbf{b}_{1}\right) & \ldots]\rfloor_{3,4}\left[\begin{array}{c}\left(a_{1}-\mathbf{f}_{1}+\mathbf{b}_{1}\right)\left(a_{1}+\mathbf{f}_{1}+\mathbf{b}_{1}\right) \\ \vdots\end{array}\right], ~\end{array}\right.\right.$
$=\left\lfloor\left[a_{1}^{2}+a_{1} \mathbf{f}_{1}+a_{1} \mathbf{b}_{1}-\mathbf{f}_{1} a_{1}-\mathbf{f}_{1}^{2}-\mathbf{f}_{1} \mathbf{b}_{1}+\mathbf{b}_{1} a_{1}+\mathbf{b}_{1} \mathbf{f}_{1}+\mathbf{b}_{1}^{2} \quad \ldots\right]\right]_{3,4} \ldots$

$$
=\left\lfloor\left[\begin{array}{ll}
a_{1}^{2}-\mathbf{f}_{1}^{2}+\mathbf{b}_{1}^{2} & \ldots \tag{87}
\end{array}\right]\right\rfloor_{3,4} \ldots
$$

We note 1) $\mathbf{b}^{2}=(b I)^{2}=-b^{2}$ and 2) $\mathbf{f}^{2}=-E_{1}^{2}-E_{2}^{2}-E_{3}^{2}+B_{1}^{2}+B_{2}^{2}+B_{3}^{2}+$ $4 e_{0} e_{1} e_{2} e_{3}\left(E_{1} B_{1}+E_{2} B_{2}+E_{3} B_{3}\right)$

$$
\begin{equation*}
=\left\lfloor\left[a_{1}^{2}-b_{1}^{2}+E_{1}^{2}+E_{2}^{2}+E_{3}^{2}-B_{1}^{2}-B_{2}^{2}-B_{3}^{2}-4 e_{0} e_{1} e_{2} e_{3}\left(E_{1} B_{1}+E_{2} B_{2}+E_{3} B_{3}\right) \quad \ldots\right]\right\rfloor_{3,4} \ldots \tag{89}
\end{equation*}
$$

We note that the terms are now complex numbers, which we rewrite as $\operatorname{Re}(z)=$ $a_{1}^{2}-b_{1}^{2}+E_{1}^{2}+E_{2}^{2}+E_{3}^{2}-B_{1}^{2}-B_{2}^{2}-B_{3}^{2}$ and $\operatorname{Im}(z)=-4\left(E_{1} B_{1}+E_{2} B_{2}+E_{3} B_{3}\right)$

$$
\begin{align*}
& =\left\lfloor\left[\begin{array}{lll}
z_{1} & \ldots & z_{2}
\end{array}\right]_{3,4}\left[\begin{array}{c}
z_{n} \\
\vdots \\
z_{n}
\end{array}\right]\right.  \tag{90}\\
& =\left[\begin{array}{lll}
z_{1}^{\dagger} & \ldots & z_{2}^{\dagger}
\end{array}\right]\left[\begin{array}{c}
z_{n} \\
\vdots \\
z_{n}
\end{array}\right]  \tag{91}\\
& =z_{1}^{\ddagger} z_{1}+\cdots+z_{n}^{\ddagger} z_{n} \tag{92}
\end{align*}
$$

Which is always non-negative.
We now define the $\operatorname{Spin}^{c}(3,1)$-valued wavefunction, which is valued is the even sub-algebra of $\mathrm{GA}(3,1)$ :
Definition $16\left(\operatorname{Spin}^{c}(3,1)\right.$-valued Wavefunction).

$$
|\psi\rangle\rangle=\left[\begin{array}{c}
e^{\frac{1}{4}\left(a_{1}+\mathbf{f}_{1}+\mathbf{b}_{1}\right)}  \tag{93}\\
\vdots \\
e^{\frac{1}{4}\left(a_{n}+\mathbf{f}_{n}+\mathbf{b}_{n}\right)}
\end{array}\right]=\left[\begin{array}{c}
\sqrt[4]{\rho_{1}} R_{1} B_{1} \\
\vdots \\
\sqrt[4]{\rho_{n}} R_{n} B_{n}
\end{array}\right]
$$

where $R_{i}$ is a rotor and $B_{i}$ is a phase.

The evolution operator of the partition function becomes:
Definition $17\left(\operatorname{Spin}^{c}(3,1)\right.$ Flow).

$$
\begin{equation*}
e^{-\frac{1}{4} \tau(\mathbf{f}(q)+\mathbf{b}(q))} \tag{94}
\end{equation*}
$$

In turn, this leads to a Schrödinger equation:
Definition $18\left(\operatorname{Spin}^{c}(3,1)\right.$ Flow Generating Schrödinger equation).

$$
\begin{equation*}
\frac{\mathrm{d}}{\mathrm{~d} \tau} \psi(\tau)=-\frac{1}{2}(\mathbf{f}+\mathbf{b}) \psi(\tau) \tag{95}
\end{equation*}
$$

We will now demonstrate that the theory contains the $\mathrm{U}(1), \mathrm{SU}(2)$, and $\mathrm{SU}(3)$ gauge symmetries, which play a fundamental role in the standard model of particle physics. To show the invariance of these groups and conservation of the charge density, we will utilize the $\gamma_{0}$ basis.

Theorem 9 (U(1) Invariance and Charge Density Conservation). [7, 8]

$$
\begin{equation*}
\left.\left.\langle\psi(q)| \gamma_{0} \psi(q)|\psi(q)| \gamma_{0} \psi(q)\right\rangle=\left\langle e^{\frac{1}{2} \mathbf{b}} \psi(q)\right| \gamma_{0} e^{\frac{1}{2} \mathbf{b}} \psi(q)\left|e^{\frac{1}{2} \mathbf{b}} \psi(q)\right| \gamma_{0} e^{\frac{1}{2} \mathbf{b}} \psi(q)\right\rangle \tag{96}
\end{equation*}
$$

Proof.

$$
\begin{align*}
&\left\langle e^{\frac{1}{2} \mathbf{b}}\right. \psi(q)\left|\gamma_{0} e^{\frac{1}{2} \mathbf{b}} \psi(q)\right| e^{\frac{1}{2} \mathbf{b}} \psi(q)\left|\gamma_{0} e^{\frac{1}{2} \mathbf{b}} \psi(q)\right\rangle  \tag{97}\\
& \quad=\left\lfloor\psi(q)^{\ddagger} e^{\frac{1}{2} \mathbf{b}} \gamma_{0} e^{\frac{1}{2} \mathbf{b}} \psi(q)\right\rfloor_{3,4} \psi(q)^{\ddagger} e^{\frac{1}{2} \mathbf{b}} \gamma_{0} e^{\frac{1}{2} \mathbf{b}} \psi(q)  \tag{98}\\
& \quad=\left\lfloor\psi(q)^{\ddagger} \gamma_{0} e^{-\frac{1}{2} \mathbf{b}} e^{\frac{1}{2} \mathbf{b}} \psi(q)\right\rfloor_{3,4} \psi(q)^{\ddagger} \gamma_{0} e^{-\frac{1}{2} \mathbf{b}} e^{\frac{1}{2} \mathbf{b}} \psi(q)  \tag{99}\\
& \quad=\left\lfloor\psi(q)^{\ddagger} \gamma_{0} \psi(q)\right\rfloor_{3,4} \psi(q)^{\ddagger} \gamma_{0} \psi(q)  \tag{100}\\
&\left.\quad=\langle\psi(q)| \gamma_{0} \psi(q)|\psi(q)| \gamma_{0} \psi(q)\right\rangle \tag{101}
\end{align*}
$$

Theorem 10 (SU(2) Invariance and Charge Density Conservation). [7, 8]

$$
\begin{equation*}
\left.\left.\langle\psi(q)| \gamma_{0} \psi(q)|\psi(q)| \gamma_{0} \psi(q)\right\rangle=\left\langle e^{\frac{1}{2} \mathbf{f}} \psi(q)\right| \gamma_{0} e^{\frac{1}{2} \mathbf{f}} \psi(q)\left|e^{\frac{1}{2} \mathbf{f}} \psi(q)\right| \gamma_{0} e^{\frac{1}{2} \mathbf{f}} \psi(q)\right\rangle \tag{102}
\end{equation*}
$$

implies $\mathbf{f}=\theta_{1} \gamma_{0} \gamma_{1}+\theta_{2} \gamma_{0} \gamma_{2}+\theta_{3} \gamma_{0} \gamma_{3}$, which generates $\mathrm{SU}(2)$.
Proof.

$$
\begin{align*}
& \left.\left\langle e^{\frac{1}{2} \mathbf{f}} \psi(q)\right| \gamma_{0} e^{\frac{1}{2} \mathbf{f}} \psi(q)\left|e^{\frac{1}{2} \mathbf{f}} \psi(q)\right| \gamma_{0} e^{\frac{1}{2} \mathbf{f}} \psi(q)\right\rangle  \tag{103}\\
& \quad=\left\lfloor\psi(q)^{\ddagger} e^{-\frac{1}{2} \mathbf{f}} \gamma_{0} e^{\frac{1}{2} \mathbf{f}} \psi(q)\right\rfloor_{3,4} \psi(q)^{\ddagger} e^{-\frac{1}{2} \mathbf{f}} \gamma_{0} e^{\frac{1}{2} \mathbf{f}} \psi(q) \tag{104}
\end{align*}
$$

We can now identify that the condition to preserve the equality reduces to this expression:

$$
\begin{equation*}
e^{-\frac{1}{2} \mathbf{f}} \gamma_{0} e^{\frac{1}{2} \mathbf{f}}=\gamma_{0} \tag{105}
\end{equation*}
$$

We further note that moving the left most term to the right yields:

$$
\begin{align*}
& e^{-\theta_{1} \gamma_{0} \gamma_{1}-\theta_{2} \gamma_{0} \gamma_{2}-\theta_{3} \gamma_{0} \gamma_{3}-B_{1} \gamma_{2} \gamma_{3}-B_{2} \gamma_{1} \gamma_{3}-B_{3} \gamma_{1} \gamma_{2}} \gamma_{0} e^{\frac{1}{2} \mathbf{f}}  \tag{106}\\
& \quad=\gamma_{0} e^{-\theta_{1} \gamma_{0} \gamma_{1}-\theta_{2} \gamma_{0} \gamma_{2}-\theta_{3} \gamma_{0} \gamma_{3}+B_{1} \gamma_{2} \gamma_{3}+B_{2} \gamma_{1} \gamma_{3}+B_{3} \gamma_{1} \gamma_{2}} e^{\frac{1}{2} \mathbf{f}} \tag{107}
\end{align*}
$$

Therefore, the product $e^{-\frac{1}{2} \mathbf{f}} \gamma_{0} e^{\frac{1}{2} \mathbf{f}}$ reduces to $\gamma_{0}$ if and only if $B_{1}=B_{2}=B_{3}=0$, leaving $\mathbf{f}=\theta_{1} \gamma_{0} \gamma_{1}+\theta_{2} \gamma_{0} \gamma_{2}+\theta_{3} \gamma_{0} \gamma_{3}$ :

Finally, we note that $e^{\theta_{1} \gamma_{0} \gamma_{1}+\theta_{2} \gamma_{0} \gamma_{2}+\theta_{3} \gamma_{0} \gamma_{3}}$ generates $\mathrm{SU}(2)$.
Theorem 11 (SU(3) invariance). [7, 8]

$$
\begin{equation*}
\left.\left.\langle\psi(q)| \gamma_{0} \psi(q)|\psi(q)| \gamma_{0} \psi(q)\right\rangle=\langle\mathbf{f} \psi(q)| \gamma_{0} \mathbf{f} \psi(q)|\mathbf{f} \psi(q)| \gamma_{0} \mathbf{f} \psi(q)\right\rangle \tag{108}
\end{equation*}
$$

Proof. From the above relation, we identify that the following expression must remain invariant: $-\mathbf{f} \gamma_{0} \mathbf{f}=\gamma_{0}$. Now, let $\mathbf{f}=E_{1} \gamma_{0} \gamma_{1}+E_{2} \gamma_{0} \gamma_{2}+E_{3} \gamma_{0} \gamma_{3}+$ $B_{1} \gamma_{2} \gamma_{3}+B_{2} \gamma_{1} \gamma_{3}+B_{3} \gamma_{1} \gamma_{2}$. Then:

$$
\begin{equation*}
-\left(E_{1} \gamma_{0} \gamma_{1}+E_{2} \gamma_{0} \gamma_{2}+E_{3} \gamma_{0} \gamma_{3}+B_{1} \gamma_{2} \gamma_{3}+B_{2} \gamma_{1} \gamma_{3}+B_{3} \gamma_{1} \gamma_{2}\right) \gamma_{0} \mathbf{f} \tag{109}
\end{equation*}
$$

The first three terms anticommute with $\gamma_{0}$, while the last three commute with $\gamma_{0}$ :

$$
\begin{equation*}
=\gamma_{0}\left(E_{1} \gamma_{0} \gamma_{1}+E_{2} \gamma_{0} \gamma_{2}+E_{3} \gamma_{0} \gamma_{3}-B_{1} \gamma_{2} \gamma_{3}-B_{2} \gamma_{1} \gamma_{3}-B_{3} \gamma_{1} \gamma_{2}\right) \mathbf{f} \tag{110}
\end{equation*}
$$

This can be written as:

$$
\begin{align*}
& \gamma_{0}(\mathbf{E}-\mathbf{B})(\mathbf{E}+\mathbf{B})  \tag{111}\\
& \quad=\gamma_{0}\left(\mathbf{E}^{2}+\mathbf{E B}-\mathbf{B E}-\mathbf{B}^{2}\right) \tag{112}
\end{align*}
$$

where $\mathbf{E}=E_{1} \gamma_{0} \gamma_{1}+E_{2} \gamma_{0} \gamma_{2}+E_{3} \gamma_{0} \gamma_{3}$ and $\mathbf{B}=B_{1} \gamma_{2} \gamma_{3}+B_{2} \gamma_{1} \gamma_{3}+B_{3} \gamma_{1} \gamma_{2}$.
Thus, for $-\mathbf{f} \gamma_{0} \mathbf{f}=\gamma_{0}$, we require: 1) $\mathbf{E}^{2}-\mathbf{B}^{2}=1$ and 2) $\mathbf{E B}=\mathbf{B E}$. The second requirement means that $\mathbf{E}$ and $\mathbf{B}$ must commute (and thus be isomorphic to three complex numbers), and the first implies:

$$
\begin{equation*}
\mathbf{E}^{2}-\mathbf{B}^{2}=\left(E_{1}^{2}+B_{1}^{2}\right)+\left(E_{2}^{2}+B_{2}^{2}\right)+\left(E_{3}^{2}+B_{3}^{2}\right)=1 \tag{113}
\end{equation*}
$$

which are the defining conditions for the $\mathrm{SU}(3)$ symmetry group.
We now investigate the metric tensor. The construction of the metric tensor as a quantum observable relies on the self-adjointness of the gamma matrices within the multilinear form:

Theorem 12 (Metric Tensor). The metric tensor is the expectation value of the $\gamma_{\mu}$ and $\gamma_{\nu}$ observables:

$$
\begin{equation*}
\frac{\left.\langle\psi(q)| \gamma_{\mu} \psi(q)|\psi(q)| \gamma_{\nu} \psi(q)\right\rangle}{\langle\psi(q)| \psi(q)|\psi(q)| \psi(q)\rangle}=\left\langle g_{\mu \nu}(q)\right\rangle \tag{114}
\end{equation*}
$$

Proof. To improve the legibility of the proof, we have dropped the parametrization in $(q)$.

$$
\begin{align*}
& \left.\langle\psi| \gamma_{\mu} \psi|\psi| \gamma_{\nu} \psi\right\rangle  \tag{115}\\
& \quad=\left\lfloor\sqrt[4]{\rho} \tilde{R} B \gamma_{\mu} \sqrt[4]{\rho} R B\right\rfloor_{3,4} \sqrt[4]{\rho} \tilde{R} B \gamma_{\nu} \sqrt[4]{\rho} R B \tag{116}
\end{align*}
$$

We note that $B \gamma_{\mu} B=\gamma_{\mu}$, because the pseudoscalar anticommutes with vectors. Finally, since $\mathbf{e}_{\mu}=\tilde{R} \hat{\mathbf{x}}_{\mu} R$, we have:

$$
\begin{equation*}
=\rho \mathbf{e}_{\mu} \mathbf{e}_{\nu} \tag{117}
\end{equation*}
$$

For basis vectors, the geometric product is the same as the dot product, yielding:

$$
\begin{equation*}
=\rho g_{\mu \nu} \tag{118}
\end{equation*}
$$

For completeness, we also investigate the self-adjoint:

$$
\begin{align*}
& \left.\left\langle\gamma_{\mu} \psi\right| \psi\left|\gamma_{\nu} \psi\right| \psi\right\rangle  \tag{119}\\
& \quad=\left\lfloor\sqrt[4]{\rho} \tilde{R} B\left(-\gamma_{\mu}\right) \sqrt[4]{\rho} R B\right\rfloor_{3,4} \sqrt[4]{\rho} \tilde{R} B\left(-\gamma_{\nu}\right) \sqrt[4]{\rho} R B  \tag{120}\\
& \quad=\left\lfloor\sqrt[4]{\rho} \tilde{R} B \gamma_{\mu} \sqrt[4]{\rho} R B\right\rfloor_{3,4} \sqrt[4]{\rho} \tilde{R} B \gamma_{\nu} \sqrt[4]{\rho} R B  \tag{121}\\
& \quad=\rho g_{\mu \nu} \tag{122}
\end{align*}
$$

As one can swap $\gamma_{\mu}$ with $\gamma_{\nu}$ and obtain the same metric tensor, the multilinear form guarantees that $g_{\mu \nu}$ is symmetric. Finally, dividing the above results with $\langle\psi| \psi|\psi| \psi\rangle=\rho$ yields $\rho g_{\mu \nu} / \rho=g_{\mu \nu}$. Finally, since $\left.\left\langle\gamma_{\mu} \psi(q)\right| \psi(q)\left|\gamma_{\nu} \psi(q)\right| \psi(q)\right\rangle=$ $\left.\langle\psi(q)| \gamma_{\mu} \psi(q)|\psi(q)| \gamma_{\nu} \psi(q)\right\rangle$, then $\gamma_{\mu}$ and $\gamma_{\nu}$ are self-adjoint within the multilinear form, entailing the interpretation of $g_{\mu \nu}$ as a quantum observable.

The following theorem provides a general expression for the interference pattern arising from the superposition the $\operatorname{Spin}^{c}(3,1)$-valued wavefunction, which generalizes the complex interference commonly found in standard quantum mechanics. This interference leads to a sum over geometries within the probability measure:
Theorem 13 (Multivector Superposition and Interference).
Proof. Let $|V\rangle\rangle=\frac{1}{\sqrt{2}}\left[\begin{array}{l}\mathbf{u}_{1} \\ \mathbf{u}_{2}\end{array}\right]$. Now suppose an Hadamard transformation yielding $\left.\left|V^{\prime}\right\rangle\right\rangle=\frac{1}{2}\left[\begin{array}{l}\mathbf{u}_{1}+\mathbf{u}_{2} \\ \mathbf{u}_{1}-\mathbf{u}_{2}\end{array}\right]$. The general form of geometric interference for two-state system is as follows. Let us take the state $\mathbf{u}_{1}+\mathbf{u}_{2}$ as an example (dropping the multiplication scalars for legibility):

$$
\begin{align*}
& \left\lfloor\left(\mathbf{u}_{1}+\mathbf{u}_{2}\right)^{\ddagger}\left(\mathbf{u}_{1}+\mathbf{u}_{2}\right)\right\rfloor_{3,4}\left(\mathbf{u}_{1}+\mathbf{u}_{2}\right)^{\ddagger}\left(\mathbf{u}_{1}+\mathbf{u}_{2}\right)  \tag{123}\\
& \quad=\left\lfloor\left(\mathbf{u}_{1}^{\ddagger}+\mathbf{u}_{2}^{\ddagger}\right)\left(\mathbf{u}_{1}+\mathbf{u}_{2}\right)\right\rfloor_{3,4}\left(\mathbf{u}_{1}^{\ddagger}+\mathbf{u}_{2}^{\ddagger}\right)\left(\mathbf{u}_{1}+\mathbf{u}_{2}\right)  \tag{124}\\
& \quad=\left\lfloor\left(\mathbf{u}_{1}^{\ddagger} \mathbf{u}_{1}+\mathbf{u}_{1}^{\ddagger} \mathbf{u}_{2}+\mathbf{u}_{2}^{\ddagger} \mathbf{u}_{1}+\mathbf{u}_{2}^{\ddagger} \mathbf{u}_{2}\right)\right\rfloor_{3,4}\left(\mathbf{u}_{1}^{\ddagger} \mathbf{u}_{1}+\mathbf{u}_{1}^{\ddagger} \mathbf{u}_{2}+\mathbf{u}_{2}^{\ddagger} \mathbf{u}_{1}+\mathbf{u}_{2}^{\ddagger} \mathbf{u}_{2}\right)  \tag{125}\\
& \quad=\lfloor\underbrace{\left.\mathbf{u}_{1}^{\ddagger} \mathbf{u}_{1}\right\rfloor_{3,4} \mathbf{u}_{1}^{\ddagger} \mathbf{u}_{1}}_{\rho_{1}}+\lfloor\underbrace{\left.\mathbf{u}_{2}^{\ddagger} \mathbf{u}_{2}\right\rfloor_{3,4} \mathbf{u}_{2}^{\ddagger} \mathbf{u}_{2}}_{\rho_{2}}+\underbrace{\left\lfloor\mathbf{u}_{1}^{\ddagger} \mathbf{u}_{1}\right\rfloor_{3,4} \mathbf{u}_{1}^{\ddagger} \mathbf{u}_{2}+13 \text { terms }}_{\text {geometric interference pattern }} \tag{126}
\end{align*}
$$

The metric fluctuations are defined as follows:
Definition 19 (Metric Fluctuations).

$$
\begin{equation*}
\sigma\left(g_{\mu \nu}\right)^{2}=\left\langle g_{\mu \nu}^{2}\right\rangle-\left\langle g_{\mu \nu}\right\rangle^{2} \tag{127}
\end{equation*}
$$

The absence of superposition reduces to a spacetime without fluctuations:
Theorem 14 (Reduction to Smooth spacetime). In the absence of superpositions, the fluctuations reduces to 0 .

Proof. Based on the definition of the metric tensor as an expectation value, for $\left\langle g_{\mu \nu}\right\rangle^{2}$, we have:

$$
\begin{equation*}
\left(\frac{\left\lfloor\psi(q)^{\ddagger} \gamma_{\mu} \psi(q)\right\rfloor_{3,4} \psi(q)^{\ddagger} \gamma_{\nu} \psi(q)}{\left\lfloor\psi(q)^{\ddagger} \psi(q)\right\rfloor_{3,4} \psi(q)^{\ddagger} \psi(q)}\right)^{2}=\mathbf{e}_{\mu} \mathbf{e}_{\nu} \mathbf{e}_{\mu} \mathbf{e}_{\nu}=-\mathbf{e}_{\mu} \mathbf{e}_{\mu} \mathbf{e}_{\nu} \mathbf{e}_{\mu}=-g_{\mu \mu} g_{\nu \nu} \tag{128}
\end{equation*}
$$

and to construct $\left\langle g_{\mu \nu}^{2}\right\rangle$, we first expand $\rho(q) g_{\mu \nu}(q)$ :

$$
\begin{equation*}
\left\lfloor\psi(q)^{\ddagger} \gamma_{\mu} \psi(q)\right\rfloor_{3,4} \psi(q)^{\ddagger} \gamma_{\nu} \psi(q)=\rho(q) \mathbf{e}_{\mu}(q) \mathbf{e}_{\nu}(q) \tag{129}
\end{equation*}
$$

Now to get $\left\langle g_{\mu \nu}^{2}\right\rangle$, we must square $\mathbf{e}_{\mu} \mathbf{e}_{\nu}$, yielding $\rho(q)\left(\mathbf{e}_{\mu}(q) \mathbf{e}_{\nu}(q)\right)^{2}$ which is the correct expression for $\left\langle g_{\mu \nu}^{2}\right\rangle$. However, since $\left(\mathbf{e}_{\mu}(q) \mathbf{e}_{\nu}(q)\right)^{2}=-g_{\mu \mu} g_{\nu \nu}$, it follows that $\left\langle g_{\mu \nu}^{2}\right\rangle-\left\langle g_{\mu \nu}\right\rangle^{2}=0$.

In summary, multivector-valued amplitudes offer a powerful framework for describing the fundamental interactions of particles and spacetime geometry, naturally incorporating $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ gauge symmetries and associated charge density conservation, the metric tensor as a quantum mechanical object and retaining invariance with respect to the $\mathrm{SO}(3,1)$ and $\operatorname{Spin}^{c}(3,1)$ group.

### 2.3 Dimensional Obstructions

In this section, we explore the dimensional obstructions that arise when attempting to extend the multivector amplitude formalism to dimensions other than $3+1$. We begin by examining the self-products associated with low-dimensional geometric algebras.
Definition 20. From the results of [6], the self-products associated with lowdimensional geometric algebras are:

$$
\begin{align*}
\mathrm{CL}(0,1): & \varphi^{\dagger} \varphi  \tag{130}\\
\mathrm{CL}(2,0): & \varphi^{\ddagger} \varphi  \tag{131}\\
\mathrm{CL}(3,0): & \left\lfloor\varphi^{\ddagger} \varphi\right\rfloor_{3} \varphi^{\ddagger} \varphi  \tag{132}\\
\mathrm{CL}(3,1): & \left\lfloor\varphi^{\ddagger} \varphi\right\rfloor_{3,4} \varphi^{\ddagger} \varphi  \tag{133}\\
\mathrm{CL}(4,1): & \left(\left\lfloor\varphi^{\ddagger} \varphi\right\rfloor_{3,4} \varphi^{\ddagger} \varphi\right)^{\dagger}\left(\left\lfloor\varphi^{\ddagger} \varphi\right\rfloor_{3,4} \varphi^{\ddagger} \varphi\right) \tag{134}
\end{align*}
$$

From Theorem 6, and the results obtained in the previous sections, we have seen that in the CL $(3,1)$ case, the self-product corresponds to the determinant of the matrix representation of the corresponding geometric algebra and can be interpreted as a probability measure associated with many physical phenomena. However, when we investigate other dimensions, we encounter several obstructions that prevent the construction of a consistent and physically meaningful probability measure.

The first obstruction arises in the case of $\mathrm{CL}(0,1), \mathrm{CL}(3,0)$, and higher odd-dimensional geometric algebras, where the determinant of the matrix representation is complex-valued and, consequently, cannot represent a probability.

Theorem 15. For $\mathrm{CL}(0,1)$, $\mathrm{CL}(3,0)$, and higher odd-dimensional geometric algebras, the determinant of the matrix representation is complex-valued and, consequently, cannot represent a probability.

Proof. The probabilities in the POP framework are defined by the determinant of a matrix. 3D geometric algebra is represented by 2 x 2 complex matrices, and the determinant of such matrices is complex, not real. Hence, the probabilities are complex-valued, not real-valued, making the solution unphysical in 3D. In $0+1 \mathrm{D}$, the GA is isomorphic to the complex numbers, and the determinant of a complex number is the complex number itself. Since odd-dimensional geometric algebras map to complex-valued matrices, this is also the case with 5D geometric algebra and higher odd-dimensional spaces.

This theorem highlights the fundamental issue with odd-dimensional geometric algebras, where the complex-valued determinant of the matrix representation cannot be interpreted as a physically meaningful probability measure.

The second obstruction concerns the lack of a corresponding geometric algebra formulation for certain matrix dimensions, which limits the ability to define a wavefunction in terms of multivectors, necessary for defining an amplitude.

Theorem 16. For $1 \times 1,3 \times 3$, or any higher odd-dimensional matrices, there is no corresponding geometric algebra formulation. It is, therefore, not possible to represent the determinant as a self-product of multivectors, which limits the ability to define a wavefunction.

Proof. All geometric algebras, regardless of signature or dimension, map to even-dimensional square matrices. This means that odd-dimensional square matrices, such as $3 \times 3$ matrices, do not have a corresponding geometric algebra formulation and thus cannot define an amplitude.

This theorem emphasizes the importance of having a geometric algebra formulation for the matrix representation, as it allows for the definition of a wavefunction in terms of multivectors and the construction of an amplitude based on the multivector self-product.

As we move to higher dimensions, we encounter further obstructions that prevent the construction of a consistent probability measure and the satisfaction of observables. In particular, the multivector representation of the norm in 6 D fails to extend the self-product patterns found in lower dimensions.

Conjecture 1. The multivector representation of the norm in $6 D$ cannot satisfy any observables.

Argument. In six dimensions and above, the self-product patterns found in Definition 20 collapse. The research by Acus et al.[9] in 6D geometric algebra demonstrates that the determinant, so far defined through a self-products of the multivector, fails to extend into 6 D . The crux of the difficulty is evident in the reduced case of a 6 D multivector containing only scalar and grade- 4 elements:

$$
\begin{equation*}
s(B)=b_{1} B f_{5}\left(f_{4}(B) f_{3}\left(f_{2}(B) f_{1}(B)\right)\right)+b_{2} B g_{5}\left(g_{4}(B) g_{3}\left(g_{2}(B) g_{1}(B)\right)\right) \tag{135}
\end{equation*}
$$

This equation is not a multivector self-product but a linear sum of two multivector self-products.

The full expression [9] is given in the form of a system of 4 equations, which is too long to list in its entirety. A small characteristic part is shown:

$$
\begin{align*}
& a_{0}^{4}-2 a_{0}^{2} a_{47}^{2}+b_{2} a_{0}^{2} a_{47}^{2} p_{412} p_{422}+\langle 72 \text { monomials }\rangle=0  \tag{136}\\
& b_{1} a_{0}^{3} a_{52}+2 b_{2} a_{0} a_{47}^{2} a_{52} p_{412} p_{422} p_{432} p_{442} p_{452}+\langle 72 \text { monomials }\rangle=0  \tag{137}\\
& \langle 74 \text { monomials }\rangle=0  \tag{138}\\
& \langle 74 \text { monomials }\rangle=0 \tag{139}
\end{align*}
$$

From Equation 135, it is possible to see that no observable $\mathbf{O}$ can satisfy this equation because the linear combination does not allow one to factor it out of the equation.

$$
\begin{equation*}
b_{1} \mathbf{O} B f_{5}\left(f_{4}(B) f_{3}\left(f_{2}(B) f_{1}(B)\right)\right)+b_{2} B g_{5}\left(g_{4}(B) g_{3}\left(g_{2}(B) g_{1}(B)\right)\right)=b_{1} B f_{5}\left(f_{4}(B) f_{3}\left(f_{2}(B) f_{1}(B)\right)\right)+b_{2} \mathbf{O} B g_{5}\left(g_{4}(B) g_{3}\left(g_{2}(B) g_{1}(B)\right)\right) \tag{140}
\end{equation*}
$$

Any equality of the above type between $b_{1} \mathbf{O}$ and $b_{2} \mathbf{O}$ is frustrated by the factors $b_{1}$ and $b_{2}$, forcing $\mathbf{O}=1$ as the only satisfying observable. Since the obstruction occurs within grade-4, which is part of the even sub-algebra it is questionable that a satisfactory quantum theory (with observables) be constructible in 6D.

This conjecture proposes that the multivector representation of the determinant in 6 D does not allow for the construction of non-trivial observables, which is a crucial requirement for a consistent quantum formalism. The linear combination of multivector self-products in the 6D expression prevents the factorization of observables, limiting their role to the identity operator.

Conjecture 2. The norms beyond $6 D$ are progressively more complex than the $6 D$ case, which is already obstructed.

Finally, we consider the specific case of four dimensions and show that the POP method requires a $3+1 \mathrm{D}$ signature to maintain consistency with the previously established results.

Theorem 17. The POP method in four dimensions specifically requires a $3+1 D$ signature.

Proof. Starting with 4 x 4 real matrices as our solution, we are restricted to choosing a geometric algebra isomorphic to it. In 4D, the options are:

1. $\mathrm{GA}(3,1)$ is isomorphic to the algebra of $4 \times 4$ real matrices, denoted as $\mathrm{M}(4, \mathbb{R})$.
2. $\mathrm{GA}(1,3)$ is isomorphic to the algebra of $2 \times 2$ quaternionic matrices, denoted as $\mathrm{M}(2, \mathbb{H})$ or $\mathbb{H}(2)$.
3. $\mathrm{GA}(4,0)$ is isomorphic to the direct sum of two copies of the algebra of $2 \times 2$ real matrices, denoted as $\mathrm{M}(2, \mathbb{R}) \oplus \mathrm{M}(2, \mathbb{R})$.
4. $\mathrm{GA}(2,2)$ is isomorphic to the algebra of $4 \times 4$ real matrices, denoted as $\mathrm{M}(4, \mathbb{R})$.
5. $\mathrm{GA}(0,4)$ is isomorphic to the algebra of $2 \times 2$ quaternionic matrices, denoted as $\mathrm{M}(2, \mathbb{H})$ or $\mathbb{H}(2)$.

This leaves only the choice of either GA $(3,1)$ or $\mathrm{GA}(2,2)$ as signatures of interest.

Conjecture 3 (Obstruction in GA(2,2)). The maximization problem introduces a single Lagrange multiplier $\tau$, governing the time evolution of systems, leading to possible obstructions when applied to a spacetime with multiple time dimensions, such as $\mathrm{GA}(2,2)$.

Conjecture 4 (Obstruction in GA(4, 0) and GA( 0,4$)$ and GA(2, 0)). The maximization problem introduces a single Lagrange multiplier $\tau$, governing the time evolution of systems, leading to obstructions when applied to a spacetime with no time dimensions, such as $\mathrm{GA}(0,4)$, $\mathrm{GA}(4,0)$ or $\mathrm{GA}(2,0)$.

Theorem 18 (Obstruction in $1+1 \mathrm{D}$ ). We repeat the obstruction found in $1+1 D$, leading to negative probabilities because the bilinear norm resolves to $a^{2}-b^{2}$.

These theorems and conjectures provide additional insights into the unique role of the $3+1 \mathrm{D}$ signature in the POP method. It suggests a plausible mechanism for the specific dimensional arrangement of the universe deeply linked to the mathematical good behavior of multivector amplitudes.

## 3 Discussion

## The Geometric Anti-Constraint as the Sole Axiom

The geometric anti-constraint, given by $0=\frac{1}{d} \operatorname{tr} \sum_{q \in \mathbb{Q}} \rho(q) \mathbf{M}(q)$, serves as the sole axiom of the theory in our formulation. This constraint shapes
the optimization problem and determines the structure of the resulting quantum theory. Just as the average energy constraint $\bar{E}=\sum_{q \in \mathbb{Q}} \rho(q) E(q)$ in statistical mechanics yields the Gibbs measure, and the phase anti-constraint $0=\operatorname{tr} \sum_{q \in \mathbb{Q}} \rho(q)\left[\begin{array}{cc}0 & -E(q) \\ E(q) & 0\end{array}\right]$, a special case of the geometric anti-constraint, leads to the five canonical axioms of quantum mechanics in our previous work, the geometric anti-constraint resolves into a quantum theory that naturally incorporates multivector amplitudes.

The power of multivector amplitudes lies in their ability to encapsulate the essential features of both particle physics and gravitation within a single framework. The gauge symmetries of the Standard Model, namely $\mathrm{SU}(3) \times$ $\mathrm{SU}(2) \times \mathrm{U}(1)$, along with charge density conservation arise naturally from the time invariance of the probability measure. Similarly, the geometry of spacetime emerges through the metric tensor, defined as a quantum observable by using the gamma matrices as operators.

This remarkable property suggests that the geometric anti-constraint contains the necessary information to describe the fundamental interactions of particles and fields, as well as the geometry of spacetime, without the need for ad hoc assumptions or additional postulates. As the POP framework contains a sole axiom, namely the geometric anti-constraint, its parsimony is unmatched.

Addressing the Relativistic Nature of the Schrödinger Equation
A common objection to the relativistic nature of our theory arises from the use of the Schrödinger equation or Schrödinger-like time evolution. However, it is crucial to distinguish between the general Schrödinger equation itself and the non-relativistic single-particle Schrödinger equation. The Schrödinger equation, given by $|\psi(t)\rangle=e^{-i t H / \hbar}|\psi(0)\rangle$, is relativistic provided H is relativistic. This formulation is equivalent to the Feynman path integral representation, which is manifestly compatible with relativity.

To illustrate this point, let's consider the example of a free scalar field. In the Feynman path integral representation, the action for a free scalar field $\phi(x)$ is given by:

$$
\begin{equation*}
S[\phi]=\int d^{4} x\left(\frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi-\frac{1}{2} m^{2} \phi^{2}\right) \tag{141}
\end{equation*}
$$

where $m$ is the mass of the scalar field. The path integral is then defined as:

$$
\begin{equation*}
Z=\int \mathcal{D} \phi e^{i S[\phi] / \hbar} \tag{142}
\end{equation*}
$$

which sums over all possible field configurations weighted by the exponential of the action.

In the Hamiltonian formulation, the Schrödinger equation for the free scalar field is given by:

$$
\begin{equation*}
i \hbar \frac{\partial}{\partial t}|\Psi(t)\rangle=\hat{H}|\Psi(t)\rangle \tag{143}
\end{equation*}
$$

where the Hamiltonian $\hat{H}$ is obtained from the Legendre transform of the Lagrangian:

$$
\begin{equation*}
\hat{H}=\int d^{3} x\left(\frac{1}{2} \hat{\pi}^{2}+\frac{1}{2}(\nabla \hat{\phi})^{2}+\frac{1}{2} m^{2} \hat{\phi}^{2}\right) \tag{144}
\end{equation*}
$$

with $\hat{\pi}=\frac{\partial \mathcal{L}}{\partial \dot{\phi}}$ being the canonical momentum operator.
The solution to the Schrödinger equation is given by:

$$
\begin{equation*}
|\Psi(t)\rangle=e^{-i t \hat{H} / \hbar}|\Psi(0)\rangle \tag{145}
\end{equation*}
$$

which describes the time evolution of the quantum state $|\Psi(t)\rangle$.
The relativistic compatibility of both the Feynman path integral representation and the Hamiltonian formulation using the Schrödinger equation are dependant on the Lagrangian or Hamiltonian used, and not on the choice of representation.

The POP methodology resolves to the Schrödinger picture, yet this does not prevent it from being relativistic.

Probability Density
Let us now extend the entropy maximization problem from the discreet $\Sigma$ to the continuum $\int$, using a Riemann sum:
$\mathcal{L}=-\lim _{n \rightarrow \infty}\left(\sum_{i=1}^{n} \rho\left(x_{i}\right) \ln \frac{\rho\left(x_{i}\right)}{p\left(x_{i}\right)}+\lambda\left(1-\sum_{i=1}^{n} \rho\left(x_{i}\right)\right)+\tau\left(\operatorname{tr} \sum_{i=1}^{n} \rho\left(x_{i}\right) \frac{1}{\varepsilon\left(x_{i}\right)}\left[\begin{array}{cc}0 & -E\left(x_{i}\right) \\ E\left(x_{i}\right) & 0\end{array}\right]\right)\right) \Delta x$
where

- n is the number of subintervals,
- $\Delta x=(b-a) / n$ is the width of each subinterval,
- $x_{i}$ is a point within the i-th subinterval $\left[x_{i-1}, x_{i}\right]$, often chosen to be the midpoint $\left(x_{i-1}+x_{i}\right) / 2$.
- $1 / \varepsilon\left(x_{i}\right)$ is a factor required to transform the energy $E(x)$ into an energy density $\mathcal{E}(x)=E(x) / \varepsilon(x)$, required for integration.
which yields an integral:

$$
\mathcal{L}=-\int_{a}^{b} \rho(x) \ln \frac{\rho(x)}{p(x)} \mathrm{d} x+\lambda\left(1-\int_{a}^{b} \rho(x) \mathrm{d} x\right)+\tau\left(\operatorname{tr} \int_{a}^{b} \rho(x) \frac{1}{\varepsilon(x)}\left[\begin{array}{cc}
0 & -E(x)  \tag{147}\\
E(x) & 0
\end{array}\right] \mathrm{d} x\right)
$$

Solving this optimization problem yields a probability measure parametrized over the continuum.

We can extend this formulation to multivector amplitudes by using the geometric anti-constraint and parametrized over a world manifold $X^{4}$ :

$$
\begin{equation*}
\mathcal{L}=-\int_{a}^{b} \rho\left(x^{\mu}\right) \ln \frac{\rho\left(x^{\mu}\right)}{p\left(x^{\mu}\right)} \sqrt{-g} \mathrm{~d}^{4} x+\lambda\left(1-\int_{a}^{b} \rho\left(x^{\mu}\right) \sqrt{-g} \mathrm{~d}^{4} x\right)+\tau\left(\operatorname{tr} \int_{a}^{b} \frac{1}{4} \rho\left(x^{\mu}\right) \frac{1}{\varepsilon\left(x^{\mu}\right)} \mathbf{M}\left(x^{\mu}\right) \sqrt{-g} \mathrm{~d}^{4} x\right) \tag{148}
\end{equation*}
$$

The solution to this optimization problem is a probability density:

$$
\begin{equation*}
\frac{\partial \mathcal{L}(\rho, \lambda, t)}{\partial \rho}=0 \Longrightarrow \rho\left(x^{\mu}\right)=\underbrace{\frac{1}{\int_{a}^{b} p\left(x^{\mu}\right) \exp \left(-\frac{1}{4} \tau \frac{1}{\varepsilon\left(x^{\mu}\right)} \operatorname{tr} \mathbf{M}\left(x^{\mu}\right)\right) \sqrt{-g} \mathrm{~d}^{4} x}}_{\text {Geometrically Invariant Ensemble }} \underbrace{\exp \left(-\frac{1}{4} \tau \frac{1}{\varepsilon\left(x^{\mu}\right)} \operatorname{tr} \mathbf{M}\left(x^{\mu}\right)\right)}_{\text {Geometric Born Rule }} \underbrace{p\left(x^{\mu}\right)}_{\text {Initial State }} \tag{149}
\end{equation*}
$$

This formulation extends the multivector amplitude framework to the continuum, allowing for the description of continuous systems while preserving the geometric structure and invariance properties of the theory.

## Double copy gauge theory

The $\mathrm{U}(1), \mathrm{SU}(2)$, and $\mathrm{SU}(3)$ invariances of the multilinear form lead to a double copy structure of gauge theories, as each side of the multilinear form can evolve independently. In the previous result section, we have derived the $\mathrm{SU}(3)$ gauge symmetry using a single bivector $\mathbf{f}$, however the multilinear form is able to support two gauges $\mathbf{f}_{1}$ and $\mathbf{f}_{2}$. For instance in the $\mathrm{SU}(3)$ case:

$$
\begin{equation*}
\left.\left.\langle\psi(q)| \gamma_{0} \psi(q)|\psi(q)| \gamma_{0} \psi(q)\right\rangle=\left\langle\mathbf{f}_{1} \psi(q)\right| \gamma_{0} \mathbf{f}_{1} \psi(q)\left|\mathbf{f}_{2} \psi(q)\right| \gamma_{0} \mathbf{f}_{2} \psi(q)\right\rangle \tag{150}
\end{equation*}
$$

This results in two separately conserved $\mathrm{SU}(3)$ gauge theories:

$$
\begin{align*}
& -\mathbf{f}_{1} \gamma_{0} \mathbf{f}_{1}=\gamma_{0} \Longrightarrow \mathrm{SU}(3) \text { as copy } 1  \tag{151}\\
& -\mathbf{f}_{2} \gamma_{0} \mathbf{f}_{2}=\gamma_{0} \Longrightarrow \text { another } \mathrm{SU}(3) \text { as copy } 2 \tag{152}
\end{align*}
$$

This argument also holds for $\mathrm{U}(1)$ and $\mathrm{SU}(2)$. In this case, the gravitational theory is a "double copy" of a gauge theory.

A potential future research direction could be to investigate whether this double copy structure is connected to the double copy theory[10], which aims to express gravity as a double copy of a gauge theory. Exploring this relationship may provide further insights into the interaction picture of quantum gravity.

## 4 Conclusion

In conclusion, this paper advances the 'Prescribed Observation Problem' (POP) into a multivector quantum theory, seamlessly bridging the realms of quantum mechanics and spacetime geometry. Our findings reveal the POP's exceptional
ability to generate a mathematically well-behaved theory that generalizes quantum probabilities through the introduction of the multivector probability measure, a generalization of the Born rule. This measure is invariant under a wide range of geometric transformations, including those generated the gauge groups of the standard model, and leading to the metric tensor as a quantum mechanical observables, without the need for additional assumptions beyond the geometric anti-constraint. Remarkably, multivector amplitudes are found to be consistent only with a $3+1 \mathrm{D}$ spacetime, encountering obstructions in other dimensional configurations. This finding aligns with the observed dimensionality and gauge symmetries of the universe and suggests a possible explanation for its specificity. This research represents a significant step in reconciling quantum mechanics with general relativity, challenging and expanding conventional methodologies in theoretical physics, and potentially paving the way for new insights in the field.

## Statements and Declarations

- Competing Interests: The author declares that he has no competing financial or non-financial interests that are directly or indirectly related to the work submitted for publication.
- Data Availability Statement: No datasets were generated or analyzed during the current study.
- During the preparation of this manuscript, we utilized a Large Language Model (LLM), for assistance with spelling and grammar corrections, as well as for minor improvements to the text to enhance clarity and readability. This AI tool did not contribute to the conceptual development of the work, data analysis, interpretation of results, or the decision-making process in the research. Its use was limited to language editing and minor textual enhancements to ensure the manuscript met the required linguistic standards.


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