Fundamental Physics as the General Solution to a Maximization Problem on the Shannon Entropy of All Measurements

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Abstract

We present a novel approach to quantum theory construction that involves solving a maximization problem on the Shannon entropy of all possible measurements of a system relative to its initial preparation. By constraining the maximization problem with a phase that vanishes under measurements, we obtain quantum mechanics (vanishing U(1)-valued phase), relativistic quantum mechanics (vanishing $\mathrm{Spin}^{\mathrm{c}}(3,1)$-valued phase), and quantum gravity (also a vanishing $\mathrm{Spin}^{\mathrm{c}}(3,1)$-valued phase, but with a non-vanishing dilation part). The first two cases are equivalent to established theory, whereas the latter case yields a quantum theory of arbitrary frame fields, in which a quantized version of the Einstein field equation lives. Specifically, the spacetime interval is promoted to an observable, effectively building the metric tensor from the underlying quantum structure. Moreover, the $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ gauge symmetries of the Standard Model arise naturally without additional assumptions. Finally, the solution is consistent only with 3+1 spacetime dimensions, as it encounters obstructions in all other dimension configurations. This framework integrates quantum mechanics, relativistic quantum mechanics, quantum gravity, spacetime dimensionality, and particle physics gauge symmetries from a simple entropy maximization problem constrained by a vanishing phase.

1 Introduction

The canonical formalism of quantum mechanics (QM) is based on five principal axioms[1, 2]:

QM Axiom 1 of 5 **State Space:** Each physical system corresponds to a complex Hilbert space, with the system's state represented by a ray in this space.

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QM Axiom 2 of 5 **Observables:** Physical observables correspond to Hermitian operators within the Hilbert space.

QM Axiom 3 of 5 **Dynamics:** The time evolution of a quantum system is dictated by the Schrödinger equation, where the Hamiltonian operator signifies the system's total energy.

QM Axiom 4 of 5 **Measurement:** The act of measuring an observable results in the system's transition to an eigenstate of the associated operator, with the measurement value being one of the eigenvalues.

QM Axiom 5 of 5 **Probability Interpretation:** The likelihood of a specific measurement outcome is determined by the squared magnitude of the state vector's projection onto the relevant eigenstate.

Contrastingly, statistical mechanics (SM), the other statistical pillar of physics, derives its probability measures through entropy maximization, constrained by the following expression:

SM Constraint 1 of 1: **Average Energy Constraint:** The average of energy measurements of a system at thermodynamic equilibrium converge to a specific value ($\overline{E}$):

$$\overline{E} = \sum_{q \in \mathbb{Q}} \rho(q) E(q) \tag{1}$$

To maximize entropy while satisfying this constraint, the theory uses a Lagrange multiplier approach.

Definition 1 (Fundamental Lagrange Multiplier Equation of SM)**.**

$$\mathcal{L}(\rho, \lambda, \beta) = \underbrace{-k_B \sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q)}_{\textit{Boltzmann entropy}} + \underbrace{\lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q)\right)}_{\textit{Normalization Constraint}} + \underbrace{\beta \left(\overline{E} - \sum_{q \in \mathbb{Q}} \rho(q) E(q)\right)}_{\textit{Average Energy Constraint}} \tag{2}$$

where λ and β are the Lagrange multipliers.

Theorem 1 (Gibbs Measure)**.** *The solution to the Lagrange multiplier equation of SM, is the well-known Gibbs measure.*

$$\rho(q) = \underbrace{\frac{1}{\sum_{r \in \mathbb{Q}} \exp(-\beta E(r))}}_{\textit{Microcanonical Ensemble}} \exp(-\beta E(q)) \tag{3}$$

Proof. This is an well-known result by E. T. Jaynes [3, 4]. As a convenience, we replicate the proof in Annex A. ∎

As evident from E. T. Jaynes' methodological innovation, SM relies on a single constraint related to the nature of the measurements under consideration, which allows the formulation of an optimization problem sufficient to derive the relevant probability measure. This is an exceptionally parsimonious formulation of a physical theory.

We propose a generalization of E. T. Jaynes' approach to the realms of Quantum Mechanics (QM), Relativistic Quantum Mechanics (RQM), and Quantum Gravity (QG). For each of these three domains, we will introduce a single constraint related to measurements, formulate a corresponding entropy maximization problem, and present a main theorem that fully encapsulates the theory. This formulation reduces fundamental physics to its most parsimonious expression, deriving the core theories as optimal solutions to a well-defined entropy maximization problem.

1.1 Quantum Mechanics

To reformulate QM as the solution to an entropy maximization problem, we propose the following constraint:

QM Constraint 1 of 1 **Vanishing Complex-Phase:** Quantum measurements admit a vanishing complex phase. The constraint is:

$$0 = \operatorname{tr} \sum_{q \in \mathbb{Q}} \rho(q) \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{4}$$

where the matrix representation engenders the complex phase, and the trace will cause it to vanish under measurement.

which associates to the follow equation:

Definition 2 (Fundamental Lagrange Multiplier Equation of QM)**.**

$$\mathcal{L}(\rho, \lambda, \tau) = \underbrace{-\sum_{q \in \mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)}}_{\textit{Relative Shannon Entropy}} + \underbrace{\lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q)\right)}_{\substack{\textit{Normalization} \\ \textit{Constraint}}} + \underbrace{\tau \left(-\operatorname{tr} \sum_{q \in \mathbb{Q}} \rho(q) \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right)}_{\textit{Vanishing Complex-Phase}} \tag{5}$$

where λ and τ are the Lagrange multipliers.

The *relative* Shannon entropy[5, 6] is utilized because we are solving for the least biased theory that connects an initial preparation $p(q)$ to its final measurement $\rho(q)$.

Theorem 2. *The least biased theory that connects an initial preparation $p(q)$ to its final measurement $\rho(q)$, under the constraint of the vanishing complex-phase,* *is:*

$$\rho(q) = \underbrace{\frac{1}{\sum_{r\in\mathbb{Q}} p(r)\|\exp(-itE(r)/\hbar)\|}}_{\textit{Unitarily Invariant Ensemble}} \underbrace{\|\exp(-itE(q)/\hbar)\|}_{\textit{Born Rule}} \underbrace{p(q)}_{\textit{Initial Preparation}} \tag{6}$$

where we have defined $\tau = t/\hbar$ (analogous to $\beta = 1/(k_B T)$ in SM).

The proof of this theorem will be presented in the results section. We will show that this solution entails the five axioms of QM, which are now promoted to theorems, yielding a parsimonious formulation of QM.

1.2 Relativistic Quantum Mechanics

Before we can discuss RQM, we first need to introduce some notation. Let $\mathbf{u} = a + \mathbf{x} + \mathbf{f} + \mathbf{v} + \mathbf{b}$, where a is a scalar, $\mathbf{x}$ is a vector, $\mathbf{f}$ is a bivector, $\mathbf{v}$ is a pseudo-vector and $\mathbf{b}$ is a pseudo-scalar, be a multivector of the geometric algebra $\mathrm{GA}(3,1)$, and let $\mathbf{M_u}$ be its matrix representation. Then, the fundamental constraint of RQM is:

RQM Constraint 1 of 1 **Vanishing Relativistic Phase:** Our formulation of RQM is based around a vanishing phase spanning the $\mathrm{Spin}^c(3,1)$ group. The constraint is:

$$0 = \mathrm{tr}\,\frac{1}{2}\sum_{q\in\mathbb{Q}} \rho(q)\mathbf{M_u}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{v}\to 0} \tag{7}$$

where $\mathbf{M_u}(q)$ is the matrix representation of the multivector $\mathbf{u}$ of $\mathrm{GA}(3,1)$, using the real Majorana representation of the gamma matrices:

$$\mathbf{M_u}|_{a\to 0,\mathbf{x}\to 0,\mathbf{v}\to 0} = \begin{bmatrix} f_{02} & b - f_{13} & -f_{01} + f_{12} & f_{03} + f_{23} \\ -b + f_{13} & f_{02} & f_{03} + f_{23} & f_{01} - f_{12} \\ -f_{01} - f_{12} & f_{03} - f_{23} & -f_{02} & -b - f_{13} \\ f_{03} - f_{23} & f_{01} + f_{12} & b + f_{13} & -f_{02} \end{bmatrix} \tag{8}$$

The matrix representation engenders the $\mathrm{Spin}^c(3,1)$-phase and the trace will cause it to vanish under measurement.

The Lagrange multiplier equation is as follows:

Definition 3 (Fundamental Lagrange Multiplier Equation of RQM)**.**

$$\mathcal{L}(\rho,\lambda,\zeta) = \underbrace{-\sum_{q\in\mathbb{Q}} \rho(q)\ln\frac{\rho(q)}{p(q)}}_{\substack{\textit{Relative Shannon}\\ \textit{Entropy}}} + \underbrace{\lambda\left(1 - \sum_{q\in\mathbb{Q}}\rho(q)\right)}_{\substack{\textit{Normalization}\\ \textit{Constraint}}} + \underbrace{\zeta\left(-\,\mathrm{tr}\,\frac{1}{2}\sum_{q\in\mathbb{Q}}\rho(q)\mathbf{M_u}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{v}\to 0}\right)}_{\textit{Vanishing Relativistic Phase}} \tag{9}$$

where λ and ζ are the Lagrange multipliers.

Theorem 3. *The least biased theory that connects an initial preparation $p(q)$ to its final measurement $\rho(q)$, under the constraint of the vanishing relativistic phase, is:*

$$\rho(q) = \underbrace{\frac{1}{\sum_{r\in\mathbb{Q}} p(r)\det\exp\left(-\zeta\frac{1}{2}\mathbf{M}_{\mathbf{u}}(r)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0}\right)}}_{\textit{Spin}^c\textit{(3,1) Invariant Ensemble}} \underbrace{\det\exp\left(-\zeta\frac{1}{2}\mathbf{M}_{\mathbf{u}}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0}\right)}_{\textit{Spin}^c\textit{(3,1) Born Rule}} \underbrace{p(q)}_{\textit{Initial Preparation}} \tag{10}$$

In the results section, we aim to demonstrate that this solution represents a quantum mechanical theory of inertial reference frames, where ζ is a one-parameter generator of boosts, rotations, and phase transformations. This theory allows for measurements, superpositions, and interference between inertial reference frames, providing the arena in which RQM lives. The formulation thus lays the foundation for the forthcoming development of quantum gravity through the introduction of quantum frames of reference.

1.3 Quantum Gravity

Our formulation of QG is based on a quantum theory of frame fields. To formulate the maximization problem whose resolution automatically yields the theory, we utilize the same vanishing phase constraint as in the RQM case, but we add dilations:

Definition 4 (Fundamental Lagrange Multiplier Equation of QG)**.**

$$\mathcal{L}(\rho,\lambda,\zeta) = \underbrace{-\sum_{q\in\mathbb{Q}}\rho(q)\ln\frac{\rho(q)}{p(q)}}_{\substack{\textit{Relative Shannon}\\ \textit{Entropy}}} + \underbrace{\lambda\left(1-\sum_{q\in\mathbb{Q}}\rho(q)\right)}_{\textit{Normalization Constraint}} + \underbrace{\zeta\left(\overline{2a} - \operatorname{tr}\frac{1}{2}\sum_{q\in\mathbb{Q}}A(q)\mathbf{M}(q)|_{\mathbf{x}\to 0,\mathbf{v}\to 0}\right)}_{\substack{\textit{Vanishing Relativistic Phase,}\\ \textit{with Dilations}}} \tag{11}$$

where λ and ζ are the Lagrange multipliers.

Using the real Majorana representation of the gamma matrices, $\mathbf{M}_{\mathbf{u}}(q)|_{\mathbf{x}\to 0,\mathbf{v}\to 0}$ is:

$$\mathbf{M}_{\mathbf{u}}|_{\mathbf{x}\to 0,\mathbf{v}\to 0} = \begin{bmatrix} a+f_{02} & b-f_{13} & -f_{01}+f_{12} & f_{03}+f_{23} \\ -b+f_{13} & a+f_{02} & f_{03}+f_{23} & f_{01}-f_{12} \\ -f_{01}-f_{12} & f_{03}-f_{23} & a-f_{02} & -b-f_{13} \\ f_{03}-f_{23} & f_{01}+f_{12} & b+f_{13} & a-f_{02} \end{bmatrix} \tag{12}$$

Theorem 4. *The least biased theory which connects an initial preparation $p(q)$ to its final measurement $\rho(q)$, under the constraint of the vanishing linear phase* *with dilations, is:*

$$\rho(q) = \underbrace{\frac{1}{\sum_{r\in\mathbb{Q}} p(q) \det\exp\left(-\frac{1}{2}\zeta\mathbf{M}(q)|_{\mathbf{x}\to 0,\mathbf{v}\to 0}\right)}}_{\textit{Geometrically Invariant Ensemble}} \underbrace{\det\exp\left(-\zeta\frac{1}{2}\mathbf{M}(q)|_{\mathbf{x}\to 0,\mathbf{v}\to 0}\right)}_{\textit{Geometric Born Rule}} \underbrace{p(q)}_{\textit{Initial Preparation}} \tag{13}$$

In the results section, we aim to demonstrate that the solution entails a quantum theory of frame fields. This theory defines the arena in which QG operates. The solution will admit the spacetime interval as an observable, enabling the construction of the metric tensor, valid for metrics of any curvature. This allows us to derive the quantized Einstein field equations.

1.4 Dimensional Obstructions

We end the results section with a number of theorems showing that the formalism, except for SM (no vanishing phase) and QM (vanishing complex phase), is found to be consistent only with 3+1-dimensional spacetime (vanishing $\text{Spin}^c(3,1)$ phase), encountering various obstructions in all other dimension configurations, and we discuss the implications.

2 Results

2.1 Quantum Mechanics

In statistical mechanics, the founding observation is that energy measurements of a thermally equilibrated system tend towards an average value. Comparatively, in QM, the founding observation involves the interplay between the systematic elimination of complex phases in measurement outcomes and the presence of interference effects in repeated measurement outcomes. To represent this observation, we introduce the *Vanishing Complex-Phase* Anti-Constraint:

$$0 = \operatorname{tr}\sum_{q\in\mathbb{Q}} \rho(q) \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{14}$$

where $E(q)$ are scalar-valued functions of $\mathbb{Q}$. The usage of the matrix generates a U(1) phase, and the trace causes it to vanish under specific circumstances (which will correspond to measurements).

At first glance, this expression may seem to reduce to a tautology equating zero with zero, suggesting it imposes no restriction on energy measurements. However, this appearance is deceptive. Unlike a conventional constraint that limits the solution space, this expression serves as a formal device to expand it, allowing for the incorporation of complex phases into the probability measure. The expression's role in broadening, rather than restricting, the solution space leads to its designation as an "anti-constraint."

In general, usage of anti-constraints expand classical probability measures into larger domains, such as quantum probabilities.

Its significance will become evident upon the completion of the optimization problem. For the moment, this expression can be conceptualized as the correct expression that, when incorporated as an anti-constraint within an entropy-maximization problem, resolves into the axioms of quantum mechanics.

Our next procedural step involves solving the corresponding Lagrange multiplier equation, mirroring the methodology employed in statistical mechanics by E. T. Jaynes. We utilize the relative Shannon entropy because we wish to solve for the least biased measure that connects an initial preparation $p(q)$ to its final measurement $\rho(q)$. For that, we deploy the following Lagrange multiplier equation:

$$\mathcal{L} = \underbrace{-\sum_{q\in\mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)}}_{\substack{\text{Relative Shannon} \\ \text{Entropy}}} + \underbrace{\lambda \left(1 - \sum_{q\in\mathbb{Q}} \rho(q)\right)}_{\substack{\text{Normalization} \\ \text{Constraint}}} + \underbrace{\tau \left(\operatorname{tr} \sum_{q\in\mathbb{Q}} \rho(q) \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right)}_{\text{Vanishing Complex-Phase}} \tag{15}$$

Where λ and τ are the Lagrange multipliers.

We solve the maximization problem as follows:

$$\frac{\partial \mathcal{L}(\rho, \lambda, \tau)}{\partial \rho(q)} = -\ln \frac{\rho(q)}{p(q)} - p(q) - \lambda - \tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{16}$$

$$0 = \ln \frac{\rho(q)}{p(q)} + p(q) + \lambda - \tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{17}$$

$$\implies \ln \frac{\rho(q)}{p(q)} = -p(q) - \lambda - \tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{18}$$

$$\implies \rho(q) = p(q) \exp(-p(q) - \lambda) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right) \tag{19}$$

$$= \frac{1}{Z(\tau)} p(q) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right) \tag{20}$$

The partition function, is obtained as follows:

$$1 = \sum_{r\in\mathbb{Q}} p(r) \exp(-p(q) - \lambda) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix}\right) \tag{21}$$

$$\implies (\exp(-p(q) - \lambda))^{-1} = \sum_{r\in\mathbb{Q}} p(r) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix}\right) \tag{22}$$

$$Z(\tau) := \sum_{r\in\mathbb{Q}} p(r) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix}\right) \tag{23}$$

Finally, the least biased theory that connects an initial preparation $p(q)$ to its final measurement $\rho(q)$, under the constraint of the vanishing complex phase, is:

$$\rho(q) = \frac{1}{\sum_{r\in\mathbb{Q}} p(r) \exp\left(-\tau \operatorname{tr} \left[\begin{smallmatrix} 0 & -E(r) \\ E(r) & 0 \end{smallmatrix}\right]\right)} \exp\left(-\tau \operatorname{tr} \left[\begin{smallmatrix} 0 & -E(q) \\ E(q) & 0 \end{smallmatrix}\right]\right) p(q) \quad (24)$$

Though initially unfamiliar, this form effectively establishes a comprehensive formulation of quantum mechanics, as we will demonstrate.

Upon examination, we find that phase elimination is manifestly evident in the probability measure: since the trace evaluates to zero, the probability measure simplifies to classical probabilities, aligning precisely with the Born rule's exclusion of complex phases:

$$\rho(q) = \frac{p(q)}{\sum_{r\in\mathbb{Q}} p(r)} \quad (25)$$

However, the significance of this phase elimination extends beyond this mere simplicity. As we will soon see, the partition function Z gains unitary invariance, allowing for the emergence of interference patterns and other quantum characteristics under appropriate basis changes.

We will begin by aligning our results with the conventional quantum mechanical notation. As such, we transform the representation of complex numbers from $\left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right]$ to $a + ib$. For instance, the exponential of a complex matrix is:

$$\exp \left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right] = r \left[\begin{smallmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{smallmatrix}\right], \text{ where } r = \exp a \quad (26)$$

Then, we associate the exponential trace to the complex norm using $\exp \operatorname{tr} \mathbf{M} \equiv \det \exp \mathbf{M}$:

$$\begin{aligned}
\exp \operatorname{tr} \left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right] = \det \exp \left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right] &= r^2 \det \left[\begin{smallmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{smallmatrix}\right], \text{ where } r = \exp a && (27)\\
&= r^2(\cos^2(b) + \sin^2(b)) && (28)\\
&= \|r(\cos(b) + i\sin(b))\| && (29)\\
&= \|r \exp(ib)\| && (30)
\end{aligned}$$

Finally, substituting $\tau = t/\hbar$ analogously to $\beta = 1/(k_B T)$, and applying the complex-norm representation to both the numerator and to the denominator, consolidates the Born rule, normalization, and initial prepration into :

$$\rho(q) = \underbrace{\frac{1}{\sum_{r\in\mathbb{Q}} p(r)\|\exp(-itE(r)/\hbar)\|}}_{\text{Unitarily Invariant Partition Function}} \underbrace{\|\exp(-itE(q)/\hbar)\|}_{\text{Born Rule}} \underbrace{p(q)}_{\text{Initial Preparation}} \quad (31)$$

We are now in a position to explore the solution space.

The wavefunction is delineated by decomposing the complex norm into a complex number and its conjugate. It is then visualized as a vector within a complex n-dimensional Hilbert space. The partition function acts as the inner product. This relationship is articulated as follows:

$$\sum_{r\in\mathbb{Q}} p(r)\|\exp(-itE(r)/\hbar)\| = Z = \langle\psi|\psi\rangle \tag{32}$$

where

$$\begin{bmatrix}\psi_1(t)\\ \vdots \\ \psi_n(t)\end{bmatrix} = \begin{bmatrix}\exp(-itE(q_1)/\hbar) & & \\ & \ddots & \\ & & \exp(-itE(q_n)/\hbar)\end{bmatrix}\begin{bmatrix}\psi_1(0)\\ \vdots \\ \psi_n(0)\end{bmatrix} \tag{33}$$

We clarify that $p(q)$ represents the probability associated with the initial preparation of the wavefunction, where $p(q_i) = \langle\psi_i(0)|\psi_i(0)\rangle$.

We also note that Z is invariant under unitary transformations.

Let us now investigate how the axioms of quantum mechanics are recovered from this result:

- The entropy maximization procedure inherently normalizes the vectors $|\psi\rangle$ with $1/Z = 1/\sqrt{\langle\psi|\psi\rangle}$. This normalization links $|\psi\rangle$ to a unit vector in Hilbert space. Furthermore, as the POP formulation of QM associates physical states with its probability measure, and the probability is defined up to a phase, we conclude that physical states map to Rays within Hilbert space. This demonstrates QM Axiom 1 of 5.

- In Z, an observable must satisfy:

$$\overline{O} = \sum_{r\in\mathbb{Q}} p(r)O(r)\|\exp(-itE(r)/\hbar)\| \tag{34}$$

  Since $Z = \langle\psi|\psi\rangle$, then any self-adjoint operator satisfying the condition $\langle\mathbf{O}\psi|\phi\rangle = \langle\psi|\mathbf{O}\phi\rangle$ will equate the above equation, simply because $\langle\mathbf{O}\rangle = \langle\psi|\,\mathbf{O}\,|\psi\rangle$. This demonstrates QM Axiom 2 of 5.

- Upon transforming Equation 33 out of its eigenbasis through unitary operations, we find that the energy, $E(q)$, typically transforms in the manner of a Hamiltonian operator:

$$|\psi(t)\rangle = \exp(-it\mathbf{H}/\hbar)\,|\psi(0)\rangle \tag{35}$$

  The system's dynamics emerge from differentiating the solution with respect to the Lagrange multiplier. This is manifested as:

$$\begin{aligned}\frac{\partial}{\partial t}|\psi(t)\rangle &= \frac{\partial}{\partial t}(\exp(-it\mathbf{H}/\hbar)\,|\psi(0)\rangle) && (36)\\ &= -i\mathbf{H}/\hbar\exp(-it\mathbf{H}/\hbar)\,|\psi(0)\rangle && (37)\\ &= -i\mathbf{H}/\hbar\,|\psi(t)\rangle && (38)\\ \implies \mathbf{H}\,|\psi(t)\rangle &= i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle && (39)\end{aligned}$$

  Which is the Schrödinger equation. This demonstrates QM Axiom 3 of 5.

- From Equation 33 it follows that the possible microstates $E(q)$ of the system correspond to specific eigenvalues of $\mathbf{H}$. An observation can thus be conceptualized as sampling from $\rho(q,t)$, with the measured state being the occupied microstate q of $\mathbb{Q}$. Consequently, when a measurement occurs, the system invariably emerges in one of these microstates, which directly corresponds to an eigenstate of $\mathbf{H}$. Measured in the eigenbasis, the probability measure is:

$$\rho(q,t) = \frac{1}{\langle\psi|\psi\rangle}(\psi(q,t))^\dagger\psi(q,t). \tag{40}$$

  In scenarios where the probability measure $\rho(q,\tau)$ is expressed in a basis other than its eigenbasis, the probability $P(\lambda_i)$ of obtaining the eigenvalue λ_i is given as a projection on a eigenstate:

$$P(\lambda_i) = |\langle\lambda_i|\psi\rangle|^2 \tag{41}$$

  Here, $|\langle\lambda_i|\psi\rangle|^2$ signifies the squared magnitude of the amplitude of the state $|\psi\rangle$ when projected onto the eigenstate $|\lambda_i\rangle$. As this argument hold for any observables, this demonstrates QM Axiom 4 of 5.

- Finally, since the probability measure (Equation 31) replicates the Born rule, QM Axiom 5 of 5 is also demonstrated.

Revisiting quantum mechanics with this perspective offers a coherent and unified narrative. Specifically, the vanishing complex phase constraint (Equation 14) is sufficient to entail the foundations of quantum mechanics (Axiom 1, 2, 3, 4 and 5) through the principle of entropy maximization. Equation 14 becomes the formulation's new singular foundation, and Axioms 1, 2, 3, 4, and 5 are now theorems.

2.2 RQM in 2D

In this section, we investigate RQM in 2D. Although all dimensional configurations except 3+1D contain obstructions, which will be discussed later in this section, the 2D case provides a valuable starting point before addressing the more complex 3+1D case. In RQM 2D, the fundamental Lagrange Multiplier Equation is:

$$\mathcal{L}(\rho,\lambda,\theta) = \underbrace{-\sum_{q\in\mathbb{Q}}\rho(q)\ln\frac{\rho(q)}{p(q)}}_{\substack{\text{Relative Shannon}\\ \text{Entropy}}} + \underbrace{\lambda\left(1-\sum_{q\in\mathbb{Q}}\rho(q)\right)}_{\substack{\text{Normalization}\\ \text{Constraint}}} + \underbrace{\theta\left(-\operatorname{tr}\frac{1}{2}\sum_{q\in\mathbb{Q}}\rho(q)\mathbf{M}_{\mathbf{u}}(q)|_{a\to 0,\mathbf{x}\to 0}\right)}_{\text{Vanishing Relativistic Phase}} \tag{42}$$

where λ and θ are the Lagrange multipliers, and where $\mathbf{M}_{\mathbf{u}}(q)$ is the matrix representation of a multivector $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$ of GA(2), where a is a scalar, $\mathbf{x}$

is a vector and $\mathbf{b}$ is a bivector:

$$\begin{bmatrix} a+x & y-b \\ y+b & a-x \end{bmatrix} \cong a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}} \tag{43}$$

where the basis elements are defined as:

$$\hat{\mathbf{x}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \hat{\mathbf{y}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \hat{\mathbf{x}} \wedge \hat{\mathbf{y}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{44}$$

If we take $a \to 0, \mathbf{x} \to 0$ then $\mathbf{M_u}$ reduces as follows:

$$\mathbf{u} = a + \mathbf{x} + \mathbf{b}|_{a\to 0, \mathbf{x}\to 0} = \mathbf{b} \implies \mathbf{M_u}|_{a\to 0,\mathbf{x}\to 0} = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \tag{45}$$

The Lagrange multiplier equation can be solved as follows:

$$\frac{\partial \mathcal{L}(\rho, \lambda, \theta)}{\partial \rho(q)} = 0 = -\ln \frac{\rho(q)}{p(q)} - p(q) - \lambda - \theta \operatorname{tr} \frac{1}{2} \left[\begin{smallmatrix} 0 & -b(q) \\ b(q) & 0 \end{smallmatrix} \right] \tag{46}$$

$$0 = \ln \frac{\rho(q)}{p(q)} + p(q) + \lambda + \theta \operatorname{tr} \frac{1}{2} \left[\begin{smallmatrix} 0 & -b(q) \\ b(q) & 0 \end{smallmatrix} \right] \tag{47}$$

$$\implies \ln \frac{\rho(q)}{p(q)} = -p(q) - \lambda - \theta \operatorname{tr} \frac{1}{2} \left[\begin{smallmatrix} 0 & -b(q) \\ b(q) & 0 \end{smallmatrix} \right] \tag{48}$$

$$\implies \rho(q) = p(q) \exp(-p(q) - \lambda) \exp\left(-\theta \operatorname{tr} \frac{1}{2} \left[\begin{smallmatrix} 0 & -b(q) \\ b(q) & 0 \end{smallmatrix} \right] \right) \tag{49}$$

$$= \frac{1}{Z(\theta)} p(q) \exp\left(-\theta \operatorname{tr} \frac{1}{2} \left[\begin{smallmatrix} 0 & -b(q) \\ b(q) & 0 \end{smallmatrix} \right] \right) \tag{50}$$

The partition function $Z(\theta)$, serving as a normalization constant, is determined as follows:

$$1 = \sum_{r \in \mathbb{Q}} p(r) \exp(-p(q) - \lambda) \exp\left(-\theta \operatorname{tr} \frac{1}{2} \left[\begin{smallmatrix} 0 & -b(q) \\ b(q) & 0 \end{smallmatrix} \right] \right) \tag{51}$$

$$\implies (\exp(-p(q) - \lambda))^{-1} = \sum_{r \in \mathbb{Q}} p(r) \exp\left(-\theta \operatorname{tr} \frac{1}{2} \left[\begin{smallmatrix} 0 & -b(q) \\ b(q) & 0 \end{smallmatrix} \right] \right) \tag{52}$$

$$Z(\theta) := \sum_{r \in \mathbb{Q}} p(r) \exp\left(-\theta \operatorname{tr} \frac{1}{2} \left[\begin{smallmatrix} 0 & -b(q) \\ b(q) & 0 \end{smallmatrix} \right] \right) \tag{53}$$

Consequently, the least biased theory that connects an initial preparation $p(q)$ to a final measurement $\rho(q)$, under the constraint of the vanishing relativistic phase in 2D is:

$$\rho(q) = \underbrace{\frac{1}{\sum_{r\in\mathbb{Q}} p(r) \det \exp\left(-\frac{1}{2}\theta \left[\begin{smallmatrix} 0 & -b(q) \\ b(q) & 0 \end{smallmatrix} \right] \right)}}_{\text{Spin(2) Invariant Ensemble}} \underbrace{\det \exp\left(-\frac{1}{2}\theta \left[\begin{smallmatrix} 0 & -b(q) \\ b(q) & 0 \end{smallmatrix} \right] \right)}_{\text{Spin(2) Born Rule}} \underbrace{p(q)}_{\text{Initial Preparation}} \tag{54}$$

where $\det \exp M = \exp \operatorname{tr} M$.

In 2D, the Lagrange multiplier θ correspond to an angle of rotation, and in 1+1D it would correspond to the rapidity ζ:

$$\text{2D}: \quad \exp \theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta \text{ is the angle of rotation} \tag{55}$$

$$1+1\text{D}: \quad \exp \zeta \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cosh\zeta & \sinh\zeta \\ \sinh\zeta & \cosh\zeta \end{bmatrix} \quad \zeta \text{ is the rapidity} \tag{56}$$

The 2D solution may appear equivalent to the QM case because they are related by an isomorphism $\mathrm{Spin}(2) \cong \mathrm{SO}(2) \cong \mathrm{U}(1)$ and under the replacement $\theta \to \tau$. However, an isomorphism is not an equality, and in Spin(2) we gain extra structures related to a relativistic description, which are not available in the QM case.

To investigate the solution in more detail, we introduce the multivector conjugate, also known as the Clifford conjugate, which generalizes the concept of complex conjugation to multivectors.

Definition 5 (Multivector conjugate (a.k.a Clifford conjugate))**.** *Let* $\mathbf{u} = a + \mathbf{x} + \mathbf{b}$ *be a multi-vector of the geometric algebra over the reals in two dimensions* GA(2)*. The multivector conjugate is defined as:*

$$\mathbf{u}^\ddagger = a - \mathbf{x} - \mathbf{b} \tag{57}$$

The determinant of the matrix representation of a multivector can be expressed as a self-product:

Theorem 5 (Determinant as a Multivector Self-Product)**.**

$$\mathbf{u}^\ddagger \mathbf{u} = \det \mathbf{M_u} \tag{58}$$

Proof. Let $\mathbf{u} = a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}}$, and let $\mathbf{M_u}$ be its matrix representation $\left[\begin{smallmatrix} a+x & y-b \\ y+b & a-x \end{smallmatrix}\right]$. Then:

$$\begin{aligned}
1: \quad & \mathbf{u}^\ddagger \mathbf{u} && (59)\\
& = (a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}})^\ddagger (a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}}) && (60)\\
& = (a - x\hat{\mathbf{x}} - y\hat{\mathbf{y}} - b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}})(a + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + b\hat{\mathbf{x}} \wedge \hat{\mathbf{y}}) && (61)\\
& = a^2 - x^2 - y^2 + b^2 && (62)\\
2: \quad & \det \mathbf{M_u} && (63)\\
& = \det \left[\begin{smallmatrix} a+x & y-b \\ y+b & a-x \end{smallmatrix}\right] && (64)\\
& = (a+x)(a-x) - (y-b)(y+b) && (65)\\
& = a^2 - x^2 - y^2 + b^2 && (66)
\end{aligned}$$

□

Building upon the concept of the multivector conjugate, we introduce the multivector conjugate transpose, which serves as an extension of the Hermitian conjugate to the domain of multivectors.

Definition 6 (Multivector Conjugate Transpose). *Let* $|V\rangle\!\rangle \in (\mathrm{GA}(2))^n$*:*

$$|V\rangle\!\rangle = \begin{bmatrix} a_1 + \mathbf{x}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{x}_n + \mathbf{b}_n \end{bmatrix} \tag{67}$$

The multivector conjugate transpose of $|V\rangle\!\rangle$ *is defined as first taking the transpose and then the element-wise multivector conjugate:*

$$\langle\!\langle V| = \begin{bmatrix} a_1 - \mathbf{x}_1 - \mathbf{b}_1 & \dots & a_n - \mathbf{x}_n - \mathbf{b}_n \end{bmatrix} \tag{68}$$

Definition 7 (Bilinear Form). *Let* $|V\rangle\!\rangle$ *and* $|W\rangle\!\rangle$ *be two vectors valued in* $\mathrm{GA}(2)$*. We introduce the following bilinear form:*

$$\langle\!\langle V|W\rangle\!\rangle = (a_1 - \mathbf{x}_1 - \mathbf{b}_1)(a_1 + \mathbf{x}_1 + \mathbf{b}_1) + \dots (a_n - \mathbf{x}_n - \mathbf{b}_n)(a_n + \mathbf{x}_n + \mathbf{b}_n) \tag{69}$$

Theorem 6 (Inner Product). *Restricted to the even sub-algebra of* $\mathrm{GA}(2)$*, the bilinear form is an inner product.*

Proof.

$$\langle\!\langle V|W\rangle\!\rangle_{\mathbf{x}\to 0} = (a_1 - \mathbf{b}_1)(a_1 + \mathbf{b}_1) + \dots (a_n - \mathbf{b}_n)(a_n + \mathbf{b}_n) \tag{70}$$

This is isomorphic to the inner product of a complex Hilbert space, with the identification $i \cong \hat{\mathbf{x}} \wedge \hat{\mathbf{y}}$. $\square$

Definition 8 (Spin(2)-valued Wavefunction).

$$|\psi\rangle\!\rangle = \begin{bmatrix} e^{\frac{1}{2}(a_1 + \mathbf{b}_1)} \\ \vdots \\ e^{\frac{1}{2}(a_n + \mathbf{b}_n)} \end{bmatrix} = \begin{bmatrix} \sqrt{\rho_1} R_1 \\ \vdots \\ \sqrt{\rho_2} R_2 \end{bmatrix} \tag{71}$$

where $\sqrt{\rho_i} = e^{\frac{1}{2}a_i}$ *representing the square root of the probability and* $R_i = e^{\frac{1}{2}\mathbf{b}_i}$ *representing a rotor in 2D (or boost in 1+1D).*

The partition function of the probability measure can be expressed using the bilinear form applied to the Spin(2)-valued Wavefunction:

Theorem 7 (Partition Function). $Z = \langle\!\langle \psi|\psi\rangle\!\rangle$

Proof.

$$\langle\!\langle \psi|\psi \rangle\!\rangle = \sum_{q\in\mathbb{Q}} \psi(q)^\ddagger \psi(q) = \sum_{q\in\mathbb{Q}} \rho(q) R(q)^\ddagger R(q) = \sum_{q\in\mathbb{Q}} \rho(q) = Z \tag{72}$$

□

Thus, the Spin(2)-valued wavefunction $|\psi\rangle\!\rangle$ is a linear object whose inner product reduces to the partition function.

Definition 9 (Spin(2)-valued Evolution Operator)**.**

$$T = \begin{bmatrix} e^{-\frac{1}{2}\theta \mathbf{b}_1} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\theta \mathbf{b}_n} \end{bmatrix} \tag{73}$$

Theorem 8. *The partition function is invariant with respect to the Spin(2)-valued evolution operator.*

Proof.

$$\langle\!\langle T\psi|T\psi \rangle\!\rangle = \sum_{q\in\mathbb{Q}} \det(T(q)\psi(q)) = \sum_{q\in\mathbb{Q}} \det T(q) \det \psi(q) = \sum_{q\in\mathbb{Q}} \det \psi(q) = \langle\!\langle \psi|\psi \rangle\!\rangle \tag{74}$$

where $\det T(q) = 1$, because $e^{-\frac{1}{2}\theta \mathbf{b}(q)}$ is traceless. □

We note that since the even sub-algebra of GA(2) is closed under addition and multiplication, and the bilinear form constitutes an inner product, it follows that it can be employed to construct a Hilbert space, in this case a Spin(2)-valued Hilbert space. The primary difference between a wavefunction living in a complex Hilbert space and one living in a Spin(2) Hilbert space relates to the subject matter of the theory. In the present case, the subject matter is a quantum theory of inertial reference frames in 2D.

The dynamics of reference frame transformations follow from the Schrödinger equation, which is obtained by taking the derivative of the wavefunction with respect to the Lagrange multiplier θ. Each element of the wavefunction represents an inertial reference frame, whose transformation is generated by the θ angle (for instance, the change of angle experienced by an inertial observer).

Definition 10 (Spin(2)-valued Schrödinger Equation)**.**

$$\frac{\mathrm{d}}{\mathrm{d}\theta} \begin{bmatrix} \psi_1(\theta) \\ \vdots \\ \psi_n(\theta) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\mathbf{b}_1 & & \\ & \ddots & \\ & & -\frac{1}{2}\mathbf{b}_n \end{bmatrix} \begin{bmatrix} \psi_1(\theta) \\ \vdots \\ \psi_n(\theta) \end{bmatrix} \tag{75}$$

The Spin(2)-valued Schrödinger Equation can be parametrized in space

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\psi(\theta,x,y) = -\frac{1}{2}\mathbf{b}(x,y)\psi(\theta,x,y) \tag{76}$$

In this case θ represents a global one-parameter evolution parameter akin to time, which is able to transform the wavefunction under the Spin(2), locally across 2D space. This is an extremely general equation that captures all transformations that can be done consistently with the evolution group of the wavefunction.

Definition 11 (Reference Frame Measurement)**.** *The QM Axiom 5 of 5, regarding the measurement postulates, is derived as a theorem in the RQM case as well (for the same reason as it is in the QM case). This allows us to measure the wavefunction $|\psi\rangle$ into one of its states q according to probability $\rho(q)$. Here the post-measurement state q corresponds to picking a specific inertial reference frame q from $\mathbb{Q}$.*

We note that, as a linear system, linear combinations of the wavefunction (such as $\psi(q) = \lambda_1\psi_1(q) + \lambda_2\psi_2(q)$) will also be solutions. This can introduce interference patterns between inertial reference frames:

Theorem 9 (Reference Frame Superpositions and Interference)**.**

Proof. Let $T = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$, and $|\psi\rangle\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{\rho_1}R_1\\ \sqrt{\rho_2}R_2\end{bmatrix}$, then:

$$T\,|\psi\rangle\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}\sqrt{\rho_1}R_1\\ \sqrt{\rho_1}R_2\end{bmatrix} \tag{77}$$

$$= \frac{1}{2}\begin{bmatrix}\sqrt{\rho_1}R_1 + \sqrt{\rho_2}R_2\\ \sqrt{\rho_1}R_1 - \sqrt{\rho_2}R_2\end{bmatrix} \tag{78}$$

$$= \frac{1}{2}(\sqrt{\rho_1}R_1 + \sqrt{\rho_2}R_2)\,|0\rangle\rangle + \frac{1}{2}(\sqrt{\rho_1}R_1 - \sqrt{\rho_2}R_2)\,|1\rangle\rangle \tag{79}$$

Then the probability can be computed as follows:

$$|\langle\langle 0|\psi\rangle\rangle|^2 = \frac{1}{2}(\sqrt{\rho_1}R_1 + \sqrt{\rho_2}R_2)^\ddagger(\sqrt{\rho_1}R_1 + \sqrt{\rho_2}R_2) \tag{80}$$

$$= \frac{1}{2}\rho_1 + \frac{1}{2}\rho_2 + \frac{1}{2}\sqrt{\rho_1\rho_2}(R_1^\ddagger R_2 + R_2^\ddagger R_1) \tag{81}$$

$$= \frac{1}{2}\rho_1 + \frac{1}{2}\rho_2 + \underbrace{\frac{1}{2}\sqrt{\rho_1\rho_2}\cos(\theta b_1 - \theta b_2)}_{\text{Spin(2)-valued Interference}} \tag{82}$$

Since Spin(2)≅U(1), then Spin(2)-valued interference is isomorphism to complex interference. ☐

Definition 12 (David Hestenes' Formulation)**.** *In 3+1D, the David Hestenes' formulation [7] of the wavefunction is $\psi = \sqrt{\rho}Re^{ib/2}$, where $R = e^{\mathbf{f}/2}$ is a*

Lorentz boost or rotation and where $e^{ib/2}$ is a phase. In 2D, as the algebra only admits a bivector, his formulation would reduce to $\psi = \sqrt{\rho}R$, which is identical to what we recovered.

The definition of the Dirac current applicable to our wavefunction follows the formulation of David Hestenes:

Definition 13 (Dirac Current)**.** *Given the basis $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$, the Dirac current is defined as:*

$$J_1 \equiv \psi(q)^\ddagger \hat{\mathbf{x}} \psi(q) = \rho(q) R(q)^\ddagger \hat{\mathbf{x}}(q) R(q) = \rho(q)\mathbf{e}_1 \tag{83}$$

$$J_2 \equiv \psi(q)^\ddagger \hat{\mathbf{y}} \psi(q) = \rho(q) R(q)^\ddagger \hat{\mathbf{y}}(q) R(q) = \rho(q)\mathbf{e}_2 \tag{84}$$

where $\mathbf{e}_1$ and $\mathbf{e}_2$ are a Spin(2) rotated frame field.

2.2.1 Obstructions

We identify two obstructions:

1. **In 1+1D:** The 1+1D theory results in a split-complex quantum theory due to the bilinear form $(a - b\hat{\mathbf{t}} \wedge \hat{\mathbf{x}})(a + b\hat{\mathbf{t}} \wedge \hat{\mathbf{x}})$, which yields negative probabilities: $a^2 - b^2 \in \mathbb{R}$ for certain wavefunction states, in contrast to the non-negative probabilities $a^2 + b^2 \in \mathbb{R}^{\geq 0}$ obtained in the Euclidean 2D case. (This is why we had to use 2D instead of 1+1D in this two-dimensional introduction...)

2. **In 1+1D and in 2D:** The basis vectors ($\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ in 2D, and $\hat{\mathbf{t}}$ and $\hat{\mathbf{x}}$ in 1+1D) are not self-adjoint. Although used in the context defining the Dirac current, their non-self-adjointness prevents the construction of the spacetime interval (or in 2D, the Euclidean distance) as a quantum observable. The benefits of having the basis vectors self-adjoint will become obvious in the 3+1D case, where we will be able to construct the metric tensor from spacetime interval measurements. Specifically, in 2D:

$$(\hat{\mathbf{x}}_\mu \mathbf{u})^\ddagger \mathbf{u} \neq \mathbf{u}^\ddagger \hat{\mathbf{x}}_\mu \mathbf{u} \tag{85}$$

because $(\hat{\mathbf{x}}_\mu \mathbf{u})^\ddagger \mathbf{u} = \mathbf{u}^\ddagger \hat{\mathbf{x}}_\mu^\ddagger \mathbf{u} = \mathbf{u}^\ddagger(-\hat{\mathbf{x}}_\mu)\mathbf{u}$.

In the following section, we will explore the obstruction-free 3+1D case.

2.3 RQM in 3+1D

In this section, we extend the concepts and techniques developed for multivector amplitudes in 2D to the more physically relevant case of 3+1D dimensions. The Lagrange multiplier equation is as follows:

$$\mathcal{L}(\rho, \lambda, \tau) = \underbrace{-\sum_{q\in\mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)}}_{\substack{\text{Relative Shannon} \\ \text{Entropy}}} + \underbrace{\lambda \left(1 - \sum_{q\in\mathbb{Q}} \rho(q)\right)}_{\substack{\text{Normalization} \\ \text{Constraint}}} + \underbrace{\zeta \left(-\operatorname{tr} \frac{1}{2} \sum_{q\in\mathbb{Q}} \rho(q) \mathbf{M}_{\mathbf{u}}(q)|_{a\to 0, \mathbf{x}\to 0, \mathbf{v}\to 0}\right)}_{\text{Vanishing Relativistic-Phase}} \tag{86}$$

The solution (proof in Annex B) is obtained using the same step-by-step process as the 2D case, and yields:

$$\rho(q)=\underbrace{\frac{1}{\sum_{r\in\mathbb{Q}}p(r)\det\exp\left(-\zeta\frac{1}{2}\mathbf{M}_{\mathbf{u}}(r)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0}\right)}}_{\text{Spin}^c(3,1)\text{ Invariant Ensemble}}\underbrace{\det\exp\left(-\zeta\frac{1}{2}\mathbf{M}_{\mathbf{u}}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0}\right)}_{\text{Spin}^c(3,1)\text{ Born Rule}}\underbrace{p(q)}_{\text{Initial Preparation}} \tag{87}$$

where ζ is a "twisted-phase" rapidity. (If the invariance group was Spin(3,1) instead of Spin^c(3,1), obtainable by posing $\mathbf{b}\to 0$, then it would simply be the rapidity).

Our initial goal will be to express the partition function as a self-product of elements of the vector space. As such, we begin by defining a general multivector in the geometric algebra GA$(3,1)$.

Definition 14 (Multivector)**.** *Let* $\mathbf{u}$ *be a multivector of* GA$(3,1)$*. Its general form is:*

$$\begin{aligned}
\mathbf{u}=a & \qquad (88)\\
&+x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}+t\hat{\mathbf{t}} \qquad (89)\\
&+f_{01}\hat{\mathbf{t}}\wedge\hat{\mathbf{x}}+f_{02}\hat{\mathbf{t}}\wedge\hat{\mathbf{y}}+f_{03}\hat{\mathbf{t}}\wedge\hat{\mathbf{z}}+f_{12}\hat{\mathbf{x}}\wedge\hat{\mathbf{y}}+f_{13}\hat{\mathbf{x}}\wedge\hat{\mathbf{z}}+f_{23}\hat{\mathbf{y}}\wedge\hat{\mathbf{z}} \qquad (90)\\
&+p\hat{\mathbf{x}}\wedge\hat{\mathbf{y}}\wedge\hat{\mathbf{z}}+q\hat{\mathbf{t}}\wedge\hat{\mathbf{y}}\wedge\hat{\mathbf{z}}+v\hat{\mathbf{t}}\wedge\hat{\mathbf{x}}\wedge\hat{\mathbf{z}}+w\hat{\mathbf{t}}\wedge\hat{\mathbf{x}}\wedge\hat{\mathbf{y}} \qquad (91)\\
&+b\hat{\mathbf{t}}\wedge\hat{\mathbf{x}}\wedge\hat{\mathbf{y}}\wedge\hat{\mathbf{z}} \qquad (92)
\end{aligned}$$

where $\hat{\mathbf{t}},\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}$ *are the basis vectors in the real Majorana representation.*

A more compact notation for $\mathbf{u}$ *is*

$$\mathbf{u}=a+\mathbf{x}+\mathbf{f}+\mathbf{v}+\mathbf{b} \tag{93}$$

where a *is a scalar,* $\mathbf{x}$ *a vector,* $\mathbf{f}$ *a bivector,* $\mathbf{v}$ *is pseudo-vector and* $\mathbf{b}$ *a pseudo-scalar.*

This general multivector can be represented by a 4×4 real matrix using the real Majorana representation:

Definition 15 (Matrix Representation $\mathbf{M_u}$ of $\mathbf{u}$)**.**

$$\mathbf{M_u}=\begin{bmatrix}
a+f_{02}-q-z & b-f_{13}+w-x & -f_{01}+f_{12}-p+v & f_{03}+f_{23}+t+y\\
-b+f_{13}+w-x & a+f_{02}+q+z & f_{03}+f_{23}-t-y & f_{01}-f_{12}-p+v\\
-f_{01}-f_{12}+p+v & f_{03}-f_{23}+t-y & a-f_{02}+q-z & -b-f_{13}-w-x\\
f_{03}-f_{23}-t+y & f_{01}+f_{12}+p+v & b+f_{13}-w-x & a-f_{02}-q+z
\end{bmatrix} \tag{94}$$

To manipulate and analyze multivectors in GA$(3,1)$, we introduce several important operations, such as the multivector conjugate, the 3,4 blade conjugate, and the multivector self-product.

Definition 16 (Multivector Conjugate (in 4D))**.**

$$\mathbf{u}^\ddagger = a - \mathbf{x} - \mathbf{f} + \mathbf{v} + \mathbf{b} \tag{95}$$

Definition 17 (3,4 Blade Conjugate)**.** *The 3,4 blade conjugate of* $\mathbf{u}$ *is*

$$\lfloor \mathbf{u} \rfloor_{3,4} = a + \mathbf{x} + \mathbf{f} - \mathbf{v} - \mathbf{b} \tag{96}$$

The results of Lundholm[8], demonstrates that the multivector norms in the following definition, are the *unique* forms which carries the properties of the determinants such as $N(\mathbf{uv}) = N(\mathbf{u})N(\mathbf{v})$ to the domain of multivectors:

Definition 18. *The self-products associated with low-dimensional geometric algebras are:*

$$\mathrm{GA}(0,1): \qquad \varphi^\dagger \varphi \tag{97}$$
$$\mathrm{GA}(2,0): \qquad \varphi^\ddagger \varphi \tag{98}$$
$$\mathrm{GA}(3,0): \qquad \lfloor \varphi^\ddagger \varphi \rfloor_3 \varphi^\ddagger \varphi \tag{99}$$
$$\mathrm{GA}(3,1): \qquad \lfloor \varphi^\ddagger \varphi \rfloor_{3,4} \varphi^\ddagger \varphi \tag{100}$$
$$\mathrm{GA}(4,1): \qquad (\lfloor \varphi^\ddagger \varphi \rfloor_{3,4} \varphi^\ddagger \varphi)^\dagger (\lfloor \varphi^\ddagger \varphi \rfloor_{3,4} \varphi^\ddagger \varphi) \tag{101}$$

We can now express the determinant of the matrix representation of a multivector via the self-product $\lfloor \varphi^\ddagger \varphi \rfloor_{3,4} \varphi^\ddagger \varphi$. This choice is not arbitrary, but the unique choice with allows us to represent the determinant of the matrix representation of a multivector within $\mathrm{GA}(3,1)$:

Theorem 10 (Determinant as a Multivector Self-Product)**.**

$$\lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u} = \det \mathbf{M}_\mathbf{u} \tag{102}$$

Proof. Please find a computer assisted symbolic proof of this equality in Annex C. ☐

Definition 19 ($\mathrm{GA}(3,1)$-valued Vector)**.**

$$|V\rangle\!\rangle = \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} = \begin{bmatrix} a_1 + \mathbf{x}_1 + \mathbf{f}_1 + \mathbf{v}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{x}_n + \mathbf{f}_n + \mathbf{v}_n + \mathbf{b}_n \end{bmatrix} \tag{103}$$

These constructions allow us to express the partition function in terms of the multivector self-product.

Definition 20 (Multilinear Form)**.**

$$\langle\!\langle V|V|V|V\rangle\!\rangle = \left\lfloor \begin{bmatrix} \mathbf{u}_1^\ddagger & \dots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n \end{bmatrix} \right\rfloor_{3,4} \begin{bmatrix} \mathbf{u}_1^\ddagger & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n^\ddagger \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} \tag{104}$$

Theorem 11 (Partition Function)**.** $Z = \langle\!\langle V|V|V|V\rangle\!\rangle$

Proof.

$$\begin{aligned}
&\langle\!\langle V|V|V|V\rangle\!\rangle && (105)\\
&= \lfloor \begin{bmatrix} \mathbf{u}_1^\ddagger & \dots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} \mathbf{u}_1^\ddagger & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \mathbf{u}_n^\ddagger \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n \end{bmatrix} && (106)\\
&= \lfloor \begin{bmatrix} \mathbf{u}_1^\ddagger \mathbf{u}_1 & \dots & \mathbf{u}_n \mathbf{u}_n \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} \mathbf{u}_1^\ddagger \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_n^\ddagger \mathbf{u}_n \end{bmatrix} && (107)\\
&= \lfloor \mathbf{u}_1^\ddagger \mathbf{u}_1 \rfloor_{3,4} \mathbf{u}_1^\ddagger \mathbf{u}_1 + \dots + \lfloor \mathbf{u}_n^\ddagger \mathbf{u}_n \rfloor_{3,4} \mathbf{u}_n^\ddagger \mathbf{u}_n && (108)\\
&= \sum_{i=1}^{n} \det \mathbf{M}_{\mathbf{u}_i} && (109)\\
&= Z && (110)
\end{aligned}$$

□

Theorem 12 (Non-negative inner product)**.** *The multilinear form, applied to the even sub-algebra of* $\mathrm{GA}(3,1)$ *is awlays non-negative.*

Proof. Let $|V\rangle\!\rangle = \begin{bmatrix} a_1 + \mathbf{f}_1 + \mathbf{b}_1 \\ \vdots \\ a_n + \mathbf{f}_n + \mathbf{b}_n \end{bmatrix}$. Then,

$$\begin{aligned}
&\langle\!\langle V|V|V|V\rangle\!\rangle && (111)\\
&= \lfloor \begin{bmatrix} (a_1 + \mathbf{f}_1 + \mathbf{b}_1)^\ddagger (a_1 + \mathbf{f}_1 + \mathbf{b}_1) & \dots \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} (a_1 + \mathbf{f}_1 + \mathbf{b}_1)^\ddagger (a_1 + \mathbf{f}_1 + \mathbf{b}_1) \\ \vdots \end{bmatrix} && (112)\\
&= \lfloor \begin{bmatrix} (a_1 - \mathbf{f}_1 + \mathbf{b}_1)(a_1 + \mathbf{f}_1 + \mathbf{b}_1) & \dots \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} (a_1 - \mathbf{f}_1 + \mathbf{b}_1)(a_1 + \mathbf{f}_1 + \mathbf{b}_1) \\ \vdots \end{bmatrix} && (113)\\
&= \lfloor \begin{bmatrix} a_1^2 + a_1\mathbf{f}_1 + a_1\mathbf{b}_1 - \mathbf{f}_1 a_1 - \mathbf{f}_1^2 - \mathbf{f}_1\mathbf{b}_1 + \mathbf{b}_1 a_1 + \mathbf{b}_1\mathbf{f}_1 + \mathbf{b}_1^2 & \dots \end{bmatrix} \rfloor_{3,4} \dots && (114)\\
&= \lfloor \begin{bmatrix} a_1^2 - \mathbf{f}_1^2 + \mathbf{b}_1^2 & \dots \end{bmatrix} \rfloor_{3,4} \dots && (115)
\end{aligned}$$

We note 1) $\mathbf{b}^2 = (bI)^2 = -b^2$ and 2) $\mathbf{f}^2 = -E_1^2 - E_2^2 - E_3^2 + B_1^2 + B_2^2 + B_3^2 + 4e_0e_1e_2e_3(E_1B_1 + E_2B_2 + E_3B_3)$

$$= \lfloor \begin{bmatrix} a_1^2 - b_1^2 + E_1^2 + E_2^2 + E_3^2 - B_1^2 - B_2^2 - B_3^2 - 4e_0e_1e_2e_3(E_1B_1 + E_2B_2 + E_3B_3) & \dots \end{bmatrix} \rfloor_{3,4} \dots \quad (116)$$

We note that the terms are now complex numbers, which we rewrite as $\mathrm{Re}(z) = a_1^2 - b_1^2 + E_1^2 + E_2^2 + E_3^2 - B_1^2 - B_2^2 - B_3^2$ and $\mathrm{Im}(z) = -4(E_1B_1 + E_2B_2 + E_3B_3)$

$$= \lfloor \begin{bmatrix} z_1 & \dots & z_2 \end{bmatrix} \rfloor_{3,4} \begin{bmatrix} z_n \\ \vdots \\ z_n \end{bmatrix} \tag{117}$$

$$= \begin{bmatrix} z_1^\dagger & \dots & z_2^\dagger \end{bmatrix} \begin{bmatrix} z_n \\ \vdots \\ z_n \end{bmatrix} \tag{118}$$

$$= z_1^\ddagger z_1 + \dots + z_n^\ddagger z_n \tag{119}$$

Which is always non-negative. □

We now define the $\mathrm{Spin}^c(3,1)$-valued wavefunction, which is valued in the even sub-algebra of $\mathrm{GA}(3,1)$:

Definition 21 ($\mathrm{Spin}^c(3,1)$-valued Wavefunction)**.**

$$|\psi\rangle\rangle = \begin{bmatrix} e^{\frac{1}{2}(a_1 + \mathbf{f}_1 + \mathbf{b}_1)} \\ \vdots \\ e^{\frac{1}{2}(a_n + \mathbf{f}_n + \mathbf{b}_n)} \end{bmatrix} = \begin{bmatrix} \sqrt{\rho_1} R_1 B_1 \\ \vdots \\ \sqrt{\rho_n} R_n B_n \end{bmatrix} \tag{120}$$

where R_i is a rotor, B_i is a phase, and $\sum_{q \in \mathbb{Q}} \rho(q) = 1$.

The evolution operator, leaving the partition function invariant, becomes:

Definition 22 ($\mathrm{Spin}^c(3,1)$ Evolution Operator)**.**

$$T = \begin{bmatrix} e^{-\frac{1}{2}\zeta(\mathbf{f}_1 + \mathbf{b}_1)} & & \\ & \ddots & \\ & & e^{-\frac{1}{2}\zeta(\mathbf{f}_n + \mathbf{b}_n)} \end{bmatrix} \tag{121}$$

In turn, this leads to a Schrödinger equation obtained by taking the derivative of the wavefunction with respect to the Lagrange multiplier ζ:

Definition 23 ($\mathrm{Spin}^c(3,1)$-valued Schrödinger equation)**.**

$$\frac{\mathrm{d}}{\mathrm{d}\zeta} \begin{bmatrix} \psi_1(\zeta) \\ \vdots \\ \psi_n(\zeta) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(\mathbf{f}_1 + \mathbf{b}_1) & & \\ & \ddots & \\ & & -\frac{1}{2}(\mathbf{f}_n + \mathbf{b}_n) \end{bmatrix} \begin{bmatrix} \psi_1(\zeta) \\ \vdots \\ \psi_n(\zeta) \end{bmatrix} \tag{122}$$

The $\mathrm{Spin}^c(3,1)$-valued Schrödinger Equation can be parametrized in spacetime

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\psi(\theta,t,x,y,z) = -\frac{1}{2}\left(\mathbf{f}(t,x,y,z) + \mathbf{b}(t,x,y,z)\right)\psi(\theta,t,x,y,z) \tag{123}$$

In this case ζ represents a global one-parameter evolution parameter akin to time, which is able to transform the wavefunction under the $\mathrm{Spin}^c(3,1)$, locally across spacetime. This is an extremely general equation that captures all transformations that can be done consistently with the evolution group of the wavefunction.

Definition 24 (David Hestenes' Formulation)**.** *Our* $\mathrm{Spin}^c(3,1)$*-valued wavefunction is identical to David Hestenes'[7] formulation of the wavefunction within GA(3,1). Both contain a rotor* $R = e^{-\mathbf{f}/2}$*, a phase* $B = e^{-\mathbf{b}/2}$ *and the probability term* $\sqrt{\rho}$*.*

Definition 25 (Dirac Current)**.** *The definition employed in the 2D case (same as Hestenes') applies here as well:*

$$J \equiv \psi^\ddagger \gamma_\mu \psi = \rho R^\ddagger B^\ddagger \gamma_\mu B R = \rho R^\ddagger \gamma_\mu B^{-1} B R = \rho \mathbf{e}_\mu \tag{124}$$

We will now demonstrate that the multilinear form is invariant with respect to the U(1), SU(2), and SU(3) gauge symmetries, which play a fundamental role in the standard model of particle physics. Using the γ_0 basis to enforce the invariance means that we are interested in a transformation that preserves a charge density in time, rather than that of a charge current in space $(\gamma_1, \gamma_2, \gamma_3)$.

Theorem 13 (U(1) Invariance)**.** *[9, 10]*

$$\langle \psi(q)|\gamma_0\psi(q)|\psi(q)|\gamma_0\psi(q)\rangle = \langle e^{\frac{1}{2}\mathbf{b}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)|e^{\frac{1}{2}\mathbf{b}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)\rangle \tag{125}$$

Proof.

$$\begin{aligned}
&\langle e^{\frac{1}{2}\mathbf{b}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)|e^{\frac{1}{2}\mathbf{b}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)\rangle && (126)\\
&\quad = \lfloor \psi(q)^\ddagger e^{\frac{1}{2}\mathbf{b}}\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)\rfloor_{3,4}\psi(q)^\ddagger e^{\frac{1}{2}\mathbf{b}}\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q) && (127)\\
&\quad = \lfloor \psi(q)^\ddagger \gamma_0 e^{-\frac{1}{2}\mathbf{b}} e^{\frac{1}{2}\mathbf{b}}\psi(q)\rfloor_{3,4}\psi(q)^\ddagger \gamma_0 e^{-\frac{1}{2}\mathbf{b}} e^{\frac{1}{2}\mathbf{b}}\psi(q) && (128)\\
&\quad = \lfloor \psi(q)^\ddagger \gamma_0 \psi(q)\rfloor_{3,4}\psi(q)^\ddagger \gamma_0 \psi(q) && (129)\\
&\quad = \langle \psi(q)|\gamma_0\psi(q)|\psi(q)|\gamma_0\psi(q)\rangle && (130)
\end{aligned}$$

□

Theorem 14 (SU(2) Invariance)**.** *[9, 10]*

$$\langle \psi(q)|\gamma_0\psi(q)|\psi(q)|\gamma_0\psi(q)\rangle = \langle e^{\frac{1}{2}\mathbf{f}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)|e^{\frac{1}{2}\mathbf{f}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)\rangle \tag{131}$$

implies $\mathbf{f} = \theta_1\gamma_0\gamma_1 + \theta_2\gamma_0\gamma_2 + \theta_3\gamma_0\gamma_3$*, which generates* SU(2)*.*

Proof.

$$\begin{aligned}
&\langle e^{\frac{1}{2}\mathbf{f}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)|e^{\frac{1}{2}\mathbf{f}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)\rangle && (132)\\
&\quad = \lfloor \psi(q)^\ddagger e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)\rfloor_{3,4}\psi(q)^\ddagger e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q) && (133)
\end{aligned}$$

We can now identify that the condition to preserve the equality reduces to this expression:

$$e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}} = \gamma_0 \tag{134}$$

We further note that moving the left most term to the right yields:

$$e^{-\theta_1\gamma_0\gamma_1-\theta_2\gamma_0\gamma_2-\theta_3\gamma_0\gamma_3-B_1\gamma_2\gamma_3-B_2\gamma_1\gamma_3-B_3\gamma_1\gamma_2}\gamma_0 e^{\frac{1}{2}\mathbf{f}} \tag{135}$$

$$= \gamma_0 e^{-\theta_1\gamma_0\gamma_1-\theta_2\gamma_0\gamma_2-\theta_3\gamma_0\gamma_3+B_1\gamma_2\gamma_3+B_2\gamma_1\gamma_3+B_3\gamma_1\gamma_2} e^{\frac{1}{2}\mathbf{f}} \tag{136}$$

Therefore, the product $e^{-\frac{1}{2}\mathbf{f}}\gamma_0 e^{\frac{1}{2}\mathbf{f}}$ reduces to γ_0 if and only if $B_1 = B_2 = B_3 = 0$, leaving $\mathbf{f} = \theta_1\gamma_0\gamma_1 + \theta_2\gamma_0\gamma_2 + \theta_3\gamma_0\gamma_3$:

Finally, we note that $e^{\theta_1\gamma_0\gamma_1+\theta_2\gamma_0\gamma_2+\theta_3\gamma_0\gamma_3}$ generates SU(2). □

Theorem 15 (SU(3) invariance)**.** *[9, 10]*

$$\langle\psi(q)|\gamma_0\psi(q)|\psi(q)|\gamma_0\psi(q)\rangle = \langle\mathbf{f}\psi(q)|\gamma_0\mathbf{f}\psi(q)|\mathbf{f}\psi(q)|\gamma_0\mathbf{f}\psi(q)\rangle \tag{137}$$

Proof. From the above relation, we identify that the following expression must remain invariant: $-\mathbf{f}\gamma_0\mathbf{f} = \gamma_0$. Now, let $\mathbf{f} = E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3 + B_1\gamma_2\gamma_3 + B_2\gamma_1\gamma_3 + B_3\gamma_1\gamma_2$. Then:

$$-(E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3 + B_1\gamma_2\gamma_3 + B_2\gamma_1\gamma_3 + B_3\gamma_1\gamma_2)\gamma_0\mathbf{f} \tag{138}$$

The first three terms anticommute with γ_0, while the last three commute with γ_0:

$$= \gamma_0(E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3 - B_1\gamma_2\gamma_3 - B_2\gamma_1\gamma_3 - B_3\gamma_1\gamma_2)\mathbf{f} \tag{139}$$

This can be written as:

$$\gamma_0(\mathbf{E} - \mathbf{B})(\mathbf{E} + \mathbf{B}) \tag{140}$$

$$= \gamma_0(\mathbf{E}^2 + \mathbf{EB} - \mathbf{BE} - \mathbf{B}^2) \tag{141}$$

where $\mathbf{E} = E_1\gamma_0\gamma_1 + E_2\gamma_0\gamma_2 + E_3\gamma_0\gamma_3$ and $\mathbf{B} = B_1\gamma_2\gamma_3 + B_2\gamma_1\gamma_3 + B_3\gamma_1\gamma_2$.

Thus, for $-\mathbf{f}\gamma_0\mathbf{f} = \gamma_0$, we require: 1) $\mathbf{E}^2 - \mathbf{B}^2 = 1$ and 2) $\mathbf{EB} = \mathbf{BE}$. The second requirement means that $\mathbf{E}$ and $\mathbf{B}$ must commute (and thus be isomorphic to three complex numbers), and the first implies:

$$\mathbf{E}^2 - \mathbf{B}^2 = (E_1^2 + B_1^2) + (E_2^2 + B_2^2) + (E_3^2 + B_3^2) = 1 \tag{142}$$

which are the defining conditions for the SU(3) symmetry group. □

We have now demonstrated that the solution to the entropy maximization problem offers a powerful framework that naturally incorporates SU(3)×SU(2)×U(1) gauge symmetries, retains invariance with respect to the Spin^c(3, 1) group, includes the Dirac current and equation, and introduces the notion of the metric tensor via spacetime interval measurements. The specificity of these gauges is attributable to the set of all time-invariant gauges supported by the multilinear form in GA(3, 1), and cannot be different.

2.4 Quantum Gravity in 3+1D

In the previous section, we developed a quantum theory of inertial reference frames valued in $\text{Spin}^c(3{,}1)$, in which RQM lives. Our goal in this section is to extend the methodology to arbitrary frame fields, in which *General Relativity* (GR) lives. To formulate the theory, we will exploit the features of the multilinear form, which will allow us to formulate the spacetime interval as an observable from which the metric tensor can be constructed.

2.4.1 Initial Investigation

The multilinear form supports more operation than are possible with a bilinear form:

Definition 26 (Double-Copy)**.** *Let ψ and φ be two $\text{Spin}^c(3{,}1)$-valued wavefunctions. Then, the double copy*

$$\underbrace{\lfloor \psi(q)^\ddagger \psi(q) \rfloor_{3,4}}_{\textit{copy 1}} \underbrace{\varphi(q)^\ddagger \varphi(q)}_{\textit{copy 2}} = e^{ib}\rho_\psi \rho_\varphi = e^{ib}\rho \tag{143}$$

yields a transition amplitude that satisfies the probability measure. We note that the multiplication of two probabilities measure yields a probability measure $\rho_\psi \rho_\phi = \rho$.

Furthermore, the multilinear form supports a double-basis measurement. This feature will be crucial to formulate the spacetime interval as an observable.

First, let us explore how the adjoint action of the wavefunction acts on a single basis element.

$$\begin{aligned}
\mathbf{e}_\mu &= \psi^\ddagger \gamma_\mu \psi && (144)\\
&= e^{\frac{1}{2}a} e^{-\frac{1}{2}\mathbf{f}} e^{\frac{1}{2}\mathbf{b}} \gamma_\mu e^{\frac{1}{2}\mathbf{b}} e^{\frac{1}{2}\mathbf{f}} e^{\frac{1}{2}a} && (145)\\
&= e^{\frac{1}{2}a} e^{-\frac{1}{2}\mathbf{f}} \gamma_\mu e^{-\frac{1}{2}\mathbf{b}} e^{\frac{1}{2}\mathbf{b}} e^{\frac{1}{2}\mathbf{f}} e^{\frac{1}{2}a} && (146)\\
&= \underbrace{e^{\frac{1}{2}a} \underbrace{e^{-\frac{1}{2}\mathbf{f}} \gamma_\mu e^{\frac{1}{2}\mathbf{f}}}_{\text{rotation/boost}} e^{\frac{1}{2}a}}_{\text{dilation}} && (147)
\end{aligned}$$

From this, we note that the wavefunction contains all the multivectorial components required to map a vector such as γ_μ to any other vector $\mathbf{e}_\mu$, allowing for rotations/boosts and dilations of the vector, but leaving the origin unchanged.

Comparatively, we previously defined the Dirac current as $\rho \mathbf{e}_\mu = \psi^\ddagger \gamma_\mu \psi$. The difference here is that we absorbed $e^{\frac{1}{2}a}$ into a dilation of the basis vector.

Tthe construction of the metric tensor requires the multiplication of two basis elements:

$$g_{\mu\nu} = \frac{1}{2}(\mathbf{e}_\mu \mathbf{e}_\nu + \mathbf{e}_\nu \mathbf{e}_\mu) \tag{148}$$

Constructing this object will require separate joint actions on both γ_μ and γ_ν simultaneously, which the multilinear form makes possible.

Theorem 16 (Metric Measurement)**.** *The metric measurement is the expectation value of the γ_μ and γ_ν vectors, applied to a set of* $\mathrm{Spin}^c(3,1)$*-valued wavefunctions, identified as ψ, φ, ϕ and ξ:*

$$g_{00} = \frac{1}{2}\left(\lfloor \psi^\ddagger \gamma_0 \psi \rfloor_{3,4} \psi^\ddagger \gamma_0 \psi + \lfloor \psi^\ddagger \gamma_0 \psi \rfloor_{3,4} \psi^\ddagger \gamma_0 \psi\right) \tag{149}$$

$$g_{01} = \frac{1}{2}\left(\lfloor \psi^\ddagger \gamma_0 \psi \rfloor_{3,4} \varphi^\ddagger \gamma_1 \varphi + \lfloor \varphi^\ddagger \gamma_1 \varphi \rfloor_{3,4} \psi^\ddagger \gamma_0 \psi\right) \tag{150}$$

$$g_{02} = \frac{1}{2}\left(\lfloor \psi^\ddagger \gamma_0 \psi \rfloor_{3,4} \phi^\ddagger \gamma_1 \phi + \lfloor \phi^\ddagger \gamma_1 \phi \rfloor_{3,4} \psi^\ddagger \gamma_0 \psi\right) \tag{151}$$

$$g_{03} = \frac{1}{2}\left(\lfloor \psi^\ddagger \gamma_0 \psi \rfloor_{3,4} \phi^\ddagger \gamma_1 \phi + \lfloor \phi^\ddagger \gamma_1 \phi \rfloor_{3,4} \psi^\ddagger \gamma_0 \psi\right) \tag{152}$$

$$\textit{etc.} \tag{153}$$

Proof. Without loss of generality, let us prove g_{01}. Let $\psi = e^{\frac{1}{2}a}e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}$ and $\varphi = e^{\frac{1}{2}a'}e^{\frac{1}{2}\mathbf{f}'}e^{\frac{1}{2}\mathbf{b}'}$:

$$\frac{1}{2}\left(\lfloor \psi^\ddagger \gamma_0 \psi \rfloor_{3,4} \varphi^\ddagger \gamma_1 \varphi + \lfloor \varphi^\ddagger \gamma_1 \varphi \rfloor_{3,4} \psi^\ddagger \gamma_0 \psi\right) \tag{154}$$

$$= \frac{1}{2}\left(\lfloor e^{\frac{1}{2}a}e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\gamma_0 e^{\frac{1}{2}a}e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}}\rfloor_{3,4} e^{\frac{1}{2}a'}e^{\frac{1}{2}\mathbf{f}'}e^{\frac{1}{2}\mathbf{b}'}\gamma_1 e^{\frac{1}{2}a}e^{\frac{1}{2}\mathbf{f}}e^{\frac{1}{2}\mathbf{b}} + \dots\right) \tag{155}$$

$$= \frac{1}{2}\left(\mathbf{e}_0\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_0\right) \tag{156}$$

$$= g_{01} \tag{157}$$

As one can swap γ_μ with γ_ν and obtain the same metric tensor, the multilinear form guarantees that $g_{\mu\nu}$ is symmetric. Finally, since $\lfloor \gamma_\mu \psi^\ddagger \psi \rfloor_{3,4} \gamma_\nu \psi^\ddagger \psi = \lfloor \psi^\ddagger \gamma_\mu \psi \rfloor_{3,4} \psi \gamma_\nu \psi$, then γ_μ and γ_ν are self-adjoint within the multilinear form, entailing the interpretation of $g_{\mu\nu}$ as an observable. □

In general, we can formulate the spacetime interval as an observable:

Definition 27 (Spacetime Interval Measurement)**.** *The spacetime interval measurement is the expectation value of the $\mathbf{v} = v_0\gamma_0 + v_1\gamma_1 + v_2\gamma_2 + v_3\gamma_3$ and $\mathbf{w} = w_0\gamma_0 + w_1\gamma_1 + w_2\gamma_2 + w_3\gamma_3$ vectors, with wavefunctions ψ, φ, ϕ and ξ:*

$$\mathbf{v}\cdot\mathbf{w} = \frac{1}{2}\left(\lfloor \psi^\ddagger v_0\gamma_0 \psi \rfloor_{3,4} \psi^\ddagger w_0\gamma_0 \psi + \lfloor \psi^\ddagger w_0\gamma_0 \psi \rfloor_{3,4} \psi^\ddagger v_0\gamma_0 \psi\right) \tag{158}$$

$$+ \frac{1}{2}\left(\lfloor \psi^\ddagger v_0\gamma_0 \psi \rfloor_{3,4} \varphi^\ddagger w_1\gamma_1 \varphi + \lfloor \varphi^\ddagger w_1\gamma_1 \varphi \rfloor_{3,4} \psi^\ddagger v_0\gamma_0 \psi\right) \tag{159}$$

$$\vdots \tag{160}$$

2.4.2 The Lagrange Multiplier Equation

Following this initial heuristic investigation, we now define the problem formally via a Lagrange multiplier equation. First, we raise an interpretational observation regarding the scalar term $e^{\frac{1}{2}a}$ of ψ. In the previous sections on

QM and RQM, this term was associated with the square root of the probability $e^{\frac{1}{2}a} = \sqrt{\rho}$. However, as we noted in Theorem 16, it here associates with a dilation factor. Specifically, the frame field absorbed this term within its curvilinear transformation. Finally, we note that a probability varies between $[0, 1]$, but dilations can vary between $]0, \infty[$ (in the case of an orientable manifold).

Understanding the correspondence between dilations and probabilities came from dimensional analysis. Specifically, to construct the entries of the metric tensor from the wavefunction, the scalar terms ends up being multiplied four times (twice per gamma matrix). The 4-volume density of the metric, given by the square root of the metric determinant $\sqrt{-|g|}$, thus scales as e^{4a}. Significantly, e^{2a} is the square root of the 4-volume e^{4a}, indicating that the probabilistic weight of a quantum state grows with the area (or square root of the 4-volume) associated with the metric it defines.

The constraint, which permits dilations, is:

$$\overline{2a} = \frac{1}{2} \operatorname{tr} \sum_{q \in \mathbb{Q}} \rho(q) \mathbf{M_u}(q)|_{\mathbf{x} \to 0, \mathbf{v} \to 0} \tag{161}$$

where $\overline{2a}$ is the average dilation scale.

The Lagrange multiplier equation is as follows:

Definition 28 (The Fundamental Lagrange Multiplier Equation of QG)**.**

$$\mathcal{L}(\rho, \lambda, \zeta) = \underbrace{-\sum_{q \in \mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)}}_{\textit{Relative Shannon Entropy}} + \underbrace{\lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q)\right)}_{\textit{Normalization Constraint}} + \underbrace{\zeta \left(\overline{2a} - \frac{1}{2} \operatorname{tr} \sum_{q \in \mathbb{Q}} \rho(q) \mathbf{M_u}(q)|_{\mathbf{x} \to 0, \mathbf{v} \to 0}\right)}_{\textit{Vanishing Relativistic Phase, with Dilation}} \tag{162}$$

where $\rho(q)$ is the measure, $p(q)$ is the initial preparation, $\mathbf{M_u}(q)$ maps q to a 4×4 matrix, and where ζ is the Lagrange multiplier.

The solution to this optimization problem is obtained as follows:

Theorem 17. *The least biased theory which connects an initial preparation $p(q)$ to its final measurement $\rho(q)$, under the constraint of the vanishing relativistic phase, is:*

$$\rho(q) = \underbrace{\frac{1}{\sum_{r \in \mathbb{Q}} p(r) \det \exp\left(-\frac{1}{2}\zeta \mathbf{M_u}(r)|_{\mathbf{x} \to 0, \mathbf{v} \to 0}\right)}}_{\textit{Geometrically Invariant Ensemble}} \underbrace{\det \exp\left(-\frac{1}{2}\zeta \mathbf{M_u}(q)|_{\mathbf{x} \to 0, \mathbf{v} \to 0}\right)}_{\textit{Geometric Born Rule}} \underbrace{p(q)}_{\textit{Initial Preparation}} \tag{163}$$

Proof. The Lagrange multiplier equation can be solved as follows:

$$\frac{\partial \mathcal{L}(\rho, \lambda, \zeta)}{\partial \rho(q)} = 0 = -\ln \frac{\rho(q)}{p(q)} - 1 - \lambda - \zeta \operatorname{tr} \frac{1}{2} \mathbf{M}_{\mathbf{u}}(q)|_{\mathbf{x}\to 0, \mathbf{b}\to 0} \tag{164}$$

$$0 = \ln \frac{\rho(q)}{p(q)} + 1 + \lambda + \zeta \operatorname{tr} \frac{1}{2} \mathbf{M}_{\mathbf{u}}(q)|_{\mathbf{x}\to 0, \mathbf{b}\to 0} \tag{165}$$

$$\implies \ln \frac{\rho(q)}{p(q)} = -1 - \lambda - \zeta \operatorname{tr} \frac{1}{2} \mathbf{M}_{\mathbf{u}}(q)|_{\mathbf{x}\to 0, \mathbf{b}\to 0} \tag{166}$$

$$\implies \rho(q) = p(q) \exp(-1-\lambda) \exp\left(-\zeta \operatorname{tr} \frac{1}{2} \mathbf{M}_{\mathbf{u}}(q)|_{\mathbf{x}\to 0, \mathbf{b}\to 0}\right) \tag{167}$$

$$= \frac{1}{Z(\zeta)} p(q) \exp\left(-\zeta \operatorname{tr} \frac{1}{2} \mathbf{M}_{\mathbf{u}}(q)|_{\mathbf{x}\to 0, \mathbf{b}\to 0}\right) \tag{168}$$

The partition function $Z(\zeta)$, serving as a normalization constant, is determined as follows:

$$1 = \sum_{r \in \mathbb{Q}} p(r) \exp(-1-\lambda) \exp\left(-\zeta \operatorname{tr} \frac{1}{2} \mathbf{M}_{\mathbf{u}}(q)|_{\mathbf{x}\to 0, \mathbf{b}\to 0}\right) \tag{169}$$

$$\implies (\exp(-1-\lambda))^{-1} = \sum_{r \in \mathbb{Q}} p(r) \exp\left(-\zeta \operatorname{tr} \frac{1}{2} \mathbf{M}_{\mathbf{u}}(q)|_{\mathbf{x}\to 0, \mathbf{b}\to 0}\right) \tag{170}$$

$$Z(\zeta) := \sum_{r \in \mathbb{Q}} p(r) \exp\left(-\zeta \operatorname{tr} \frac{1}{2} \mathbf{M}_{\mathbf{u}}(q)|_{\mathbf{x}\to 0, \mathbf{b}\to 0}\right) \tag{171}$$

□

2.4.3 Dynamics

The dynamics are governed by the metric Schrödinger equation. It is able to generate all possible metrics as a continuous one-parameter flow from the initial preparation. The equation is obtained by taking the derivative of the wavefunction with respect to the Lagrange multiplier ζ:

Definition 29 (Metric Schrödinger Equation)**.**

$$\frac{\mathrm{d}}{\mathrm{d}\zeta} \begin{bmatrix} \psi_1(\zeta) \\ \vdots \\ \psi_n(\zeta) \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} a_1 + \mathbf{f}_1 + \mathbf{b}_1 & & \\ & \ddots & \\ & & a_n + \mathbf{f}_n + \mathbf{b}_n \end{bmatrix} \begin{bmatrix} \psi_1(\zeta) \\ \vdots \\ \psi_n(\zeta) \end{bmatrix} \tag{172}$$

where a_i accounts for dilation changes, $\mathbf{f}_i$ accounts for Spin(3,1) transformations, and $\mathbf{b}_i$ for phase transformations. When parametrized in (t, x, y, z), the equation performs an arbitrary metric transformation at every event.

2.4.4 Multilinear Observables

Theorem 18 (Multilinear Observable)**.**

$$\frac{1}{2}\left(\langle\!\langle\psi|A\phi|\varphi|B\xi\rangle\!\rangle + \langle\!\langle\psi|B\phi|\varphi|A\xi\rangle\!\rangle\right) = \frac{1}{2}\left(\langle\!\langle A\psi|\phi|B\varphi|\xi\rangle\!\rangle + \langle\!\langle B\psi|\phi|A\varphi|\xi\rangle\!\rangle\right) \tag{173}$$

$$\implies A^\ddagger = \pm A, B^\ddagger = \pm B \tag{174}$$

Proof.

$$1 : \langle\!\langle\psi|A\phi|\varphi|B\xi\rangle\!\rangle \tag{175}$$

$$= \begin{bmatrix}\psi_1 & \psi_2\end{bmatrix}\begin{bmatrix}a_{00} & a_{01}\\ a_{10} & a_{11}\end{bmatrix}\begin{bmatrix}\phi_1 & \\ & \phi_2\end{bmatrix}\begin{bmatrix}\varphi_1 & \\ & \varphi_2\end{bmatrix}\begin{bmatrix}b_{00} & b_{01}\\ b_{10} & b_{11}\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} \tag{176}$$

$$2 : \langle\!\langle A\psi|\phi|B\varphi|\xi\rangle\!\rangle \tag{177}$$

$$= \begin{bmatrix}\psi_1 & \psi_2\end{bmatrix}\begin{bmatrix}a^\ddagger_{00} & a^\ddagger_{01}\\ a^\ddagger_{10} & a^\ddagger_{11}\end{bmatrix}\begin{bmatrix}\phi_1 & \\ & \phi_2\end{bmatrix}\begin{bmatrix}\varphi_1 & \\ & \varphi_2\end{bmatrix}\begin{bmatrix}b^\ddagger_{00} & b^\ddagger_{10}\\ b^\ddagger_{01} & b^\ddagger_{11}\end{bmatrix}\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} \tag{178}$$

$$\implies A^\ddagger = \pm A \text{ and } B^\ddagger = \pm B \tag{179}$$

This permits the measurement of various geometric objects constructed from multivectors. The plus/minus signs follow from the double copy which eliminates $(-1)^2$. □

In their eigenbasis, multilinear observables are expressed as follows:

$$DAD^{-1} = \begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix} \tag{180}$$

where $\lambda_1, \ldots, \lambda_n$ are multivector valued, and where $\lambda_i^\ddagger = \pm\lambda_i$. For instance, a metric measurement involves these observables:

$$\hat{\gamma}_\mu = \begin{bmatrix}\gamma_\mu & & \\ & \ddots & \\ & & \gamma_\mu\end{bmatrix}, \text{ and } \hat{\gamma}_\nu = \begin{bmatrix}\gamma_\nu & & \\ & \ddots & \\ & & \gamma_\nu\end{bmatrix} \tag{181}$$

Since,

$$\hat{\gamma}^\ddagger_\mu = \begin{bmatrix}-\gamma_\mu & & \\ & \ddots & \\ & & -\gamma_\mu\end{bmatrix}, \text{ and } \hat{\gamma}^\ddagger_\nu = \begin{bmatrix}-\gamma_\nu & & \\ & \ddots & \\ & & -\gamma_\nu\end{bmatrix} \tag{182}$$

then the observables meet the requirement $\lambda_i^\ddagger = \pm\lambda_i$.

In general, all observables A and B whose eigenvalues are vector-valued, will yield the value of the inner product between the eigenvalues of A and of B, within the multilinear measurement equation: $\frac{1}{2}\left(\langle\!\langle\psi|A\psi|\psi|B\psi\rangle\!\rangle + \langle\!\langle\psi|B\psi|\psi|A\psi\rangle\!\rangle\right)$

Definition 30 (Metric Operator)**.**

$$\langle \hat{g}_{\mu\nu} \rangle = \frac{1}{2} \left(\langle\!\langle \psi_\mu | \hat{\gamma}_\mu \psi_\mu | \psi_\nu | \hat{\gamma}_\nu \psi_\nu \rangle\!\rangle + \langle\!\langle \psi_\nu | \hat{\gamma}_\nu \psi_\nu | \psi_\mu | \hat{\gamma}_\mu \psi_\mu \rangle\!\rangle \right) \tag{183}$$

where

$$\hat{\gamma}_\mu = \begin{bmatrix} \gamma_\mu & & \\ & \ddots & \\ & & \gamma_\mu \end{bmatrix} \qquad \hat{\gamma}_\nu = \begin{bmatrix} \gamma_\nu & & \\ & \ddots & \\ & & \gamma_\nu \end{bmatrix} \tag{184}$$

where ψ and g are parametrized in (t, x, y, z).

2.4.5 Quantum Einstein Field Equations

To study the EFE within the present framework, we must express the Einstein-Hilbert action (EH):

$$S(g_{\mu\nu}) = \frac{c^4}{16\pi G} \int R\sqrt{-g}\mathrm{d}^4 x \tag{185}$$

in terms of the metric operator $\langle \hat{g}_{\mu\nu} \rangle$ (Definition 30):

Definition 31 (EH)**.** *The Einstein-Hilbert action expressed in terms of $\langle \hat{g}_{\mu\nu} \rangle$ is:*

$$S(\langle \hat{g}_{\mu\nu} \rangle) = \frac{c^4}{16\pi G} \int R\sqrt{-\det\langle \hat{g}_{\mu\nu} \rangle}\mathrm{d}^4 x \tag{186}$$

Varying this action with respect to $\langle \hat{g}_{\mu\nu} \rangle$ yields the EFE, simply because $g_{\mu\nu} = \langle \hat{g}_{\mu\nu} \rangle$.

Definition 32 (Quantum EFE)**.** *The quantum EFE is obtained by varying the action with respect to $\langle \hat{g}_{\mu\nu} \rangle$:*

$$R_{\mu\nu} - \frac{1}{2} R \langle \hat{g}_{\mu\nu} \rangle = 0 \tag{187}$$

We obtain an equation which constrains the expectation value of the metric operator, the metric tensor, to the EFE.

We note the symmetries of this theory. The metric expectation value contains a SO(3,1) symmetry at the quantum level of the metric operator, and a diffeomorphism symmetry at the EFE level of the metric expectation value.

In a future paper, we will investigate this quantum EFE in more details.

2.5 Dimensional Obstructions

In this section, we explore the dimensional obstructions that arise when attempting to extend the multivector amplitude formalism to other dimensional configurations. We found that all dimensional configurations except those we have explored here (e.g. GA(0), GA(0, 1) and GA(3, 1)) are obstructed:

Dimensions	*Obstruction*	
GA(0)	**Unobstructed** $\implies$ statistical mechanics	(188)
GA(0, 1)	**Unobstructed** $\implies$ quantum mechanics	(189)
GA(1, 0)	Negative probabilities in the RQM	(190)
GA(2, 0)	No metric measurement $\implies$ Geometry not observationally complete	(191)
GA(1, 1)	Negative probabilities in the RQM	(192)
GA(0, 2)	Not isomorphic to a real matrix algebra	(193)
GA(3, 0)	Not isomorphic to a real matrix algebra	(194)
GA(2, 1)	Not isomorphic to a real matrix algebra	(195)
GA(1, 2)	Not isomorphic to a real matrix algebra	(196)
GA(0, 3)	Not isomorphic to a real matrix algebra	(197)
GA(4, 0)	Not isomorphic to a real matrix algebra	(198)
GA(3, 1)	**Unobstructed** $\implies$ quantum gravity $\wedge$ SU(3) $\times$ SU(2) $\times$ U(1)	(199)
GA(2, 2)	Negative probabilities in the RQM	(200)
GA(1, 3)	Not isomorphic to a real matrix algebra	(201)
GA(0, 4)	Not isomorphic to a real matrix algebra	(202)
GA(5, 0)	Not isomorphic to a real matrix algebra	(203)
$\vdots$	$\vdots$	
GA(6, 0)	No multilinear form as a self-product	(204)
$\vdots$	$\vdots$	
∞		(205)

Let us now demonstrate the obstructions mentioned above.

Theorem 19 (Not isomorphic to a real matrix algebra)**.** *The determinant of the matrix representation of the geometric algebras in this category is either complex-valued or quaternion-valued, making them unsuitable as a probability.*

Proof. These geometric algebras are classified as follows:

$$\mathrm{GA}(0,2) \cong \mathbb{H} \tag{206}$$
$$\mathrm{GA}(3,0) \cong \mathbb{M}_2(\mathbb{C}) \tag{207}$$
$$\mathrm{GA}(2,1) \cong \mathbb{M}_2^2(\mathbb{R}) \tag{208}$$
$$\mathrm{GA}(1,2) \cong \mathbb{M}_2(\mathbb{C}) \tag{209}$$
$$\mathrm{GA}(0,3) \cong \mathbb{H}^2 \tag{210}$$
$$\mathrm{GA}(4,0) \cong \mathbb{M}_2(\mathbb{H}) \tag{211}$$
$$\mathrm{GA}(1,3) \cong \mathbb{M}_2(\mathbb{H}) \tag{212}$$
$$\mathrm{GA}(0,4) \cong \mathbb{M}_2(\mathbb{H}) \tag{213}$$
$$\mathrm{GA}(5,0) \cong \mathbb{M}_2^2(\mathbb{H}) \tag{214}$$

The determinant of these objects, when such a thing exists, is valued in $\mathbb{C}$ or in $\mathbb{H}$, where $\mathbb{C}$ are the complex numbers, and where $\mathbb{H}$ are the quaternions. □

Theorem 20 (Negative Probabilities in the RQM)**.** *The even sub-algebra, which associates to the RQM part of the theory, of these dimensional configurations allows for negative probabilities, making them unsuitable as a RQM.*

Proof. We note three cases:

GA(1, 0): Let $\psi(q) = a + be_1$, then:

$$(a + be_1)^\ddagger(a + be_1) = (a - be_1)(a + be_1) = a^2 - b^2 e_1 e_1 = a^2 - b^2 \tag{215}$$

which is valued in $\mathbb{R}$.

GA(1, 1): Let $\psi(q) = a + be_0e_1$, then:

$$(a + be_0e_1)^\ddagger(a + be_0e_1) = (a - be_0e_1)(a + be_0e_1) = a^2 - b^2 e_0e_1e_0e_1 = a^2 - b^2 \tag{216}$$

which is valued in $\mathbb{R}$.

GA(2, 2): Let $\psi(q) = a + be_0e_\emptyset e_1e_2$, where $e_0^2 = -1, e_\emptyset^2 = -1, e_1^2 = 1, e_2^2 = 1$, then:

$$\lfloor(a + \mathbf{b})^\ddagger(a + \mathbf{b})\rfloor_{3,4}(a + \mathbf{b})^\ddagger(a + \mathbf{b}) \tag{217}$$
$$= \lfloor a^2 + 2a\mathbf{b} + \mathbf{b}^2\rfloor_{3,4}(a^2 + 2a\mathbf{b} + \mathbf{b}^2) \tag{218}$$

We note that $\mathbf{b}^2 = b^2 e_0e_\emptyset e_1e_2e_0e_\emptyset e_1e_2 = b^2$, therefore:

$$= (a^2 + b^2 - 2a\mathbf{b})(a^2 + b^2 + 2a\mathbf{b}) \tag{219}$$
$$= (a^2 + b^2)^2 - 4a^2\mathbf{b}^2 \tag{220}$$
$$= (a^2 + b^2)^2 - 4a^2b^2 \tag{221}$$

which is valued in $\mathbb{R}$.

In all of these cases the RQM probability can be negative. □

We repeat the following self-products[8] (Definition 18), which will help us demonstrate the next theorem:

$$\mathrm{GA}(0,1): \qquad \varphi^\dagger\varphi \tag{222}$$

$$\mathrm{GA}(2,0): \qquad \varphi^\ddagger\varphi \tag{223}$$

$$\mathrm{GA}(3,0): \qquad \lfloor\varphi^\ddagger\varphi\rfloor_3\varphi^\ddagger\varphi \tag{224}$$

$$\mathrm{GA}(3,1): \qquad \lfloor\varphi^\ddagger\varphi\rfloor_{3,4}\varphi^\ddagger\varphi \tag{225}$$

$$\mathrm{GA}(4,1): \qquad (\lfloor\varphi^\ddagger\varphi\rfloor_{3,4}\varphi^\ddagger\varphi)^\dagger(\lfloor\varphi^\ddagger\varphi\rfloor_{3,4}\varphi^\ddagger\varphi) \tag{226}$$

Theorem 21 (No Metric Measurements)**.** *This obstruction applies to* GA$(2,0)$. *Multilinear forms of at least four self-products are required for the theory to be observationally complete with respect to the geometry.*

Proof. A metric measurement requires a multilinear form of 4 self products because the metric tensor is defined using 2 self-products of the gamma matrices:

$$g_{\mu\nu} = \frac{1}{2}(\mathbf{e}_\mu\mathbf{e}_\nu + \mathbf{e}_\nu\mathbf{e}_\mu) \tag{227}$$

Each pair of wavefunction products fixes one basis elements. Thus, two pairs of wavefunction products are required to fix the geometry from the wavefunction. As multilinear forms of four self-products begin to appear in 3D, then the GA$(2,0)$ cannot produce a metric measurement as a quantum observable, thus its geometry is not observationally complete. □

Conjecture 1 (No multilinear form as a self-product (in 6D))**.** *The multivector representation of the norm in 6D cannot satisfy any observables.*

Argument. In six dimensions and above, the self-product patterns found in Definition 18 collapse. The research by Acus et al.[11] in 6D geometric algebra demonstrates that the determinant, so far defined through a self-products of the multivector, fails to extend into 6D. The crux of the difficulty is evident in the reduced case of a 6D multivector containing only scalar and grade-4 elements:

$$s(B) = b_1Bf_5(f_4(B)f_3(f_2(B)f_1(B))) + b_2Bg_5(g_4(B)g_3(g_2(B)g_1(B))) \tag{228}$$

This equation is not a multivector self-product but a linear sum of two multivector self-products[11].

The full expression is given in the form of a system of 4 equations, which is too long to list in its entirety. A small characteristic part is shown:

$$a_0^4 - 2a_0^2a_{47}^2 + b_2a_0^2a_{47}^2p_{412}p_{422} + \langle 72 \text{ monomials}\rangle = 0 \tag{229}$$

$$b_1a_0^3a_{52} + 2b_2a_0a_{47}^2a_{52}p_{412}p_{422}p_{432}p_{442}p_{452} + \langle 72 \text{ monomials}\rangle = 0 \tag{230}$$

$$\langle 74 \text{ monomials}\rangle = 0 \tag{231}$$

$$\langle 74 \text{ monomials}\rangle = 0 \tag{232}$$

From Equation 228, it is possible to see that no observable $\mathbf{O}$ can satisfy this equation because the linear combination does not allow one to factor it out of the equation.

$$b_1\mathbf{O}Bf_5(f_4(B)f_3(f_2(B)f_1(B))) + b_2Bg_5(g_4(B)g_3(g_2(B)g_1(B))) = b_1Bf_5(f_4(B)f_3(f_2(B)f_1(B))) + b_2\mathbf{O}Bg_5(g_4(B)g_3(g_2(B)g_1(B))) \tag{233}$$

Any equality of the above type between $b_1\mathbf{O}$ and $b_2\mathbf{O}$ is frustrated by the factors b_1 and b_2, forcing $\mathbf{O} = 1$ as the only satisfying observable. Since the obstruction occurs within grade-4, which is part of the even sub-algebra it is questionable that a satisfactory quantum theory (with observables) be constructible in 6D. □

This conjecture proposes that the multivector representation of the determinant in 6D does not allow for the construction of non-trivial observables, which is a crucial requirement for a consistent quantum formalism. The linear combination of multivector self-products in the 6D expression prevents the factorization of observables, limiting their role to the identity operator.

Conjecture 2 (No multilinear form as a self-product (above 6D))**.** *The norms beyond 6D are progressively more complex than the 6D case, which is already obstructed.*

These theorems and conjectures provide additional insights into the unique role of the unobstructed 3+1D signature in our proposal.

It is also interesting that our proposal is able to rule out $\mathrm{GA}(1,3)$ even if in relativity, the signature of the metric $(+,-,-,-)$ versus $(-,-,-,+)$ does not influence the physics. However, in geometric algebra, $\mathrm{GA}(1,3)$ represents 1 space dimension and 3 time dimensions. Therefore, it is not the signature itself that is ruled out but rather the specific arrangement of 3 time and 1 space dimensions, as this configuration yields quaternion-valued "probabilities" (i.e. $\mathrm{GA}(1,3) \cong \mathbb{M}_2(\mathbb{H})$ and $\det \mathbb{M}_2(\mathbb{H}) \in \mathbb{H}$).

Consequently, 3+1D is the only dimensional configuration (other than the "non-geometric" configurations of $\mathrm{GA}(0) \cong \mathbb{R}$ and $\mathrm{GA}(0,1) \cong \mathbb{C}$) in which a 'least biased' solution to the problem of maximizing the Shannon entropy of quantum measurements relative to an initial preparation, exists. This is an extremely strong claim regarding the possible spacetime configurations of the universe, and our ability (or inability) to construct a least biased theory to explain it.

3 Discussion

3.1 Maximizing The Relative Shannon Entropy

The principle of maximum entropy[3] states that the probability measure that best represents the current state of knowledge about a system is the one with the largest entropy, constrained by prior data.

In QM, an experiment begins with an initial preparation, followed by some transformations, and concludes with a final measurement of the system, yielding the result of the experiment. Consistent with the maximum entropy principle, our aim is to derive the 'least biased' theory that connects the initial preparation $p(q)$ to its final measurement $\rho(q)$, thereby formulating the theory as a solution to a maximization problem, rather than merely by axiomatic stipulation.

Using this methodology, fundamental physics can be formulated as the general solution to a maximization problem involving the Shannon entropy of all possible measurements of an arbitrary system relative to its initial preparation, under the constraint of a vanishing phase. As such, the structure of the inferred theory is determined by the nature and generality of the employed constraint. In this paper, we have investigated these four entropy maximization problems:

Constraint	*Vanishing Phase*	*Inferred Theory*	*Wavefunction*
$\overline{E} = \sum_{q\in\mathbb{Q}} \rho(q)E(q)$	none	SM	$\mathbb{R}$
$0 = \operatorname{tr} \sum_{q\in\mathbb{Q}} \rho(q)\begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}$	U(1)	QM	$\mathbb{C}$
$0 = \operatorname{tr} \sum_{q\in\mathbb{Q}} \rho(q)\mathbf{M_u}\vert_{a\to 0,\mathbf{x}\to 0,\mathbf{v}\to 0}$	$\mathrm{Spin}^c(3,1)$	RQM	$\mathbb{R}\times\mathrm{Spin}^c(3,1)$
$\overline{2a} = \operatorname{tr} \sum_{q\in\mathbb{Q}} A(q)\mathbf{M_u}\vert_{\mathbf{x}\to 0,\mathbf{v}\to 0}$	$\mathrm{Spin}^c(3,1)$, with dilations	QG	$\mathbb{R}\times\mathrm{Spin}^c(3,1)$

Despite the differences in constraints, all four theories hereso formulated share a common logical genesis, adhere to the same principle of maximum entropy, and qualify as the least biased theory for their given constraint.

3.2 The Multilinear Form

David Hestenes' work on the representation of the relativistic wavefunction within $\mathrm{GA}(3,1)$ was instrumental in the development of this research. His results served as a milestone, confirming the validity of our approach at various stages. Hestenes' wavefunction, $\psi = e^{\frac{1}{2}(a+\mathbf{f}+\mathbf{b})} = \sqrt{\rho}Re^{-ib/2}$, contains the same geometric structures as the $\mathrm{Spin}^c(3,1)$ wavefunction in our theory.

However, it is noteworthy that Hestenes' work does not include a fully satisfactory probability measure. To illustrate the difficulty, let us investigate a few options.

1. Multiplying the wavefunction with its reverse yields:

$$\tilde{\psi}\psi = \rho\tilde{R}e^{-ib/2}Re^{-b/2} = \rho e^{-ib} \tag{234}$$

   The result ρe^{-ib} does contains ρ, but it also includes a phase factor e^{-ib}. As such, it is not a proper probability measure.

2. Applying a joint action to the γ_μ basis, yields the Dirac current:

$$J = \tilde{\psi}\gamma_\mu\psi = \rho e_\mu \tag{235}$$

This approach eliminates the phase contribution because $e^{-ib/2}\gamma_\mu e^{-ib/2} = \gamma_\mu e^{ib/2}e^{-ib/2} = \gamma_\mu$. However, as it contains a basis e_μ, the Dirac current is not a proper probability measure (nor is it designed to be).

3. To construct an adapted Born rule that directly yields the probability when applied to the wavefunction, one might be tempted to apply the conjugate to ψ in addition to the reverse:

$$\tilde{\psi}^{\ddagger}\psi = \rho\tilde{R}e^{ib/2}Re^{-ib/2} = \rho \tag{236}$$

In this case one indeeds maps ψ to ρ, however, this approach disrupts the definition of the Dirac current: $\tilde{\psi}^{\ddagger}\gamma_\mu\psi = \rho\tilde{R}\gamma_\mu e^{ib/2}Re^{-ib/2} = \rho e_\mu e^{-ib/2} \neq J$.

4. Finally, the proposal retained by David Hestenes is to define probability measure as

$$\langle\psi,\psi\rangle = \langle\psi\psi^{\ddagger}\rangle_0 = \rho \tag{237}$$

where $\langle\mathbf{u}\rangle_0$ retains only the scalar part (grade 0) of the multivector.

However, such a definition is not the solution to an entropy maximization problem, and therefore does not represent the least biased probability measure for the situation. Specifically, it erases some of the features required to fully describe the system.

To correctly incorporate all the necessary features, including both the Dirac current and a probability measure yielding the probability density, and to retain all the geometric features of the formulation, the multilinear form must be employed. Transitioning from bilinear forms to multilinear forms involving <u>four</u> self-products of ψ represents a significant conceptual leap. The strength of the entropy maximization problem lies in its ability to automatically reveal the appropriate form to use. Specifically:

1. The multilinear form maps ψ to a probability measure:

$$\begin{aligned}
\lfloor\psi^{\ddagger}\psi\rfloor_{3,4}\psi^{\ddagger}\psi &= \lfloor\sqrt{\rho}\tilde{R}e^{-ib/2}\sqrt{\rho}Re^{-ib/2}\rfloor_{3,4}\sqrt{\rho}\tilde{R}e^{-ib/2}\sqrt{\rho}Re^{-ib/2} && (238)\\
&= \rho^2\tilde{R}R\tilde{R}Re^{ib/2}e^{ib/2}e^{-ib/2}e^{-ib/2} && (239)\\
&= \rho^2 && (240)
\end{aligned}$$

2. The definition of the Dirac current is retained:

$$\begin{aligned}
\psi^\ddagger\gamma_\mu\psi &= \sqrt{\rho}R^\ddagger e^{-ib/2}\gamma_\mu\sqrt{\rho}e^{ib/2}R && (241)\\
&= \rho\tilde{R}\gamma_\mu R && (242)\\
&= \rho e_\mu && (243)\\
&= J && (244)
\end{aligned}$$

3. In the multilinear form the "Dirac current" (i.e. sandwiching the gamma matrices within the form) is upgraded to a metric measurement:

$$\frac{1}{2}\Big(\lfloor\psi_\mu^\ddagger\gamma_\mu\psi_\mu\rfloor_{3,4}\psi_\nu^\ddagger\gamma_\nu\psi_\nu+\lfloor\psi_\nu^\ddagger\gamma_\nu\psi_\nu\rfloor_{3,4}\psi_\mu^\ddagger\gamma_\mu\psi_\mu\Big)=g_{\mu\nu} \qquad (245)$$

   In general the multilinear form permits arbitrary spacetime interval measurement, which is used as the foundation to our proposal for a quantum theory of gravity.

4. The multilinear form admits invariances with respect to:

$$U(1):\quad \langle\psi(q)|\gamma_0\psi(q)|\psi(q)|\gamma_0\psi(q)\rangle=\langle e^{\frac{1}{2}\mathbf{b}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)|e^{\frac{1}{2}\mathbf{b}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{b}}\psi(q)\rangle \qquad (246)$$

$$SU(2):\quad \langle\psi(q)|\gamma_0\psi(q)|\psi(q)|\gamma_0\psi(q)\rangle=\langle e^{\frac{1}{2}\mathbf{f}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)|e^{\frac{1}{2}\mathbf{f}}\psi(q)|\gamma_0 e^{\frac{1}{2}\mathbf{f}}\psi(q)\rangle \qquad (247)$$

$$SU(3):\quad \langle\psi(q)|\gamma_0\psi(q)|\psi(q)|\gamma_0\psi(q)\rangle=\langle\mathbf{f}\psi(q)|\gamma_0\mathbf{f}\psi(q)|\mathbf{f}\psi(q)|\gamma_0\mathbf{f}\psi(q)\rangle \qquad (248)$$

5. Finally, the multilinear form leads to an obstruction-free quantum theory only in 3+1D.

3.3 Interpretation

The Born rule is the least biased probability measure for a complex Hilbert space (Theorem 2). However, when extending to 3+1D, this is no longer the case. The least biased probability measure becomes the multilinear form (Theorem 3). It is because of the increased geometric flexibility of the multilinear form that the results we have obtained are possible, notably a quantum description of 3+1D spacetime in the form of a quantum theory of frame fields.

4 Conclusion

In conclusion, this paper presents a novel approach to physical theory construction by solving a maximization problem on the Shannon entropy of all possible measurements of a system relative to its initial preparation, under the constraint of a vanishing phase. By appropriately selecting the group of the vanishing phase, the solution resolves to quantum mechanics, relativistic quantum mechanics, or a theory of quantum gravity. Our findings reveal the exceptional ability of this approach to generate a mathematically well-behaved theory that generalizes quantum probabilities through the introduction of vanishing phases. The resulting measure is invariant under a wide range of geometric transformations, including those generated by the gauge groups of the Standard Model, those associated to general relativity, and leads to the metric tensor as a quantum mechanical observable, without the need for additional assumptions beyond the vanishing phase. This finding aligns with the observed dimensionality and gauge symmetries of the universe and suggests a possible explanation for its specificity. By reducing fundamental physics to the optimal solution to an entropy maximization problem, the framework integrates statistical mechanics, quantum mechanics, relativistic quantum mechanics, and quantum gravity, while also accounting for the dimensionality of spacetime and the gauge symmetries of particle physics, under the same conceptual and mathematical basis.

Statements and Declarations

- Competing Interests: The author declares that he has no competing financial or non-financial interests that are directly or indirectly related to the work submitted for publication.
- Data Availability Statement: No datasets were generated or analyzed during the current study.
- During the preparation of this manuscript, we utilized a Large Language Model (LLM), for assistance with spelling and grammar corrections, as well as for minor improvements to the text to enhance clarity and readability. This AI tool did not contribute to the conceptual development of the work, data analysis, interpretation of results, or the decision-making process in the research. Its use was limited to language editing and minor textual enhancements to ensure the manuscript met the required linguistic standards.

A SM

Here, we solve the Lagrange multiplier equation of SM.

$$\mathcal{L}(\rho,\lambda,\beta) = \underbrace{-k_B \sum_{q\in\mathbb{Q}} \rho(q)\ln\rho(q)}_{\substack{\text{Boltzmann}\\ \text{Entropy}}} + \underbrace{\lambda\left(1-\sum_{q\in\mathbb{Q}}\rho(q)\right)}_{\substack{\text{Normalization}\\ \text{Constraint}}} + \underbrace{\beta\left(\overline{E}-\sum_{q\in\mathbb{Q}}\rho(q)E(q)\right)}_{\text{Average Energy Constraint}} \tag{249}$$

We solve the maximization problem as follows:

$$\frac{\partial \mathcal{L}(\rho, \lambda, \beta)}{\partial \rho(q)} = 0 = -\ln \rho(q) - 1 - \lambda - \beta E(q) \tag{250}$$

$$0 = \ln \rho(q) + 1 + \lambda + \beta E(q) \tag{251}$$

$$\implies \ln \rho(q) = -1 - \lambda - \beta E(q) \tag{252}$$

$$\implies \rho(q) = \exp(-1-\lambda) \exp\left(-\beta E(q)\right) \tag{253}$$

$$= \frac{1}{Z(\tau)} \exp\left(-\beta E(q)\right) \tag{254}$$

The partition function, is obtained as follows:

$$1 = \sum_{r \in \mathbb{Q}} \exp(-1-\lambda) \exp\left(-\beta E(q)\right) \tag{255}$$

$$\implies \left(\exp(-1-\lambda)\right)^{-1} = \sum_{r \in \mathbb{Q}} \exp\left(-\beta E(q)\right) \tag{256}$$

$$Z(\tau) := \sum_{r \in \mathbb{Q}} \exp\left(-\beta E(q)\right) \tag{257}$$

Finally, the probability measure is:

$$\rho(q) = \frac{1}{\sum_{r \in \mathbb{Q}} \exp\left(-\beta E(q)\right)} \exp\left(-\beta E(q)\right) \tag{258}$$

B RQM in 3+1D

$$\mathcal{L}(\rho, \lambda, \tau) = \underbrace{-\sum_{q \in \mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)}}_{\substack{\text{Relative Shannon} \\ \text{Entropy}}} + \underbrace{\lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q)\right)}_{\substack{\text{Normalization} \\ \text{Constraint}}} + \underbrace{\zeta \left(-\operatorname{tr} \frac{1}{2} \sum_{q \in \mathbb{Q}} \rho(q) \mathbf{M}_{\mathbf{u}}(q)|_{a \to 0, \mathbf{x} \to 0, \mathbf{v} \to 0}\right)}_{\substack{\text{Vanishing Relativistic-Phase} \\ \text{Anti-Constraint}}} \tag{259}$$

The solution is obtained using the same step-by-step process as the 2D case, and yields:

$$\rho(q) = \underbrace{\frac{1}{\sum_{r \in \mathbb{Q}} p(r) \det \exp\left(-\zeta \frac{1}{2} \mathbf{M}_{\mathbf{u}}(r)|_{a \to 0, \mathbf{x} \to 0, \mathbf{b} \to 0}\right)}}_{\text{Spin}^c(3,1) \text{ Invariant Ensemble}} \underbrace{\det \exp\left(-\zeta \frac{1}{2} \mathbf{M}_{\mathbf{u}}(q)|_{a \to 0, \mathbf{x} \to 0, \mathbf{b} \to 0}\right)}_{\text{Spin}^c(3,1) \text{ Born Rule}} \underbrace{p(q)}_{\text{Initial Preparation}} \tag{260}$$

Proof. The Lagrange multiplier equation can be solved as follows:

$$\frac{\partial \mathcal{L}(\rho,\lambda,\zeta)}{\partial \rho(q)} = 0 = -\ln\frac{\rho(q)}{p(q)} - p(q) - \lambda - \zeta \operatorname{tr}\frac{1}{2}\mathbf{M}_{\mathbf{u}}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0} \tag{261}$$

$$0 = \ln\frac{\rho(q)}{p(q)} + p(q) + \lambda + \zeta \operatorname{tr}\frac{1}{2}\mathbf{M}_{\mathbf{u}}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0} \tag{262}$$

$$\implies \ln\frac{\rho(q)}{p(q)} = -p(q) - \lambda - \zeta \operatorname{tr}\frac{1}{2}\mathbf{M}_{\mathbf{u}}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0} \tag{263}$$

$$\implies \rho(q) = p(q)\exp(-p(q)-\lambda)\exp\left(-\zeta \operatorname{tr}\frac{1}{2}\mathbf{M}_{\mathbf{u}}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0}\right) \tag{264}$$

$$= \frac{1}{Z(\zeta)}p(q)\exp\left(-\zeta \operatorname{tr}\frac{1}{2}\mathbf{M}_{\mathbf{u}}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0}\right) \tag{265}$$

The partition function $Z(\zeta)$, serving as a normalization constant, is determined as follows:

$$1 = \sum_{r\in\mathbb{Q}} p(r)\exp(-p(q)-\lambda)\exp\left(-\zeta \operatorname{tr}\frac{1}{2}\mathbf{M}_{\mathbf{u}}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0}\right) \tag{266}$$

$$\implies (\exp(-p(q)-\lambda))^{-1} = \sum_{r\in\mathbb{Q}} p(r)\exp\left(-\zeta \operatorname{tr}\frac{1}{2}\mathbf{M}_{\mathbf{u}}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0}\right) \tag{267}$$

$$Z(\zeta) := \sum_{r\in\mathbb{Q}} p(r)\exp\left(-\zeta \operatorname{tr}\frac{1}{2}\mathbf{M}_{\mathbf{u}}(q)|_{a\to 0,\mathbf{x}\to 0,\mathbf{b}\to 0}\right) \tag{268}$$

□

C SageMath program showing $\lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u} = \det \mathbf{M}_{\mathbf{u}}$

```
from sage.algebras.clifford_algebra import CliffordAlgebra
from sage.quadratic_forms.quadratic_form import QuadraticForm
from sage.symbolic.ring import SR
from sage.matrix.constructor import Matrix

# Define the quadratic form for GA(3,1) over the Symbolic Ring
Q = QuadraticForm(SR, 4, [-1, 0, 0, 0, 1, 0, 0, 1, 0, 1])

# Initialize the GA(3,1) algebra over the Symbolic Ring
algebra = CliffordAlgebra(Q)

# Define the basis vectors
e0, e1, e2, e3 = algebra.gens()
```

```
# Define the scalar variables for each basis element
a = var('a')
t, x, y, z = var('t x y z')
f01, f02, f03, f12, f23, f13 = var('f01 f02 f03 f12 f23 f13')
v, w, q, p = var('v w q p')
b = var('b')

# Create a general multivector
udegree0=a
udegree1=t*e0+x*e1+y*e2+z*e3
udegree2=f01*e0*e1+f02*e0*e2+f03*e0*e3+f12*e1*e2+f13*e1*e3+f23*e2*e3
udegree3=v*e0*e1*e2+w*e0*e1*e3+q*e0*e2*e3+p*e1*e2*e3
udegree4=b*e0*e1*e2*e3
u=udegree0+udegree1+udegree2+udegree3+udegree4

u2 = u.clifford_conjugate()*u

u2degree0 = sum(x for x in u2.terms() if x.degree() == 0)
u2degree1 = sum(x for x in u2.terms() if x.degree() == 1)
u2degree2 = sum(x for x in u2.terms() if x.degree() == 2)
u2degree3 = sum(x for x in u2.terms() if x.degree() == 3)
u2degree4 = sum(x for x in u2.terms() if x.degree() == 4)
u2conj34 = u2degree0+u2degree1+u2degree2-u2degree3-u2degree4


I = Matrix(SR, [[1, 0, 0, 0],
                [0, 1, 0, 0],
                [0, 0, 1, 0],
                [0, 0, 0, 1]])

#MAJORANA MATRICES
y0 = Matrix(SR, [[0, 0, 0, 1],
                 [0, 0, -1, 0],
                 [0, 1, 0, 0],
                 [-1, 0, 0, 0]])

y1 = Matrix(SR, [[0, -1, 0, 0],
                 [-1, 0, 0, 0],
                 [0, 0, 0, -1],
                 [0, 0, -1, 0]])

y2 = Matrix(SR, [[0, 0, 0, 1],
                 [0, 0, -1, 0],
                 [0, -1, 0, 0],
                 [1, 0, 0, 0]])
```

```
y3 = Matrix(SR, [[-1, 0, 0, 0],
                 [0, 1, 0, 0],
                 [0, 0, -1, 0],
                 [0, 0, 0, 1]])

mdegree0 = a
mdegree1 = t*y0+x*y1+y*y2+z*y3
mdegree2 = f01*y0*y1+f02*y0*y2+f03*y0*y3+f12*y1*y2+f13*y1*y3+f23*y2*y3
mdegree3 = v*y0*y1*y2+w*y0*y1*y3+q*y0*y2*y3+p*y1*y2*y3
mdegree4 = b*y0*y1*y2*y3
m=mdegree0+mdegree1+mdegree2+mdegree3+mdegree4

print(u2conj34*u2 == m.det())
```

The program outputs

```
True
```

showing, by computer assisted symbolic manipulations, that the determinant of the real Majorana representation of a multivector u is equal to the multilinear form: $\det \mathbf{M_u} = \lfloor \mathbf{u}^\ddagger \mathbf{u} \rfloor_{3,4} \mathbf{u}^\ddagger \mathbf{u}$.

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