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Completeness and Correspondence in Chellas-Segerberg Semantics

Abstract. We investigate a lattice of conditional logics described by a Kripke type semantics, which was suggested by Chellas and Segerberg – Chellas-Segerberg (CS) semantics – plus 30 further principles. We (i) present a non-trivial frame-based completeness result, (ii) a translation procedure which gives one corresponding trivial frame conditions for arbitrary formula schemata, and (iii) non-trivial frame conditions in CS semantics which correspond to the 30 principles.

Keywords: Chellas-Segerberg Semantics; Standard Segerberg Frame Completeness; Correspondence; Non-trivial Frame Condition; Conditional Logic.

1. Introduction

In this paper we focus on Chellas-Segerberg (CS) semantics, a possible worlds semantics for a two-place necessity operator $\Box\!\!\rightarrow$ that goes back to Chellas [5] and Segerberg [21], where a formula $\alpha \Box\!\!\rightarrow \beta$ is interpreted in terms of a conditional of sort 'if α then β'. CS semantics uses a three-place accessibility relation R_X between pairs of possible worlds relativized to sets of possible worlds X. In this semantics a two-place necessity formula (or conditional) $\alpha \Box\!\!\rightarrow \beta$ is true at a world w iff β is true at all worlds accessible from w by $R_{||\alpha||}$ where $||\alpha||$ is the set of possible worlds at which α is true according to a valuation function V. CS semantics is a relativized Kripke possible worlds semantics insofar as $\alpha \Box\!\!\rightarrow \beta$ can be unofficially read as a necessity statement of the form $[\alpha \Box\!\!\rightarrow]\beta$ where $[\alpha \Box\!\!\rightarrow]$ represents a necessity operator for each antecedent α (cf., [5, p. 138]; [21, p. 157]).

We present here (i) a non-trivial frame-based completeness result for a lattice of systems described by 30 principles discussed in the conditional logic literature, (ii) a translation procedure that gives us corresponding trivial frame conditions for arbitrary formula schemata, and (iii) non-trivial frame conditions in CS semantics which correspond to the 30 principles. The completeness property envisaged in (i) is non-trivial insofar as *not* all conditional logics are strongly complete with respect to (w.r.t.) some class of frames. The completeness result is based on structural, frame-based as opposed to model-based conditions which were not established in [5] and [21].

CS semantics resembles set-selection semantics, which can be described in terms of an accessibility relation R_α that is relativized to formulas rather than sets of possible worlds, and where a conditional $\alpha \mathrel{\Box\!\!\rightarrow} \beta$ is true at a world w iff β is true at all worlds accessible by R_α (e.g., Priest [17, p. 85]; Lewis [13, p. 60]). Whereas set-selection semantics does not face the difficulties described in Sect. 5, it does not allow for a characterization in terms of structural, frame-based conditions, since the accessibility relations R_α eventually have to be interpreted in terms of valuation functions.

Note that the present results extend the results described in Unterhuber [24] and are *not* implied by the investigations of multi-modal extensions of Kripke frames by Blackburn et al. [4, p. 20], Fine and Schurz [6], and Gabbay et al. [7, p. 20f].

2. Languages

We discuss CS semantics as described by the modal language $\mathcal{L}_{\mathrm{CL}}$ ('CL' for 'Conditional Logic'), which contains atomic propositional variables $p, p_0, \ldots \in \mathcal{AV}$ ($\mathcal{AV}$ is the set of atomic variables) and is closed under truth-functional propositional operators and the two-place modal operator $\mathrel{\Box\!\!\rightarrow}$ plus its dual $\mathrel{\Diamond\!\!\rightarrow}$. We use letters $\alpha, \alpha_0, \ldots, \beta, \beta_0, \ldots$ to refer to formulas of the language $\mathcal{L}_{\mathrm{CL}}$. The primitive logical symbols of $\mathcal{L}_{\mathrm{CL}}$ are $\neg$ ("negation"), $\vee$ ("disjunction") and $\mathrel{\Box\!\!\rightarrow}$ ("conditional operator") as well as the defined logical symbols $\wedge$ ("conjunction"), $\rightarrow$ ("material implication"), $\leftrightarrow$ ("logical equivalence"), $\top$ ("verum"), $\bot$ ("falsum") and $\mathrel{\Diamond\!\!\rightarrow}$ ("dual of the conditional operator") which are defined as usual (in particular, $(\alpha \mathrel{\Diamond\!\!\rightarrow} \beta) =_{\mathrm{df}} \neg(\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta)$). We also use the expression $\mathcal{L}$ to refer to the set of all formulas of a language $\mathcal{L}$.

In this paper we explicate a notion of non-trivial frame completeness which is based on a notion of non-trivial *correspondence* between logical axioms α formulated in the modal propositional language $\mathcal{L}_{\mathrm{CL}}$ and frame conditions C_α expressed in a suitable predicate logic language. There are two possible ways to express these frame conditions: (i) either in the meta-language (i.e., the same language which is used to describe the semantics), or (ii) in a second suitably specified object language. In this paper we use the second approach and call this second object language $\mathcal{L}_{\mathrm{FC}}$ ("FC" or "Frame Conditions"). Inductive proofs concerning translation functions between the two languages are easier to handle in the second approach. (In the first approach, we would have to use quotation marks in order to allow the meta-language to speak about itself, which is possible but unusual in philosophical logic.)[1]

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[1] Van Benthem [25] is not explicit about this choice. He chooses approach (i) but uses different variables for possible worlds – 'w' for semantical usage and 'x' for frame conditions. Van Benthem's

In CS semantics – unlike in Kripke semantics – all (interesting) frame conditions can only be formulated in a language (fragment) that allows for quantification over possible worlds as well as *sets* of possible worlds due to the relativization of R to subsets of possible worlds. Therefore, we describe here frame conditions formulated in the *two-sorted* set-theoretic, predicate language $\mathcal{L}_{\mathrm{FC}}$ using the following symbols:

1. variables w, w_0, $\ldots$ of sort 1 for possible worlds (we use w, w_0, $\ldots$ as meta-language variables for worlds in W),
2. variables X, X_0, $\ldots$, Y, Y_0, $\ldots$ of sort 2 for sets of possible worlds (we use X, X_i, $\ldots$ as meta-language variables for subsets of W),
3. the constants W and $\underline{\varnothing}$ representing the set of all possible worlds and the empty set, respectively,
4. the Boolean function symbols $\underline{-}(\tau_1)$, $\underline{\cup}(\tau_1,\tau_2)$, $\underline{\cap}(\tau_1,\tau_2)$, (where τ, τ_0, $\ldots$ are meta-variables ranging over terms of sort 2) and the non-Boolean function symbol $\underline{\Box\!\!\rightarrow}(\tau_1,\tau_2)$,
5. the three-place accessibility relation symbol $\mathsf{R}(\mathsf{w}_i,\mathsf{w}_j,\tau)$ with three argument places, τ as a placeholder for terms of sort 2 and the element relation $\underline{\in}(\mathsf{w}_i,\tau)$ with two argument places one each for variables of sort 1 and terms of sort 2 as the only non-logical symbols, and
6. the undefined logical symbols $\underline{\neg}$ ("negation"), $\underline{\vee}$ ("disjunction"), $\underline{\forall}$ ("unversal qantifier"), identity symbol $\underline{=}$ (with two argument places of type 1) and the defined logical symbols $\underline{\wedge}$ ("conjunction"), $\underline{\rightarrow}$ ("material implication"), $\underline{\leftrightarrow}$ ("logical equivalence") and $\underline{\exists}$ ("existential quantifier").

We underline symbols of the language $\mathcal{L}_{\mathrm{FC}}$ or use Roman font in order to distinguish them from correspondig expressions of our meta-language. The logical symbols $\underline{\wedge}$, $\underline{\rightarrow}$ and $\underline{\leftrightarrow}$ in $\mathcal{L}_{\mathrm{FC}}$ are defined in the usual way; moreover, for $\mathcal{L}_{\mathrm{FC}}$ it holds that $\underline{\exists}\mathsf{w}_i\alpha =_{\mathrm{df}} \underline{\neg}\,\underline{\forall}\mathsf{w}_i\underline{\neg}\,\alpha$ and $\exists\mathsf{X}_j\alpha =_{\mathrm{df}} \underline{\neg}\,\underline{\forall}\mathsf{X}_j\underline{\neg}\,\alpha$, and we abbreviate '$\underline{\in}(\mathsf{w}_i,\tau)$', '$\underline{-}(\tau)$', '$\underline{\cup}(\tau_1,\tau_2)$', '$\underline{\cap}(\tau_1,\tau_2)$' and '$\mathsf{R}(\mathsf{w}_i,\mathsf{w}_j,\tau)$' by '$(\mathsf{w}_i\,\underline{\in}\,\tau)$', '$\underline{-}\,\tau$', '$(\tau_1\,\underline{\cup}\,\tau_2)$', '$(\tau_1\,\underline{\cap}\,\tau_2)$', and '$\mathsf{w}_i\mathsf{R}_\tau\mathsf{w}_j$', respectively. The fragment of $\mathcal{L}_{\mathrm{FC}}$ that we actually use contains only closed formulas of the form $\underline{\forall}X_{i_1}\ldots\underline{\forall}X_{i_n}\alpha$, where in α no quantifier symbol w.r.t. variables of sort 2 occurs.

Logical consequence over $\mathcal{L}_{\mathrm{FC}}$ formulas is defined in the ordinary way but restricted to first-order models based on first-order structures, as explained in Sect. 5. $\mathbf{L}_{\mathrm{FC}}$ denotes the logic obtained from this consequence operation, i.e. the set of theorems.

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conventions are sufficient for his purposes but not for ours.

3. Basic Semantic Notions in CS Semantics

The basic semantic notions in CS semantics can be defined as follows:

Definition 3.1. $\mathcal{F}_C = \langle W, R\rangle$ is a Chellas frame iff
(a) W is a non-empty set of possible worlds and
(b) R is a relation on $W \times W \times \text{Pow}(W)$.

Definition 3.2. Let $\mathcal{F}_C = \langle W, R\rangle$ be a Chellas frame. Then, $\mathcal{M}_C = \langle W, R, V\rangle$ is a Chellas model iff V is a valuation function from $\mathcal{AV} \times W$ to $\{0, 1\}$ and for all formulas $\alpha, \beta \in \mathcal{L}_{\text{CL}}$ and $w \in W$ it holds:

$\langle\langle W, R, V\rangle, w\rangle \models \neg\alpha$ iff $\langle\langle W, R, V\rangle, w\rangle \not\models \alpha$,
$\langle\langle W, R, V\rangle, w\rangle \models \alpha \vee \beta$ iff $\langle\langle W, R, V\rangle, w\rangle \models \alpha$ or $\langle\langle W, R, V\rangle, w\rangle \models \beta$, and
$\langle\langle W, R, V\rangle, w\rangle \models \alpha \mathrel{\Box\!\!\rightarrow} \beta$ iff $\forall w'(wR_{||\alpha||^{\mathcal{M}_C}} w' \rightarrow \langle\langle W, R, V\rangle, w'\rangle \models \beta)$.

The expressions $\langle\langle W, R, V\rangle, w\rangle \models \alpha$ and $\langle\langle W, R, V\rangle, w\rangle \not\models \alpha$ abbreviate $V(\alpha, w) = 1$ and $V(\alpha, w) \neq 1$, respectively, for a Chellas model $\langle W, R, V\rangle$ and $w \in W$, and the expression $||\alpha||^{\mathcal{M}}$ refers to the set of possible worlds at which α is true according to a model $\mathcal{M}$. (We omit reference to $\mathcal{M}$ where the context does not allow for ambiguity.)

We now define Segerberg frames which are nothing but *general* Chellas frames:

Definition 3.3. $\mathcal{F}_S = \langle W, R, P\rangle$ is a Segerberg frame iff W is a non-empty set of possible worlds, R is a relation on $W \times W \times P$ and $P \subseteq \text{Pow}(W)$ defines admissible valuations, i.e., it holds for P:
$\varnothing \in P$, (Def$_{P_\varnothing}$)
if $X \in P$ then $-X \in P$, (Def$_{P_-}$)
if $X, Y \in P$ then $X \cup Y \in P$, and (Def$_{P_\cup}$)
if $X, Y \in P$ then $\{w \in W \mid \forall w' \in W(wR_X w' \rightarrow w' \in Y)\} \in P$. (Def$_{P_{Mod}}$)

Definition 3.4. $\mathcal{M}_S = \langle W, R, P, V\rangle$ is a Segerberg model iff
(a) $\langle W, R, P\rangle$ is a Segerberg frame as described in Def. 3.3,
(b) V is a valuation function w.r.t. W and R as described in Def. 3.2, and
(c) V is admissible in $\mathcal{F}_S$, i.e. $||\alpha|| \in P$ for all $\alpha \in \mathcal{L}_{\text{CL}}$.

A Chellas model $\langle W, R, V\rangle$ is based on a Chellas frame $\langle W', R'\rangle$ iff $W = W'$ and $R = R'$. A Chellas model $\langle W, R, V\rangle$ is based on a Segerberg frame $\langle W', R', P\rangle$ iff $W = W'$, $R' = R\!\uparrow\! P$, and V is admissible in $\mathcal{F}_S$, where $R\!\uparrow\! P$ is the restriction of R to elements of P. A Segerberg model $\langle W, R, P, V\rangle$ is based on a Segerberg frame $\langle W', R', P'\rangle$ iff $W = W'$, $R = R'$, $P = P'$, and V is admissible in $\mathcal{F}_S$. Then, α is valid on a Segerberg [Chellas] frame $\mathcal{F}$ ($\mathcal{F} \models \alpha$) iff α is true at all worlds in all Segerberg [Chellas] models $\mathcal{M}$ based on $\mathcal{F}$.

4. The Lattice of Conditional Logics

In this section we describe a notion of completeness which is non-trivial and based on purely structural as opposed to model-based notions. The unproblematic case is logic **CK**, which is sound and complete w.r.t. the class of all Chellas frames (see [5, Sect. 6]), where the logic **CK** is defined by the following set of axioms and rules ('**CK**' stands for 'Conditional (System) **K**'):[2]

LLE if $\alpha \leftrightarrow \beta$ then $(\alpha \mathrel{\Box\!\!\rightarrow} \gamma) \leftrightarrow (\beta \mathrel{\Box\!\!\rightarrow} \gamma)$
RW if $\alpha \rightarrow \beta$ then $(\gamma \mathrel{\Box\!\!\rightarrow} \alpha) \rightarrow (\gamma \mathrel{\Box\!\!\rightarrow} \beta)$
AND $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \beta \wedge \gamma)$
LT $\alpha \mathrel{\Box\!\!\rightarrow} \top$

LLE, RW and LT abbreviate 'Left Logical Equivalence', 'Right Weakening' and 'Logical Truth', respectively, and $\mathbf{\Pi_{CK}}$ denotes a lattice of logics $\mathbf{L} \subseteq \mathcal{L}_{\mathrm{CL}}$ that are extensions of system **CK** which are closed under propositional consequence, the rules of **CK** (i.e., LLE, RW) and the rule of substitution. We write $\mathbf{L} + \alpha_1 + \alpha_2 + \dots$ for a logic $\mathbf{L} \in \mathbf{\Pi_{CK}}$ that is axiomatized by the additional schemata α_1, α_2, ... (i.e., that is the closure of $\mathbf{CK} \cup \{\alpha_1, \alpha_2, \dots\}$ under the rules of propositional consequence, LLE, RW and substitution). For later reference observe that the following rule RCK is logically equivalent to RW, AND and LT:

(RCK) $(\alpha \mathrel{\Box\!\!\rightarrow} \beta_1) \wedge \cdots \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \beta_n) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \gamma)$ if $\beta_1 \wedge \cdots \wedge \beta_n \rightarrow \gamma$ $(n \geq 0)$.

A Chellas/Segerberg frame $\mathcal{F}$ [model $\mathcal{M}$] is a frame [model] *for* the logic $\mathbf{L} \in \mathbf{\Pi_{CK}}$ iff all **L**-theorems are valid on $\mathcal{F}$ [in $\mathcal{M}$]. The problem for completeness proofs w.r.t. classes of Chellas frames which represent extensions of system **CK** is that the canonical model allows one in general only to specify R_X for sets of possible worlds X which can be represented by formulas of the language $\mathcal{L}_{\mathrm{CL}}$ whereas a full completeness proof requires a specification of R_X for sets of possible worlds X that cannot be represented in that way.

Segerberg [21] provided a completeness proof for a lattice of systems w.r.t. classes of Segerberg frames which – as we saw – are nothing but general Chellas frames. It is, however, a well-known fact that in Kripke semantics completeness w.r.t. classes of general frames is equivalent to completeness w.r.t. classes of Kripke models, which in turn is trivial in the sense that any normal modal logic is (strongly) complete w.r.t. some class of Kripke models (Hughes and Cresswell [9, p. 168]). It can be seen by the following observation that an analogous result holds for Segerberg frames: For each Chellas model $\mathcal{M}_C = \langle W, R, V \rangle$ a Segerberg frame $\mathcal{F}_C^{\mathcal{M}_C} = \langle W, R, P^V \rangle$ exists s.t. $P^V = \{||\alpha||^{\mathcal{M}_C} \,|\, \alpha \in \mathcal{L}_{\mathrm{CL}}\}$. Since a logic $\mathbf{L} \in \mathbf{\Pi_{CK}}$ is

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[2] System **CK** is a conditional version of the weakest normal modal logic **K** insofar as for each "modal operator" [$\alpha \mathrel{\Box\!\!\rightarrow}$], System **CK** gives us the exact same theorems as System **K** (cf. [5, p. 139]).

closed under substitution, it holds for all formulas $\alpha \in \mathbf{L}$ that α is valid in $\mathcal{M}_C$ iff α is valid on $\mathcal{F}_C^{\mathcal{M}_C}$ and that hence completeness w.r.t. Segerberg frames is trivial in the sense that any such logic **L** is (strongly) complete w.r.t. some class of Segerberg frames.

5. Trivial and Non-Trivial Notions of Completeness for CS Semantics

In this section we explicate a notion of completeness for CS semantics which is *non*-trivial and allows one to explicate logics $\mathbf{L} \in \mathbf{\Pi_{CK}}$ in terms of purely structural as opposed to model-based frame conditions. For this purpose we propose the notion of *completeness w.r.t. classes of standard Segerberg frames*. The notion of a "standard" Segerberg frame is in turn based on the notion of a non-trivial frame condition which we are now going to explicate. First of all we introduce the notion of correspondence (analogous to van Benthem's [25] definition for normal alethic modal logics):

Definition 5.1. (First-Order Structures for $\mathcal{L}_{\text{FC}}$ Corresponding to Chellas Frames and Models Based on them)
1. If $\mathcal{F}_C =_{\text{df}} \langle W, R\rangle$ is a Chellas frame, then the corresponding first order structure for $\mathcal{L}_{\text{FC}}$, abbreviated as $S(\mathcal{F}_C)$, is defined as follows: $S(\mathcal{F}_C) =_{\text{df}} \langle W, V^*\rangle$ where V^* is a valuation function over the non-logical alphabet of $\mathcal{L}_{\text{FC}}$ satisfying the following conditions: $V^*(\mathsf{R}) = R$, $V^*(\mathsf{W}) = W$, $V^*(\underline{\varnothing}) = \varnothing$, and $\underline{\in}$, $\underline{-}$, $\underline{\cap}$, $\underline{\cup}$, and $\underline{\Box\!\!\rightarrow}$ receive their standard interpretations, i.e. $\underline{\in}$ the set-theoretic element relation $V^*(\underline{\in}) = \in$, the Boolean operators have their standard definitions, and $\underline{\Box\!\!\rightarrow}$ is interpreted as follows: $V^*(\mathsf{X} \underline{\Box\!\!\rightarrow} \mathsf{Y}) = \{w \in W \mid \forall w'(wR_{V^*(\mathsf{X})}w' \rightarrow w' \in V^*(\mathsf{Y}))\}$.
2. A first-order model $\langle W, V^*\rangle$ based on $S(\mathcal{F}_C)$ agrees with $S(\mathcal{F}_C)$ on $\mathcal{L}_{\text{FC}}$'s non-logical alphabet and in addition assigns valuations V^* to the *variables* of $\mathcal{L}_{\text{FC}}$ according to the following condition: $V^*(\mathsf{w}_i) \in W$ and $V^*(\mathsf{X}_j) \subseteq W$, for all variables w_i and X_j of sort 1 and sort 2 of the language $\mathcal{L}_{\text{FC}}$, respectively. While the structure $S(\mathcal{F}_C)$ fixes the truth value of closed formulas (sentences) of $\mathcal{L}_{\text{FC}}$, models based on $S(\mathcal{F}_C)$ are needed to define truth values for quantified sentences of L_{FC} by assigning truth values to open formulas, according to the usual conditions:

$\langle W, V^*\rangle \models \underline{\forall}\, \mathsf{w}_i\alpha$ iff for all $w \in W$, $\langle W, V^*[\mathsf{w}_i{:}w]\rangle \models \alpha$
$\langle W, V^*\rangle \models \underline{\forall}\, \mathsf{X}_j\alpha$ iff for all $X \subseteq W$, $\langle W, V^*[\mathsf{X}_j{:}X]\rangle \models \alpha$

where $V^*[\mathsf{w}_i{:}w]$ and $V^*[\mathsf{X}_j{:}X]$ are like V^* except that they assign w to w_i, and X to X_j, respectively. Which particular possible worlds and subsets of possible worlds are assigned by V^* to w_i and X_j, respectively, for the first-order model $\langle W, V^*\rangle$ depends on the valuation function V of the corresponding Chellas model $\langle W, R, V\rangle$ as defined in Lemmata 6.5 and 6.6.

In the following we define correspondence for formula (or axiom) *schemata*. In the formulation of formula schemata we use schematic letters A, A_0, ... which are placeholders for formulas α, α_0, $\ldots \in \mathcal{L}_{\text{CL}}$. An expression of the form $\alpha(A_1, \ldots, A_n)$ denotes a formula schema (e.g., an axiom schema) *for* $\mathcal{L}_{\text{CL}}$ where $A_1, \ldots, A_n$ are all schematic letters that occur in α, numerically ordered according to their first occurrence from left to right. $\alpha(\alpha_1/A_1, \ldots, \alpha_n/A_n)$ denotes an instance of the axiom schema $\alpha(A_1, \ldots, A_n)$ in the language $\mathcal{L}_{\text{CL}}$, i.e. a formula of $\mathcal{L}_{\text{CL}}$ which results from replacing schematic letters A_i by formulas α_i of $\mathcal{L}_{\text{CL}}$ (for all $1 \leq i \leq n$). That a formula schema $\alpha(A_1, \ldots, A_n)$ is valid on a Chellas [Segerberg] frame $\mathcal{F}$ means by definition that all instances of $\alpha(A_1, \ldots, A_n)$ are valid on $\mathcal{F}$.

We can now define correspondence as follows:

Definition 5.2. (Correspondence of Axioms of Logics $\mathbf{L} \in \mathbf{\Pi_{CK}}$)
A formula $\alpha \in \mathcal{L}_{\text{CL}}$ corresponds to a frame condition C_α of $\mathcal{L}_{\text{FC}}$ iff for every Chellas frame $\mathcal{F}_C$ it holds: $\mathcal{F}_C \models \alpha$ iff $S(\mathcal{F}_C) \models C_\alpha$.

The crucial property of a trivial frame condition is that it contains type 2 variables which are used to quantify over propositions but which do not occur as a relatum place of the three-place accessibility relation. Such trivial frame conditions are no longer structural – they make truth-assertions about propositons (in certain worlds) which are not used to index the accessibility relation. The following definitions intend to make this intuition precise. In order to avoid "smuggling in" superfluous type 2 variables by logical tautologies, we first need to say when a type 2 variable occurs essentially in an $\mathcal{L}_{\text{FC}}$ formula. Throughout the following, $\phi(\mathsf{X}_{i_1}, \ldots, \mathsf{X}_{i_n})$ denotes a formula ϕ of $\mathcal{L}_{\text{FC}}$ which contains as variables of sort 2 exactly $X_{i_1}, \ldots, X_{i_n}$ (ordered from left to right according to their first occurrence).

Definition 5.3. (Frame Condition)
1. A frame condition is a formula of $\mathcal{L}_{\text{FC}}$ which has the form $\underline{\forall}\, \mathsf{X}_{i_1} \ldots \underline{\forall}\, \mathsf{X}_{i_n} \alpha(\mathsf{X}_{i_1}, \ldots, \mathsf{X}_{i_n})$ (for some $n \in \mathbb{N}$), where $\alpha(\mathsf{X}_{i_1}, \ldots, \mathsf{X}_{i_n})$ is a formula that contains no quantifier over a variable of sort 2. We abbreviate such a frame condition by $C(\mathsf{X}_{i_1}, \ldots, \mathsf{X}_{i_n})$.
2. Variable X_{i_j} ($1 \leq j \leq n$) occurs essentially in frame condition $C_1(\mathsf{X}_{i_1}, \ldots, \mathsf{X}_{i_n}) \in \mathcal{L}_{\text{FC}}$ iff there exists no $\mathbf{L}_{\text{FC}}$-logically equivalent frame condition $C_2(\mathsf{X}_{i_1}, \ldots, \mathsf{X}_{i_{j-1}}, \mathsf{X}_{i_{j+1}}, \ldots, \mathsf{X}_{i_n})$.
3. A formula $\alpha(\mathsf{X}_{i_1}, \ldots, \mathsf{X}_{i_n}) \in \mathcal{L}_{\text{FC}}$ is irreducibly formulated iff every variable X_{i_j} of sort 2 ($1 \leq j \leq n$) occurs essentially in α.
4. A frame condition $C \in \mathcal{L}_{\text{FC}}$ is non-trivial iff (a) C is irreducibly formulated and (b) in C no variable X_i occurs unless X_i occurs also in the third argument place of some occurrence of the relation symbol R in C.

Observe that Def. 5.3(4) excludes non-trivial frame conditions which contain terms and subformulas, such as $(\mathsf{X} \underline{\cap} (\mathsf{Y} \underline{\cup} \underline{-} \mathsf{Y}))$ and $(\mathsf{X} \underline{\cup} (\mathsf{Y} \underline{\cap} \underline{-} \mathsf{Y}))$ on the one hand and $(\mathsf{wR_Yw_1} \underline{\vee} \underline{\neg} \mathsf{wR_Yw_1})$ and $(\mathsf{w} \underline{\in} \mathsf{Y} \underline{\vee} \underline{\neg} (\mathsf{w} \underline{\in} \mathsf{Y}))$ on the other, respectively. If two terms τ_1 and τ_2 are logically identical in $\mathbf{L}_{\mathrm{FC}}$ then $\mathsf{w_1 R_{\tau_1} w_2}$ and $\mathsf{w_1 R_{\tau_2} w_2}$ on the one hand and '$(\mathsf{w_1} \underline{\in} \tau_1)$' and '$(\mathsf{w_1} \underline{\in} \tau_2)$' on the other are logically equivalent in $\mathbf{L}_{\mathrm{FC}}$.

In Kripke semantics there exists a standard procedure for translating modal formula schemata to corresponding frame conditions formulated in a language that quantifies over sets of possible worlds (e.g., van Benthem [25, pp. 326–329]). We define an analogous translation procedure for CS semantics in Sect. 6. To motivate our Def. 5.3(4) let us for the present moment discuss the following two frame conditions which both C-correspond to axiom schema T (i.e., $(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow \beta$):

C_{T} $\quad \underline{\forall}\mathsf{X}\,\underline{\forall}\mathsf{w}(\mathsf{wR_Xw})$

C_{triv} $\quad \underline{\forall}\mathsf{X}\,\underline{\forall}\mathsf{Y}\,\underline{\forall}\mathsf{w}(\underline{\forall}\mathsf{w_1}(\mathsf{wR_Xw_1} \underline{\rightarrow} \mathsf{w_1} \underline{\in} \mathsf{Y}) \underline{\rightarrow} \mathsf{w} \underline{\in} \mathsf{Y})$

Frame condition C_{triv} results from T by a standard translation for CS semantics, while C_{T} does not. This difference is reflected in Def. 5.3(4) insofar as C_{triv} is trivial according to Def. 5.3(4) whereas C_{T} is not. A non-standard Segerberg frame $\mathcal{F}_S = \langle W, R, P\rangle$ that satisfies C_{triv} can be specified as follows: Define $W = \{w_1, w_2\}$, $R = \{\langle w_1, w_2, \varnothing\rangle, \langle w_2, w_1, \varnothing\rangle, \langle w_1, w_2, W\rangle, \langle w_2, w_1, W\rangle\}$ and $P = \{\varnothing, W\}$. T is valid on $\mathcal{F}_S$ – that is T is valid in any Chellas model based on $\mathcal{F}_S$, whereas C_{triv} and not C_{T} holds for any such model.

Based on all this, we can now define the notion of standard Segerberg frames:

Definition 5.4. (Standard Segerberg Frame)
A standard Segerberg frame $\mathcal{F}_S^{\mathrm{st}} = \langle W, R, P\rangle$ for a logic $\mathbf{L} = \mathbf{CK} + \alpha_1 + \alpha_2 + \ldots$ is a Segerberg frame that satisfies a non-trivial frame condition C_i that C-corresponds to α_i for each axiom schema α_i for $\mathcal{L}_{\mathrm{CL}}$ $(i = 1, 2, \ldots)$.

The notion of standard Segerberg frames excludes non-standard Segerberg frames as described above.

6. A Translation Procedure for Arbitrary Formula Schemata

In this section we specify a translation procedure which produces C-corresponding frame conditions based on arbitrary formulas and formula schemata for the language $\mathcal{L}_{\mathrm{CL}}$. We will reduce the translation of formula schemata to the translation of formulas as follows. For each formula schema $\alpha(A_1, \ldots, A_n)$ we define the corresponding *skeleton* as $\alpha(p_1, \ldots, p_n)$, i.e. the formula of $\mathcal{L}_{\mathrm{CL}}$ resulting from $\alpha(A_1, \ldots, A_n)$ by a uniform substitution of schematic letters A_i by propositional variables p_i $(1 \leq i \leq n)$. We say a formula schema is true at a world w of a model

$\mathcal{M}$ iff the corresponding skeleton is true at w in $\mathcal{M}$. We now specify three translation functions T, t_{w_j} and t for formulas of $\mathcal{L}_{\mathrm{CL}}$ where w_j is a variable of sort 1 of the language $\mathcal{L}_{\mathrm{FC}}$. We define the translation for formula schemata $\alpha(A_1, \ldots, A_n)$ by the translation of the corresponding skeletons, i.e. $T(\alpha(A_1, \ldots, A_n)) = T(\alpha(p_1, \ldots, p_n))$ (and likewise for t_{w_j} and t).

The first translation function T translates a formula $\alpha \in \mathcal{L}_{\mathrm{CL}}$ into a term $T(\alpha)$ of type 2 of $\mathcal{L}_{\mathrm{FC}}$, where in the corresponding first-order model the term $T(\alpha)$ denotes exactly the proposition $||\alpha||$, i.e., the set of worlds in this model as defined in Lemma 6.6 at which $\alpha \in \mathcal{L}_{\mathrm{CL}}$ is true. The second function t_{w_j} translates a formula $\alpha \in \mathcal{L}_{\mathrm{CL}}$ which is true at the world w_j of a Chellas model $\mathcal{M}$, where w_j is denoted by w_j (i.e., $V^*(\mathsf{w}_j) = w_j$) into a corresponding first-order formula $t_{\mathsf{w}_j}(\alpha) \in \mathcal{L}_{\mathrm{FC}}$ which is true *of* the world (individual) w_j (denoted by w_j) in the corresponding first-order model $\langle W, V^* \rangle$. The third function t, finally, forms the universal closure of $t_{\mathsf{w}_j}(\alpha)$; it is used to express the correspondence between the validity of a formula of $\mathcal{L}_{\mathrm{CL}}$ on a Chellas frame $\mathcal{F}$ and the truth of formula $t(\alpha) \in \mathcal{L}_{\mathrm{FC}}$ in the corresponding first-order structure for $\mathcal{L}_{\mathrm{FC}}$.

Definition 6.1. Let $\alpha \in \mathcal{L}_{\mathrm{CL}}$ and let $\mathit{Terms}^2_{\mathcal{L}_{\mathrm{FC}}}$ be the set of type 2 terms of $\mathcal{L}_{\mathrm{FC}}$. Then, the translation function T: $\mathcal{L}_{\mathrm{CL}} \rightarrow \mathit{Terms}^2_{\mathcal{L}_{\mathrm{FC}}}$ is defined the following way:

(a) if $\alpha = p_i$ then $T(\alpha) = \mathsf{X}_i$ (for every atomic propositional variable p_i of $\mathcal{L}_{\mathrm{CL}}$ and type 2 variable X_i of $\mathcal{L}_{\mathrm{FC}}$),
(b) if $\alpha = \top$ then $T(\alpha) = \mathsf{W}$, and if $\alpha = \bot$ then $T(\alpha) = \underline{\emptyset}$,
(c) if $\alpha = \neg\beta$ then $T(\alpha) = \underline{-}\,T(\beta)$,
(d) if $\alpha = (\beta \vee \gamma)$ then $T(\alpha) = (T(\beta) \,\underline{\cup}\, T(\gamma))$, and
(e) if $\alpha = (\beta \Box\!\!\rightarrow \gamma)$ then $T(\alpha) = (T(\beta) \,\underline{\Box\!\!\rightarrow}\, T(\gamma))$.

Note that by definition of T, $T(\alpha)$ contains exactly one variable of sort 2, X_i, for every propositional variable p_i occurring in α (or the corresponding schematic letter A_i if we translate the formula schema).

Building on this we now define our translation function t_{w_j} as follows.

Definition 6.2. Let $\alpha \in \mathcal{L}_{\mathrm{CL}}$. Then, for any w_j the translation function t_{w_j}: $\mathcal{L}_{\mathrm{CL}} \rightarrow \mathcal{L}_{\mathrm{FC}}$ is defined as follows:

(I) If α does not contain '$\Box\!\!\rightarrow$' then $t_{\mathsf{w}_j}(\alpha) = (\mathsf{w}_j \,\underline{\in}\, T(\alpha))$, and otherwise
(II.a) if $\alpha = \neg\beta$ then $t_{\mathsf{w}_j}(\alpha) = \underline{\neg}\, t_{\mathsf{w}_j}(\beta)$,
(II.b) if $\alpha = (\beta \vee \gamma)$ then $t_{\mathsf{w}_j}(\alpha) = (t_{\mathsf{w}_j}(\beta) \,\underline{\vee}\, t_{\mathsf{w}_j}(\gamma))$, and
(II.c) if $\alpha = (\beta \Box\!\!\rightarrow \gamma)$ then $t_{\mathsf{w}_j}(\alpha) = \underline{\forall}\, \mathsf{w}_k(\mathsf{w}_j \mathsf{R}_{T(\beta)} \mathsf{w}_k \,\underline{\rightarrow}\, t_{\mathsf{w}_k}(\gamma))$, where w_k is a new variable of type 1.

Note that by definition of t_{w_j}, $t_{\mathsf{w}_j}(\alpha)$ contains exactly one free variable of type 1, namely w_j, and moreover, for every propositional variable p_i occurring in α (or the corresponding schematic letter A_i if we translate the formula schema) exactly one variable of sort 2, i.e. X_i. Since an axiom schema holds on a frame exactly iff every instance of it holds on that frame, we define the translation function t as the universal closure of the result of translation function t_{w_j} over $t_{\mathsf{w}_j}(\alpha)$'s type 2 variables and the type 1 variable w_j as follows:

Definition 6.3. Let $\alpha(p_{i_1}, \ldots, p_{i_n}) \in \mathcal{L}_{\mathrm{CL}}$ and t: $\mathcal{L}_{\mathrm{CL}} \to \mathcal{L}_{\mathrm{FC}}$. Then, $t(\alpha) = \underline{\forall}\,\mathsf{X}_{i_1} \ldots \underline{\forall}\,\mathsf{X}_{i_n}\,\underline{\forall}\,\mathsf{w}_j(t_{\mathsf{w}_j}(\alpha(p_{i_1}, \ldots, p_{i_n})))$.

The next theorem establishes that t determines corresponding frame conditions for formula schemata for $\mathcal{L}_{\mathrm{CL}}$.

Theorem 6.4. *Let $\mathcal{F}_C$ be an arbitrary Chellas frame and let $\alpha \in \mathcal{L}_{\mathrm{CL}}$. Then, $\forall\alpha\,\forall\mathcal{F}_C(\mathcal{F}_C \models \alpha$ iff $S(\mathcal{F}_C) \models t(\alpha))$.*

Proof. Let $\mathcal{F}_C = \langle W, R\rangle$ be a Chellas frame and let $\alpha \in \mathcal{L}_{\mathrm{CL}}$.

"$\Leftarrow$": Suppose that $\langle W, R\rangle \not\models \alpha$. Then, for some $w \in W$ and some valuation function V, s.t. $\langle W, R, V\rangle$ is a Chellas model, it holds that $\langle\langle W, R, V\rangle, w\rangle \not\models \alpha$. By Lemma 6.5, there exists a first-order model $\langle W, V^*\rangle$ based on the first-order structure $S(\mathcal{F}_C)$ which falsifies $t_{\mathsf{w}_j}(\alpha)$. So by Def. 6.3, $S(\mathcal{F}_C) \not\models t(\alpha)$.

"$\Rightarrow$": Suppose that $S(\langle W, R\rangle) \not\models t(\alpha)$, where $\alpha = \alpha(p_{i_1}, \ldots, p_{i_n})$. Then by Def. 6.3, $t(\alpha)$ has the form $\underline{\forall}\,\mathsf{X}_{i_1} \ldots \underline{\forall}\,\mathsf{X}_{i_n}\,\underline{\forall}\,\mathsf{w}_j t_{\mathsf{w}_j}(\alpha)$. Hence $S(\langle W, R\rangle) \not\models \forall X_{i_1} \ldots \forall X_{i_n} \forall \mathsf{w}_j t_{\mathsf{w}_j}(\alpha)$. So, by Lemma 6.5 there exists a model $\langle W, R, V\rangle$ and a world $w \in W$ such that $\langle\langle W, R, V\rangle, w\rangle \not\models \alpha$. Hence, $\langle W, R\rangle \not\models \alpha$. ■

Lemma 6.5. *Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a Chellas model based on frame $\langle W, R\rangle$. Then, for every formula of $\mathcal{L}_{\mathrm{CL}}$, $\phi = \phi(p_{i_1}, \ldots, p_{i_n})$ and for every first-order model $\langle W, V^*\rangle$ based on the corresponding first-order structure $S(\langle W, R\rangle)$ which satisfies*

$$V^*(\mathsf{X}_{i_j}) = V(p_{i_j}) =_{\mathrm{df}} \|p_{i_j}\|^{\mathcal{M}_C} \text{ for every } 1 \leq j \leq n$$

it holds for every type 1 variable w_k that $\langle\langle W, R, V\rangle,\ V^*(\mathsf{w}_k)\rangle \models \phi$ iff $\langle W, V^*\rangle \models t_{\mathsf{w}_k}(\phi)$.

Proof. Assume $V^*(\mathsf{w}_k) =_{\mathrm{df}} w$. The proof is by induction on the construction of formulas.

(a) If $\phi = p_{i_j}$ then $V(\phi, w) = 1$ iff $V^*((\mathsf{w}_k \,\underline{\in}\, \mathsf{X}_{i_j})) = 1$ by definition of the first-order structure and assumptions on V^*, where $(\mathsf{w}_k \,\underline{\in}\, \mathsf{X}_{i_j}) = (\mathsf{w}_k \,\underline{\in}\, T(p_{i_j})) = t_{\mathsf{w}_k}(\phi)$ by Def. 6.2(I) and Def. 6.1 ($1 \leq j \leq n$).

(b) If $\phi = \neg\beta$ then $V(\phi, w) = 1$ iff $V(\beta, w) = 0$ iff (by the induction hypothesis) $V^*(t_{\mathsf{w}_k}(\beta)) = 0$ iff $V^*(\underline{\neg}\, t_{\mathsf{w}_k}(\beta)) = 1$ iff (by Def. 6.2(II.a)) $V^*(t_{\mathsf{w}_k}(\phi)) = 1$.

(c) If $\phi = (\beta \vee \gamma)$ then $V(\phi, w) = 1$ iff $V(\beta, w) = 1$ or $V(\gamma, w) = 1$ iff (by the induction hypothesis) $V^*(t_{\mathsf{w}_k}(\beta)) = 1$ or $V^*(t_{\mathsf{w}_k}(\gamma)) = 1$ iff $V^*((t_{\mathsf{w}_k}(\beta) \,\underline{\vee}\, t_{\mathsf{w}_k}(\gamma))) = 1$

iff (by Def. 6.2(II.b)) $V^*(t_{\mathsf{w}_k}(\phi)) = 1$.
(d) If $\phi = (\beta \mathrel{\Box\!\!\to} \gamma)$ then $V(\phi, w) = 1$ iff $\forall w'(wR_{||\beta||}w' \rightarrow V(\gamma, w') = 1)$ iff (by Lemma 6.6) $\forall w' \in W(wR_{V^*(T(\beta))}w' \rightarrow V(\gamma, w') = 1)$ iff (by the induction hypothesis) $\forall w' \in W$: $\langle W, V^*[\mathsf{w}_r{:}w']\rangle \models \mathsf{w}_k\mathsf{R}_{T(\beta)}\mathsf{w}_r \rightarrow \langle W, V^*[\mathsf{w}_r{:}w']\rangle \models t_{\mathsf{w}_r}(\gamma))$ where w_r is a new type 1 variable iff (by definition of $\langle W, V^*\rangle$) $\langle W, V^*\rangle \models \underline{\forall}\, \mathsf{w}_r(\mathsf{w}_k\mathsf{R}_{T(\beta)}\mathsf{w}_r \underline{\rightarrow}\, t_{\mathsf{w}_r}(\gamma))$ iff (by Def. 6.2(II.c)) $\langle W, V^*\rangle \models t_{\mathsf{w}_k}(\phi)$. ■

Observe that in point (a) of the proof of Lemma 6.5 the clause I of Def. 6.2 is applicable, since t_{w_k} refers to a formula $p_{i_j} \in \mathcal{L}_{\mathrm{CL}}$ in which $\Box\!\!\to$ cannot occur.

Lemma 6.6. *Let $\mathcal{M}_C = \langle W, R, V\rangle$ be a Chellas model based on the frame $\langle W, R\rangle$. Then, for every formula of $\mathcal{L}_{\mathrm{CL}}$, $\phi = \phi(p_{i_1}, \ldots, p_{i_n})$ and for every first-order model $\langle W, V^*\rangle$ based on the corresponding first-order structure $S(\langle W, R\rangle)$ which satisfies*

$$V^*(\mathsf{X}_{i_j}) = V(p_{i_j}) =_{\mathrm{df}} ||p_{i_j}||^{\mathcal{M}_C} \text{ for every } 1 \leq j \leq n$$

it holds that $V(\phi) =_{\mathrm{df}} ||\phi||^{\mathcal{M}_C} = V^*(T(\phi))$.

Proof. The proof goes by induction on the construction of formulas.
(a) If $\phi = p_{i_j}$ $(1 \leq j \leq n)$ then $V^*(T(\phi)) = V^*(X_{i_j}) = ||p_{i_j}||^{\mathcal{M}_C}$ by assumption and Def. 6.1(a).
(b) If $\phi = \top$ then $T(\phi) = \mathsf{W}$, so $V^*(T(\phi)) = V^*(\mathsf{W}) = ||\top||^{\mathcal{M}_C}$, and if $\phi = \bot$ then $T(\phi) = \underline{\varnothing}$, so $V^*(T(\phi)) = V^*(\underline{\varnothing}) = ||\bot||^{\mathcal{M}_C}$ (both by Def. 6.1(b)).
(c) If $\phi = \neg\gamma$ then $V^*(T(\phi)) = V^*(\underline{-}\, T(\gamma)) = -V^*(T(\gamma)) = -||\gamma||^{\mathcal{M}_C} = ||\neg\gamma||^{\mathcal{M}_C}$ (by Def. 6.1(c) and the induction hypothesis).
(d) If $\phi = (\gamma \vee \delta)$ then $V^*(T(\phi)) = V^*((T(\gamma) \mathbin{\underline{\cup}} T(\delta))) = V^*(T(\gamma)) \cup V^*(T(\delta)) = ||\gamma||^{\mathcal{M}_C} \cup ||\delta||^{\mathcal{M}_C} = ||(\gamma \vee \delta)||^{\mathcal{M}_C}$ (by Def. 6.1(d) and the induction hypothesis).
(e) If $\phi = (\gamma \mathrel{\Box\!\!\to} \delta)$ then $V^*(T(\phi)) = V^*((T(\gamma) \mathrel{\underline{\Box\!\!\to}} T(\delta))) = V^*(T(\gamma)) \mathrel{\Box\!\!\to} V^*(T(\delta)) = ||\gamma||^{\mathcal{M}_C} \mathrel{\Box\!\!\to} ||\delta||^{\mathcal{M}_C} = ||(\gamma \mathrel{\Box\!\!\to} \delta)||^{\mathcal{M}_C}$ (by Def. 6.1(e) and the induction hypothesis). ■

Most though not all corresponding frame conditions obtained by the translation schema are trivial. This shows that finding a non-trivial frame condition that corresponds to an axiom schema of a conditional logic $\mathbf{L} \in \mathbf{\Pi_{CK}}$ is indeed a non-trivial enterprise, and even more, proving frame completeness with respect to standard frames, i.e., frame classes that are definable by non-trivial frame conditions. In the next section we present a couple of correspondence and completeness results of this sort.

7. Non-Trivial Correspondence and Completeness

In this section we focus on the lattice of conditional logic systems defined by System **CK** plus 30 principles discussed in the conditional logic literature (see Table 1)

Table 1. Axiom Schemata for Conditional Logics

System P	
$\alpha \mathrel{\Box\!\!\rightarrow} \alpha$	(Refl)
$(\alpha \mathrel{\Box\!\!\rightarrow} \gamma) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma)$	(CM)
$(\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \gamma)$	(CC)
$(\alpha_0 \mathrel{\Box\!\!\rightarrow} \alpha_1) \wedge \ldots \wedge (\alpha_{k-1} \mathrel{\Box\!\!\rightarrow} \alpha_k) \wedge (\alpha_k \mathrel{\Box\!\!\rightarrow} \alpha_0) \rightarrow (\alpha_0 \mathrel{\Box\!\!\rightarrow} \alpha_k)$ $(k \geq 2)$	(Loop)
$(\alpha \mathrel{\Box\!\!\rightarrow} \gamma) \wedge (\beta \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\alpha \vee \beta \mathrel{\Box\!\!\rightarrow} \gamma)$	(Or)
$(\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma))$	(S)
$(\neg\alpha \mathrel{\Box\!\!\rightarrow} \alpha) \rightarrow (\beta \mathrel{\Box\!\!\rightarrow} \alpha)$	(MOD)
Extensions of System P	
$(\alpha \mathrel{\Box\!\!\rightarrow} \gamma) \wedge (\alpha \mathrel{\Diamond\!\!\rightarrow} \beta) \rightarrow (\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma)$	(RM)
$(\alpha \mathrel{\Box\!\!\rightarrow} (\beta \rightarrow \gamma)) \wedge (\alpha \mathrel{\Diamond\!\!\rightarrow} \beta) \rightarrow (\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma)$	(RM′)
$(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \vee (\alpha \mathrel{\Box\!\!\rightarrow} \neg\beta)$	(CEM)
Weak Probability Logics (Threshold Logics)	
$\neg(\top \mathrel{\Box\!\!\rightarrow} \bot)$	(P-Cons)
$(\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma) \wedge (\alpha \wedge \neg\beta \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \gamma)$	(WOR)
Monotonic Systems	
$(\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \delta) \wedge (\gamma \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\alpha \wedge \gamma \mathrel{\Box\!\!\rightarrow} \delta)$	(Cut)
$(\alpha \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma)$	(Mon)
$(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \wedge (\beta \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \gamma)$	(Trans)
$(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\neg\beta \mathrel{\Box\!\!\rightarrow} \neg\alpha)$	(CP)
Bridge Principles	
$(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\alpha \rightarrow \beta)$	(MP)
$\alpha \wedge \beta \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \beta)$	(CS)
$(\neg\alpha \mathrel{\Box\!\!\rightarrow} \alpha) \rightarrow \alpha$	(TR)
$(\top \mathrel{\Box\!\!\rightarrow} \alpha) \rightarrow \alpha$	(Det)
$\alpha \rightarrow (\top \mathrel{\Box\!\!\rightarrow} \alpha)$	(Cond)
Collapse Conditions Material Implication	
$\beta \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \beta)$	(VEQ)
$\neg\alpha \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} \beta)$	(EFQ)
Traditional Extensions	
$(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\alpha \mathrel{\Diamond\!\!\rightarrow} \beta)$	(D)
$(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow \beta$	(T)
$\alpha \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} (\alpha \mathrel{\Diamond\!\!\rightarrow} \beta))$	(B)
$(\alpha \mathrel{\Box\!\!\rightarrow} \beta) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} (\alpha \mathrel{\Box\!\!\rightarrow} \beta))$	(4)
$(\alpha \mathrel{\Diamond\!\!\rightarrow} \beta) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} (\alpha \mathrel{\Diamond\!\!\rightarrow} \beta))$	(5)
Iterated Principles	
$(\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma) \rightarrow (\alpha \mathrel{\Box\!\!\rightarrow} (\beta \mathrel{\Box\!\!\rightarrow} \gamma))$	(Ex)
$(\alpha \mathrel{\Box\!\!\rightarrow} (\beta \mathrel{\Box\!\!\rightarrow} \gamma)) \rightarrow (\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma)$	(Im)

For abbreviations see text.

Table 2. Axioms and Corresponding Frame Conditions in CS Semantics

System P	
Refl*	$\underline{\forall} w \underline{\forall} w_1 (wR_X w_1 \underline{\rightarrow} w_1 \underline{\in} X)$
CM	$\underline{\forall} w (\underline{\forall} w_1 (wR_X w_1 \underline{\rightarrow} w_1 \underline{\in} Y) \underline{\rightarrow} \underline{\forall} w_1 (wR_{X \underline{\cap} Y} w_1 \underline{\rightarrow} wR_X w_1))$
CC	$\underline{\forall} w (\underline{\forall} w_1 (wR_X w_1 \underline{\rightarrow} w_1 \underline{\in} Y) \underline{\rightarrow} \underline{\forall} w_1 (wR_X w_1 \underline{\rightarrow} wR_{X \underline{\cap} Y} w_1))$
Loop*	$\underline{\forall} w (\underline{\forall} w_1 (wR_{X_0} w_1 \underline{\rightarrow} w_1 \underline{\in} X_1) \underline{\land} \ldots \underline{\land} \underline{\forall} w_1 (wR_{X_{k-1}} w_1 \underline{\rightarrow} w_1 \underline{\in} X_k) \underline{\land}$
	$\underline{\forall} w_1 (wR_{X_k} w_1 \underline{\rightarrow} w_1 \underline{\in} X_0) \underline{\rightarrow} \underline{\forall} w_1 (wR_{X_0} w_1 \underline{\rightarrow} w_1 \underline{\in} X_k))$ $(k \geq 2)$
Or	$\underline{\forall} w \underline{\forall} w_1 (wR_{X \underline{\cup} Y} w_1 \underline{\rightarrow} wR_X w_1 \underline{\lor} wR_Y w_1)$
S	$\underline{\forall} w \underline{\forall} w_1 (wR_X w_1 \underline{\land} w_1 \underline{\in} Y \underline{\rightarrow} wR_{X \underline{\cap} Y} w_1)$
MOD*	$\underline{\forall} w (\underline{\forall} w_1 (wR_{\underline{-} X} w_1 \underline{\rightarrow} w_1 \underline{\in} X) \underline{\rightarrow} \underline{\forall} w_1 (wR_Y w_1 \underline{\rightarrow} w_1 \underline{\in} X))$
Extensions of System P	
RM	$\underline{\forall} w (\underline{\exists} w_1 (wR_X w_1 \underline{\land} w_1 \underline{\in} Y) \underline{\rightarrow} \underline{\forall} w_1 (wR_{X \underline{\cap} Y} w_1 \underline{\rightarrow} wR_X w_1))$
RM′	$\underline{\exists} w_1 (wR_X w_1 \underline{\land} w_1 \underline{\in} Y) \underline{\rightarrow} \underline{\forall} w_1 (wR_{X \underline{\cap} Y} w_1 \underline{\rightarrow} wR_X w_1 \underline{\land} w_1 \underline{\in} Y)$
CEM	$\underline{\forall} w \underline{\forall} w_1 \underline{\forall} w_2 (wR_X w_1 \underline{\land} wR_X w_2 \underline{\rightarrow} w_2 \underline{=} w_1)$
Weak Probability Logics (Threshold Logics)	
P-Cons	$\underline{\forall} w \underline{\exists} w_1 (wR_W w_1)$
WOR	$\underline{\forall} w \underline{\forall} w_1 (wR_X w_1 \underline{\rightarrow} wR_{X \underline{\cap} Y} w_1 \underline{\lor} wR_{X \underline{\cap} \underline{-} Y} w_1)$
Monotonic Systems	
Cut	$\underline{\forall} w (\underline{\forall} w_1 (wR_Z w_1 \underline{\rightarrow} w_1 \underline{\in} Y) \underline{\rightarrow} \underline{\forall} w_1 (wR_{X \underline{\cap} Z} w_1 \underline{\rightarrow} wR_{X \underline{\cap} Y} w_1))$
Mon	$\underline{\forall} w \underline{\forall} w_1 (wR_{X \underline{\cap} Y} w_1 \underline{\rightarrow} wR_X w_1)$
Trans	$\underline{\forall} w (\underline{\forall} w_1 (wR_X w_1 \underline{\rightarrow} w_1 \underline{\in} Y) \underline{\rightarrow} \underline{\forall} w_1 (wR_X w_1 \underline{\rightarrow} wR_Y w_1))$
CP*	$\underline{\forall} w (\underline{\forall} w_1 (wR_X w_1 \underline{\rightarrow} w_1 \underline{\in} Y) \underline{\rightarrow} \underline{\forall} w_1 (wR_{\underline{-} Y} w_1 \underline{\rightarrow} w_1 \underline{\in} \underline{-} X))$
Bridge Principles	
MP	$\underline{\forall} w (w \underline{\in} X \underline{\rightarrow} wR_X w)$
CS	$\underline{\forall} w (w \underline{\in} X \underline{\rightarrow} \underline{\forall} w_1 (wR_X w_1 \underline{\rightarrow} w_1 \underline{=} w))$
TR*	$\underline{\forall} w (\underline{\forall} w_1 (wR_{\underline{-} X} w_1 \underline{\rightarrow} w_1 \underline{\in} X) \underline{\rightarrow} w \underline{\in} X)$
Det	$\underline{\forall} w (wR_W w)$
Cond	$\underline{\forall} w \underline{\forall} w_1 (wR_W w_1 \underline{\rightarrow} w_1 \underline{=} w)$
Collapse Conditions Material Implication	
VEQ	$\underline{\forall} w \underline{\forall} w_1 (wR_X w_1 \underline{\rightarrow} w_1 \underline{=} w)$
EFQ	$\underline{\forall} w (w \underline{\in} \underline{-} X \underline{\rightarrow} \underline{\neg} \underline{\exists} w_1 (wR_X w_1))$
Traditional Extensions	
D	$\underline{\forall} w \underline{\exists} w_1 (wR_X w_1)$
T	$\underline{\forall} w (wR_X w)$
B	$\underline{\forall} w \underline{\forall} w_1 (wR_X w_1 \underline{\rightarrow} w_1 R_X w)$
4	$\underline{\forall} w \underline{\forall} w_1 \underline{\forall} w_2 (wR_X w_1 \underline{\land} w_1 R_X w_2 \underline{\rightarrow} wR_X w_2)$
5	$\underline{\forall} w \underline{\forall} w_1 \underline{\forall} w_2 (wR_X w_1 \underline{\land} wR_X w_2 \underline{\rightarrow} w_1 R_X w_2)$
Iteration Principles	
Ex	$\underline{\forall} w \underline{\forall} w_1 \underline{\forall} w_2 (wR_X w_1 \underline{\land} w_1 R_Y w_2 \underline{\rightarrow} wR_{X \underline{\cap} Y} w_2)$
Im	$\underline{\forall} w \underline{\forall} w_1 (wR_{X \underline{\cap} Y} w_1 \underline{\rightarrow} \underline{\exists} w_2 (wR_X w_2 \underline{\land} w_2 R_Y w_1))$

For better readability outer universal quantifiers over variables $X, X_0 \ldots, Y, Y_0 \ldots$ have been omitted.

and their C-corresponding non-trivial frame conditions in the sense of Def. 5.3(4) (see Table 2). The frame conditions for Refl, MP, and Or on the one hand and CM, RM, RM′, S, Det and Cond on the other are nothing but the frame conditions identified already by [5, p. 142f] and [21, p. 163], respectively.[3] It is important to note that the frame conditions for Refl, Loop, MOD, CP and TR are actually given by the general translation procedure specified in Def. 6.3. (We marked these principles with an asterix.) Def. 6.3, however, does not always give one non-trivial frame conditions in the sense of Def. 5.3(4), as we saw in the previous section.
The conditional logic axioms described in Table 1 are the following: Refl ("Reflexivity"), CM ("Cautious Monotonicity"), CC ("Cautious Cut"), Loop ("Loop of Antecedent and Consequent Formulas"), Or, S, MOD ("Modality"), RM ("Rational Monotonicity"), RM′ (variant of RM), CEM ("Conditional Excluded Middle"), P-Cons ("Probabilistic Consistency"), WOR ("Weak Or"), Cut, Mon ("Monotonicity"), Trans ("Transitivity"), CP ("Contraposition"), MP ("Modus Ponens"), CS ("Conjunctive Sufficiency"), TR ("Total Reflexivity"), Det ("Detachment"), Cond ("Conditionalization"), VEQ ("Verum Ex Quodlibet"), EFQ ("Ex Falso Quodlibet"), D, T, B, 4, 5, Ex ("Exportation") and Im ("Importation"). The names of the principles follow Kraus et al. [10] and Lehmann and Magidor [12] rather than, for example, [15]. The principles are, for instance, discussed in Adams [1, 2], Arló-Costa [3], [10], [12], [13], [15], McGee [14], Schurz [18, 19] and Stalnaker [22]. Note that Principles D, T, B, 4 and 5 correspond to frame conditions which are generalizations from serial, reflexive, symmetrical, transitive and Euclidian frame conditions in Kripke semantics, respectively.

We can prove the following results:

THEOREM 7.1. *Every formula schema α in Table 1 C-corresponds to the respective non-trivial frame condition C_α in Table 2.*

The proof of C-correspondence for Principles Refl, Loop, MOD, CP and TR is immediately given by Theorem 6.4. In Appendix A we prove C-correspondence for Principles CM and Im. We chose both principles, since their correspondence proofs were among the more interesting ones (see also below). For the remaining proofs of C-correspondence (except for Principle RM′) we refer to [24, Ch. 5].

THEOREM 7.2. *All logics* $\mathbf{L} = \mathbf{CK} + \alpha_1 + \alpha_2 + \ldots$ *where* $\alpha_1, \alpha_2, \ldots$ *are axiom schemata from Table 1 are strongly complete w.r.t. the class of standard Segerberg frames for which* $C_{\alpha_1}, C_{\alpha_2}, \ldots$ *in Table 2 hold.*

In Appendix B we prove Theorem 7.2 for the sublattice of systems described by System **CK** and Principles CM and Im. We picked both principles, since their

[3] [21, p. 163] also describes a frame condition for a variant of (MOD), his axioms #2.

canonicity proofs were typical and among the more interesting ones, respectively. For proofs of the remaining principles (except Principle RM′) we refer to [24, Ch. 6].

8. Final Remark

By the axioms in Table 1 we can axiomatize a range of well-known conditional logic systems, such as Lewis' [13] Systems **V** and **VC** (see also Leahy et al. [11]), Stalnaker's [22] System, and also Kraus et al.'s [10] Systems **C**, **CL**, and **P**, Kraus und Lehmann's [12] System **R** (see also Hawthorne [8] and Pfeifer and Kleiter [16]) and Adams' [1, 2] **P**-Systems formulated in the language $\mathcal{L}_{\mathrm{CL}}$ (see [24, Ch. 7]).[4]

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9. Appendix A

In this appendix we prove C-correspondence for Principles CM and Im.

Axiom Schema CM. ("$\Leftarrow$") Let $\langle W, R\rangle$ be a Chellas frame s.t. C_{CM} holds. So, for all $w \in W$ and $X, Y \subseteq W$ it is the case that (i) $\forall w'(wR_Xw' \rightarrow w' \in Y) \rightarrow \forall w'(wR_{X\cap Y}w' \rightarrow wR_Xw')$. Let $\langle W, R, V\rangle$ be a Chellas model such that $\langle\langle W, R, V\rangle, w\rangle \models (\alpha \mathrel{\Box\!\!\rightarrow} \gamma) \wedge (\alpha \mathrel{\Box\!\!\rightarrow} \beta)$. Then, (ii) $\forall w'(wR_{||\alpha||}w' \rightarrow w' \in ||\gamma||)$ and (iii) $\forall w'(wR_{||\alpha||}w' \rightarrow w' \in ||\beta||)$ and from (i) we can infer $\forall w'(wR_{||\alpha||}w' \rightarrow w' \in ||\beta||) \rightarrow \forall w'(wR_{||\alpha||\cap||\beta||}w' \rightarrow wR_{||\alpha||}w')$. The latter and (iii) imply $\forall w'(wR_{||\alpha||\cap||\beta||}w' \rightarrow wR_{||\alpha||}w')$ and, hence, $\forall w'(wR_{||\alpha\wedge\beta||}w' \rightarrow wR_{||\alpha||}w')$. Due to (ii) we get $\forall w'(w R_{||\alpha\wedge\beta||}w' \rightarrow w' \in ||\gamma||)$ and $\langle\langle W, R, V\rangle, w\rangle \models \alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$.

("$\Rightarrow$") Let $\langle W, R\rangle$ be a Chellas frame s.t. C_{CM} does not hold. So, there are $w, w' \in W$ and $X, Y \subseteq W$ such that $\forall w''(wR_Xw'' \rightarrow w'' \in Y)$, $wR_{X\cap Y}w'$ and not wR_Xw'. Let $\langle W, R, V\rangle$ be a Chellas model s.t. $X = ||\alpha||$, $Y = ||\beta||$, (i) $\forall w''(wR_{||\alpha||}w'' \rightarrow w'' \in ||\gamma||)$ and $w' \notin ||\gamma||$. It follows that (ii) $\forall w''(wR_{||\alpha||}w'' \rightarrow w'' \in ||\beta||)$, $wR_{||\alpha||\cap||\beta||}w'$ and not $wR_{||\alpha||}w'$. This assignment is always possible, as by assumption

[4] For a comparison of System **P** with other conditional logic systems in terms of the accuarcy of probabilistic interferences see Schurz and Thorn [20] and Thorn and Schurz [23].

w' is not among the w''s s.t. $wR_{||\alpha||}w''$. By (i) and (ii) we obtain $\langle\langle W, R, V\rangle, w\rangle \models \alpha \mathrel{\Box\!\!\rightarrow} \gamma$ and $\langle\langle W, R, V\rangle, w\rangle \models \alpha \mathrel{\Box\!\!\rightarrow} \beta$. Since $wR_{||\alpha||\cap||\beta||}w'$ and $w' \notin ||\gamma||$, we have $wR_{||\alpha\wedge\beta||}w'$ and, hence, $\langle\langle W, R, V\rangle, w\rangle \not\models \alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$. ■

Axiom Schema Im. ("⇐") Let $\langle W, R\rangle$ be a Chellas frame s.t. C_{Im} holds. So, for all $w \in W$ and $X, Y \subseteq W$ it is the case that (i) $\forall w'(wR_{X\cap Y}w' \rightarrow \exists w''(wR_X w'' \wedge w''R_Y w'))$. Let $\langle W, R, V\rangle$ be a Chellas model s.t. $\langle\langle W, R, V\rangle, w\rangle \models \alpha \mathrel{\Box\!\!\rightarrow} (\beta \mathrel{\Box\!\!\rightarrow} \gamma)$. We have $\forall w'(wR_{||\alpha||}w' \rightarrow \forall w''(w'R_{||\beta||}\ w'' \rightarrow w'' \in ||\gamma||))$ and, hence, (ii) $\forall w'\forall w''(wR_{||\alpha||}w' \wedge w'R_{||\beta||}w'' \rightarrow w'' \in ||\gamma||)$. By (i) we get (iii) $\forall w'(wR_{||\alpha||\cap||\beta||}w' \rightarrow \exists w''(wR_{||\alpha||}w'' \wedge w''R_{||\beta||}w'))$. Let $wR_{||\alpha||\cap||\beta||}w'$ be the case. Then, by (iii) there exists some $w'' \in W$ such that $wR_{||\alpha||}w''$ and $w''R_{||\beta||}w'$. By (ii) we get $w' \in ||\gamma||$. This implies $\forall w'(wR_{||\alpha||\cap||\beta||}w' \rightarrow w' \in ||\beta||)$ and $\forall w'(wR_{||\alpha\wedge\beta||}w' \rightarrow w' \in ||\beta||)$. So, we obtain $\langle\langle W, R, V\rangle, w\rangle \models \alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$.

("⇒") Let $\langle W, R\rangle$ be a Chellas frame s.t. C_{Im} does not hold. So, there exist $w, w' \in W$ and $X, Y \subseteq W$ s.t. $wR_{X\cap Y}w'$ and $\neg\exists w''\ (wR_X w'' \wedge w''R_Y w')$. Let $\langle W, R, V\rangle$ be a Chellas model s.t. $X = ||\alpha||$, $Y = ||\beta||$, (i) $\forall w''(wR_{||\alpha||}w'' \rightarrow \forall w'''(w''R_{||\beta||}w''' \rightarrow w''' \in ||\gamma||))$ and $w' \notin ||\gamma||$. Then, $wR_{||\alpha||\cap||\beta||}w'$, and (ii) $\neg\exists w''(wR_{||\alpha||}w'' \wedge w''R_{||\beta||}w')$. This assignment is always possible, since by (ii) if there is some $w'' \in W$ s.t. $wR_{||\alpha||}\ w''$ then not $w''R_{||\beta||}w'$. By (i) $\langle\langle W, R, V\rangle, w\rangle \models \alpha \mathrel{\Box\!\!\rightarrow} (\beta \mathrel{\Box\!\!\rightarrow} \gamma)$ follows. As $wR_{||\alpha||\cap||\beta||}w'$ and $w' \notin ||\gamma||$, we get $wR_{||\alpha\wedge\beta||}w'$ and $\langle\langle W, R, V\rangle, w\rangle \not\models \alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma$. ■

Appendix B

We prove here strong standard Segerberg frame completeness (see Theorem 7.2) for the lattice of systems described by system **CK** and Principles CM and Im. We use the canonical model technique (see [21, p. 162]) and canonicity proofs for Principles CM and Im – i.e. proofs that show that the frame of the canonical model for the given logic $\mathbf{L} \in \mathbf{\Pi_{CK}}$ is a frame that satisfies the corresponding non-trivial frame condition. For a completeness result, we also require that the frame conditions are non-trivial in the sense of Def. 5.3(4) and C-correspond to the respective axiom schemata (see Def. 5.4) in addition to canonicity.

Definition 9.1. A Segerberg model $\langle W^c, R^c, P^c, V^c\rangle$ is the canonical model for $\mathbf{L} \in \mathbf{\Pi_{CK}}$ iff

(a) W^c is the class of all maximally **L**-consistent formula sets, (Def$_{W^c}$)
(b) $P^c = \{X \subseteq W^c \mid \exists\alpha\, X = |\alpha|\}$, (Def$_{P^c}$)
(c) $\forall\alpha\forall w\forall w' \in W^c\ \forall X \in P^c$ s.t. $X = |\alpha|$: $wR^c_X w'$ iff $\forall\beta(\alpha \mathrel{\Box\!\!\rightarrow} \beta \in w \rightarrow \beta \in w')$, and (Def$_{R^c}$)
(d) $\forall p \in \mathcal{AV}, w \in W^c$: $V^c(p, w) = 1$ iff $p \in w$. (Def$_{V^c}$)

The expression $|\alpha|$ refers to the set of possible worlds which have α as an element in a given canonical model for a logic $\mathbf{L} \in \mathbf{\Pi_{CK}}$.

LEMMA 9.2. *For a canonical model* $\mathcal{M}_C = \langle W^c, R^c, P^c, V^c \rangle$ *for* $\mathbf{L} \in \mathbf{\Pi_{CK}}$ *and for all formulas* $\alpha \in \mathcal{L}_{\text{CL}}$ *and all* $w \in W^c$ *it holds:* $\alpha \in w$ *iff* $\langle\langle W, R, V\rangle, w\rangle\alpha$.

PROOF. The proof is by induction on the construction of formulas. We describe here only the modal case.

"⇒": Suppose $\alpha \;\Box\!\!\rightarrow\; \beta \in w$. By Lemma 9.3 it holds for all $w' \in W^c$ that $wR^c_{|\alpha|}w' \rightarrow \beta \in w'$. By the induction hypothesis we have $|\alpha| = ||\alpha||$ and $\forall w \in W^c$: $\beta \in w$ iff $\langle\langle W^c, R^c, V^c\rangle, w\rangle \models \beta$. Hence, $\langle\langle W^c, R^c, V^c\rangle, w\rangle \models \alpha \;\Box\!\!\rightarrow\; \beta$ is obtained.

"⇐": Suppose $\alpha \;\Box\!\!\rightarrow\; \beta \notin w$. Then, by Lemma 9.3 there exists a world $w' \in W^c$ such that $wR^c_{|\alpha|}w'$ and $\beta \notin w'$. By the induction hypothesis there exists a world $w' \in W^c$ such that $wR^c_{||\alpha||}w'$ and $\langle\langle W^c, R^c, V^c\rangle, w'\rangle \not\models \beta$ and, hence, $\langle\langle W^c, R^c, V^c\rangle, w\rangle \not\models \alpha \;\Box\!\!\rightarrow\; \beta$. ■

LEMMA 9.3. *For a canonical model* $\langle W^c, R^c, P^c, V^c\rangle$ *for* $\mathbf{L} \in \mathbf{\Pi_{CK}}$ *and all formulas* $\alpha, \beta \in \mathcal{L}_{\text{CL}}$ *and all* $w \in W^c$ *it holds:* $\alpha \;\Box\!\!\rightarrow\; \beta \in w$ *iff* $\forall w' \in W^c(wR^c_{|\alpha|}w' \rightarrow \beta \in w')$.

PROOF. "⇒": Suppose that $\neg\forall w' \in W^c(wR^c_{|\alpha|}w' \rightarrow \beta \in w')$. Then, there exists a $w' \in W^c$ s.t. $wR^c_{|\alpha|}w'$ and $\beta \notin w'$ and by Def_{R^c} we obtain $\alpha \;\Box\!\!\rightarrow\; \beta \notin w$.

"⇐": Suppose that $\alpha \;\Box\!\!\rightarrow\; \beta \notin w$. Since w is maximal $\mathbf{L}$-consistent we have $\neg(\alpha \;\Box\!\!\rightarrow\; \beta) \in w$ and by the Lindenbaum Lemma and Lemma 9.4 there exists a world $w' \in W^c$ s.t. $\{\gamma \,|\, \alpha \;\Box\!\!\rightarrow\; \gamma \in w\} \cup \{\neg\beta\} \subseteq w'$. Thus, $\forall\gamma(\alpha \;\Box\!\!\rightarrow\; \gamma \in w \rightarrow \gamma \in w')$ and by Def_{R^c} we have $wR^c_{|\alpha|}w'$ and $\beta \notin w'$. ■

LEMMA 9.4. *If* Γ *is* $\mathbf{L}$*-consistent for* $\mathbf{L} \in \mathbf{\Pi_{CK}}$ *and* $\neg(\alpha \;\Box\!\!\rightarrow\; \beta) \in \Gamma$ *then* $\{\gamma \,|\, \alpha \;\Box\!\!\rightarrow\; \gamma \in \Gamma\} \cup \{\neg\beta\}$ *is* $\mathbf{L}$*-consistent.*

PROOF. Suppose that $\{\gamma \,|\, \alpha \;\Box\!\!\rightarrow\; \gamma \in \Gamma\} \cup \{\neg\beta\}$ is $\mathbf{L}$-inconsistent. Then, there exists a $\mathbf{L}$-inconsistent set $\{\gamma_1, \ldots, \gamma_n, \neg\beta\}$ for $\gamma_1, \ldots, \gamma_n \in \{\gamma \,|\, \alpha \;\Box\!\!\rightarrow\; \gamma \in \Gamma\}$. It follows that $\vdash_{\mathbf{L}} \neg(\gamma_1 \wedge \ldots \wedge \gamma_n \wedge \neg\beta)$ and, thus, $\vdash_{\mathbf{L}} \gamma_1 \wedge \ldots \wedge \gamma_n \rightarrow \beta$. Since $\mathbf{L} \in \mathbf{\Pi_{CK}}$, rule RCK is derivable in $\mathbf{L}$ (see Sect. 5). $\vdash_{\mathbf{L}} (\alpha \;\Box\!\!\rightarrow\; \gamma_1) \wedge \ldots \wedge (\alpha \;\Box\!\!\rightarrow\; \gamma_n) \rightarrow (\alpha \;\Box\!\!\rightarrow\; \beta)$ follows and, hence, $\vdash_{\mathbf{L}} \neg((\alpha \;\Box\!\!\rightarrow\; \gamma_1) \wedge \ldots \wedge (\alpha \;\Box\!\!\rightarrow\; \gamma_n) \wedge \neg(\alpha \;\Box\!\!\rightarrow\; \beta))$. So, $\{\alpha \;\Box\!\!\rightarrow\; \gamma_1, \ldots, \alpha \;\Box\!\!\rightarrow\; \gamma_n\} \cup \{\neg(\alpha \;\Box\!\!\rightarrow\; \beta)\}$ is $\mathbf{L}$-inconsistent and, thus, $\Gamma \cup \{\neg(\alpha \;\Box\!\!\rightarrow\; \beta)\}$ as well. ■

We give here canonicity proofs for the Principles CM and Im:

Axiom Schema CM. Let $\langle W^c, R^c, P^c, V^c\rangle$ be the canonical model for $\mathbf{L} \in \mathbf{\Pi_{CK}}$ s.t. CM $\in \mathbf{L}$. Assume that CM is not canonical. Then, there exist $X, Y \in P^c$ and $w, w' \in W^c$ such that $\forall w''(wR^c_X w'' \rightarrow w'' \in Y)$, $wR^c_{X\cap Y}w'$, and $\neg wR^c_X w'$. By Def_{P^c} there are formulas α and β s.t. $X = |\alpha|$ and $Y = |\beta|$. Hence, (i) $\forall w''(wR^c_{|\alpha|}w'' \rightarrow$

$w'' \in |\beta|$), $wR^c_{|\alpha|\cap|\beta|}w'$ and $\neg wR^c_{|\alpha|}w'$ hold and, thus, $wR^c_{|\alpha\wedge\beta|}w'$. By Lemma 9.3, condition (i) implies that $\alpha \mathrel{\Box\!\!\rightarrow} \beta \in w$. By Def$_{R^c}$ and $\neg wR^c_{|\alpha|}w'$ there is a formula γ s.t. $\alpha \mathrel{\Box\!\!\rightarrow} \gamma \in w$ and $\gamma \notin w'$. Since $wR^c_{|\alpha\wedge\beta|}w'$, by Lemma 9.3 it holds that $\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \gamma \notin w$. As $\alpha \mathrel{\Box\!\!\rightarrow} \beta \in w$ and $\alpha \mathrel{\Box\!\!\rightarrow} \gamma \in w$, this contradicts Def$_{W^c}$ by Axiom Schema CM. ■

Axiom Schema Im. Let $\langle W^c, R^c, P^c, V^c\rangle$ be the canonical model for $\mathbf{L} \in \mathbf{\Pi_{CK}}$ s.t. Im $\in \mathbf{L}$. Assume that Im is not canonical. Then, there exist $X, Y \in P^c$ and $w, w' \in W^c$ s.t. $wR^c_{X\cap Y}w'$ and $\neg\exists w''$ $(wR^c_X w'' \wedge w''R^c_Y w')$. By Def$_{P^c}$ there are formulas α, β s.t. $X = |\alpha|$ and $Y = |\beta|$. Thus, $wR^c_{|\alpha|\cap|\beta|}w'$ and (ii) $\neg\exists w''(wR^c_{|\alpha|}w'' \wedge w''R^c_{|\beta|}w')$ and, thus, $wR^c_{|\alpha\wedge\beta|}w'$. By Def$_{W^c}$ there is a formula γ s.t. $\gamma \in w'$ and $\gamma \notin w''$ for any other world $w'' \in W^c$. By (ii) no world w'' s.t. $wR^c_{|\alpha|}w''$ can see w' by $R^c_{|\beta|}$. As w' is the only world in W^c s.t. $\gamma \in w'$, due to Lemma 9.3 $\beta \mathrel{\Box\!\!\rightarrow} \neg\gamma \in w''$ holds. Since this holds for all possible worlds w'' s.t. $wR^c_{|\alpha|}w''$, we get $\alpha \mathrel{\Box\!\!\rightarrow} (\beta \mathrel{\Box\!\!\rightarrow} \neg\gamma) \in w$. As $wR^c_{|\alpha\wedge\beta|}w'$ and $\gamma \in w'$, it follows by Lemma 9.3 that $\alpha \wedge \beta \mathrel{\Box\!\!\rightarrow} \neg\gamma \notin w$. This contradicts Def$_{W^c}$ by Axiom Schema Im.

■

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