THE SEMANTICS OF ENTAILMENT

By

Alasdair Ian Fenton Urquhart

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FOREWORD

I am heavily indebted to several logicians. Professors J. Michael Dunn and Robert K. Meyer are to be given joint credit for the important counter-example of §6. Section 11 also owes much to Professor Dunn, as it could not have been written without the inspiration given me by his forthcoming paper on Analytic Implication; he also provided the proof of T9. Professor Richard Routley brought to my attention the facts which underly the discussion of semantical extensions in §8. To all three logicians mentioned above I am indebted for the stimulus of their correspondence and conversation. My greatest debts are to my teacher, Professor Alan Ross Anderson and to my teacher and adviser, Professor Nuel D. Belnap, Jr. Most of the ideas developed below have their source either in their published work or in their seminars, lectures and discussions. I would like to express my sincere thanks to both for the inspiration of their writings, the influence of their teaching and their friendly encouragement and advice.

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1.0 INTRODUCTION

The aim of the present study is the semantical analysis of various non-classical theories of implication or entailment: the systems E_I, R_I and related logics, Parry's system of analytic implication and Łukasiewicz's many-valued logics. For a detailed and profound study of the problem of entailment in both its technical and philosophical aspects the reader is referred to the papers of Anderson and Belnap given in the list of references, and to the comprehensive work Anderson and Belnap *. For background in many-valued logics the reader may consult McCall 1967 and Rescher 1969.

Several of the ideas in the present essay have been discovered independently by others. The semantical concepts and analyses of §§1 to 4 were conceived independently by Richard Routley (Routley **), as was the analysis of negation in §8 (Routley*, Routley and Routley*). The interpretation of many-valued logics in §10 was discovered by Dana Scott (Scott*) at a somewhat earlier date than the author.

In the logical systems to be discussed below we use P_o, P_1, P_2, ...etc. as propositional variables. The formulas of the language of propositional logic are built up as usual, using connectives such as $\rightarrow$, &, $\vee$, $\sim$. We employ the conventions of Church 1956 in abbreviating formulas. A, B, C, ... etc. are used as syntactical variables ranging over formulas. The set of non-negative integers is denoted by N. If S is a non-empty set a function F defined on N taking subsets F_k of S as values is a _value function on S_. Standard algebraic concepts (e.g. _semilattice_, _monoid_, _partial ordering_, _distributive lattice_) are used throughout. For definitions of these concepts the reader is referred to Birkhoff and Bartee 1970, from which we have also taken our set-theoretical notation.

2.0 RELEVANT IMPLICATION

2.1 Pieces of information

We begin with the concept of a "piece of information" as it is basic in most of the ensuing discussions. Let us suppose we have a particular subject under consideration, and a language in which to formulate discourse about this subject. It is to be assumed that from the sentences of this language we can isolate the basic or atomic sentences from which logically complex sentences are formed by operations such as conjunction, disjunction and implication. Thus if the subject under consideration were number theory the atomic sentences would be numerical equations, if physics they would be simple statements of experimental results, and so forth. A "piece of information" is to be thought of as an arbitrary set of basic sentences. Such a set may be given as a finite list, or may be listed mechanically if infinite, possibly even defined in a non-mechanical manner.

This concept is to be contrasted with two less general concepts, those of an "evidential situation" and of a "possible world". The former concept is one suitable for an analysis of intuitionistic logic (Kripke 1965), the latter for an analysis of modal logic (Kripke 1959, 1963). An evidential situation is to be considered as a set of facts established as true during the course of some investigation. It must therefore satisfy the requirement of consistency. The concept of a possible world is still narrower. As a "possible world" is intended to be a total description of a situation, it must satisfy not only the consistency requirement, but also one of completeness.

Let us suppose we are given a family S of pieces of information about some topic. What can we say about the structure of S? At least, it would appear, we would wish to include 0, the empty piece of information, in S; further, it seems clear that if x and y are in S so is $x \cup y$, that is, the piece of information which is the union of x and y. Thus S has the structure of a semilattice with a zero element. It seems reasonable to go further and require S also to be closed under intersection, so that it would have the structure of a distributive lattice. However, since we shall not make any use of the intersection operation, we shall consider S only in its semilattice aspect.

It is to be noted that a semilattice structure can also be imposed on a set of evidential situations. If x and y are both sets of statements established as true during the investigation of a fixed subject, then x and y are jointly consistent, so that $x \cup y$ is again an evidential situation. It is this fact which explains why intuitionistic logic may be analysed as a special case of the semantics for relevant logic. On the other hand, the union operation makes no sense when considered as applying to a set of possible worlds. In fact, one would expect in general two distinct possible worlds to be jointly inconsistent.

Before we can say anything useful about a semilattice S of informational quanta, we need one further concept, namely a primitive notion of consequence or entailment. A piece of information x will entail or have as consequence certain atomic sentences p; we write "$x \models p$" if this relationship holds. For instance, we might have:

$\{1+1 = 2,\ 2= 1\times 1\} \models 1+1 = 1\times 1$,

{John is a bachelor} $\models$ John is unmarried, and so on. This consequence relation is essentially logic free, that is, it holds by virtue not of the logical complexities of the sentences involved, but by virtue of (a) the meanings of the predicates and descriptions in the basic sentences and (b) certain background facts assumed to be true ((a) and (b) may not be entirely separable). This fact shows that there is not a vicious circularity involved in defining a consequence relation for logically complex sentences in terms of the basic consequence relation. Finally, we do not postulate that if $x \models p$ then $x \cup y \models p$. The reason is that the consequence relation may be interpreted in such a way that $x \models p$ may fail if x is supplemented by additional irrelevant statements.

The notion of consequence for complex statements can now be defined recursively, given S and the basic consequence relation relative to S. Let us suppose for the moment that the language is purely implicational. If the consequence relation has been extended to A and B what are the truth conditions for $(A \rightarrow B)$? Well, since the $\rightarrow$ connective represents the notion of logical consequence, we wish it to be the case that $(A \rightarrow B)$ is a consequence of x whenever B is a consequence of x and A together. We could then write: $x \models A \rightarrow B$ if $x \cup \{A\} \models B$. However, this will not do as a formal definition. The sentence A may be logically complex, so that $x \cup \{A\}$ would not be a piece of information. We can nevertheless reproduce the intention of the definition in a more general form; $x \models A \rightarrow B$ if for any y in S, whenever $y \models A$ then $x \cup y \models B$. This statement, which seems to reproduce exactly the sense in which $\rightarrow$

represents deducibility, we take as the recursive consequence definition.

Those familiar with the literature of relevant logic will recognize the provenance of this definition. It is simply the subscripting requirement of the Fitch-style formulations in Anderson 1960, recast in slightly more general form. What we have been trying to emphasize above is the naturalness and philosophical plausibility of Anderson's condition.

We shall now restate these ideas in terms of a formal semantics for a pure implicational language.

2.2 Semantics for R_I

$Q=(S,F)$ is an r-model

if

(i) S is a semilattice with zero element (0),

(ii) F is a value function on S.

We define the truth of a formula A at x in Q as follows.

1. $x \models_Q P_k$ iff $x \varepsilon F_k$,

2. $x \models_Q A \rightarrow B$ iff for all y in S either not $y \models_Q A$ or $x \cup y \models_Q B$.

A formula A is <u>true</u> in Q if $0 \models_Q A$, <u>r-valid</u> if true in all r-models.

2.3 Axiomatization of R_I.

R_I is axiomatized by the schemata

1. $A \rightarrow A$

2. $A \rightarrow B \rightarrow . C \rightarrow A \rightarrow . C \rightarrow B$

3. $(A \rightarrow . B \rightarrow C) \rightarrow . B \rightarrow . A \rightarrow C$

4. $(A \rightarrow . A \rightarrow B) \rightarrow . A \rightarrow B$

with <u>modus ponens</u> (from A and $A \rightarrow B$ infer B) as sole rule of inference.

Let $(A_1,\ldots,A_n)$ be a finite sequence of formulas of R_I. Then $(A_1,\ldots,A_n)\vdash B$ is defined to hold if $\vdash_{R_I} A_1 \rightarrow . \ldots A_n \rightarrow B$. We now list some derived rules in terms of this definition.

$$\text{DR1} \quad \frac{\alpha \vdash A}{\alpha^1 \vdash A},$$

where α^1 is any permutation of α.

$$\text{DR2} \quad \frac{\alpha \vdash A \rightarrow B \qquad \beta \vdash A}{\alpha,\beta \vdash B},$$

$$\text{DR3} \quad \frac{\alpha, A, A, \beta \vdash B}{\alpha, A, \beta \vdash B},$$

$$\text{DR4} \quad \frac{\alpha, A \vdash B}{\alpha, A \rightarrow A \rightarrow A \vdash B},$$

$$\text{DR5} \quad \frac{\alpha, A \rightarrow A \rightarrow A \vdash B}{\alpha, A \vdash B},$$

where α, β are finite sequences of formulas. We do not establish these rules here, instead referring the reader to the convenient subproof formulation of R_I in Anderson and Belnap 1961a, in which they are easily demonstrated.

2.4 Completeness of R_I

The semantic consistency of R_I, that is, the fact that every theorem of R_I is r-valid, is easily proved. We prove semantic completeness by construction of a canonical model for R_I.

Let S be the semilattice of all finite sets of formulas of R_I (that is, $x \cup y$ is set union, 0 the empty set). Let x be in F_k if $\overline{x} \vdash P_k$ for some sequence $\overline{x}$ in which each element of x occurs once and only once.

Let Q be the r-model (S,F).

LEMMA $x \models_Q A$ iff $\overline{x} \vdash A$ for some $\overline{x}$.

Proof. The lemma holds for propositional variables by definition. Assume the lemma for A and B. Let it be the case that $\overline{x} \vdash A \rightarrow B$. Then if $y \models_Q A$, $\overline{y} \vdash A$ by induction hypothesis, so that $\overline{x}, \overline{y} \vdash B$ by DR2. By DR1 and DR3, repetitions in $(\overline{x},\overline{y})$ may be eliminated, so that $\overline{x \cup y} \vdash B$, hency $x \cup y \models_Q B$. Hence, $x \models_Q A \rightarrow B$. Now assume conversely that $x \models_Q A \rightarrow B$. Define $N^0 A$ to be A, $N^{k+1} A$ to be $N^k A \rightarrow N^k A \rightarrow N^k A$. Let m be the least k such that $N^k A$ is not in x. Now $\{N^m A\} \models_Q A$ by axiom 1, DR4 and the induction hypothesis, so $x \cup \{N^m A\} \models_Q B$, by assumption. By induction hypothesis, $\overline{x \cup \{N^m A\}} \vdash B$. Since $N^m A$ is not in x, $\overline{x \cup \{N^m A\}}$ is $(\overline{y}, N^m A, \overline{z})$, $x = y \cup z$. Thus $\overline{x}, A \vdash B$ by DR1 and DR5, that is, $\overline{x} \vdash A \rightarrow B$.

As a corollary to the lemma we have: $\vdash_{R_I} A$ iff $0 \models_Q A$, so that the completeness of R_I is proved.

2.5 Modifications of r-models

Before proceeding to the discussion of other calculuses related to R_I, some easy modifications of the semantics may be noted.

Firstly, we can weaken the condition on an r-model that S be closed under the union operation. Accordingly, an r^1-model is defined to be a quadruple $Q = (S,\leqslant,0,F)$ where

(i) S is a set partially ordered by $\leqslant$, with 0 the least element of S, such that whenever a finite subset X of S has an upper bound in S it has a least upper bound $\bigcup X$ in S,

(ii) F is a value function on S. We then define the consequence relation as follows.

1. $x \models_Q P_k$ iff $x \varepsilon F_k$,

2. $x \models_Q A \rightarrow B$ iff for all y such that $\bigcup\{x,y\}$ exists either

not $y \models_Q A$ or $\cup\{x,y\} \models_Q B$.

Truth in a model and validity are defined as in 1.1. Completeness follows by the results of 1.3.

Secondly, we can exploit the fact that the full semilattice structure is not required for the proof of semantic consistency for R_I. For instance, to validate $(A \rightarrow . A \rightarrow B) \rightarrow . A \rightarrow B$ we do not need the condition $x \cup x = x$; all that is needed is the weaker condition that if $x \cup x \models_Q A$ then $x \models_Q A$.

With these facts in mind, we define an r''-model to be a triple $Q = (S, \leqslant, F)$ where

(i) S is a commutative monoid,

(ii) $x \cup x \leqslant x$ for all x in S, and if $x \leqslant y$ then $x \cup z \leqslant y \cup z$,

(iii) F is a valuation function on S such that if $x \leqslant y$ and $x \varepsilon F_k$ then $y \varepsilon F_k$. Truth and validity are defined as in 1.1. Completeness again follows from the results of 1.3, since $\leqslant$ may be taken to be the identity relation in an r-model.

These two modifications of the semantics are not interesting from a philosophical standpoint, but as we shall see below they reduce the technical difficulties of a completeness proof when R_I is extended by the addition of certain new connectives.

3.0 GENERALIZATIONS OF RELEVANT IMPLICATION

3.1 Generalized Semantics

Although from the classical point of view R_I is a very weak calculus it can equally well be regarded as a very strong one, the limiting case in a family of relevant implicational logics. We now consider the consequences of weakening the semi-lattice structure imposed upon an r-model. The calculuses defined by the resulting semantics seem interesting and well-motivated (at least in some cases); furthermore, some light is thrown on the exact semantical conditions which underpin the validity of theorems in R_I.

We define an r∅-model to be a structure Q=(S,F), where $0 \in S$, and $\cup$ is a binary operation defined on S. The notions of <u>consequence</u>, <u>truth</u> and <u>validity</u> are defined as in §1.1. For future reference we list the semilattice conditions.

1. $0 \cup x = x$
2. $x \cup 0 = x$
3. $(x \cup y) \cup z = x \cup (y \cup z)$
4. $x \cup y = y \cup x$
5. $x \cup x = x$

Let w be a subsequence of 1, 2, 3, 4, 5. Then an r∅-model satisfying the set {w} of conditions will be referred to as an rw-model. The set of formulas true in all rw-models will be referred to as R_w. For instance, R_I is $R_{12345I} = R_{1345I}$.

3.2 The System $R_{\emptyset I}$

$R_{\emptyset I}$ is not very interesting, as it has no theorems at all. Let A be an arbitrary formula of $R_{\emptyset I}$. Consider $Q=(S,F)$, where S, $\cup$ and F are defined as in 1.1, and $O=\{P_k\}$ where P_k is a variable which does not occur in A. Then Q is an r345-model; A is falsified in Q by the variable-sharing theorem for R_I (Belnap 1960b). From this it follows that if w is contained in (3, 4, 5), then R_w is devoid of theorems.

3.3 The System R_{1I}

R_{1I} is axiomatized by the schema

2. $A \rightarrow A$

with the rules of inference

R1. From A and $A \rightarrow B$ infer B

R2. From $A \rightarrow B$ infer $C \rightarrow A \rightarrow . C \rightarrow B$

R3. From $A \rightarrow B$ infer $B \rightarrow C \rightarrow . A \rightarrow C$

To show completeness, consider $Q=(S,F)$, where S is the set of all sets of formulas, $O=\{A: \vdash_{R_{1I}} A\}$, $x \cup y=\{B: (\exists A)(A \rightarrow B \varepsilon x \ \& \ A \varepsilon y)\}$, $x \varepsilon F_k$ iff $P_k \varepsilon x$. Now if $A \varepsilon O \cup x$, then for some B, $B \rightarrow A \varepsilon O$, $B \varepsilon x$. But if $B \rightarrow A \varepsilon O$, B is identical with A, hence $A \varepsilon x$. Conversely, if $A \varepsilon x$, $A \rightarrow A \varepsilon O$, hence $A \varepsilon O \cup x$. Thus $O \cup x=x$ for all x in S, so Q is an r1-model. The simple proof that $x \models_Q A$ iff $A \varepsilon x$ is left to the reader.

3.4 The System R_{14I}

To axiomatize R_{14I} we add to the preceding axiom system the schema

2. $A \rightarrow . A \rightarrow B \rightarrow B$.

To show completeness, we consider the model Q given by the definitions:

(i) S is the set of all sets of formulas x such that if $A \rightarrow B \varepsilon 0$, $A \varepsilon x$ then $B \varepsilon x$,

(ii) $0=\{A: \vdash_{R_{14I}} A\}$,

(iii) $x \cup y=\{B: (\exists A)(A \rightarrow B \varepsilon x \ \& \ A \varepsilon y)\}$

(iv) $x \varepsilon Fk$ iff $P_k \varepsilon x$.

The rule R3 ensures that S is closed under $\cup$; the schemata 1 and 2 ensure that Q satisfies conditions 1 and 4. Thus Q is an r14-model.

To conclude the completeness proof, we must show that $x \models_Q A$ iff $A \varepsilon x$. This holds for propositional variables by construction. Now assume that it holds for A and B. If $A \rightarrow B \varepsilon x$ then if $y \models_Q A$, $A \varepsilon y$ by inductive assumption so $B \varepsilon x \cup y$, hence $x \cup y \models_Q B$; hence $x \models_Q A \rightarrow B$. Conversely, let $x \models_Q A \rightarrow B$. Consider $y=\{C: A \rightarrow C \varepsilon 0\}$. This set is in S by R3. Now by inductive hypothesis $y \models_Q A$, so $x \cup y \models_Q B$. Then for some C, $C \rightarrow B \varepsilon x$ and $A \rightarrow C \varepsilon 0$. Then by R3, $C \rightarrow B \rightarrow . A \rightarrow B \varepsilon 0$, so $A \rightarrow B \varepsilon x$.

3.5 The System R_{123I}

R_{123I} is axiomatized by the schemata

1. $A \rightarrow A$

2. $A \rightarrow B \rightarrow . C \rightarrow A \rightarrow . C \rightarrow B$

with the rules of inference

R1. From A and $A \rightarrow B$ infer B,

R2. From A infer $A \rightarrow B \rightarrow B$.

Let x by a finite sequence of formulas. We define $x \vdash B$ to hold if $\vdash_{R_{123I}} A_1 \rightarrow \ldots . A_n \rightarrow B$, where x is $(A_1, \ldots, A_n)$.

<u>LEMMA 1</u>. If x, $B \vdash C$ and $\vdash B$, then $x \vdash C$.

<u>Proof</u>: If $\vdash B$ then by R2 $\vdash B \rightarrow C \rightarrow C$. Hence by repeated use of schema 2, $\vdash (A_1 \rightarrow \ldots A_n \rightarrow . B \rightarrow C) \rightarrow (A_1 \rightarrow \ldots A_n \rightarrow C)$ so by R1, $A_1, \ldots, A_n \vdash C$, where

$x=(A_1,\ldots,A_n)$.

LEMMA 2. If $x \vdash A \to B$ and $y \vdash A$ then $x,y \vdash B$.

Proof: Let $x=(C_1,\ldots,C_m)$ and $y=(D_1,\ldots,D_n)$. Assume that $C_1,\ldots,C_m \vdash A \to B$, $D_1,\ldots,D_n \vdash A$. By axiom 2, and R1, $\vdash C_1 \to \ldots C_m \to (D_1 \to \ldots D_n \to A) \to (D_1 \to \ldots D_n \to B)$, so by Lemma 1, $(C_1,\ldots,C_m,D_1,\ldots,D_n) \vdash B$.

Now let S be the set of all finite sequences of formulas, 0 the empty sequence, $x \cup y$ the sequence (x,y) and let $x \varepsilon Fk$ iff $x \vdash Pk$. $Q=(S,F)$ is an r123-model. It remains to be shown that $x \models_Q A$ iff $x \vdash A$ for all x in S. This follows easily from Lemma 2, so the completeness of R_{123I} is proved.

3.6 The System R_{1234I}

R_{1234I} is axiomatized by simply omitting the fourth schema from the axiomatization of 1.2.

Let $x=(A_1,\ldots,A_n)$ be a finite sequence of formulas. $x \vdash B$ is defined to hold if $\vdash_{R_{1234I}} A_1 \to \ldots A_n \to B$. For any two finite sequences x and y let $x \equiv y$ hold if x is a permutation of y.

LEMMA 1. If $x \vdash A \to B$ and $y \vdash A$ then $x,y \vdash B$.

Proof: As for Lemma 2 in the preceding section.

LEMMA 2. If $x \vdash A$ and $x \equiv y$ then $y \vdash A$.

Proof: If suffices to prove that if x, A, B, $y \vdash C$ then x, B, A, $y \vdash C$, as may be shown from the schemata 1 and 2..

Now let S be the set of all equivalence class [x] of finite sequences x under the relation $\equiv$, 0 the equivalence class of the null sequence, $[x] \cup [y]=[xy]$ and let $x \varepsilon F_k$ iff $x \vdash P_k$. Note that $\cup$ is well-defined since $x \equiv y$, $z \equiv w$ implies $xz \equiv yw$, and F is well-defined by Lemma 2. $Q=(S,F)$ is evidently an r1234-model, and is a characteristic model for R_{1234I} by

Lemma 1.

R_{1234I} has been treated in earlier literature. It appears in Smiley 1959 and is discussed in Meredith and Prior 1963 with the name BCI.

3.7 Systems with weakening and mingle

Systems with the "weakening" axiom $A \to . B \to A$ may be considered as resulting from specializations of the semilattice semantics. The condition needed to validate this axiom we shall designate by K;

if $x \varepsilon F_k$ then $x \cup y \varepsilon F_k$.

Of the systems with the weakening axiom only two seem worthy of comment, namely R_{1234K} and R_{12345K}. The latter is simply the implicational fragment of intuitionistic logic; the former is the system BCK of Meredith and Prior 1963. R_{1234K} also appears in Tarski 1956, Paper XIV where it is proved that if an implicational logic contains R_{1234K} then it has only one consistent complete extension, if any.

A more restricted condition related to K is the condition M ("mingle"): if $x \varepsilon F_k$ and $y \varepsilon F_k$ then $x \cup y \varepsilon F_k$. RM_I is axiomatized by adding $A \to . A \to A$ to the axiomatization of R_I (see Anderson and Belnap *, Dunn 1970).

3.8 Cut free formulations

Several of the systems discussed above have elegant Gentzen-style formulations. The most interesting case is possibly R_{123I} which can be formulated as follows. The axioms are all of the form $A \vdash A$. There are two rules of inference:

$$\to\vdash \quad \frac{\beta \vdash A \qquad\qquad \alpha, B, \gamma \vdash C}{\alpha, A \to B, \beta, \gamma \vdash C}$$

$$\vdash\rightarrow \quad \frac{\alpha,\ A\vdash B}{\alpha\vdash A\rightarrow B}$$

To show that this formulation is equivalent to the axiomatic formulation of R_{123I}, we must prove for the Gentzen version the Elimination Theorem (ET): If (1)$\alpha\vdash A$ and (2) $\beta,A,\gamma\vdash B$ are provable. so is (3) $\beta,\alpha,\gamma\vdash B$.

Proof: Let the definitions of parameter, constituent, parametric ancestor be as in Curry 1963, p. 199. The rank of a constituent A of a sequent is the number of parametric ancestors of A less 1.

Example: in the derivation

$$\frac{\dfrac{A\vdash A \qquad B\vdash B}{A\rightarrow B, A\vdash B}}{A\rightarrow B\vdash A\rightarrow B}$$

the rank of the lefthand constituent in the last sequent is 1, that of the righthand constituent 0.

Let m be the rank of A in (1), n the rank of A in (2). We first show that if either (1) or (2) are axioms then the ET holds. If (1) is an axiom then α consists of A alone, so the theorem follows trivially. If (2) is an axiom A is the same formula as C so ET again is trivial.

Now assume that the ET holds for m, n and a given formula A. To show that it holds for the same A, m and n + 1, we observe that the last line of the proof of (2) must be an instance of a rule in which A is parametric. The conclusion follows from the fact that an instance of either rule remains an instance of the same rule upon replacement of the parametric A in premisses and conclusion by α.

Now assume ET for m, n and a fixed A. We aim to show that it holds for m + 1, n and the same A. If the rank of A in (1) is m + 1, then (1) must be immediately derived by $\rightarrow\vdash$. Let us suppose the derivation to end as follows:

$$\frac{\varepsilon\vdash C \qquad \delta,D,\mu\vdash A}{\delta,C\rightarrow D,\varepsilon,\ \mu\vdash A}$$

By assumption, $\beta,\delta,D,\mu,\gamma\vdash B$ is derivable, hence by $\rightarrow\vdash$

$$\frac{\varepsilon\vdash C \qquad \beta,\delta,D,\mu,\gamma\vdash B}{\beta,\delta,C\rightarrow D,\varepsilon,\mu,\gamma\vdash B},$$

That is, $\beta,\alpha,\gamma\vdash B$ is derivable.

Finally, assume that the ET holds generally when the eliminated constituents are C and D. We shall show that it holds for $C\rightarrow D$ when m=n=o. In view of the preceding remarks we may suppose that neither (1) nor (2) are axioms so that the last parts of the derivations of (1) and (2) are applications of $\vdash\rightarrow$ and $\rightarrow\vdash$ respectively:

(3)

$$\frac{\alpha,C\vdash D}{\alpha\vdash C\rightarrow D}$$

(4) (5)

$$\frac{\gamma_1\vdash C \qquad \beta,D,\gamma_2\vdash B}{\beta,C\rightarrow D,\gamma_1,\ \gamma_2,\vdash B}$$

From (3) and (5) by the ET, $\beta,\alpha,C,\ \gamma_2\vdash B$ is derivable, hence by the ET from (4) $\beta,\alpha,\gamma_1,\gamma_2\vdash B$ is derivable. This completes the proof of the elimination theorem.

It is now easy, using the results of 2.5, to show the two formulations of R_{123I} equivalent, in the sense that A is provable in R_{123I} iff $\vdash A$ is provable in the Gentzen system. Thus R_{123I} corresponds to a rather interesting cut-free formulation -- one in which there are no structural rules whatever.

Gentzen formulations of R_{1234I} and R_I are obtained by adding rules of permutation and contraction.

4.0 TICKET ENTAILMENT

4.1 Introduction

Pieces of information may be ordered in a variety of ways. One way which seems especially interesting is connected with the varying degrees of generality which a piece of information may possess. Let us write $x \leq y$ if the piece of information x is more general than y, or alternatively, y contains at least as much information as x. The relation $\leq$ must possess some of the properties of set inclusion, for instance $x \leq x \cup y$ for any x,y. On the other hand, it would not appear to be correct to identify $\leq$ with set inclusion in general. Thus it would seem plausible that

{John's wife is sick}$\leq${John has a wife}, while the corresponding statement of set inclusion is of course false.

It would seem that $A \rightarrow B$ is a more general statement than A or B. We therefore postulate that $A \rightarrow B$ follows from x if the appropriate deducibility condition is satisfied by all pieces of information as general as, or less general than x. This idea leads to the system T_I of ticket entailment.

4.2 Semantics for T_I

$Q=(S,\leq,F)$ is a t-model if

(i) S is a semi-lattice with O.

(ii) $\leq$ is a transitive binary relation defined on S such that for any x and y in S if $x \leq y$ then $x \cup z \leq y \cup z$, and for any x in S, $O \leq x$.

(iii) F is a value function on S. The truth of a formula A at x in Q is defined as follows.

1. $x \models_Q Pk$ iff $x \in Fk$,

2. $x \models_Q A \to B$ iff for all y in Q such that $x \leq y$ either not $y \models_Q A$ or $x \cup y \models_Q B$.

A formula A is true in Q iff $0 \models_Q A$, t-valid if true in all t-models.

4.3 Axiomatization of T_I

T_I is axiomatized by the schemata

1. $A \to A$

2. $A \to B \to . B \to C \to . A \to C$

3. $A \to B \to . C \to A \to . C \to B$

4. $(A \to . A \to B) \to . A \to B$

with modus ponens as sole rule of inference.

4.4 Completeness of T_I

Let T be the set of all finite sets of positive integers, $\mathcal{U}$ the set of all formulas of T_I. For x,y in T the relation $x \leq y$ is defined to hold if max (x) is less than or equal to max (y), where max (x) is the greatest member of x if $x \neq \emptyset$ and max $(\emptyset)=0$. Members of $\mathcal{U} \times T$ will be referred to as terms. A term may be written as a formula with a subscript, for instance $A \to C_{\{1,2\}}$. Let S be the set of all finite subsets of $\mathcal{U} \times T$. For x in S, s(x) is defined to be the union of all subscripts of terms in x. For x in S, and A in $\mathcal{U}$, a proof of A from x is defined to be a sequence of terms $B_{1_{x_1}}, \ldots, B_{n_{x_n}}$ such that B_n is A and x_n is s(x) and for each $B_{i_{x_i}}$ either (i) $B_{i_{x_i}}$ is in x or (ii) B_i is a theorem of T_I and x_i is empty or (iii) $B_{i_{x_i}}$ is $D_{y \cup z}$, $y \leq z$ and for some C, $C \to D_y$ and C_z occur earlier in the sequence. We write $x \vdash A$ if there is a proof of A from x. Finally, let $x \leq y$ hold for x,y in S if $s(x) \leq s(y)$.

LEMMA 1. If $x \vdash A \to B$, $y \vdash A$ and $x \leq y$ then $x \cup y \vdash B$.

Proof: Immediate from definitions.

LEMMA 2. If $x \cup \{A_{\{k\}}\} \vdash B$, where $k > s(x)$ then $x \vdash A \to B$.

Proof: The proof of this lemma is a straightforward adaptation of the method used by Anderson and Belnap * §6 to show the equivalence of the axiomatic and subproof formulations of T_I.

Now consider $Q=(S,\leq,F)$, where S is the set semilattice of all finite sets of terms, $\leq$ is defined as above on S and $x \in F_k$ iff $x \vdash P_k$. We prove by induction on the complexity of a formula A that $x \models_Q A$ iff $x \vdash A$, using Lemmas 1 and 2. Thus Q is a t-model which is characteristic for T_I.

4.5 T_I minus contraction

As in the case of R_I the semantics of T_I may be generalized by weakening the semilattice conditions. One logic which may be treated semantically in this way is T_I-W, T_I minus contraction (Anderson and Belnap *, §8.11), which is axiomatized by omitting the fourth schema from the axiom system of 3.2. We define a t1234-model in the same way as a t-model, save that S is required merely to be a commutative monoid with zero.

For x,y arbitrary sequences let $x \equiv y$ hold if x is a permutation of y. Let T be the set of equivalence classes [a] under $\equiv$, where a is a finite sequence of positive integers; let $\leq$ be defined on T as in the preceding section. Members of the set $\mathcal{U} \times T$ will be referred to as terms, where $\mathcal{U}$ is as in 3.3; a term (A,[a]) may be written A_a, with the understanding that for instance $A_{12}=A_{21}$, while on the other hand $A_{12} \neq A_{122}$. Now let S be the set of all equivalence classes [x] under $\equiv$ where x is a finite sequence of terms; s is a commutative monoid with zero when $[x] \cup [y] = [xy]$. The function $s(x)$ and the relation $x \leq y$ are defined for x and y in s by analogy with 3.3, as is the notion "proof of A from x", where x is a finite sequence of terms. For such an x, let $[x] \vdash A$ hold if there is a proof of A from x.

LEMMA 1. If $[x] \vdash A \rightarrow B$, $[x] \leq [y]$ and $[y] \vdash A$ then $[xy] \vdash A$

Proof: As in 3.3.

LEMMA 2. If $[x] \cup [A_k] \vdash B$, where $k > s(x)$, then $[x] \vdash A \rightarrow B$.

Proof: By hypothesis there is a sequence of terms $C_{1_{a_1}}, \ldots, C_{n_{a_n}}$ which is a proof of B from x, A_k. We proceed in two stages to convert it into a proof of $A \rightarrow B$ from x. First, delete from the sequence all $C_{i_{a_i}}$ where k occurs more than once in a_i. The resulting sequence $\{D_{i_{a_i}}\}$ is still a proof of B from x; for k can occur at most once in any term from which $D_{n_{a_n}} = Bs(x) \cup [k]$ is derived. The second part of the conversion then proceeds as in Anderson & Belnap * §6.

Let $Q = (S, \leq, F)$ where $[x] \varepsilon F_k$ iff $[x] \vdash P_k$. Q is shown to be a characteristic model for T_I-W by use of the two lemmas.

4.6 T_I minus contraction and permutation

$Q = (S, \leq, F)$ is defined to be a t123-model if S is a monoid, F a value function on S and $\leq$ a binary relation on S such that for all x,y,z in s, $0 \leq x$ and if $x \cup y \leq z$ then $y \leq z$ and $x \leq y \dot{\cup} z$.

Truth and validity are defined as usual.

T_{123I} is axiomatized by the schemata

1. $A \rightarrow A$

2. $A \rightarrow B \rightarrow . C \rightarrow A \rightarrow . C \rightarrow B$

with the rules of inference

R1. From A and $A \to B$ infer B

R2. From $A \to B$ infer $B \to C \to . A \to C$

Let $\mathcal{U}$ be the set of all finite sequences of positive integers, and for x,y in $\mathcal{U}$ let $x \leq y$ hold iff either x is empty or y is non-empty and xy is a consecutive sequence. Let S be the set of all finite sequences of terms in $F \times \mathcal{U}$ and for x in S let s(x) be the sequence obtained by concatenating subscripts in the order in which they occur in x. For x,y in s let $x \leq y$ hold if $s(x) \leq s(y)$. A proof of A from x may be defined as in 3.4. Now let $x \vdash A$ hold if there is a proof of A from x and s(x) is a consecutive sequence.

LEMMA 1. If $x \vdash A \to B$, $y \vdash A$ and $x \leq y$ then $xy \vdash B$.

LEMMA 2. If x, $A_k \vdash B$ and $s(x) \leq k$, then $x \vdash A \to B$.

Proof: The proof is parallel to that of Lemma 2, 3.4.

Now let $x \varepsilon F_k$ iff $x \vdash P_k$. $Q = (S, \leq, F)$ is a t123-model, and by the two lemmas above, characteristic for T_{123I}.

One final system related to T_I may be mentioned here, that which arises by adding the requirement $x \leq y$ iff $x \cup y = y$ to the definition of a t-model. This condition validates all theorems of $S4_I$; we conjecture that the condition characterizes $S4_I$.

5.0 ENTAILMENT

5.1 Introduction

To motivate the semantics of entailment, let us return to the considerations of 1.0. We supposed given a primitive, logic-free consequence relation holding between pieces of information and atomic sentences, which holds by virtue of (a) the meanings of words in the basic sentences and (b) certain background facts. Now in R_I and T_I the set of background facts is tacit; it is considered fixed or invariable. However, if we take into account the idea that there may be alternate sets of background facts the picture changes. For instance, the statement "Richard Nixon likes Billy Graham" follows from the piece of information {The President of the U.S.A. likes Billy Graham}, given the background of facts of actual events in 1971, but it would not follow against a background in which, say, Hubert Humphrey was President. In other words, the fundamental notion for entailment is not simply logical consequence, but logical consequence relative to a set of background facts. In what follows we shall use the term "possible world" instead of the clumsier "set of background facts."

Given a class W of possible worlds, one further concept is required to determine the truth conditions of complex statements, namely a relation R of relative possibility defined on W. We now proceed to a formal definition of validity.

5.2 Semantics for E_I

$Q=(S,W,R,w,F)$ is an e-model if

(i) S is a semilattice,

(ii) W is a set with $w \in W$, R a transitive reflexive relation on W,

(iii) F is a value function on $S \times W$.

The truth of a purely implicational formula at $x \in S$, $u \in W$ is defined as follows.

1. $x, u \models_Q P_k$ iff $(x,u) \in F_k$,

2. $x, u \models_Q A \rightarrow B$ iff for all y in S, v in W, if uRv and $y, v \models_Q A$ then $x \cup y, v \models_Q B$.

A formula A is e-valid if $0, w \models_Q A$ in all e-models.

5.3 Axiomatization of E_I

A convenient axiomatization of E_I is provided by the axiom schemata

1. $A \rightarrow A \rightarrow B \rightarrow B$

2. $A \rightarrow B \rightarrow . B \rightarrow C \rightarrow . A \rightarrow C$

3. $(A \rightarrow . A \rightarrow B) \rightarrow . A \rightarrow B$

with modus ponens as sole rule of inference. As usual we leave it to the reader to prove semantic consistency.

Let S be the set of all finite subsets of N and T the set of all terms A_x with A a formula of E_I and x in S. Now let W be the set of all u such that

(a) u is contained in T,

(b) The union of all subscripts in u is finite,

(c) If $\vdash_{E_I} A$ then $A_\emptyset \in u$

(d) If $A \rightarrow B_x$, $A_y \in u$ then $B_{x \cup y} \in u$.

For $u, v \in W$ let uRv hold if for all A, B, x if $A \rightarrow B_x$ is in u then $A \to B_x$ is in v.

For $\mathcal{X}$ a finite set of terms let a proof of a term A_y from $\mathcal{X}$ be defined as a sequence of terms such that each term in the sequence is either in $\mathcal{X}$ or is C_ϕ where $\vdash_{E_I} C$ or is derived from preceding terms according to the rule (d) above, and such that the last term is A_y. Now for u in W let u' be the set of terms $A \to B_x$ in u; let k be a number greater than any occurring in any subscript in u. Define $P(u, A_{\{k\}})$ to be $\{B_y$: There is a proof of B_y from some $\mathcal{X} \subseteq u' \cup \{A_{\{k\}}\}\}$.

<u>LEMMA</u>: If $B_{x \cup \{k\}}$ is in $P(u, A_{\{k\}})$ then $A \to B_x$ is in u.

<u>Proof</u>: The proof is a variation on the proof of equivalence of the axiomatic and subproof formulations of E_I in Anderson & Belnap 1961a, *.

Let w be $\{A_\emptyset : \vdash_{E_I} A\}$, and let $(x,u) \in F_k$ iff $P_{k_x} \in u$. Then $Q=(S,W,R,w,F)$ is an e-model. By an inductive argument using the lemma we show that $x,u \models_Q A$ iff $A_x \in u$, so that Q is a characteristic model for E_I.

5.4 Variations on a modal theme.

E_I is only one of a family of logics. One group arises by leaving the semilattice requirement fixed but altering the conditions on the relation R. For instance, we may simply require that R be reflexive. The resulting implicational logic, EM_I, appears to be rather tricky to axiomatize. Like its classical counterpart M_I, EM_I seems to require complex rules of inference (see Hacking 1963), as many implicational principles are present in it only in the form of rules. Rather more interesting is the logic $E5_I$ which arises by requiring R to be an equivalence relation on W. The schema $A \to .(A \to .B \to C) \to .B \to C$ is valid in $E5_I$ though not in E_I. It is conjectured that the addition of this schema to E_I results in an axiomatization of $E5_I$.

Evidently, we may also vary the semilattice conditions freely and independently. The author has not investigated the resulting family of logics in detail, and leaves such experimentation to the reader.

6.0 CONJUNCTION

6.1 Classical Conjunction

To treat classical conjunction we define relative to an r- or t-model:

$x \models_Q A\&B$ iff $x \models_Q A$ and $x \models_Q B$, and relative to an e-model:

$x,u \models_Q A\&B$ iff $x,u \models_Q A$ and $x,u \models_Q B$.

The systems R_{IC}, P_{IC} are axiomatized by adding to the axiomatizations of 1.2 and 3.2 the schemata:

5. $A\&B \rightarrow A$
6. $A\&B \rightarrow B$
7. $(A \rightarrow B \& A \rightarrow C) \rightarrow (A \rightarrow . B\&C)$

and the rule of adjunction (from A and B to infer A&B). The same extension of the implicational fragment provides a complete axiomatization of the IC fragment of all logics axiomatized in §§1-3 with the exception of $R_{\emptyset I}$. In each case the completeness proof is a straightforward modification of the proof for the I fragment. It is to be noted that in the case of R_{1IC} the inference rules of 2.2, which are redundant in R_{1I} are essential in the extended calculus.

In the case of E_{IC} we must add in addition to schemata 5-7 and the rule of adjunction the schema

8. $(NA\&NB) \rightarrow N(A\&B)$, where $NA =df A \rightarrow A \rightarrow A$. With this addition, a modification of the argument of the 4.2 suffices to prove completeness.

6.2 Intensional Conjunction

The semantics of R_{IC} extend to include "intensional conjunction" or "joint consistency" by the consequence definition:

$x \models_Q AoB$ iff for some y,z,x=y$\cup$z and $y \models_Q A$ and $z \models_Q B$.

This definition validates the schema

9. $(A \rightarrow . B \rightarrow C) \leftrightarrow (AoB \rightarrow C)$

where $A \leftrightarrow B =$df $(A \rightarrow B) \& (B \rightarrow A)$. The author conjectures that the addition of 9 gives a complete system. We can give an easy completeness result by using the modified models of 1.4. Relative to an r′′-model Q let the valuation rule for & be as usual, and let

$x \models_Q AoB$ iff for some y,z,$y \cup z \leq x$ and $y \models_Q A$ and $z \models_Q B$.

It is also convenient to introduce a constant t, for which the rule is:

$x \models_Q t$ iff $0 \leq x$.

For the constant t we have the valid schema:

10. $A \leftrightarrow (Aot)$

The set of valid formulas in $\rightarrow$,&,o and t we shall denote by R''_{ICkt}. We claim that this set is axiomatized by adding the schemata 9 and 10 to R_{IC}. Before giving the proof, we list some theorems derivable in the axiomatization, whose proofs we leave to the reader.

T1. $(A \leftrightarrow B \& C \leftrightarrow D) \rightarrow (AoC \leftrightarrow BoD)$

T2. $Ao(BoC) \leftrightarrow (AoB)oC$

T3. $AoB \leftrightarrow BoA$

T4. $A \rightarrow AoA$

T5. $(A \rightarrow B) \rightarrow (AoC \rightarrow BoC)$

T6. $(A \rightarrow B \& C \rightarrow D) \rightarrow (AoC \rightarrow BoD)$

Now let S be the set of all non-empty equivalence classes of formulas of R''_{ICkt} under provable equivalence. Let the equivalence class of A be $\overline{A}$. We define $\overline{A} \cup \overline{B}$ as $\overline{AoB}$; this is a valid definition by T1. Let 0 be $\overline{t}$, $\overline{A} \leq \overline{B}$ iff $\vdash B \rightarrow A$ and $\overline{A} \in F_k$ iff $\vdash A \rightarrow P_k$. Schema 8, together with T2-T6, ensures that $Q=(S,\leq,F)$ is an r´´-model. The completeness proof is concluded by showing that for any A, B, $\bar{A} \models_Q B$ iff $\vdash A \rightarrow B$, the proof of which we omit.

The semantics for E_{IC} extend similarly by the rule:

$x, u \models_Q A \circ B$ iff for some $y,z, x=y \cup z$ and $y,u \models_Q A$ and $z, u \models_Q B$.

With this rule, T1-T6 are validated, as is $(A \rightarrow .B \rightarrow C) \rightarrow .A \circ B \rightarrow C$. The converse of this last schema is invalid.

7.0 DISJUNCTION

It would seem that the extension of the semantics to include disjunction is as unproblematic as the extension to conjunction. We define in an r- or t-model:

$x \models_Q A \vee B$ iff $x \models_Q A$ or $x \models_Q B$,

and in an e-model:

$x,u \models_Q A \vee B$ iff $x,u \models_Q A$ or $x,u \models_Q B$

These rules validate the schemata:

11. $A \rightarrow . A \vee B$

12. $B \rightarrow . A \vee B$

13. $(A \rightarrow C \& B \rightarrow C) \rightarrow . A \vee B \rightarrow C$

14. $(A \& (B \vee C)) \rightarrow . (A \& B) \vee C$

Thus all theorems derivable from the negation-free axioms of E, R and T are valid in the appropriate senses. It seems plausible to conjecture that these negation-free fragments are complete with respect to the semantics. Surprisingly, this conjecture is false. A counter-example, due to the joint effort of J. Michael Dunn and Robert K. Meyer is provided by the schema

15. $[(A \rightarrow A) \& (A \& B \rightarrow C) \& (A \rightarrow . B \vee C)] \rightarrow . A \rightarrow C$ which is r-, t- and e-valid, but not provable in R. Its independence in R may be shown by the matrices in Anderson and Belnap *, §22.1.3; give A the value +3, B the value +0 and C the value -0, so that 15 takes on the undesignated value -3. A slightly simpler schema:

16. $[(A \rightarrow . B \vee C) \& (B \rightarrow D)] \rightarrow (A \rightarrow . D \vee C)$ is also valid under all three definitions but may be shown independent in R by using the same matrices and giving A, B and C the same values as before and D the

value -0. The first schema is easily deduced from the second in the context of T with the help of the distribution axiom.

The program of semantical analysis thus breaks down in the presence of disjunction, and this failure appears irreparable. There seems to be no plausible substitute for the obvious evaluation rule for disjunction. It follows that the evaluation rule for implication, though completely successful where implication alone is concerned, must be basically altered if the full systems R, T and E are to be treated.

8.0 A GENTZEN SYSTEM

Although we have failed to provide a correct analysis of any of the major intensional logics, the systems defined by the semantics appear independently interesting and well-motivated. The problem of axiomatizing these systems when disjunction is included has not been solved in any instance. However, as we now show, the set of r-valid formulas in $\rightarrow$, & and v can be provided with a kind of "Gentzen formulation."

The basic formal objects of the system which we shall call RD are <u>sequents</u> $\alpha \vdash \beta$ where α and β are sequences (possibly empty) of terms. The axioms of RD are all sequents of the form

$$\alpha,\ A_x,\ \beta \vdash \gamma,\ A_x,\ \delta.$$

The rules of RD follow.

$$\rightarrow\vdash \quad \frac{\alpha, A\rightarrow B_x,\ \beta \vdash \gamma,\ A_y \qquad \alpha, A\rightarrow B_x,\ \beta,\ B_{x\cup y} \vdash \gamma}{\alpha,\ A\rightarrow B_x,\ \beta \vdash \gamma}$$

$$\vdash\rightarrow \quad \frac{\alpha,\ A_{\{k\}} \vdash \beta,\ A\rightarrow B_x,\ \gamma,\ B_{x\cup\{k\}}}{\alpha \vdash \beta,\ A\rightarrow B_x,\ \gamma}$$

where k does not occur in x or in any subscript in α, β or γ.

$$\&\vdash \quad \frac{\alpha,\ A\&B_x,\ \beta,\ A_x,\ B_x \vdash \gamma}{\alpha,\ A\&B_x,\ \beta \vdash \gamma}$$

$$\vdash\& \quad \frac{\alpha \vdash \beta,\ A\&B_x,\ \gamma,\ A_x \qquad \alpha \vdash \beta,\ A\&B_x\ \gamma,\ B_x}{\alpha \vdash \beta,\ A\&B_x,\ \gamma}$$

$$\mathrm{v}\vdash \quad \frac{\alpha,\ A\mathrm{v}B_x,\ \beta,\ A_x \vdash \gamma \qquad \alpha,\ A\mathrm{v}B_x,\ \beta,\ B_x \vdash \gamma}{\alpha,\ A\mathrm{v}B_x,\ \beta \vdash \gamma}$$

$$\vdash\mathrm{v} \quad \frac{\alpha \vdash \beta,\ A\mathrm{v}B_x,\ \gamma,\ A_x,\ B_x}{\alpha \vdash \beta,\ A\mathrm{v}B_x,\ \gamma}$$

Our main object is to prove the following completeness theorem.

T1. $\vdash A_{\emptyset}$ is provable in RD iff A is r-valid.

Proof: We first show that a proof of $\vdash A_{\emptyset}$ in RD shows A to be r-valid. This is done by correlating the sequents of RD with statements about validity of formulas in r-structures. In a sequent $\alpha \vdash \beta$ replace each term $A_{\{a,\ldots,m\}}$ by the statement $x_a \cup \ldots \cup x_m \models_Q A$. Then let $S(\alpha \vdash \beta)$ be the statement "For any r-structure Q, if all statements in α are true, then at least one statement in β is true." We may now prove that if $\alpha \vdash \beta$ is provable in RD then $S(\alpha \vdash \beta)$ is true, hence that if $\vdash A_{\emptyset}$ is provable, A is r-valid.

To complete the proof, an algorithm is described which consists of a systematic attempt to construct a proof of a sequent. If the attempt fails the algorithm produces a falsifying r-model. The idea is to start with a given sequent $\vdash A_{\emptyset}$, then apply the rules of RD in reverse, thus producing a growing tree of sequents. First, some definitions. A term A_x occurring in a sequent $\alpha \vdash \beta$ is said to be discharged if one of the eight conditions hold: (1) $\alpha \vdash \beta$ is an axiom of RD, (2) A_x is $B \rightarrow C_x$, occurs in β and for some k, $B_{\{k\}}$ is in α and $C_{x \cup \{k\}}$ is in β, (3) A_x is $B \rightarrow C_x$, is in α and for all y contained in w, either B_y occurs in β or $C_{x \cup y}$ is in α, where w is the union of all subscripts in $\alpha \vdash \beta$, (4) A_x is $B\&C_x$, is in α and B_x and C_x are both in α, (5) A_x is $B\&C_x$, is in β and either B_x or C_x is in β, (6) A_x is BvC_x, is in α and either B_x or C_x is in α (7) A_x is BvC_x, is in β and both B_x and C_x are in β, (8) A is a propositional variable. Otherwise, A_x is undischarged. The rules for the algorithm follow.

Stage 0. Start with $\vdash A_{\emptyset}$ as the origin of the tree.

Stage 2n. Apply the appropriate rule of RD in reverse to the leftmost undischarged term in the tree which is not an implication on the right of a sequent. For instance, let $A\&B_x$ be the leftmost such term, occurring in α in $\alpha \vdash \beta$; we add to the end of the branch a new sequent which is a premiss for an instance of $\& \vdash$ in which $A\&B_x$ is the principal constituent. This stage must eventually produce a tree in which all terms are discharged, with the possible exception of arrow terms on the right of sequents. The reason for this is that no new subscripts are introduced in the stage, while applications of the rules in reverse produce only formulas of lesser complexity than the formula in the term to which they are applied. When this occurs, proceed to stage 2n+1.

<u>Stage 2n+1</u>: Apply $\vdash\rightarrow$ in reverse to the leftmost undischarged term. Proceed to Stage 2n+2.

Three possibilities arise. First, the algorithm may terminate, producing a tree in which all end sequents are axioms. Second, the algorithm may terminate, but in the resulting tree, at least one end sequent is not an axiom. Third, the algorithm may fail to terminate (for an example, apply it to $\vdash A \rightarrow B \rightarrow A \rightarrow A_{\emptyset}$). In the first case, the tree is a proof of $\vdash A_{\emptyset}$ in RD. In the last two cases it must be shown that the resulting tree (in the last case, an infinite tree) provides an r-model in which it is not the case that $0 \models_{Q} A$.

In the second case at least one branch of the tree ends in a sequent which is not an axiom, but in which all terms are discharged. In the third case, it follows by König's lemma that the tree has at least one infinitely long branch. Let us call either type of branch a <u>full</u> branch. A term is said to occur <u>on the left</u> (<u>on the right</u>) of a branch if it occurs on the left (on the right) of some sequent in the branch. Let S(B) be the union of all subscripts occurring on a branch B.

LEMMA 1 If B is a full branch then (1) No term occurs on both the left and right of B, (2) if $A \rightarrow B_x$ occurs on the left of B then for all y contained in S(B) either A_y occurs on the right of B or $B_{x \cup y}$ occurs on the left of B, (3) if $A \rightarrow B_x$ occurs on the right of B then for some y contained in S(B) A_y occurs on the left and $B_{x \cup y}$ on the right, (4) if $A\&B_x$ occurs on the left, both A_x and B_x occur on the left, (5) if $A\&B_x$ occurs on the right either A_x or B_x occur on the right, (6) if AvB_x occurs on the left either A_x or B_x occurs on the left, (7) if AvB_x occurs on the right both A_x and B_x occur on the right.

Proof: Part (1) follows from the fact that terms occurring on either side of sequents occur on the same sides of dominating sequents. If B is finite (2) to (7) follow from the fact that all terms in the end sequent of B are discharged. Assume that B is infinite. Let $A \rightarrow B_x$ occur on the left of B and let y be contained in S(B); then there is an earliest stage n at which $A \rightarrow B_x$ occurs on the left of a partial branch B′ contained in B and y is contained in S(B′). Then since at the next even stage after n $A \rightarrow B_x$ is discharged, either A_y occurs on the right or $B_{x \cup y}$ occurs on the left of B. Let $A \rightarrow B_x$ occur on the right of B. Then since B is infinite, at some stage 2n+1 $A \rightarrow B_x$ is the leftmost undischarged term in a branch B′ contained in B, so that it is discharged at stage 2n+1, proving (3). Parts (4) to (7) in this case follow from the fact that all terms save arrow terms on the right are discharged during even stages.

Now let S be the set of all finite subsets of S(B), and for x in S, let $x \varepsilon F_k$ iff P_{k_x} occurs on the left of B. Then $Q_B=(S,F)$ is an r-model.

LEMMA 2. If A_x occurs on the left of B, $x \models_{Q_B} A$; if A_x occurs on the right of B, not $x \models_{Q_B} A$.

Proof: For propositional variables the result holds by definition; the inductive step follows by Lemma 1.

Hence, if the algorithm fails to produce a proof of $\vdash A_\emptyset$ in RD, then $\vdash A_\emptyset$ appears on the right of a full branch B, so $0 \models_{Q_B} A$ is false. This proves T1.

The success of the Gentzen formulation with respect to completeness, however, does not seem to bring us nearer to providing a standard axiomatization of the set of all r-valid formulas. The difficulty is that there is no apparent way to translate the sequents of RD into the language of R. (A translation is, however, possible where implication alone is concerned -- see Urquhart *).

9.0 NEGATION

Negation, after implication, poses the most interesting problems in the context of intensional logics. It is involved essentially in two of the most notorious "paradoxes of material implication", $(A\&\bar{A})\rightarrow B$ and $A\rightarrow(Bv\bar{B})$. Any account of negation in intensional logics must at the very least avoid rendering these schemata valid.

This fact immediately rules out what could be called the classical valuation rule for negation:

($\sim$C) $x \models_{Q} \sim A$ iff it is not the case that $x \models_{Q} A$, which validates both paradoxes. Their validity can be traced to two consequences implicit in the classical rule. If this rule truly describes the behaviour of the negation connective then all pieces of information must be both consistent and complete. Neither property however, necessarily attaches to a piece of information, as was argued informally in 1.0.

We therefore have to provide a rule for negation which is consistent with the existence of inconsistent and incomplete pieces of information. At the same time, if we wish to follow the lines of the axiomatization of intensional logics provided by Ackermann, Anderson, Belnap and others, we must strive to preserve many laws ordinarily considered characteristic of classical negation (for instance, the law of double negation). This aim is partially fulfilled by an extension of the concept of an r-model.

Let Q=(S,*,F) be an r*-model if (1) (S,F) is an r-model and

(2) * is a function defined on S such that $0^*=0$ and for any x in S, $x^{**}=x$. Then if Q is an r*-model, negation is treated by the rule:

($\sim$*): $x \models_Q \sim A$ iff it is not the case that $x^* \models_Q A$.

Similarly, Q=(S,*,W,R,w,F) is an e*-model if (1) (S,W,R,w,F) is an e-model and (2) * is a function defined on S satisfying the above conditions. The rule for $\sim$ in an e*-model is: $x, u \models_Q \sim A$ iff it is not the case that $x^*, u \models_Q A$.

These definitions have many of the right features. For instance, let S={0,a,b} and let $\cup$ and * be defined by the tables

$\cup$	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

*	
0	0
a	b
b	a

Let $F_1=\{a\}$, $F_2=\{b\}$. As may easily be checked, Q=(S,F) is an r*-model, where F is otherwise arbitrary. Furthermore, Q falsifies both paradoxes mentioned above, as well as the principle of the disjunctive syllogism $(\sim P_1 \& (P_1 v P_2)) \rightarrow P_2$.

We can actually state something stronger. The zero- and first-degree entailments which are r*-valid are exactly those provable in R, or equivalently, in E or T. This fact can be proved by introducing an auxiliary concept. Q=(S,0,*,F) is a *-model if $0 \in S$, * is a function on S such that $0^*=0$, $x^{**}=x$ and F is a value function on S. The recursive valuation rules for &, v, $\sim$ are as for r*-models, but $\rightarrow$ is dealt with by the rule: $x \models_Q A \rightarrow B$ iff for all y in S, if $y \models_Q A$ then $y \models_Q B$. A formula A is *-valid if $0 \models_Q A$ for all *-models. A first-degree formula is one which contains no nested arrows.

<u>Theorem</u>. If A is a first-degree formula, then $\vdash_R A$ iff A is *-valid.

<u>Proof</u>: A proof of this theorem is to be found in Routley *. It may also be derived by combining the completeness results of Belnap 1967 with the representation theorem of Białnycki-Birula and Rasiowa 1957.

In view of the above theorem all that remains to be shown is that a fdf (first-degree formula) is r*-valid iff *-valid. The implication from right to left is trivial. Now let A be a *-invalid fdf, so that A is falsified in a *-model $Q=(S,0,*,F)$. Let f be any one-one map of $(S,0)$ into a set semilattice S'. For $x \varepsilon f(S)$, let $x^{*'} = f((f^{-1}(x))^*)$, $x \varepsilon F'_k$ iff $f^{-1}(x) \varepsilon F_k$. For x in $S'-f(S)$, let $x^*=x$, $x \varepsilon F'_k$ iff $0 \varepsilon F'_k$. Then $Q'=(S',*',F')$ is an r*-model. We observe that for B a zero-degree formula, $x \models_Q B$ iff $fx \models_{Q'} B$ and for x in $S'-f(S)$, $x \models_{Q'} B$ iff $0 \models_Q B$. It follows that if B is a fdf, $0 \models_Q B$ iff $0 \models_{Q'} B$. Hence, the *-invalid formula A is falsified in Q'. Thus we have shown that the r*-valid fdfs coincide exactly with those provable in R.

However, when we go beyond the first-degree fragment even semantic consistency fails. The schemata $A \rightarrow \bar{B} \rightarrow . B \rightarrow \bar{A}$ (contraposition) and $A \rightarrow \bar{A} \rightarrow \bar{A}$ (reductio) are both r*-invalid, though they are both axiom schemata of T. Let $S=\{0,a,b,c\}$, let $\cup$ and * be defined by the tables

$\cup$	0	a	b	c
0	0	a	b	c
a	a	a	c	c
b	b	c	b	c
c	c	c	c	c

*	
0	0
a	b
b	a
c	c

and let $F_1=\{b\}$, $F_2=\{c\}$. Then $Q=(S,*,F)$ is an r*-model which falsifies both $P_2 \rightarrow \sim P_1 \rightarrow . P_1 \rightarrow \sim P_2$ and $P_1 \rightarrow \sim P_1 \rightarrow \sim P_1$.

Of course, it is possible to postulate a variety of negation for which these two principles are assured. $Q=(S,C,F)$ is an rf-model if (S,F) is an r-model and C is a subset of S. The valuation rule for negation is:

(∿f): $x \models_Q \sim A$ iff for all y in S either

not $y \models_Q A$ or $x \cup y$ is in C.

The rf-valid formulas in which only → and ∿ appear can be axiomatized by adding reductio and contraposition to R_I. Let S, F be as in 1.3, and let x be in C iff for some A, A is in x and for some ordering $\overline{x\sim\{A\}}$ of $x\sim\{A\}$, $\overline{x\sim\{A\}} \vdash \sim A$. It is then easy to show that (S,C,F) is a characteristic model for this axiomatization.

The rule (∿f) invalidates principles like ∿∿A→A, A∨∿A and ∿(A&B)→.∿A∨∿B; in general, no formula is rf-valid which is not provable in the minimal logic of Johansson. This fact is incompatible with logics like E and R which contain theorems pointing towards a classical or non-constructive interpretation of negation.

We have so far failed to find an extension of the concept of r-validity suitable for such logics as the full system R. If could be that we have not been ingenious enough. However, it is possible to show that on certain plausible assumptions this is not the case. A simple argument shows that if the class of possible extensions is limited in a fairly weak way no suitable extension exists. Let us say that a model (Q,T) is an <u>expansion</u> of an r-model if Q is an r-model and T is a sequence of relations and functions defined on S, the domain of Q. Now let us suppose that the concept of r-validity has been extended in the following sense: a class of models -- call them <u>rn-models</u> -- has been defined, each rn-model being an expansion of an r-model, and the consequence relation relative to an rn-model has been defined, leaving the consequence definitions unchanged for the positive connectives.

Further, let it be the case that each r-model has an expansion which is an rn-model. Then not all theorems of R are rn-valid. The reason is that if the concept of r-validity is extended as above, then every rn-valid negation-free formula is also r-valid. However, if all theorems of R are rn-valid, since the schema (D) must also be rn-valid, then (by some simple manipulations) the schema $((A \& B \rightarrow C) \& (D \rightarrow B)) \rightarrow (A \& D \rightarrow C)$ must be rn-valid. It is not, however, r-valid, which contradicts the original supposition. Hence, any extension of the positive semantics satisfying the stated conditions must fail to validate some theorems of R.

So far we have discussed only the technical aspects of an attempt to extend the notion of consequence to include negation. A philosophical analysis may disclose some of the deeper reasons underlying the purely formal difficulties discussed above. Let us call the negation operators whose valuation rules are ($\sim$c), ($\sim$*) and ($\sim$f) c-negation (classical negation), *-negation and f-negation (constructive negation) respectively. Of these three, f-negation has an immediate intuitive interpretation. The meaning of the constructive negation operator may be summed up by saying that $\sim$A is a consequence of x just in case A is <u>refutable</u> from x (Curry 1963, Ch. 6). A sentence A is refutable from a piece of information x just in case an obvious absurdity (e.g. "1=2") can be deduced from A and x jointly. What is considered to be obviously absurd will depend on context. In any case, if we read "x is in C" as "an obvious absurdity is deducible from C" then the above discussion exactly describes the content of the rule ($\sim$f).

It is important not to confuse refutability with non-deducibility. For instance, it is clear that

$0 \models$ Xanofon Xatjoules likes lobster is false. On the other hand we cannot say that Xanofon is not fond of lobster, given no information about Xanofon and his preferences. This is a distinction familiar from intuitionism -- from the constructive point of view an undecided conjecture is neither true (provable) nor false (refutable).

If we keep this distinction in mind, the other two varieties of negation seem by contrast to be lacking in intuitive content, at least within the framework of the informal interpretation which we have developed in conjunction with the formal semantics. C-negation interprets negation in terms of non-deducibility. It is clear from the example of the last paragraph that this is quite implausible in the present framework. Similar criticism applies to *-negation. The requirement $0^*=0$ (which is needed to validate classical tautologies like $A \vee \sim A$) shows that at least in the case of 0, refutability is interpreted as non-deducibility, so the argument again applies.

This philosophical analysis does appear to provide a partial explanation for the results of our purely formal investigations. The philosophical analysis, however, is only as good as the set of concepts and presuppositions in which it is conducted. It is possible that a quite different array of informal concepts exists within which *-negation would appear as the most natural analysis of negation.

If the basic semantical concept is that of a possible world, then the rule of c-negation is the only possible choice. The rule (*∿) seems to be closer to the classical than the constructive approach. This fact suggests reinterpreting the elements of an r*-model as possible worlds rather than pieces of information, 0 being interpreted as the actual world and "$x \models A$" being read as "A is true in world x." Or rather, one wishes to say that the elements of an r*-model are almost possible worlds; those elements x for which $x^*=x$ are possible in the sense of being actualisable, while those for which $x^* \neq x$ are conceivable if not actualisable, as they follow certain regular laws in spite of being inconsistent or incomplete. This, at least, corresponds to the interpretation given to the rule (*∿) in Routley & Routley*. However, although this interpretation has a certain plausibility, it is difficult to see what meaning can be attached to the $\cup$ or * operators. Nor is it easy to see in what sense inconsistent "worlds" are conceivable.

Routley and Meyer, using a valuation rule for implication which is a generalization of the rules used above and the rule (∿*) for negation, have succeeded in providing completeness proofs for R, T and other relevant logics. The reader is referred to Routley and Meyer *, **, *** for the details of these and other important results.

10.0 PARADOXES OF IMPLICATION

The systems of intensional logic were expressly devised to avoid the so-called "paradoxes of material implication." A semantical analysis of these systems should throw light on these paradoxes, and on the reasons underlying their acceptance or rejection.

Anderson and Belnap * distinguish fallacies of relevance and fallacies of modality. The archetypical examples of the former and the latter respectively are:

#1. $A \rightarrow . B \rightarrow A$

#2. $A \rightarrow . A \rightarrow A \rightarrow A$

In terms of our semantics a fallacy of relevance might be defined as a theorem of intuitionistic logic which depends essentially on the condition K(§2.7). A fallacy of modality may be defined as a formula which is valid only when the structure of possible worlds is ignored. Further categories of fallacy can be distinguished: fallacies of <u>consistency</u> could be defined as those formulas which would be valid only if inconsistent pieces of information were not admitted, for instance $(A \& \sim A) \rightarrow B$. Fallacies of completeness could be defined as those which depend on the condition that all pieces of information -- or possible worlds -- have as consequence every statement or its negation; an example would be $A \rightarrow . B \vee \sim B$.

These categories of fallacies are mutually independent. We can add a consistency requirement to the definition of an rf-model, thus producing a logic which is a relevant version of intuitionistic rather than the Kolmogorov/Johansson minimal logic. This logic has none of the categories of fallacy save those of consistency and modality. A similar version of ef-models would produce a logic with fallacies of consistency alone. In possible-world semantics for modal logic we may replace the classical negation rule by (*∿), thus producing a logic which lacks paradoxes of consistency and completeness, but contains fallacies of relevance. Routley * carries out this proposal in detail for the case of the modal logic S3. Noting that the set of rf-valid formulas contains fallacies of modality without any from the other categories, we may conclude that each variety of fallacy with the possible exception of fallacies of completeness can be incorporated in a coherent system which excludes all other categories of fallacy.

In any case, the analysis given above indicates that there are at least two intuitions underlying the rejection of the fallacies. That is, the rejection of the rule K as governing the consequence relation, and the acceptance of "badly behaved" (i.e. inconsistent or incomplete) elements in a model appear to be distinct commitments.

Our view on the validity of the paradoxes, then, may depend on what we require our logic to do. In mathematical argument, for instance, we make the idealising assumption that our reasoning does not lead to contradictions, so ruling out contradictory pieces of information. However, if we were investigating the formal logic of knowledge and belief, inconsistent pieces of information would be of great interest and importance. The paradoxes of relevance similarly may or may not be viewed as important depending on our purpose. From the classical point of view a mathematical theorem is a self-sufficient truth -- how we establish its truth is of no importance provided the arguments are sound. From this point of view to infer "If B then A" from "A" is not unsound, though somewhat pointless. However, there is another point of view on mathematical statements, that of intuitionism. Intuitionistically, a mathematical statement is not a self-sufficient entity -- it is rather a highly compressed and inadequate shorthand indication that a certain construction has been effected (Heyting 1956, p. 8). The classical view of logic is that it is made up of statements which are true under all possible circumstances; in contrast, the logical truths of intuitionism represent proofs and methods of construction applicable in all contexts. If, then, we regard the purpose of a logical calculus as the representation of constructions or proofs rather than truths, it is clear that the stronger a logic is (from the classical point of view) the less adequate it is from our point of view. If we allow the rule of weakening, we "blur" the representation of proofs so far as to obscure what premisses are actually used in an argument, and form its core. It can be plausibly argued along these lines that the most adequate logic, from the point of view of proof representation, would be something like R_{123}. Although excessively weak from the classical standpoint, R_{123} exactly represents proofs in the sense that a theorem of this logic displays exactly the number and order of the non-logical premisses used in a proof. To put it another way, if we adopt the intuitionistic viewpoint with regard to logic, the omission of the paradoxes of relevance appears not as an arbitrary excision of odd-seeming but harmless principles, but as a natural consequence of the attempt to represent proofs and constructions as accurately as possible in a formal system.

11.0 MANY-VALUED LOGICS

11.1 Introduction

Developments in model theory over the past two decades have shown the fruitfulness in non-standard logic of the general concept of a "model-structure" consisting of "points of reference" or "indices" at which formulas are evaluated (for a general formulation see Cresswell 1971, Montague 1968). In view of these develments, it is surprising that the many-valued logics of Łukasiewicz have not been fitted into this general framework, as they were perhaps the first non-standard logics to be investigated in depth. In the present section we attempt to fill this gap by presenting a tense-logical interpretation of Łukasiewicz's matrices, incidentally revealing a surprisingly close connection between these matrices and our earlier semantical analyses of implication and negation.

11.2 Model theory

For $n \in N$ let $S_n = \{x \in N : o \leq x \leq n\}$. A <u>valuation over S_n</u> is a function F defined on N such that

(i) $F_k \subseteq S_n$

(ii) If $x \in F_k$, $y \in S_n$ then $y \in F_k$.

For x in S_n, F a valuation over S_n, A a formula in $\rightarrow$, $\sim$, the consequence relation $x \models^F_n A$ (read "A is assertable at x under F") is defined recursively as follows.

1. $x \models^F_n Pk$ iff $x \in F_k$
2. $x \models^F_n \sim A$ iff it is not the case that $(n-x) \models^F_n A$
3. $x \models^F_n A \rightarrow B$ iff for all y in S_n, if x+y is in S_n and $y \models^F_n A$,

then $(x+y) \models^F_n B$.

LEMMA. If $x \models^F_n A$ and $x \leq y$ the $y \models^F_n A$.

Proof. By induction on the complexity of A, using the fact that if $x \leq y$ then $n-y \leq n-x$ and $x+z \leq y+z$.

Note that if, following Łukasiewicz, we define $A \vee B$ as $(A \rightarrow B) \rightarrow B$ and A&B as $\sim(\sim A \vee \sim B)$ then the derived semantical conditions are

4. $x \models^F_n A \vee B$ iff $x \models^F_n A$ or $x \models^F_n B$,

5. $x \models^F_n A \& B$ iff $x \models^F_n A$ and $x \models^F_n B$.

A formula A is n-valid, $\models_n A$, if $0 \models^F_n A$ for all valuations F over S_n, and ω-valid, $\models_\omega A$, if $\models_n A$ for all n.

11.3 Informal Explanation

The elements of S_n can be regarded as moments of time, with n, the last element in S_n, some fixed future date, while 0 is interpreted as the present moment. The statement "$x \models^F_n A$" can be read "A is assertable at moment x." A proposition may or may not be assertable at a given moment; for instance, a proposition about some future event may or may not be assertable now. However, if A is assertable now it is assertable at all later times. This means that in the present context we do not think of propositions as temporally indefinite (e.g. "Abraham Lincoln is President now") but as temporally definite (e.g. "Abraham Lincoln is President in 1971 A.D.") So far, our informal explanation is in line with the philosophical motivation given in Łukasiewicz 1930. The connectives $\rightarrow$ and $\sim$, however, do not appear to correspond to the standard interpretations of implication and negation. The standard interpretation would be embodied in the rules:

$x \models^{F}_{n} A \rightarrow B$ iff for all y in Sn, if $x \leq y$ and $y \models^{F}_{n} A$ then $y \models^{F}_{n} B$;

$x \models^{F}_{n} \sim A$ iff it is not the case for some y in S_n that $y \models^{F}_{n} A$.

(Note that for not — A to be assertable at x it is not sufficient that A is not assertable at x -- it must be the case that A is never assertable.) By contrast, in the present interpretation of Łukasiewicz's systems A→B is assertable at time x iff whenever A is assertable at time y, B is assertable at time x+y, i.e. at a time x instants in the future of y. The statement ∿A is assertable at time x iff A is not assertable at the instant which is x instants in the past of the last moment in the temporal series. Thus both the "implication" and "negation" connectives of Łukasiewicz appear to differ considerably from the standard implication and negation operators. We feel that this fact goes some way towards explaining the difficulties of interpretation associated with these logics.

11.4 Equivalence of model theory and matrices

The n+2-valued matrices of Łukasiewicz, $n \epsilon N$ are defined as follows. The space of truth-values is S_{n+1}. A function v defined on the formulas in →, ∿ is a valuation in $Ł_{n+2}$ if $v(Pk) \epsilon S_{n+1}$ for all k and $v(A \rightarrow B)$ is $\min(n+1, (n+1)-v(A)+v(B))$, $v(\sim A)$ is $(n+1)-v(A)$. A formula A is valid in $Ł_{n+2}$, $Ł_{n+2} \models A$, if $v(A)=n+1$ for all valuations v in $Ł_{n+2}$. By Thm. 17(c) of Łukasiewicz and Tarski 1930, A is valid in $Ł_{\aleph_0}$, the infinite-valued logic of Łukasiewicz, if $Ł_{n+2} \models A$ for all n; we write $Ł_{\aleph_0} \models A$ when this obtains.

Theorem. Let v be a valuation in $Ł_{n+2}$. Let $F^{v}_{k} = \{x \epsilon S_n : x \geq (n+1) - v(P_k)\}$ Then for x in S_n, $x \models^{F^v}_{n} A$ if and only if $x \geq (n+1)-v(A)$.

Proof:

(i) $x \models_{n}^{F^{v}} P_k$ iff $x \varepsilon F_k^{v}$

iff $x \geqslant (n+1) - v(P_k)$.

Now assume the theorem for A and B.

(ii) $x \models_{n}^{F^{v}} \sim A$ iff not $(n-x) \models_{n}^{F^{v}} A$

iff not $(n-x) \geqslant (n+1) - v(A)$

iff $(n-x) < (n+1) - v(A)$

iff $(n-x) < v(\sim A)$

iff $x \geqslant (n+1) - v(\sim A)$

(iii) (a) Let $v(A) \leqslant v(B)$, so that $v(A \rightarrow B) = n+1$. Then for $x+y \varepsilon S_n$, if $y \models_{n}^{F^{v}} A$, then $y \geqslant (n+1) - v(A)$, so $y \geqslant (n+1) - v(B)$, hence $(x+y) \geqslant (n+1) - v(B)$, so $(x+y) \models_{n}^{F^{v}} B$. Hence for any $x \varepsilon Sn$, $x \models_{n}^{F^{v}} A \rightarrow B$, so $x \geqslant (n+1) - v(A \rightarrow B)$.

(b) Let $v(A) > (B)$. $x \models_{n}^{F^{v}} A \rightarrow B$ iff for all $y \varepsilon Sn$, if $x+y \varepsilon Sn$ and $y \geqslant (n+1) - v(A)$ then $(x+y) \geqslant (n+1) - v(B)$, iff $x \geqslant (n+1) - ((n+1) - v(A) + v(B))$ iff $x \geqslant (n+1) - v(A \rightarrow B)$.

This completes the proof.

Corollary. $\models_{n} A$ iff $Ł_{n+2} \models A$

$\models_{\omega} A$ iff $Ł_{\aleph_0} \models A$.

11.5 Generalizations.

The semantics for pure Łukasiewicz implication can be extended immediately to more general models. Let S be a non-empty subset of the non-negative reals such that if $x, y \varepsilon S$ and $x \leqslant y$ then $y - x \varepsilon S$. A valuation F over S must satisfy the condition that F_k is either empty or is $\{x \varepsilon S : x \geqslant a\}$ for some a in S. Truth at a point in S and validity in S are defined exactly as for S_n. It is then not difficult to check that all pure implicational theorems of $Ł_{\aleph_0}$ are valid in S, by using the axiomatization of Rose 1956. It may be noted that the restriction on valuations is essential. If we merely require as for S_n that if $x \varepsilon F_k$, $x \leq y$, $y \varepsilon S$ then $y \varepsilon F_k$ then not all theorems of $Ł_{\aleph_0}$ are validated. Let S be the set of all non-negative reals, $F_o = \{x : x > 1\}$, $F_1 = \{x : x \geqslant 2\}$. Then $1 \models^{F}_{s} (P_0 \to P_1) \to P_1$, but not $1 \models^{F}_{s} (P_1 \to P_0) \to P_0$, so the $Ł_{\aleph_0}$-valid formula

$((P_0 \to P_1) \to P_1) \to ((P_1 \to P_0) \to P_0)$ is falsified in S under F.

12.0 ANALYTIC IMPLICATION

12.1 Introduction

A well-known argument of C.I. Lewis (Lewis and Langford 1932 pp. 250-51) purports to derive an arbitrary proposition from a contradictory proposition. Indeed, if some apparently harmless principles of inference involving conjunction, disjunction and negation are accepted, $P\&\sim P\to Q$ must also be accepted as a law of logic. In the systems so far discussed, this conclusion has been avoided by a rejection of the law of disjunctive syllogism. As we have seen, this is not simply an *ad hoc* device, but is in fact the only choice possible in a semantical framework allowing inconsistent pieces of information.

An alternative approach is to deny the validity of the principle $A\to .A\vee B$. This seems to have much to recommend it; this principle is precisely the one which introduces the irrelevant conclusion Q into the argument. The law of disjunctive addition can be rejected according to a Kantian view of implication (definition of "analytic", CPR A7, B11). Kant's definitions are clearly meant to apply only to statements of subject/predicate form. However, we may extend the concept of analyticity to include entailment. Under this view, the consequent of an entailment $A\to B$ should simply "unpack" the antecedent, so that every concept occurring in B should also occur in A. Under this condition $A\to .A\vee B$ is indeed invalid. The paradox of relevance $A\to .B\to A$ is also invalid, as well as $A\to B\to .B\to C\to .A\to C$, and any formula $A\to B$ in which a variable occurs in B but not in A.

12.2 Parry's system and others.

A system embodying these ideas is set forth in Parry 1933, who calls it a system of "Analytic Implication". The primitive connectives are $\rightarrow$, & and $\sim$, with $\leftrightarrow$, $\vee$ and $\supset$ defined as usual. The axioms of Parry's system are all the instances of the following schemata.

A1. $A\&B\rightarrow B\&A$

A2. $A\rightarrow A\&A$

A3. $A\rightarrow\sim\sim A$

A4. $\sim\sim A\rightarrow A$

A5. $A\&(B\vee C)\rightarrow(A\&B)\vee(A\&C)$

A6. $A\vee(B\&\sim B)\rightarrow A$

A7. $(A\rightarrow B)\&(B\rightarrow C)\rightarrow.A\rightarrow C$

A8. $(A\rightarrow B\&C)\rightarrow.A\rightarrow C$

A9. $(A\rightarrow B)\&(C\rightarrow D)\rightarrow.A\&C\rightarrow B\&D$

A10. $(A\rightarrow B)\&(C\rightarrow D)\rightarrow.A\vee C\rightarrow B\vee D$

A11. $A\rightarrow B\rightarrow.A\supset B$

A12. $(A\leftrightarrow B)\&f(A)\rightarrow f(B)$

A13. $f(A)\rightarrow.A\rightarrow A$

The sole rule of inference is <u>modus</u> <u>ponens</u>. Parry 1968 adds the rule of adjunction and the further schemata

A14. $(\sim M\sim A\&(A\rightarrow B))\rightarrow\sim M\sim B$

A15. $\sim(A\supset B)\rightarrow\sim(A\rightarrow B)$,

where MA is defined as $\sim(A\rightarrow\sim A)$. The independence of A15 is proved in Dunn *.

Parry proves of his system that it satisfies what he calls the

Proscriptive Principle; no formula $A \rightarrow B$ holds if B has a variable not occurring in A. Gödel in the discussion following Parry 1933 raised the question of completeness for Parry's system:

'"p analytically implies q" can perhaps be interpreted as follows: "q is derivable from p and the axioms of logic and contains no concepts other than those in p" and it may be that after this definition has been made more precise a completeness proof for Parry's axioms could be obtained, in the sense that all propositions which are valid for the above interpretation of $\rightarrow$ are provable.'

The system consisting of the schemata A1 - A15 and the two rules of inference we shall call ASI, or the system of Analytic Strict Implication. The adjective "strict" is used because ASI is contained in Lewis' system of strict implication S4. ASI may be "demodalized" by adding the postulate

A16. $A \rightarrow . \sim A \rightarrow A$

which is not a theorem of S4. The resulting system may be called the system of analytic classical implication (ACI). These two systems conform in their separate ways to the definition of validity proposed by Gödel.

The system which forms the main object of our investigations is formed by "remodalizing" ACI by the addition of an explicit necessity operator N. We add to ACI the axiom schemata

A17. $N(A \rightarrow B) \rightarrow . NA \rightarrow NB$

A18. $N(A \& B) \rightarrow NA \& NB$

A19. $NA \rightarrow A$

A20. $NA \rightarrow NNA$

A21. $(A\rightarrow A)\rightarrow N(A\rightarrow A)$

and the rule of necessitation (from A to infer NA). To A12 we add the restriction that in f(A) there is no occurrence of A within the scope of a necessity operator. This system we shall refer to as AIN, or Analytic Implication with Necessity.

We now list some theorems of AIN, referring the reader in most cases to Dunn * for proofs.

<u>T1.</u> $B\rightarrow .A\rightarrow A$, where B is a formula in which occur all variables in A.

<u>Proof</u>: As in Dunn * T4, using A17, A21.

<u>T2</u>. A, where A is a classical tautology. Let us write $S\vdash A$ if A is derivable from S and the axioms of AIN by the rules of <u>modus ponens</u> and adjuntion.

<u>T3</u>. If $S, A\vdash B$, then $S\vdash A\supset B$

<u>Proof</u>: By T2.

<u>T4</u>. If $S, A\vdash B$ and every variable which occurs in B also occurs in A then $S\vdash A\rightarrow B$.

<u>T5</u>. $A\&B\rightarrow .A\leftrightarrow B$, provided A and B contain exactly the same set of variables.

<u>Proof</u>: By T4.

<u>T6</u>. $NA_1 \&\ldots\& NA_n\rightarrow N(A_1 \&\ldots\& A_n)$

<u>Proof</u>: By A18.

<u>T7</u>. $N(A\supset B)\supset .NA\supset NB$

<u>Proof</u>: By T3, A18.

<u>T8</u>. $\sim A\&\sim B\&(\sim A\rightarrow \sim B)\rightarrow .A\rightarrow B$

<u>Proof</u>: See Dunn T24.

T9. $A\&B\&(\sim A\rightarrow\sim B)\rightarrow.A\rightarrow B$

Proof: Assume $A\&B\&(\sim A\rightarrow\sim B)$. Then B, and $(\sim A\rightarrow\sim B)$ by the conjunction axioms. Now $\sim A\rightarrow\sim A$ by A3, A4, A7, hence $\sim A\rightarrow\sim A\&\sim B$ by A2, A7, A9. Hence $\sim A\leftrightarrow\sim A\&\sim B$ by adjunction. Now we have $A\vee B\rightarrow B$ by T4, hence $A\rightarrow B$ by A3, A4, T7 and A12. Hence T9 follows by T4. The author is indebted to Professor J.M. Dunn for this proof (personal communication).

T10. $\sim A\&B\&(A\rightarrow B)\rightarrow.\sim A\rightarrow B$

Proof: By Dunn * T26, and T9 above.

T11. $\sim A\&\sim B\&(A\rightarrow B)\rightarrow(\sim A\rightarrow\sim B)$

Proof: By T9, A3, A4.

A set S of sentences is consistent if for no B is it the case that $S\vdash B\&\sim B$, maximally consistent if S is consistent, but is not properly contained in any consistent set.

T12. Every consistent set of sentences is contained in a maximally consistent set.

Proof: By the methods of the corresponding theorem for classical logic.

T13. If M is a maximally consistent set, then for any A,B, (i) $A\epsilon M$ iff $M\vdash A$, (ii) $\sim A\epsilon M$ iff $A\not\epsilon M$, (iii) $A\vee B\epsilon M$ iff $A\epsilon M$ or $B\epsilon M$, (iv) $A\&B\epsilon M$ iff $A\epsilon M$ and $B\epsilon M$.

Proof: By T2.

11.2 Semantics for analytic implication

$Q=(I,W,\omega,\leqslant,R,F)$ is an aω-model if (i) I is a non-empty set, (ii) $\omega\epsilon W$, (iii) $\leqslant$ is a transitive reflexive relation defined on W, (iv) $R\subseteq W\times I$, (v) F is a value function on $I\cup W$. Relative to Q two "consequence" relations are defined recursively, one having I as domain, the other W.

For x in I,

$x \models_Q P_k$ iff $x \varepsilon F_k$,

$x \models_Q A\&B$ iff $x \models_Q A$ or $x \models_Q B$,

$x \models_Q {\sim}A$ iff $x \models_Q A$,

$x \models_Q A \rightarrow B$ iff $x \models_Q A$ or $x \models_Q B$,

$x \models_Q NA$ iff $x \models_Q A$.

For u in W,

$u \Vdash_Q P_k$ iff $u \varepsilon F_k$,

$u \Vdash_Q A\&B$ iff $u \Vdash_Q A$ and $u \Vdash_Q B$,

$u \Vdash_Q {\sim}A$ iff not $u \Vdash_Q A$

$u \Vdash_Q A \rightarrow B$ iff for all x in I such that uRx, if $x \models_Q B$ then $x \models_Q A$, and if $u \Vdash_Q A$ then $u \Vdash_Q B$.

$u \Vdash_Q NA$ iff for all v such that $u \leqslant v$, $v \Vdash_Q A$.

$Q=(I,\omega,F)$ is an a-model if I and F are as in an aω-model. The consequence relation relative to I is defined as above, save that the clauses for N are omitted, and the clause for $\rightarrow$ is simplified to read:

$\omega \Vdash_Q A \rightarrow B$ iff for all x in I, if $x \models_Q B$ then $x \models_Q A$, and if $\omega \Vdash_Q A$ then $\omega \Vdash_Q B$.

A formula A is <u>true</u> in a model Q if $\omega \Vdash_Q A$ and <u>aω-valid</u> (<u>a-valid</u>) if true in all aω-models (a-models).

These semantics should be understood as an attempt to make precise Gödel's informal definition of validity. In an aω-model I is to be interpreted informally as a set of concepts, while W is a set of possible worlds, ω being the real world. The relation $\leqslant$ is one of relative possibility, while for $u \varepsilon W$, $x \varepsilon I$, "uRx" is to be read as "x is a concept conceivable in world u." For x in I, "$x \models_Q A$" is to be read "concept x occurs in sentence A," while "$u \Vdash_Q A$" is to be read "A is true in possible world u." Under this interpretation the definitions are self-explanatory. The only one worthy of comment is the rule for $\rightarrow$ relative to W. According to this rule, $A \rightarrow B$ is true in world u iff (i) B follows from A in the sense of classical logic and (ii) every conceivable concept (that is, every concept conceivable in world u) which occurs in B also occurs in A. Thus the concepts of aω-validity and a-validity seem to be acceptable as a way of making Gödel's idea precise.

12.4 Completeness

Theorem 1. A formula of AIN is provable in AIN iff it is aω-valid.

Theorem 2. A formula of ACI is provable in ACI iff it is a-valid.

It is left to the reader to show semantic consistency for AIN. We now proceed to show completeness.

Let A be unprovable in AIN. Then $\{\sim A\}$ is consistent, so by T12, $\sim A$ is in some maximally consistent set M. We shall show A invalid by constructing an aω-model Q in which ω is M and for any B, $\omega \Vdash_Q B$ iff $B \varepsilon M$.

Let W be the set of all maximally consistent sets of sentences. For $u, v \varepsilon W$ let $u \leq v$ hold iff for any A if $NA \varepsilon u$ then $A \varepsilon v$. For an arbitrary set of sentences x let uIx hold iff (i) $x \subseteq u$, (ii) if $A \rightarrow B \varepsilon u$ and $A \varepsilon x$ then $B \varepsilon x$, (iii) if A, $B \varepsilon x$ then $A \& B \varepsilon x$. Let uR(v,x) hold iff u=v and vIx. Let $f_u(A)$ be A or $\sim A$ according to whether A is or is not in u (note that $f_u(A) \varepsilon u$ by T13). Let (u,x) be in F_k iff

$f_u(A)$ is not in x, and let $u \varepsilon F_k$ iff $P_k \varepsilon u$.

By A19, $\leqslant$ is reflexive, and by A20 $\leqslant$ is transitive, so $Q=(I,W,M,\leqslant,R,F)$ is an aω-model. Completeness is a consequence of the following

LEMMA. For any A, (a) $(u,x) \models_Q A$ iff $f_u(A) \notin x$, (B) $u \Vdash_Q A$ iff $A \varepsilon u$.

Proof of (a). Part (a) holds for propositional variables by definition. The inductive cases all follow from T5. We illustrate by treating the case of $\rightarrow$. Firstly, note that

$$f_u(A) \& f_u(B) \& f_u(A \rightarrow B) \rightarrow . f_u(A) \& f_u(B) \leftrightarrow f_u(A \rightarrow B)$$ is an instance of T5. Since the antecedent is in u, so is

$$f_u(A) \& f_u(B) \leftrightarrow f_u(A \rightarrow B).$$

Hence:

$$\begin{aligned}(u,x) \models_Q A \rightarrow B \text{ iff } & (u,x) \models_Q A \text{ or } (u,x) \models_Q B \\ \text{iff } & f_u(A) \notin x \text{ or } f_u(B) \notin x \\ \text{iff } & f_u(A \rightarrow B) \notin x\end{aligned}$$

Proof of (b). The basis case holds by definition. The inductive steps for & and $\sim$ follow by T13. It only remains to prove (b) for the case of $\rightarrow$ and N.

If $NA \varepsilon u$ then $u \Vdash_Q NA$ by the definition of $\leqslant$. Conversely, suppose that $NA \notin u$. Consider $S=\{B : NB \varepsilon u\}$. If $s \cup \{\sim A\}$ were inconsistent, then $S \vdash A$, so that

$\vdash B_1 \& \ldots \& B_n \supset A$, $B_i \varepsilon S$,

$\vdash N(B_1 \& \ldots \& B_n \supset A)$ by RN

$\vdash N(B_1 \& \ldots B_n) \supset NA$ by T7

$\vdash NB_1 \& \ldots \& NB_n \supset NA$ by T6.

But then $NA \varepsilon u$, contrary to assumption. Thus $S \cup \{\sim A\}$ is consistent and by T12 has a maximally consistent extension v. Hence, for some v, $u \leq v$ but not $v \models_Q A$, so that not $u \models_Q A$, as was to be proved.

Now for the case of $\rightarrow$. Firstly, let $A \rightarrow B \varepsilon u$. Then by A11, T13, either $A \not\varepsilon u$ or $B \varepsilon u$, so by inductive assumption, either not $u \models_Q A$ or $u \models_Q B$. Let uIx. We observe that for any u,

$f_u(A) \& f_u(B) \& (A \rightarrow B) \rightarrow . f_u(A) \rightarrow f_u(B)$ is provable by T9, T10. Hence if $f_u(A) \varepsilon x$, $f_u(B) \varepsilon x$ so that if $(u,x) \models_Q B$ then $(u,x) \models_Q A$, by part (a). It follows that $u \models_Q A \rightarrow B$. Secondly, let $A \rightarrow B \not\varepsilon u$, but either not $u \models_Q A$ or $u \models_Q B$. We aim to show that for some (u,x) such that uR(u,x), $(u,x) \models_Q B$ but not $(u,x) \models_Q A$. Let a be the set $\{C : f_u(A) \rightarrow C \varepsilon u\}$. By A7, A9, uIa, so uR(u,a). Evidently, $f_u(A) \varepsilon a$, so $(u,a) \models_Q A$ is false, by part (a) of the lemma. Now by T8, A16,

$f_u(A) \& f_u(B) \& (f_u(A) \rightarrow f_u(B)) \rightarrow . A \rightarrow B$ is provable, since either $f_u(A)$ is $\sim A$ or $f_u(B)$ is B. Hence, if $f_u(B) \varepsilon a$ then $f_u(A) \rightarrow f_u(B) \varepsilon u$, so $A \rightarrow B \varepsilon u$, contrary to assumption. It follows that $f_u(B)$ is not in a, so $(u,a) \models_Q B$. This concludes the proof.

Now since $\sim A \varepsilon M$, it follows that $M \models_Q A$ is false, so A is refutable in an aω-model.

The proof of Theorem 2 proceeds just as for Theorem 1, but omitting any references to W, ≤ or R.

We conclude this section with a conjecture concerning the completeness of ASI. For A a formula of ASI, we define A^N, the translation of A into the language of AIN as follows. P_k^N is P_k, $(A \& B)^N$ is $(A^N \& B^N)$, $(\sim A)^N$ is $\sim(A^N)$, $(A \rightarrow B)^N$ is $(N(A^N \rightarrow B^N))$. We conjecture that $\vdash_{ASI} A$ iff $\vdash_{AIN} A^N$, so that ASI is complete with respect to a version of the semantics of the last section.

12.5 Decidability

Decidability can be proved for AIN by the method of _filtrations_ due to Lemmon and Scott, and applied in Segerberg 1968a and 1968b. As the application of this method to AIN does not involve any new ideas in principle, we leave the proof to the reader.

12.6 Other analytic systems

It is clear that the model-theoretic ideas applied in the preceding sections are by no means limited to ACI, ASI and AIN, but are applicable to a wide variety of systems. Thus ACI and AIN are "analytic" versions of classical logic and S4 respectively. The semantics of AIN obviously generalizes to the semantics of other modal systems like M, S2, S3, S5 (Parry 1968 considers an analytic of S5). Similarly, analytic versions of E,R etc. are possible. For instance, in an analytic version of R_I, $A \rightarrow . A \rightarrow A \rightarrow A$ is provable but $A \rightarrow . A \rightarrow B \rightarrow B$ is not.

12.7 Relation to Dunn's results.

It should be emphasized that the completeness and decidability results given for ACI above are not new. Dunn in * proves a completeness result for ACI relative to certain algebras which he calls Parry matrices; he also proves decidability for ACI by algebraic matrix methods. In the final section of *, he gives a representation for Parry matrices which he credits to Robert K. Meyer. Apart from inessential details, this representation coincides with the semantical analysis of §11.2. Furthermore, the completeness proofs presented in preceding sections are essentially adaptations of Dunn's algebraic completeness proofs.

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THE SEMANTICS OF ENTAILMENT

Alasdair Ian Fenton Urquhart, Ph.D.

University of Pittsburgh, 1973

Systems of entailment designed to avoid the "paradoxes of material implication" have been intensively studied by Anderson and Belnap after initial development by Church and Ackermann. However, no systematic semantical theory was developed for these systems. The main aim of the present study is to provide such a theory for logics related to the systems E and R of Anderson and Belnap. The first part shows how the implicational fragments of E and R and other entailment logics can be analyzed in terms of valuations over a semilattice, interpreted informally as the semilattice of "pieces of information." Completeness proofs are given for these implicational fragments, and also for the fragments which include both conjunction and implication. The second part shows how these completeness results fail to extend to the positive fragments of E and R (containing disjunction) or to the fragments of these systems which contain negation. The third part shows how the many-valued logics of Łukasiewicz can be interpreted so that the semantical conditions on implication and negation bear a striking resemblance to those given earlier for entailment logics. The final section gives a semantical analysis of a different entailment logic, W.T. Parry's system of Analytic Implication. Completeness results are given which extend those of J.M. Dunn in this area.