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On the Eliminatibility of Ideal Linguistic Entities

To the memory of Jerzy Słupecki

Abstract. With reference to Polish logico-philosophical tradition two formal theories of language syntax have been sketched and then compared with each other. The first theory is based on the assumption that the basic linguistic stratum is constituted by object-tokens (concrete objects perceived through the senses) and that the types of such objects (ideal objects) are derivative constructs. The other is founded on an opposite philosophical orientation. The two theories are equivalent. The main conclusion is that in syntactic researches it is redundant to postulate the existence of abstract linguistic entities. Earlier, in a slightly different form, the idea was presented in [27] and signalled in [26] and [25].

Idealization, and so also abstraction, has become an indispensable procedure nowadays widely made use of in sciences, the science of language included. While its product are ideal entities, derivative in relation to physical objects, idealization may lead to useful fiction that facilitates considering physical objects. Still, one should also allow for another, specific idealization, e.g., mathematical or logical ascertainment. Many mathematicians and logicians are familiar with the belief that the truths of mathematical and logical theories, their axioms and theorems, are not material recordings but geometrical products, abstract objects whose representations are concrete, material recordings, that is — physical objects.

Nothing then, I believe, hinders accepting the fact that in the theory of language there exist both material linguistic objects, taking the shape of inscriptions or sounds of speech, as well as abstract linguistic entities. Such is after all, though often unconsciously managed, semiotic praxis.

From the point of view of philosophy, however, it is not indifferent whether these linguistic objects of double ontological nature are ascribed an independent existence or not and if not — to which of them the primitive existence is ascribed, and to which the derivative one. The philosophical assumptions influence also the choice of this and no other formal concept of language. Assuming, for instance, that the simplest linguistic objects are geometrical products possessing the primitive existence we may, similarly as Euclidean geometry does, postulate their existence by accepting the appropriate axioms.

Let us note that Alfred Tarski in his famous work [21] devoted to the problem of truth, while expounding the axioms of metascience, postulates that language expressions are abstract entities, intuitively understood as classes of equiform concrete inscriptions.

We will slightly modify and develop the idea of Tarski, as related to metalogic, in order to sketch in Sec. 5 (cf. [25–27]) a formal theory **T2** of language syntax, the theory deriving from certain abstract objects, namely types of inscriptions which function as primitive entities. Material inscriptions (concretes) will be defined in it. This theory will be compared with the theory **T1** which is built in Sec. 4 and presents an opposite approach (cf. [23], [24]). The effect of the comparison of the two theories (Secs. 6 and 7) explains the title of the work — a reflection of some views of Jerzy Słupecki.

At the end of his life Jerzy Słupecki inclined towards Leśniewski's nominalism; in the question of the nature of linguistic objects he accepted Kotarbiński's assumptions of ontological reism. As far as I know, Słupecki was the first to attempt to formalize certain linguistic aspects referring to concrete and abstract words. He initiated first some research in this direction [7] with reference to the theory of algorithms of A. A. Markov [16], and then inspired researches on language carried out by the author of the present work, which were crowned with a monograph [24]. A common idea of these studies was a *concretizing approach* to language, i.e., postulating the existence of inscriptions and words as concretes and ascribing derivative existence to the types of inscriptions or words treated as certain abstract products — trough linguistic abstraction.

I would like to believe that the present text successfully draws out from dimness and develops certain ideas worked out by my teacher, and in this way — by the linguistic concretization — calls him from the non-existence into derivative existence — now only intentional.

1. Non-uniform semiotic characterization of language

Certainly, one of the turning points in the twentieth-century linguistics was *Cours de linguistique générale* by Ferdinand de Saussure — a work published posthumously in 1916. It includes a postulate of scientific description of language as *la langue* — the system of wholeness of elements, signs bound by certain relations and performing certain functions, the system which is, at the same time, the mechanism serving as a tool of the communicative act between people. The postulate is by all means up-to-date. It requires a wide scientific characterization of language, taking into consideration the famous tripartition of semiotics advocated by C. Morris [17], which divides that discipline into syntax, semantics and pragmatics.

It does not mean, however, that language, in the theory of language, is not characterized in a narrower sense — exclusively syntactically, as, for example, in the epochal work by Noam Chomski — *Syntactic Structures* (1957), or at most, with the semantic component added only.

A uniform semiotic characterization of language is made difficult because of the interpretative concept of language as a product built of words. In the above-mentioned division of semiotics the concrete and abstract linguistic

entities occupy different, though equivalent places. The abstract expressions perform theoretical role. In pragmatics they serve the purpose of explaining the process of communication between people; in semiotics, by their means, such basic terms as denotation, truth, or meaning are explained; in syntactic studies they help to formulate grammatical rules. Linguistic description on the pragmatic level, which concerns the functionality of language (see e.g. [3] or [20]), is connected with the use of expressions in context, and consequently, without doubt, with linguistic concretes. Also an analysis of syntactic correctness of a given expressions and, in reference to it — making use of, for example, K. Ajdukiewicz's algorithm [1] (as a system of psycho-physical activities) demands the use of linguistic concretes.

Thus, language is a construct of a double nature: it consists of *tokens* (concretes) and *types* (ideal objects). The differentiation *types-tokens* made by C. S. Pierce (see [19]) and propagated through works by R. Carnap and Y. Bar-Hillel (see e.g. [11], [2], [3]) has been adopted for good in logic and semiotics. *Types* are generally understood here as classes of equiform (or equisounding) *tokens*. Yet it is not always so. As Witold Marciszewski[1] rightly observed they may be understood as concretes, e.g. some undetermined equiform inscriptions with data defined by means of D. Hilbert's eta-operator of indefinite description.

2. Preliminary conventions concerning language

For the purpose of the present work it seems indispensable and useful to establish certain unification of language and, consequently — some conventions. Thus:

1. Language will be characterized exclusively syntactically;

2. Language analysis will not concern spoken language[2];

3. Language will be considered in two aspects: as the language of *tokens* (*token level*) and as the language of *types* (*type level*);

4. *Tokens* will be understood as empirical objects perceived by sight; *types* — as sets of *tokens* established by equiformity relation, i.e. as some abstract products;

5. *Tokens* may, yet need not, be inscriptions on paper, table, sign-board, stone, etc. They may be some configurations of stars or colourful objects, smoke signals, or light illuminations, or the so-called "live pictures" during entertainments and shows, and so on;

6. Equiformity of *tokens* is determined by the pragmatic aim. We will assume that equiformity is an equivalence relation;

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[1] The observation was included in the review of [24].
[2] A formal concept of such a language is presented by T. Batóg in [5].

7.   The syntactic characterization of language will consider an approach referring to the theory of syntactic categories of S. Leśniewski [14] in the version modified by K. Ajdukiewicz [1] (cf. also M. J. Cresswell [12], [13] and A. Nowaczyk [18]). The idea of such an approach is to generate *concatenations* from a vocabulary of a given language which would be its *functorial expressions* (i.e. composed of the main functor and its arguments) and to assess which of them are well formed. The assessment is made with the help of *categorial indices* (types) ordered one by one (on the *token level* with the exactitude to equiformity) to every expression of a given language and precisely delimiting, at the same time, syntactic category of every language expression. It consists in checking if for every constituent of a given functorial expression the rule which expresses the *superior principle of the theory of syntactic categories* holds: the index of the main functor of a compound expression is determined by the index of this expression and indices of the arguments of its main functor. The language thus characterized is called *categorial language* (cf. [12], [13], [18], [23], [24], [25]);

8.   A complete categorial characterization of language will include the division of the set of all well-formed expressions into syntactic categories;

9.   The syntactic characterization of language will allow us to conceive it as a language generated by a *classical categorial grammar*, the idea going back to K. Ajdukiewicz [1] [3], and also as a *typed functorial language* whose precise algebraic description has been proposed by W. Buszkowski [9];.

10.   In the present work language will be characterized in a formal way by the axiomatic method (cf. [14] and [17]), within two contrastive theories: **T1** and **T2** which assume set-theoretical formalism.

3. Dual theories concepts and expressions

Theories **T1** (Sec. 4) and **T2** (Sec. 5) grasp the dual ontological approach to the syntax of language. They are presented at two levels as dualistic theories. Now **T1** provides formal foundations of categorial languages by adopting the *nominalistic (concretistic)* standpoint in the philosophy of language and assumes that *tokens*, and hence concrete objects, form the fundamental level of language, while *types* are constructs obtained in a derived analysis. The formalization of that theory is accordingly carried out first at the *token level* and yields the theory **T1**(tk), and then expanded at the *type level* it yields the

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[3] The term "categorial grammar" was introduced by Y. Bar-Hillel et al. in [4]. A historical survey of categorial grammars as well as the basic terms referring to them is given by W. Marciszewski in [15]. Categorial grammars are formal grammars developed in parallel to N. Chomsky's generative grammars. A significant share in the development of their mathematical foundations has been contributed by W. Buszkowski, who has also been popularizing the grammars in his works (see [8-10]). A contemporary formulation of categorial semantics has been developed by J. van Benthem in [22].

dual theory **T1**(tp). Now **T2** represents the opposite, *Platonic*, philosophical orientation in the syntax of language as it assumes that the study of language is based on *types*, and hence ideal objects, while *tokens* as their concrete representations are the subject matter of derived analysis. Hence that theory is constructed first as **T2**(tp) which describes objects at the *type level*, and then expanded as the *dual theory* **T2**(tk) which describes objects at the *token level.*

    Dual theories describe syntactic concepts which belong to the two different levels mentioned above. Hence the theories **T1**(tk) and **T2**(tp) as well as **T1**(tp) and **T2**(tk) are *dual,* too.

    The syntactic concepts at the *token level* include sets and relations which enable us to formally describe an arbitrary but fixed categorial language $\mathscr{L}$ as a language of expression-tokens. They are (1) sets of tokens which belong to the following system (S):

U	— the set of all tokens, that is the universe of $\mathscr{L}$,
V^1	— the vocabulary of $\mathscr{L}$,
V^2	— the auxiliary vocabulary of $\mathscr{L}$,
W^1	— the set of all words of $\mathscr{L}$,
$W^1 \backslash V^1$	— the set of all compound words of $\mathscr{L}$,
W^2	— the set of all auxiliary words of $\mathscr{L}$,
$W^2 \backslash V^2$	— the set of all compound auxiliary words of $\mathscr{L}$,
$D_1(i)$	— the domain of the relation i of indication of the indices of word-tokens,
$D_1(i) \backslash V^1$	— the set of all those compound words of $\mathscr{L}$ which have an index,
$D_2(i)$	— the counterdomain of i,
$D_2(i) \backslash V^2$	— the set of all those compound auxiliary words of $\mathscr{L}$ which are indices of words,
E_s^1	— the set of all simple expressions of $\mathscr{L}$,
E_f^1	— the set of all functorial expressions of $\mathscr{L}$,
E^1	— the set of all expressions of $\mathscr{L}$,
E_s^2	— the set of all basic index-tokens,
E_f^2	— the set of all functoral index-tokens,
E^2	— the set of all well-formed index-tokens,
E_n	— the set of all well-formed expressions of the n-th order of $\mathscr{L}$,
E	— the set of all well-formed expressions of $\mathscr{L}$,
$E \backslash E_0$	— the set of all well-formed compound expressions of $\mathscr{L}$,
B	— the set of all basic expressions of $\mathscr{L}$,
F	— the set of all functors of $\mathscr{L}$,
Ct_t	— a syntactic category with the index t,
$Ct(E^1)$	— the family of all syntactic categories of the expressions of $\mathscr{L}$,
$Ct(E)$	— the family of all syntactic categories of the expressions of E,
$Ct(B)$	— the family of all basic syntactic categories of $\mathscr{L}$,
$Ct(F)$	— the family of all functoral syntactic categories of $\mathscr{L}$;

and (2) the relations holding among the tokens from the universe U and belonging to the following system (R):

$\approx$ — the equiformity relation among tokens,

$\mathfrak{c}$ — the relation of the concatenation of tokens,

$\mathfrak{i}$ — the relation of the indication of the indices of word-tokens,

$\mathfrak{r}_1$ — the relation of the formation of functorial expressions of $\mathscr{L}$,

$\mathfrak{r}_2$ — the relation of the formation of functoral indices of word-tokens,

$(/)^n$ — the relation of replacement of n-th order constituents of expression-tokens,

$(/)$ — the relation of replacement of expression-tokens,

$\tilde{c}$ — the relation of the categorial agreement among expression-tokens.

The concepts from the systems (S) and (R) describe the theories **T1**(tk) and **T2**(tk).

The syntactic concepts at the *type level* include those sets and relations (functions) which make it possible to describe an arbitrary but fixed categorial language $\mathscr{L}$ as a language of expression-types. They are (1) concepts from the system **(S)** which is obtained from (S) by the replacement of its successive concepts by the appropriate sets of types belonging to the universe of $\mathscr{L}$ or by families of such sets, and (2) concepts from the system **(R)**, which is obtained from (R) by the replacement of the relation of equiformity $\approx$ by the relation of identity $=$ and the replacement of the remaining relations by the successive appropriate relations holding among the types of the universe of $\mathscr{L}$.

The concepts which occupy the same place in the order in (S) (resp. (R)) and **(S)** (resp. **(R)**) are termed *dual*. The terms which denote dual concepts are also called *dual*. The terms which are dual relative to one another are distinguished only by the use, in the case of the terms from **(S)**, of bold type without any change in the shape of the type used in the terms from (S), and in the case of terms from **(R)** of the single underline without any change in the shape of the type used in the terms from (R).

Let the letters (v) x, y, z, t, resp. (V) X, Y, Z, T, with or without subscripts and/or superscripts, range over set U tokens, resp. types, from the universe U, resp. U. The letters (v) resp. (V), with superscripts k, where $k = 1, 2$, are reserved for words from W^k, resp. W^k. It is also assumed that the letter A (resp. A) stands for subsets from the universe U of $\mathscr{L}$ (resp. U of $\mathscr{L}$).

Two expressions are called *dual* if one of them is recorded solely with the use of logical constants, specific terms occurring in (S) and/or (R), letters from (v) and/or the letter A, and brackets, while the other differs from the former by having the specific terms of the former replaced by dual terms (printed in bold type), lower-case letters by analogous capital letters from (V), and the letter A by the letter A. An expression dual to α is denoted by $d(\alpha)$.   ∎

4. Theory **T1**

The theory **T1** has as its primitive terms: U, $\approx$, c, V^1, V^2, i, r_1, r_2. They are at the same time the primitive terms of the fragment **T1**(tk) of **T1**. Those terms which denote the remaining concepts at the *token level* and also all those terms which denote concepts at the *type level* are defined in **T1**.

T1 refers to the theory of categorial languages presented in [24].

4.1. Formalization of **T1** at the token level; theory **T1**(tk)

4.1.1. Axioms and definitions of T1(tk)

Now **T1**(tk) is an axiomatic theory of the language $\mathscr{L}$ characterized by all primitive and derived concepts at the *token level*. The formulation of axioms and definitions will be preceded by suitable remarks in most cases pertaining to the intuitive interpretation of the concepts which categorially describe the language $\mathscr{L}$.

The *universe U of $\mathscr{L}$* is the set of all tokens, in which we distinguish certain subsets which enable us to define that language.

The *relation of equiformity* $\approx$ is a binary relation in U. Two tokens between which that relation holds are called *equiform*. The equiformity of tokens is determined by pragmatic aspects, acts in which they are used, and not by physical similarity. For instance, two inscriptions printed in different type but consisting successively of the same letters of alphabet may be equiform, whereas two nouns or two adjectives, printed in the same type, may be not equiform if one of them occurs in a sentence with an adjunct or is itself an adjunct, while the other does not or is not[4].

We adopt the following axiom characterizing equiformity:

A1a. $x \approx x$,
 b. $x \approx y \Rightarrow y \approx x$,
 c. $x \approx y \wedge y \approx z \Rightarrow x \approx z$.

The *relation of concatenation* c is a ternary relation in U. Any token z which is in the relation c with the tokens x and y, i.e., satisfies the expression $c(x, y, z)$, is called the *concatenation* of x and y. In the European ethnic languages, any inscription z obtained from an inscription equiform with x by the writing on the right of the latter, immediately after it and at the same level, of an inscription equiform with y, is a concatenation of the inscriptions x and y. In a similar way, but by writing the second inscription on the left of the first, we obtain a concatenation, e.g., in Hebrew or Arabic languages. Concatenations are not always obtained by a linear connection of two tokens, which can be seen in the case of hieroglyphs and mathematical formulas. Two equiform

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[4] If one should use a simile here, it is like having two crystal flower-vases of the same shape and cut when one is empty and the other is full of beautiful red roses, or like comparing the shape of the figure of a beautiful actress posing in exactly the same posture and background in two photos, in one of which she appears clothed, while in the other — naked.

tokens may be concatenations of the same two tokens, which shows that the relation of concatenation c is not the function. The concept of concatenation is at the basis of many formal models of language, especially the formal languages in Chomsky's sense. The concept is described in detail in [24-26].

We adopt the following axioms which describe the fundamental properties of concatenation:

A2. $\quad \exists z c(x, y, z)$,
A3. $\quad c(x, y, z) \wedge c(x', y', t) \wedge x \approx x' \wedge y \approx y' \Rightarrow z \approx t$,
A4. $\quad c(x, y, z) \wedge t \approx z \Rightarrow c(x, y, t)$.

Thus for every two tokens in $\mathscr{L}$ there is a token in U which is their concatenation; concatenations of two pairs of tokens in $\mathscr{L}$ with first and second elements pairwise equiform yield equiform tokens; a token which is equiform with the concatenation of two tokens is also their concatenation. ∎

The *vocabulary* V^1 of $\mathscr{L}$ is a set of simple word-tokens of that language. It is fixed once and for all if $\mathscr{L}$ is a formalized language, or is open and includes potential words if $\mathscr{L}$ is, for instance, a natural language. It is used to generate, by means of the relation of concatenation, the *set* W^1 of *all words* of $\mathscr{L}$. It has as its subset the *set* E of *all* its *well-formed expressions* (briefly: wfe), which determines the language $\mathscr{L}$. Hence the simplest syntactic characterization of $\mathscr{L}$ is given by the system:

$(\mathscr{L}) \quad \langle U, c, V^1; E \rangle$.

The categorial characterization of $\mathscr{L}$, which makes it possible to distinguish the set E, is done by the use of *categorial indices* assigned to the appropriate words of $\mathscr{L}$. They are tokens from U, but are not in the set W^1 of the words of $\mathscr{L}$, but are words in the metalanguage of that language. They are the so-called *auxiliary words* of $\mathscr{L}$ and are in the set W^2 of all such words. W^2 is generated from the *auxiliary vocabulary* V^2 of $\mathscr{L}$ by means of the relation c. V^2 consists of *basic indices* and auxiliary symbols, such as brackets, commas, fraction lines, etc.

It is assumed concerning the vocabularies V^k ($k = 1, 2$) that they satisfy the following axioms:

$A^k 5. \quad V^k \subseteq U$,
$A^k 6. \quad x \in V^k \wedge t \approx x \Rightarrow t \in V^k$,
$A^k 7. \quad c(x, y, z) \Rightarrow z \notin V^k$.

Thus, for $k = 1, 2$, V^k is a set of tokens; a token which is equiform with a word from V^k is also such a word; no concatenation of any pair of tokens is a word in V^k. ∎

The meaning of the terms W^k ($k = 1, 2$) is fixed by the following definitions and axioms:

$D^k 1. \quad W^k = \bigcap \{A: V^k \subseteq A \wedge \forall x, y \in A \, \forall z (c(x, y, z) \Rightarrow z \in A)\}$,
$A^k 8. \quad t \in W^k \backslash V^k \Rightarrow \exists x, y \in W^k c(x, y, t)$.

The set of words W^k $(k = 1, 2)$ is thus the smallest set of tokens containing the vocabulary V^k and closed under concatenation, while every compound word (resp. auxiliary compound word) is the concatenation of a pair of words from W^1 (resp. W^2).

Categorial indices are assigned to the appropriate words of $\mathscr{L}$ by the binary *relation* $\mathfrak{i}$ *of indication of indices of words*, that is — to use Buszkowski's terminology — by the *typization* of words.

The relation $\mathfrak{i}$ is described by the four axioms given below. In the recording of the last two axioms we use the expression of the form $\mathfrak{i}(x, y)$, which we read thus: y is the index of the word x of $\mathscr{L}$.

A9. $\mathfrak{i} \subseteq W^1 \times W^2$,

A10. $D_1(\mathfrak{i}) \cap D_2(\mathfrak{i}) = \varnothing$,

A11. $\mathfrak{i}(x^1, x^2) \wedge \mathfrak{i}(y^1, y^2) \wedge x^1 \approx y^1 \Rightarrow x^2 \approx y^2$,

A12. $\mathfrak{i}(x^1, x^2) \wedge z \approx x^1 \wedge t \approx x^2 \Rightarrow \mathfrak{i}(z, t)$.

Typization is to be used in the analysis of the syntactic correctness of the *expressions* of $\mathscr{L}$. They are in the set E^1 and can be either *simple expressions* from E_s^1, distinguished from the vocabulary V^1 and, of course, the set $D_1(\mathfrak{i})$, or compound expressions, i.e., *functorial expressions* from E_f^1, distinguished from the set $D_1(\mathfrak{i}) \backslash V^1$. The principles of the construction of functorial expressions are, self-evidently, determined by the syntactic rules of $\mathscr{L}$. In theoretical considerations we shall replace them by a single binary *relation* $\mathfrak{r}_1$ *of the formation of functorial expressions* of $\mathscr{L}$. If we assume that

$(\mathfrak{r}_1)$ $\mathfrak{r}_1(x_0^1, x_1^1, \ldots, x_n^1; x^1)$

is an expression in the theory $\mathbf{T1}(\text{tk})$, which we read: x^1 is a functorial expression consisting of the main functor x_0^1 and its successive arguments $x_1^1, \ldots, x_n^1$ $(n \geqslant 1)$, then x^1 in $(\mathfrak{r}_1)$ may be treated as a substitute of any expression of $\mathscr{L}$ which is formed of the main functor x_0^1 and its successive arguments $x_1^1, \ldots, x_n^1$, regardless of the way in which that expression in the form of the appropriate concatenation occurs in $\mathscr{L}$. Hence the same expression of the language of $\mathbf{T1}(\text{tk})$, having the form $(\mathfrak{r}_1)$, may replace expressions of $\mathscr{L}$ constructed according to various rules, for instance sentential and nominal expressions of natural language, provided that those expressions are formed of the same number of words of which one is a functor and the remaining ones are its arguments (the position occupied in the concatenation by a sentence-forming functor may obviously differ from the place which in another concatenation is occupied by a name-forming functor). The same expression of the language of $\mathbf{T1}(\text{tk})$ of the form $(\mathfrak{r}_1)$ may replace different but synonymous expressions in various languages, for instance languages of the sentential calculus. Note that the following expressions:

$$p \Leftrightarrow (q \Rightarrow r); \quad \lozenge(p, \lozenge(q, r)); \quad EpCqr,$$

recorded respectively in three notations: the one which is used in the present paper, Leśniewski's notation, and Łukasiewicz's parenthesis-free notation, are

expressions taken from the various languages of the sentential calculus but each of them consists of the functor of equivalence and its the same arguments.

The *categorial indices* by means of which the typization of the words of $\mathscr{L}$ is carried out are, as we know, auxiliary words in that language. They are in the set E^2 and are classed into *basic* ones (which are in the set E_s^2) and *functoral ones* (which are in the set E_f^2). The latter are formed of the basic ones in accordance with definite rules, which in theoretical considerations are replaced by the single binary *relation* $\mathfrak{r}_2$ *of the formation of functoral indices*. If we assume that

$(\mathfrak{r}_2)$ $\mathfrak{r}_2(x^2, x_1^2, \ldots, x_n^2; x_0^2)$

is an expression in **T1**(tk) which we read thus: x_0^2 is a functoral index formed of the index x^2 and, successively, the indices $x_1^2, \ldots, x_n^2$, then x_0^2 may be treated as a substitute for any functoral index determined by the index x^2 and the successive indices $x_1^2, \ldots, x_n^2$ ($n \geqslant 1$), regardless of the rules of concatenation of indices provided for $\mathscr{L}$. If, for instance, $V^2 = \{s, n, /, '\}$ and concatenation is to consist in right-sided linear juxtaposition, then x_0^2 equally well corresponds, for instance, to the index s/nn of a sentence-forming functor of two name arguments, and the index $s/nn//s/nn$, s/nn of a functor-forming functor which forms such a functor and also has such functors as its two arguments.

The relation $\mathfrak{r}_k$ ($k = 1, 2$) is formally described by the following axioms:

$A^k 13.$ $D_1(\mathfrak{r}_k) = \bigcup\limits_{n=2}^{\infty} D_k(\mathfrak{i})^n \wedge D_2(\mathfrak{r}_k) \subseteq D_k(\mathfrak{i}) \backslash V^k,$

$A^k 14.$ $\mathfrak{r}_k(x_0^k, x_1^k, \ldots, x_n^k; x^k) \wedge \mathfrak{r}_k(y_0^k, y_1^k, \ldots, y_m^k; y^k) \Rightarrow$

$\Rightarrow [y^k \approx x^k \Leftrightarrow m = n \wedge \forall 0 \leqslant j \leqslant n (y_j^k \approx x_j^k)],$

$A^k 15.$ $\mathfrak{r}_k(x_0^k, x_1^k, \ldots, x_n^k; x^k) \wedge \forall 0 \leqslant j \leqslant n (y_j^k \approx x_j^k) \wedge y^k \approx x^k \Rightarrow$

$\Rightarrow \mathfrak{r}_k(y_0^k, y_1^k, \ldots, y_n^k; y^k).$

Thus, the relation of the formation of functorial expressions (functoral indices) of $\mathscr{L}$ has as its domain the set of all finite Cartesian powers (greater than 1) of the set $D_1(\mathfrak{i})$ (the set $D_2(\mathfrak{i})$) of all those words of $\mathscr{L}$ which have indices (all indices of such words) and the counterdomain of $\mathfrak{r}_1$ ($\mathfrak{r}_2$) is included in the set of all compound words of $\mathscr{L}$ which have an index (compound auxiliary words of $\mathscr{L}$ which are indices of words); two functorial expressions (functoral indices) of $\mathscr{L}$ are equiform if and only if they are formed of the same number of pairwise equiform words (indices of words) of $\mathscr{L}$; a word (an auxiliary word) of $\mathscr{L}$ which is equiform with a functorial expression x^1 (functoral index x^2) of that language is a functorial expression (functoral index) formed of successive words (indices of words) which are pairwise equiform with the words (indices of words) occurring in the same order, of which the word x^1 (index x^2) is formed. ∎

The *set E_s^1 (E_s^2) of all simple expressions (all basic indices)* of $\mathscr{L}$ is defined as the set of all words of the vocabulary (auxiliary vocabulary) of that language which have an index (are indices of words). The *set E_f^1 (E_f^2) of all functorial expressions (functoral indices)* of $\mathscr{L}$ is defined as the counterdomain of the relation of the formation of functorial expression (functoral indices). The *set E^1 (E^2) of all expressions (all well-formed indices)* of $\mathscr{L}$ is defined as the sum of the sets E_s^1 and E_f^1 (E_s^2 and E_f^2). Hence the following definitions ($k = 1, 2$) oblige in **T1**(tk):

$\mathbf{D}^k\mathbf{2a}.$ $E_s^k = V^k \cap D_k(\mathfrak{i}),$

    b. $E_f^k = D_2(\mathfrak{r}_k),$

    c. $E^k = E_s^k \cup E_f^k.$ ■

The concept of the *set E of all well-formed expressions* (wfe), which is fundamental for the categorial language $\mathscr{L}$ is defined by reference to the set E_n of all such expressions of the n-th order; E_n is defined by induction:

$\mathbf{D3a}.$ $E_0 = E_s^1,$

    b. $x^1 \in E_{k+1} \Leftrightarrow x^1 \in E_k \vee \exists n \geqslant 1 \exists x_0^1, x_1^1, \ldots, x_n^1 \in E_k \big[\mathfrak{r}_1(x_0^1, x_1^1, \ldots, x_n^1; x^1) \wedge$

       $\wedge \, \forall 0 \leqslant j \leqslant n \forall x_j^2, x^2 \big(\mathfrak{i}(x_j^1, x_j^2) \wedge \mathfrak{i}(x^1, x^2) \Rightarrow \mathfrak{r}_2(x^2, x_1^2, \ldots, x_n^2; x_0^2) \big) \big].$

Thus a wfe of the 0 order in $\mathscr{L}$ is any simple expression of that language. A wfe of the $k + 1$-th order in $\mathscr{L}$ is either a wfe of the k-th order or a functorial expression formed of wfes of the k-th order: the main functor and its arguments such that any index of the main functor of that expression satisfies the rule which expresses the *superior principle of the theory of syntactic categories* (cf. Sec. 2), briefly **sptsc**, formulated as follows:

(sptsc)   *The index of the main functor of a functorial expression is a functoral index formed of the index of that expression and the successive indices of the successive arguments of that functor.*

The set E is defined as the sum of all wfes of a finite order $\geqslant 0$. Hence

$\mathbf{D3c}.$ $E = \bigcup\limits_{n=0}^{\infty} E_n.$

We also assume that

A16. $\vec{\mathfrak{i}}(E \setminus E_0) \cap E_s^2 \neq \varnothing$ [5],

A17. $\vec{\mathfrak{i}}(E) \subseteq E^2,$

which is to say that there is at least one compound wfe of $\mathscr{L}$ which has a basic index and that the indices of wfes of $\mathscr{L}$ are well-formed.

We show below that A16 guarantees the non-emptiness of U so that there is at least one token.

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[5] The expression $\vec{\mathfrak{i}}(A)$ represents the image of set A with respect to the relation $\mathfrak{i}$.

Note that the set E of all wfes of the categorial language $\mathscr{L}$ can be generated (cf. Sec. 2, p. 9) by the system

$$\mathscr{G}_{\mathscr{L}} = \langle U, \mathfrak{c}, \mathfrak{r}_1, \mathfrak{r}_2, E_s^1, E_s^2, \mathfrak{i} \rangle,$$

which may be treated as a reconstruction of the *classical categorial grammar*, whose idea going back to [1] (cf. [8–9]). That grammar is said to be *rigid* [8]: every word or expression of $\mathscr{L}$ has one categorial index assigned to it (up to equiformity). That follows from the axioms A1a and A11.

A more precise categorial characterization of the language $\mathscr{L}$ described by **T1**(tk) is thus given by the pair

$$\mathscr{L}(\mathscr{G}_{\mathscr{L}}) = \langle \mathscr{G}_{\mathscr{L}}; E \rangle.$$

The categorial analysis and the estimation of the syntactic correctness of given expression of $\mathscr{L}$ refers solely to its functorial expressions and consists in finding whether the rule **sptsc** is satisfied for every constituent of such an expression. The functorial expressions of $\mathscr{L}$ are compound expressions formed of its *basics expressions* and auxiliary expressions, that is *functors*. ∎

The *set B of all basic expressions* of $\mathscr{L}$ is defined as the set of well-formed expressions with basic indices, and the *set F of all functors* of $\mathscr{L}$ is defined as the set well-formed expressions of $\mathscr{L}$ with functoral indices. The formal definitions of those sets are as follows:

D4. $B = \{x^1 \in E: \ \forall x^2 (\mathfrak{i}(x^1, x^2) \Rightarrow x^2 \in E_s^2)\}$,

D5. $F = \{x^1 \in E: \ \forall x^2 (\mathfrak{i}(x^1, x^2) \Rightarrow x^2 \in E_f^2)\}$.   ∎

The singling out of the sets B and F from the set E does not give the complete syntactic categorial description of $\mathscr{L}$, which consists in the possibility of carrying out a logical partition of E into *syntactic categories* (see Sec. 2, p. 8).

The traditional definitions of syntactic category link it — in accordance with the ideas advanced by E. Husserl — to the set of expressions *replaceable* in any sentential contexts, or, more generally, in any well-formed ones (see [14], [21], [1], [6]). Such definitions not only do not eliminate the risk of a vicious circle (cf. [24]), but also have other undesirable consequences[6]. In any case, when carrying out a categorial analysis of a given expression it is most convenient to define its syntactic category by making use of the index of that expression, and to include in one and the same syntactic category any two expressions which have equiform indices, that is such which are *categorially in agreement*.

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[6] There exist expressions included into the same syntactic category of, for example, names, that are not replaceable in any sentence or well-formed expression. For instance, the noun "man", personal pronoun "he", or cardinal numeral "8" are names. Hence, by replacing the noun by the pronoun or numeral in the well-formed expressions: "a noble man", "John is a noble man", we obtain meaningless expressions. On the other hand, the expressions: "8 = 8" and "8 − 8" are also well-formed, though the latter emerges by replacing the sentence-forming functor "=" by the name-forming functor "−" — i.e., by a functor of another syntactic category.

Let then the *syntactic category with the index t* correspond to the set Ct_t of all those expressions of $\mathscr{L}$ whose index is equiform with t. In symbols:

D6. $Ct_t = \{x^1 \in E^1 : \forall x^2 (\mathfrak{i}(x^1, x^2) \Rightarrow x^2 \approx t)\}$,   where $t \in \vec{\mathfrak{i}}(E^1)$.

Further, let any two expressions x and y be categorially in agreement if they bear to one another the relation $\tilde{c}$ defined by the formula:

D7. $x, y \in E^1 \Rightarrow [x \tilde{c} y \Leftrightarrow \exists t(x, y \in Ct_t)]$.

Note that $\tilde{c}$ determines the logical partition of E^1 into syntactic categories, and hence also the expected logical partition of E and, consequently, of B and F (see Theorem 8 below). Those partitions are, correspondingly, families of sets $Ct(S)$, where $S \in \{E^1, E, B, F\}$, called *families of all syntactic categories of the expressions belonging to the set* S. The definitions of those families are obtained correspondingly from the schema:

D8(S). $Ct(S) = \{Ct_t : t \in \vec{\mathfrak{i}}(S)\}$,   for $S \in \{E^1, E, B, F\}$.   ∎

By adopting definition D7 we deviate from the traditional definitions of the concept of syntactic category. But we still associate that concept with the concept of replaceability of expressions, important in the theory of syntactic categories. This will be reflected in the *fundamental theorem of the theory of syntactic categories* (Theorem 9 below).

The *relation of replaceability* (/) is a four-argument relation in E^1. Its definition is based on the auxiliary concept of the *relation* (/)^n *of the replaceability of a constituent* of the *n*-th order. The latter will be defined by induction. In the recording of the definitions of both relations we shall make use of the expression $y(t/z)^n x$, which we read: an expression y of $\mathscr{L}$ is obtained from an expression x of that language by the replacement of its z constituent of the *n*-th order by an expression t. The formulation $y(t/z)x$ we read analogically with the omission of the element: of the *n*-th order.

D9a. $y(t/z)^0 x \Leftrightarrow x, y \in E^1 \wedge z \approx x \wedge t \approx y$.

  b. $y(t/z)^1 x \Leftrightarrow \exists n \geqslant 1 \; \exists x_0, x_1, \ldots, x_n \exists y_0, y_1, \ldots, y_n$
  $[\mathfrak{r}_1(x_0, x_1, \ldots, x_n; x) \wedge \mathfrak{r}_1(y_0, y_1, \ldots, y_n; y) \wedge$
  $\wedge \exists 0 \leqslant j \leqslant n(z \approx x_j \wedge t \approx y_j \wedge \forall k \neq j \wedge 0 \leqslant k \leqslant n(x_k \approx y_k))]$.

  c. $y(t/z)^{k+1} x \Leftrightarrow \exists x_1, x_2 \big(y(x_2/x_1)^k x \wedge x_2(t/z)^1 x_1 \big)$,   for $k > 0$:

  d. $y(t/z)x \Leftrightarrow \exists n \geqslant 0 \big(y(t/z)^n x \big)$.   ∎

4.1.2. Major theorems of the theory T1(tk)

As has been mentioned previously, Axiom A16 leads us in particular to the conclusion that tokens exists. This is so because it guarantees the non-emptiness of $E \backslash E_0$, and hence, by D3c, a, the non-emptiness of some set E_n ($n > 0$). Since by D3a, b, c, $D^1$2a, b, c, and $A^1$13 we have the following lemma:

(1) $E_n \subseteq E \subseteq E^1 \subseteq D_1(\mathfrak{i})$,   for all $n \geqslant 0$,

the non-emptiness of some of the sets E_n $(n > 0)$ implies 1) the non-emptiness of the sets E, E^1, $D_1(\mathfrak{i})$, and since by A9 and $D^1 1$ the following inclusion holds:

(2) $D_1(\mathfrak{i}) \subseteq W^1 \subseteq U$

it also implies 2) the non-emptiness of the sets W^1 and U. This means that there exist not only tokens in general, but, in particular, word-tokens, word-tokens which have an index, expression-tokens, well-formed expression-tokens.

One can formulate a more general theorem which guarantees the non-emptiness of sets at the *token level*, that is sets in the system (S).

To do so note first that since some E_n is non-empty, by D3b, a every set E_n is non-empty (including $E_0 = E_s^1$). By $D^1 2a$ the same applies to the vocabulary V^1. Since $E \backslash E_0 \neq \emptyset$ and since, by D3a, b, c, $D^1 2b$, $A^1 13$ and (2) we have

(3) $E \backslash E_0 \subseteq E_f^1 \subseteq D_1(\mathfrak{i}) \backslash V^1 \subseteq W^1 \backslash V^1,$

we find that the sets E_f^1, $D_1(\mathfrak{i}) \backslash V^1$, $W^1 \backslash V^1$ are non-empty. Note further that Axiom A16 also guarantees the non-emptiness of E_s^2, and the fact that $E \backslash E_0$ is non-empty guarantees the non-emptiness of E_f^2 by Definitions D3c, b, (3) and Definition $D^2 2b$. Thus $D^2 2c$ yields immediately the non-emptiness of E^2, $D^2 2a$ — the non-emptiness of V^2 and $D_2(\mathfrak{i})$; $D^2 2b$ and $A^2 13$ yield the non-emptiness of $D_2(\mathfrak{i}) \backslash V^2$. It yields the non-emptiness of $W^2 \backslash V^2$ (A9) and W^2.

The fact that B is non-empty follows from its definition (Def. D4), A16, the fact that by A11 the index of a word is determined unambiguously up to equiformity, and from the theorem stating that a word index which is equiform with a basic index is basic, too (see formula (∗) of Theorem 2 below). The fact that F is non-empty follows from D5, the non-emptiness of $E \backslash E_0$, (3), D3c, a, b and $D^2 2b$, A11, and the theorem stating that a index of word which is equiform with a functoral index is functoral, too (see formula (∗) of Theorem 2 below). It can also easily be seen that if $t \in \vec{\mathfrak{i}}(E^1)$, then the category Ct_t is non-empty by A11 and A1a and D6. By D8(S), the families $Ct(S)$ are non-empty, too, if $S \in \{E^1, E, B, F\}$, because $S \neq \emptyset$ and $\vec{\mathfrak{i}}(S) \neq \emptyset$ in view of (1) and the correctness of the inclusions $B \subseteq E$ and $F \subseteq E$.

The foregoing leads us to the following theorem:

THEOREM 1.   *All sets in the system* (S) *are non-empty.*   ∎

The next two theorems describe important properties of the relation of equiformity.

THEOREM 2.   *A token which is equiform with a token from any set* S *of the system* (S) *is also an element of* S, *i.e., for any set of tokens of* (S) *the following formula holds:*

(∗) $x \in S \wedge t \approx x \Rightarrow t \in S.$

PROOF. If $S = U$, then (∗) follows directly from the convention concerning the variables x and y (Sec. 3). If $S = V^k$ ($k = 1, 2$), the truth of (∗) is based on Axioms $A^k 6$. To substantiate (∗) when $S = W^k \backslash V^k$ and $S = W^k$ note first that the concept of concatenation bears important relations to certain sets in (S). Since it follows immediately from the definitions of the sets W^k for $k = 1, 2$ (Def. $D^k 1$) that

(4) $V^k \subseteq W^k \subseteq U,$

(5_k) $x, y \in W^k \wedge c(x, y, z) \Rightarrow z \in W^k,$

the Lemmas (5_k) and Axioms $A^k 7$ and $A^k 8$ ($k = 1, 2$) yield the relationships:

(6_k) $z \in W^k \backslash V^k \Leftrightarrow \exists x, y \in W^k c(x, y, z),$

and the formulas (4) and (6_k), the relationships:

(7_k) $z \in W^k \Leftrightarrow z \in V^k \vee \exists x, y \in W^k c(x, y, z).$

Thus the truth of (∗) for $S = W^k \backslash V^k$ follows from (6_k) and A4, and for $S = W^k$, from (7_k) and $A^k 6$ and A4. Note further that the same index corresponds to equiform words while equiform indices correspond to the same word, so that

(8) $i(x, y) \wedge t \approx x \Rightarrow i(t, y),$

(9) $i(x, y) \wedge t \approx y \Rightarrow i(x, t).$

This follows from A1a and A12. The properties (8) and (9) substantiate the correctness of (∗) for $S = D_k(i)$ ($k = 1, 2$), and hence, by A1b and $A^k 6$, its correctness for $S = D_k(i) \backslash V^k$.

The implication (∗) holds for $S = E_s^k$ ($k = 1, 2$), which follows directly from $D^k 2a$ and from the fact that it holds for $S = D_k(i)$ and $S = V^k$; the fact that it holds for E_f^k ($k = 1, 2$) follows from $D^k 2b$ and the following conclusions from A1a and $A^k 15$:

(10_k) $r_k(x_0^k, x_1^k, \ldots, x_n^k; x^k) \wedge y^k \approx x^k \Rightarrow r_k(x_0^k, x_1^k, \ldots, x_n^k; y^k).$

In view of the above (∗) holds for the sets $S = E^k$ ($k = 1, 2$) on the strength of their definitions (Def. $D^k 2c$).

The substantiation of the fact that the implication (∗) is valid for $S = E$ is based on D3c and the lemma

(11) $x \in E_n \wedge t \approx x \Rightarrow t \in E_n,$ for all $n \geqslant 0,$

whose proof by induction is in turn based on the statement that that formula holds for $E_0 = E_s^1$ (D3a) and at the inductive assumption, which is to say that its truth is assumed for $n = k$, on the statement that it holds for $n = k + 1$ on the basis of D3b, (10_k), (8) and A1b.

Now (∗) is also self-evidently correct for $S = E \backslash E_0$, which follows from D3c and (11). To complete the proof of Theorem 2 one has to prove (∗) for $S = B$, $S = F$, and $S = Ct_t$. Since (∗) is valid for $S = E$, the justification of the first two

cases is based directly on Definition D4, resp. D5, Lemma (8), and A1b. The truth of (*) for $S = Ct_t$ follows from D6, the fact that it is true for $S = E^1$, and Lemma (8). ■

THEOREM 3. *The following implication holds for every relation* R *in the system* (R):

$$(**) \qquad R(x_0, x_1, \ldots, x_n) \wedge \forall 0 \leqslant l \leqslant n(y_l \approx x_l) \Rightarrow R(y_0, y_1, \ldots, y_n), \quad \text{for } n \geqslant 1.$$

PROOF. If R equals the equiformity relation $\approx$, then (**) follows immediately from Axioms A1b, c. If $R = c$, it follows from A4 and the lemma

$$(12) \qquad c(x_0, x_1, x_2) \wedge y_0 \approx x_0 \wedge y_1 \approx x_1 \Rightarrow c(y_0, y_1, x_2).$$

The proof of (12) is based on A2 applied to the tokens y_0 and y_1, A3 and A4. If $R = i$, then (**) is a substitution of A12, and if $R = r_k$ ($k = 1, 2$), then it is a substitution of $A^k 15$[7]. If $R = (/)^n$, then the proof of that formula is by induction: for $n = 0$ it follows immediately from D9a, a substitution instance of (*) (we set E^1 for S), A1b and A1c; for $n = 1$ it follows immediately from Definition D9b, the correctness of (**) for $R = r_1$, and A1a, c; by assuming the truth of that formula for $n = k$ we arrive at stating its truth for $n = k + 1$ on the basis of D9c, A1a, and the fact that it is true for $n = 1$. If $R = (/)$, then it follows from the fact that it is true for $R = (/)^n$ by D9d. Finally, $R = \tilde{c}$, then (**) follows from D7 and D6, the truth of (*) for $S = E^1$, the Lemma (8) and A1b. ■

The successive theorems illustrate certain properties of E.

THEOREM 4. *The set E of all well-formed expression of* $\mathscr{L}$ *is the least set of tokens from the universe U of that language containing the set of all its simple expressions and satisfying the condition that it contains every functorial expression of* $\mathscr{L}$ *that satisfies the rule* **sptsc** (*cf. p.* 89).

An analogous theorem is given in [24] together with its proof. The proof of Theorem 4 is modelled on the latter. It is omitted here. ■

As has been mentioned in Sec. 4.1.1, categorial indices are not words of $\mathscr{L}$, but are words of the metalanguage of that language, namely auxiliary words. This applies in particular to the indices of wfes of $\mathscr{L}$. That fact follows from the relationships:

$$E \subseteq D_1(i) \subseteq W^1, \vec{i}(E) \subseteq D_2(i) \subseteq W^2.$$

The first of them is a direct consequence of Lemmas (1), (2), while the second follows from A17, $D^2 2c$, a, b, $A^2 13$ and A9. In view of A10 we can accordingly state that the following theorem holds:

THEOREM 5. *The set of all well-formed expressions of* $\mathscr{L}$ *is disjoint from the set of the indices of those expressions, i.e.*

$$E \cap \vec{i}(E) = \varnothing.$$

---

[7] Relations r_k ($k = 1, 2$) are $n+1$-argument relations, $n \geqslant 0$.

THEOREM 6.   *The set E of all well-formed expressions of $\mathscr{L}$ is the sum of two non-empty and disjoint sets: the set B of all basic expressions of $\mathscr{L}$ and the set F of all its functors. In symbols:*

$$E = B \cup F \wedge B \neq \varnothing \wedge F \neq \varnothing \wedge B \cap F \neq \varnothing.$$

PROOF.   Since by Lemmas (1) and (2) E is the set of those words which have indices, and by A17 the indices of such words are elements of E^2, $E = B \cup F$ by $D^2 2c$, D4, and D5. By Theorem 1 B and F are non-empty sets. They are also disjoint, which follows from their definitions, A17, $D^2 2c$, a, b, and $A^2 13$, because the indices of their expressions belong, respectively, to the disjoint sets E_s^2 and E_f^2.   ∎

Note in this connection that a theorem analogous to Theorem 5 holds for the set E^2 of all well-formed indices because for $k = 1, 2$ the following formula is valid:

(13) $E^k = E_s^k \cup E_f^k \wedge E_s^k \neq \varnothing \wedge E_f^k \neq \varnothing \wedge E_s^k \cap E_f^k \neq \varnothing.$

The sets B and F, which form a partition of the set E, have, correspondingly, common elements with the disjoint sets of expressions $E \backslash E_0$ and E_0. The fact that there is a basic expression of $\mathscr{L}$ which is a compound well-formed expression follows from A16, D4, A1a and A11, and the formula (∗) for S = B. On the other hand, as we know, $E \backslash E_0$ is a non-empty set, and by D3c, b, a there is a functorial expression of $\mathscr{L}$ and there is also such its main functor belonging to $E_0 \subseteq E$ that its arbitrary index is, by $D^2 2b$, a functoral index. The functor is, therefore, by D5, also an element of F. We accordingly have

THEOREM 7a. $(E \backslash E_0) \cap B \neq \varnothing,$
    b. $E_0 \cap F \neq \varnothing.$

In accordance with the convention 8 in Sec. 2, the categorial character of $\mathscr{L}$ should reflect a more detailed logical partition of E than Theorem 6 indicates, namely a logical partition of that set into syntactic categories. Formally this is so in fact, because the more general theorem holds:

THEOREM 8.   *If* $S \in \{E^1, E, B, F\}$, *then*
 (i) $S = \bigcup Ct(S)$ — S *is the sum of all syntactic categories of the expressions of* S
 (ii) $\forall Ct_t \in Ct(S)(Ct_t \neq \varnothing)$ — *which are non-empty*
 (iii) $\forall Ct_t, \ Ct_{t'} \in Ct(S)(Ct_t \neq Ct_{t'} \Rightarrow Ct_t \cap Ct_{t'} \neq \varnothing)$ — *and pairwise disjoint.*

PROOF.   Let $S \in \{E^1, E, B, F\}$. By Lemma (1), Definitions D4, D5, and D6, and axioms A1a and A11, an arbitrary token x from S belongs to some syntactic category with an index t (such that $i(x, t)$). The relation $\tilde{c}$ is thus reflexive on S. It follows directly from D7 that it is symmetric in that set. It is also transitive in that set, for if $x, y \in Ct_{t_1}$ and $y, z \in C_{t_2}$, then by D6 the index of the expression y is equiform with both t_1 and t_2, whence it follows that $t_1 \approx t_2$ and then $Ct_{t_1} = Ct_{t_2}$, and *eo ipso* $x, z \in Ct_{t_1}$. Since S is non-empty (Theorem 1), the equivalence relation $\tilde{c}$ determines the logical partition $S/_{\tilde{c}}$ of S into

non-empty and pairwise disjoint equivalence classes relative to $\tilde{c}$. Note further that an equivalence class relative to $\tilde{c}$ is a syntactic category whose index is the index of the expression which is a representative of that class, i.e.,

(14) $x \in S \wedge i(x, t) \Rightarrow [x]_{\tilde{c}} = Ct_t$,   for $S \in \{E^1, E, B, F\}$.

In fact, if $y \in [x]_{\tilde{c}}$, then by D7 $x, y \in Ct_{t_1}$, and since $i(x, t)$ by applying D6 we obtain $t \approx t_1$ and $Ct_t = Ct_{t_1}$, and then $y \in Ct_t$. And conversely: note that $x \in Ct_t$ because it follows by assumption and from A1a and A11 that $x \in E^1$ and if $i(x, x^2)$, then $x^2 \approx t$ for any x^2. Thus, if $y \in Ct_t$, then $x\tilde{c}y$ and $y \in [x]_{\tilde{c}}$.

Thus (14) is true, and since the index of a word-token is determined unambiguously by up to equiformity (Axiom A11) while syntactic categories with equiform indices are identical, by D8(S) the quotient family $S/_{\tilde{c}}$ is equal to the family $Ct(S)$ of all syntactic categories of expressions in S. This proves formulas (i)–(iii). ∎

Finally, we proceed to formulate the aforementioned (Sec. 4.1.1) *fundamental theorem of the theory of syntactic categories*:

THEOREM 9. (fttsc). *Two expressions of $\mathscr{L}$ belong to the same syntactic category if and only if on replacing one by the other in a well-formed expression of $\mathscr{L}$ and obtaining from it a well-formed expression of that language we find that it belongs to the same syntactic category as the former. In symbols*:

$$x, y \in E \wedge y(t/z)x \Rightarrow (t\tilde{c}z \Leftrightarrow y\tilde{c}x).$$

PROOF.   The proof of this theorem is based on the following two lemmas:

(15) $x, y \in E \wedge y(t/z)^n x \wedge t\tilde{c}z \Rightarrow y\tilde{c}x$,   for $n \geqslant 0$,
(16) $x, y \in E \wedge y(t/z)^n x \wedge y\tilde{c}x \Rightarrow t\tilde{c}z$,   for $n \geqslant 0$.

The proofs of these lemmas are carried out by induction. When $n = 0$ their truth is substantiated by reference to D9a, D7, D6, Lemma (1), and A1b, A12, and A11. The proofs for $n = 1$ are more difficult. In this case we shall prove only (15) and leave the proof of (16) to the Reader (see [24]). In the case under consideration it follows from the assumption of (15) and from D9b, D7, and D6 that

(a) $x, y \in E$
(b) $\mathfrak{r}_1(x_0, x_1, \ldots, x_n; x) \wedge \mathfrak{r}_1(y_0, y_1, \ldots, y_n; y)$
(c) $z \approx x_{j_1} \wedge t \approx y_{j_1}$,   for $0 \leqslant j_1 \leqslant n$
(d) $\forall k \neq j_1 \wedge 0 \leqslant k \leqslant n \ (x_k \approx y_k)$

and

(e) $z, t \in E^1$
(f) $\forall z^2 (i(z, z^2) \Rightarrow z^2 \approx t_1) \wedge \forall t^2 (i(t, t^2) \Rightarrow t^2 \approx t_1)$.
       It follows from (a) and (1) that
(g) $x, y \in E^1$.

To prove that $x\tilde{c}y$, which is to say that x and y are elements in the same syntactic category we assume, on the basis of (g) and (1), that x^2 and y^2 are their respective index-tokens, i.e.,

(h) $i(x, x^2) \wedge i(y, y^2)$.

We shall now demonstrate that $x^2 \approx y^2$. Note that since, in accordance with (a) and (b), x, y are compound well-formed expressions, their respective elements $x_0, x_1, \ldots, x_n$ and $y_0, y_1, \ldots, y_n$ are also well-formed expressions by D3c, b and as such have their indices (Lemma (1)). Let therefore

(i) $\forall 0 \leqslant k \leqslant n \; \big(i(x_k, x_k^2) \wedge i(y_k, y_k^2)\big)$.

Now (c), (8), and (i) yield:

(j) $i(z, x_{j_1}^2) \wedge i(t, y_{j_1}^2)$.

In view of (f) it follows from (j) and A1b, c that $x_{j_1}^2 \approx y_{j_1}^2$, and in view of (d) it follows from (i) and A11 that, for every $k \neq j_1$ and $0 \leqslant k \leqslant n$, $x_k^2 \approx y_k^2$. Hence, for every $0 \leqslant k \leqslant n$, $x_k^2 \approx y_k^2$, and in particular

(k) $x_0^2 \approx y_0^2$.

The well-formed expressions x, y are of the form (b) and as such must satisfy the rule which expresses **sptsc**. On the basis of assumptions (i) and (h) we take that rule into consideration (see D3b) in the following formula:

(l) $r_2(x^2, x_1^2, \ldots, x_n^2; x_0^2) \wedge r_2(y^2, y_1^2, \ldots, y_n^2; y_0^2)$.

Now $x^2 \approx y^2$ follows from A14 and the formulas (l) and (k).

By assuming now that $i(x, t)$ we would obtain, by (h) and A1a and A11, $t \approx x^2$. This allows us to state, by (g) and D6, that $x \in Ct_{x^2}$. Likewise we demonstrate that $y \in Ct_{y^2}$, and since $x^2 \approx y^2$, $Ct_{x^2} = Ct_{y^2}$. Now it follows that x and y belong to the same syntactic category, which allows us to state, by D7, that $x\tilde{c}y$ $(y\tilde{c}x)$.

The proofs of Lemmas (15) and (16) follow immediately, by inductive assumption, from D9c and the fact that they are true for $n = 1$.

Theorem 8 is a direct consequence of these lemmas and D9c.   ∎

4.2. Formalization of T1 at the type level; theory T1(tp)

The formalization of the theory **T1** at the *type level* consists (see Sec. 3) in the expansion of the theory **T1**(tk) in the form of its dual theory **T1**(tp), which describes all the concepts at that level, that is the concepts of the systems (S) and (R). The theory **T1**(tp) allows us to describe any fixed categorial language $\mathscr{L}$ as a language of expression-types. All concepts at the *type level* are derived constructs defined by means concepts at the *token level*. Every set **S** of types, which is an element of the system (S), except for the set Ct_T, is defined as follows by means of the dual set S of tokens:

DS $X \in \mathbf{S} \Leftrightarrow \exists x \in S(X = [x])$, i.e. $\mathbf{S} = S/\approx$.

In the above schema, and also further in the text, we use the symbol [x] for the equivalence class represented by x and determined by the equiformity relation.

The syntactic category with the index type T, that is the set Ct_T, is the family of all equivalence classes of equiform tokens belonging to the syntactic category with an index-token which is a representative of the equivalence class that determines the index T. In symbols:

DCt_T.　　$Ct_T = \{X \in E^1 : \exists x \in Ct_t (X = [x] \land T = [t])\}$.

The remaining concepts of (S), that is the families Ct(S) of all syntactic categories of expression-types from S, where $S \in \{E^1, E, B, F\}$, are defined by definitions which are dual to Definition D8(S). Hence

DCt(S).　　Ct(S) $= \{Ct_T : T \in i(S)\}$,　　for $S \in \{E^1, E, B, F\}$.

The relation $\tilde{c}$ of the categorial agreement of expression-types is defined by a definition dual to D7, namely

$D\tilde{c}$.　　$X, Y \in E^1 \Rightarrow [X \underline{\tilde{c}} Y \Leftrightarrow \exists T(X, Y \in Ct_T)]$.

Each of the remaining relations **R** from (**R**) is defined by its dual relation R from (R) in the following way:

DR.　　$R(X_0, X_1, \ldots, X_n) \Leftrightarrow \exists x_0, x_1, \ldots, x_n (X_0 = [x_0] \land X_1 = [x_1] \land \ldots$
　　　　$\ldots \land X_n = [x_n] \land R(x_0, x_1, \ldots, x_n))$,　　where $n \geqslant 1$.

Thus a relation **R** holds between types if and only if they are such equivalence classes of equiform tokens that a dual relation R holds between their representatives.

In view of the axioms and definitions of the theory **T1**(tk) and the definitions of the concepts of the systems (**S**) and (**R**) of the theory **T1**(tp) we can substantiate the following.

FACT 1.　*Every expression dual to a thesis of the theory* **T1**(tk) *is a thesis of the theory* **T1**(tp).

Fact 1 is substantiated directly by the observation that the following holds:

FACT 1a.　*Every expression dual to an axiom or definition of the theory* **T1**(tk) *is a theorem or definition of the theory* **T1**(tp).

By a *thesis* of a theory we mean in this paper its axioms, definitions and derived theorems.

Now Fact 1a follows from the fact that 1) the expressions d(A1a), d(A1b), d(A1c), d(A4), d(A12), d(A15), are theorems in the theory **T1**(tp); 2) in the theory **T1**(tp) Definitions d(D7), d(D8(S)) hold for $S \in \{E^1, E, B, F\}$; 3) the following expressions are theorems in **T1**(tp): d(A2), d(A3); d(A^k5), d(A^k6), d(A^k7), d(D^k1), d(A^k8), for $k = 1, 2$; d(A9), d(A10), d(A11); d(A^k13), d(A^k14), d(D^k2a), d(D^k2b), d(D^k2c) for $k = 1, 2$; d(D3a), d(D3b), d(D3c), d(A16), d(A17), d(D4), d(D5), d(D6), d(9a), d(9b), d(9c), d(9d).

The proofs of theorems given under 3) are fairly similar to the proofs of the corresponding theorems given in [24]. By way of example we shall give proofs of $d(A^k 8)$, $k = 1, 2$ and for $d(D6)$.

$d(A^k 8)$. $\quad T \in W^k \backslash V^k \Rightarrow \exists X, Y \in W^k \underline{c}(X, Y, Z)$.

PROOF. Let $T \in W^k \backslash V^k$ $(k = 1, 2)$. It follows from DW^k that $T = [t_1]$ and $t_1 \in W^k$, and from DV^k, that for any $x \in V^k$, $T \neq [x]$. Hence $t_1 \in W^k \backslash V^k$, and by Axioms $A^k 8$ $(k = 1, 2)$ we have that $\underline{c}(x_1, y_1, t_1)$ and $x_1, x_2 \in W^k$. Note that in accordance with DW^k we have: $[x_1], [x_1] \in W^k$, and in accordance with $D\underline{c}$ we have: $\underline{c}([x_1], [x_2], T)$. The truth of the consequent of the implication which is being proved follows immediately therefrom. ∎

$d(D6)$. $\quad Ct_T = \{X^1 \in E^1: \forall X^2 (\underline{i}(X^1, X^2) \Rightarrow X^2 = T)\}$.

PROOF. Let $X^1 \in Ct_T$. Then by DCt_T we have $X^1 \in E^1$, $X^1 = [x^1]$, $x^1 \in Ct_{t_1}$, and $T = [t_1]$. Assume additionally that $\underline{i}(X^1, X^2)$. Then by $D\underline{i}$ we have: $X^1 = [x']$, $X^2 = [x^2]$, and $\underline{i}(x', x^2)$. Since $x^1 \approx x'$, it follows from (8) and D6 that $x^2 \approx t_1$. Hence $T = X^2$. Thus the inclusion $\subseteq$ holds. To prove the converse inclusion we assume that $X^1 \in E^1$ and that for any X^2 if $\underline{i}(X^1, X^2)$, then $X^2 = T$. We want to show that $X^1 \in Ct_T$. By DE^1 we have that there is an $x^1 \in E^1$ such that $X^1 = [x^1]$, and since Lemma (1) holds, there is a t_1 such that $i(x^1, t_1)$, and in view of $D\underline{i}$ we can state that $\underline{i}(X^1, [x_1])$. It follows from the assumption that $T = [t_1]$. Hence, in order to state that $X^1 \in Ct_T$ (by applying DCt_T) it suffices to state that $x^1 \in Ct_{t_1}$. That is in fact so in view of D6, because $x^1 \in E^1$, and if $i(x^1, x^2)$, then $x^2 \approx t_1$ by A1a and A11. ∎

In the theory $T1(tp)$ we can formulate several theorems which are equivalent to expressions that are dual analogues of theses of the theory $T1(tk)$ but are not such theses themselves. Thus we have

THEOREM 10. *If* $\mathbf{R} \in \{\underline{c}, \underline{i}, \underline{r}_1, \underline{r}_2\}$, *then* $\mathbf{R}$ *is a function. The functions* $\underline{r}_1$ *and* $\underline{r}_2$ *are* $1-1$ *functions.*

PROOF. The fact that the concatenation relation $\underline{c}$ is a function follows from $d(A2)$ and $d(A3)$. Since in $T1(tp)$ the theorems $D_2(\underline{i}) \neq \emptyset$ and $d(A11)$ hold, the relation $\underline{i}$ is a function. Inasmuch as $D_2(\underline{r}_k) \neq \emptyset$ for $k = 1, 2$, $d(A^1 14)$, and $d(A^2 14)$, we immediately conclude that the relations $\underline{r}_1$ and $\underline{r}_2$ are $1-1$ functions. ∎

Writing $X^2 = \underline{i}(X^1)$ instead of $\underline{i}(X^1, X^2)$ and $X = \underline{r}_k(X_0, X_1, \ldots X_n)$, for a fixed $k = 1, 2$, for $\underline{r}_k(X_0, X_1, \ldots X_n; X)$ we can record two facts:

FACT 2. *The theorem* $d(D3b)$ *of the theory* $T1(tp)$ *is, on the basis of that theory, equivalent to the expression:*

(v) $\quad X \in E_{k+1} \Leftrightarrow X \in E_k \vee \exists n \geqslant 1 \exists X_0, X_1, \ldots, X_n \in E_k$

$\quad\quad [X = \underline{r}_1(X_0, X_1, \ldots, X_n) \wedge \underline{i}(X_0) = \underline{r}_2(\underline{i}(X), \underline{i}(X_1), \ldots, \underline{i}(X_n))]$.

Fact 3. *The theorems* d(D4), d(D5), *and* d(D6) *of the theory* **T1**(tp) *are, on the basis of that theory, equivalent respectively to the following expressions*:
  (i) $B = \{X \in E:\ \underline{i}(X) \in E_s^2\}$,
  (ii) $F = \{X \in E:\ \underline{i}(X) \in E_f^2\}$,
  (iii) $Ct_T = \{X \in E^1:\ \underline{i}(X) = T\}$.

In the proof of the equivalence of d(D3b) and (v) we avail ourselves of the lemma which is dual to (1) $(E_k \subseteq D_1(\underline{i}))$ and the theorem d(A13) $(D_2(\underline{r}_2) \subseteq D_1(\underline{i}))$. In the proof of Fact 3 we avail ourselves of the lemma which is dual to (1): $E \subseteq E^1 \subseteq D_1(\underline{i})$. ■

5. Theory T2

The theory **T2** has as its primitive terms the following symbols: U, $\underline{c}$, V^1, V^2, $\underline{i}$, $\underline{r}_1$, $\underline{r}_2$. They are at the same time the primitive terms of its fragment **T2**(tp). The terms which denote the remaining concepts at the *type level* and also all terms denoting concepts at the *token level* are defined in **T2**.

5.1. Formalization of T2 at the type level; theory T2(tp)

The theory **T2**(tp) is an axiomatic theory which describes the language $\mathfrak{L}$ characterized categorially as a language of expression-types.

The axioms and definitions of **T2**(tp) are either dual analogues of the axioms and definitions of **T1**(tk) or expressions equivalent to the latter. They are listed here. They are: Axioms d(A2), d(A3); Axioms d(A^k5), d(A^k6), d(A^k7) for $k = 1, 2$; Definitions d(D^k1) for $k = 1, 2$; Axioms d(A^k8) for $k = 1, 2$; Axioms d(A9), d(A10), d(A11); Axioms d(A^k13), d(A^k14) for $k = 1, 2$; Definitions d(D^k2a), d(D^k2b), d(D^k2c), for $k = 1, 2$; Definitions d(D3a), (v), (see Sec. 4.4.2) and d(D3c); Axioms d(A16), d(A17); Definitions (i)–(iii) (see Sec. 4.4.2) and d(D7), d(D8(S)), d(D9a), d(D9b), d(D9c), d(D9d).

On the adoption of these axioms and definitions we can prove that the relations $\underline{c}$, $\underline{i}$, $\underline{r}_1$ and $\underline{r}_2$ are functions (see Theorem 10 in Sec. 4.4.2). The concatenation of two types yields one type, a word-type has one corresponding index-type, etc. This justifies the recording of d(D3a), (v), and (i)–(iii). These definitions are, respectively, equivalent to the expressions which are dual to D3a, b, D4–D6.

Note that the following expressions are theorems in **T2**(tp): d(A1a), d(A1b), d(A1c), d(A4), d(A12), d(A15). Hence by accepting axioms and definitions of **T2**(tp) in the way described above we can state, on the one hand,

Fact 4. *Every expression dual to a thesis in* **T1**(tk) *is a thesis in* **T2**(tp), *i.e.*,

$$If\ \textbf{T1}(tk) \vdash \alpha,\ then\ \textbf{T2}(tp) \vdash d(\alpha),$$

and on the other,

Fact 5. *Every thesis in* **T2**(tp) *is either an expression dual to a thesis in* **T1**(tk) *or an expression equivalent to a dual analogue (an expression which is*

translatable into a dual analogue) of a thesis in the latter theory, that is

$$\text{If } \mathbf{T2}(\text{tp})\!\!\vdash\! \underline{\alpha}, \text{ then } \underline{\alpha} = \text{d}(\alpha) \text{ and } \mathbf{T1}(\text{tk})\!\!\vdash\! \alpha,$$

or there is a β such that ($\mathbf{T2}(\text{tp})\!\!\vdash\! \underline{\alpha}$ if and only if $\mathbf{T2}(\text{tp})\!\!\vdash\! \underline{\beta}$) and $\underline{\beta} = \text{d}(\beta)$ and $\mathbf{T1}(\text{tk})\!\!\vdash\! \beta$.

Facts 4 and 5 reveal the close connection between the theory **T1**(tk) and its dual theory **T2**(tp). From the formal point of view, if we consider only the syntactic single-level characterization of language, there is thus no essential difference between the two ontologically opposed methods of describing language by dual theories **T1**(tk) and **T2**(tp).

5.2. Formalization of T2 at the token level; theory T2(tk)

We join to **T2**(tp) two additional axioms which render certain intuitions which we associate with the concept of *type* as a non-empty class of equiform *tokens*:

A^t1. $X \neq \emptyset$ — a type is non-empty set,

A^t2. $x \in X \wedge x \in Y \Rightarrow X = Y$ — two types are equal if they have an element in common.

The formalization of **T2** at the *token level* requires a definitional expansion of the theory **T2**(tp), namely the theory **T2**^{t}(tp) by its *enrichment* with definitions of all concepts at the *token level*, defined by the appropriate dual concepts from the *type level*. The theory **T2**(tk), dual to **T2**(tp), is a fragment of **T2** which includes those definitions and describes a categorial language $\mathscr{L}$ as a language of expression-tokens.

The definitions of all sets of tokens from the system (S), except for the set Ct_t, have in **T2**(tk) the following schema:

DS. $x \in S \Leftrightarrow \exists X \in S(x \in X)$.

The set $S \neq Ct_t$ of tokens is a set of those tokens which are elements (concrete representatives) of some type that belongs to the dual set S.

Since the universe U of $\mathscr{L}$ is non-empty (Fact 1), in agreement with Axiom A^t1 and Definition DU elements of a type are tokens of U. The types are thus really sets of tokens.

The concept Ct_t of syntactic category with an index t is defined thus:

DCt_t. $Ct_t = \{x \in E_1 : \exists T \exists X \in Ct_T (x \in X \wedge t \in T)\}$.

The remaining elements of (S), that is the family $Ct(S)$, where $S \in \{E^1, E, F, R\}$, are defined as in **T1**(tk), and hence by definitions of the form D8(S).

The definitions of all relations in (R), except for $\tilde{c}$, have the following form:

DR. $R(x_0, x_1, \ldots, x_n) \Leftrightarrow \exists X_0, X_1, \ldots, X_n (x_0 \in X_0 \wedge x_1 \in X_1 \wedge \ldots \wedge x_n \in X_n \wedge$
       $\wedge R(X_0, X_1, \ldots, X_n))$,   where $n \geqslant 1$.

The relation $\tilde{c}$ is defined identically as in **T1**(tk).

Note that the definition $D \approx$ of $\approx$ can be recorded in a simpler way:

$D \approx$. $x \approx y \Leftrightarrow \exists X(x, y \in X)$.

Thus by assuming in **T2** that types are primitive entities, while tokens are derived constructs as elements of types (Definition DU), we are in a position to formally show that in accordance with our intuition any type is a set of equiform tokens. Hence in particular Definition DU in **T1**(tp) is a theorem in **T2**.

We shall discuss in greater detail that fragment **T2**(tk) of **T2** which is developed on the basis of **T2**^{t}(tp). Owing to the definitions which are valid in that fragment, namely definitions of the concept at the *token level*, it can be used to describe a categorial language $\mathscr{L}$ in a manner analogous to how it is done in **T1**(tk). This is so because we have to do with the following

FACT 6a. *Every axiom and every definition in* **T1**(tk) *is a theorem or a definition in* **T2**(tk),

and hence with

FACT 6. *Every thesis in* **T1**(tk) *is a thesis (theorem or definition) in* **T2**(tk).

The complete substantiation of Fact 6a is rather labour-consuming. The detailed substantiation of the fact that Axioms A1a, b, c $-$ A4 and $A^1 5 - A^1 8$ and Definition $D^1 1$ are theorems in **T2**(tk) is to be found in [26]. Those theorems pertain to the tokens from the universe U or its subsets V^1 and W^1. The substantiation of the fact that the analogues of the expressions $A^1 5 - A^1 8$ and $D^1 1$ pertaining to the auxiliary words in the sets V^2 and W^2, that is the expressions $A^2 5 - A^2 8$ and $D^2 1$, are theorems in **T2**(tk), is analogous. The proofs of the fact that the remaining axioms and definitions in **T1**(tk) other than D7 and D8(S), where $S \in \{E^1, E, B, F\}$ (assumed also in **T2**(tk)) are theorems in **T2**(tk) present no major problems. To show the functioning of the definitions and axioms given in this section we shall prove by way of example the expressions $A^k 15$ ($k = 1, 2$) and the simple inclusion yielding D4.

$A^k 15$. $\mathfrak{r}_k(x_0^k, x_1^k, \ldots, x_n^k; x^k) \wedge \forall 0 \leqslant j \leqslant n(y_j^k \approx x_j^k) \wedge y^k \approx x^k \Rightarrow$

$\Rightarrow \mathfrak{r}_k(y_0^k, y_1^k, \ldots, y_n^k; y^k)$.

PROOF. It follows from the first assumption of $A^k 15$ and from Dr_k ($k = 1, 2$) that there are types $X_0^k, X_1^k, \ldots, X_n^k$ such that, for any $0 \leqslant l \leqslant n$, $x_l^k \in X_l^k$, $x^k \in X^{k}$, and $R(X_0^k, X_1^k, \ldots, X_n^k; X^k)$. Since the following lemma

L1. $x \in X \wedge y \approx x \Rightarrow y \in X$

is true in view of $A^t 2$ and $D \approx$, it follows from the remaining assumptions of $A^k 15$ that, for any $0 \leqslant l \leqslant n$, $y_l^k \in X_l^k$ and $y^k \in X^k$. By availing ourselves again of Dr_k we obtain the thesis of $A^k 15$.   ∎

$D4(\subseteq)$. $B \subseteq \{x^1 \in E : \forall x^2(\mathfrak{i}(x^1, x^2) \Rightarrow x^2 \in E_s^2)\}$.

PROOF. Let $x^1 \in B$. By DB there is a type $X^1 \in B$ such that $x^1 \in X^1$. Then by Definition (i) $X^1 \in E$ and $\underline{i}(X^1) \in E_s^2$. By applying D$E$ we have $x^1 \in E$. Let us assume for the purpose of the proof that $\underline{i}(x^1, x^2)$. Then there are Y^1, Y^2 such that $x^1 \in Y^1$, $x^2 \in Y^2$, and $\underline{i}(Y^1, Y^2)$, i.e., $Y^2 = \underline{i}(Y^1)$ (Definition Di). Hence, by Axiom A^{t}2, $Y^1 = X^1$ which is to say that $x^2 \in \underline{i}(X^1)$, and by D$E_s^2$ we have $x^2 \in E_s^2$. This proves that the inclusion under consideration is true. ∎

6. The equivalence of the theories T1 and T2

The two various formalizations of the theory of the syntax of language, presented by the theories **T1** and **T2** treated in their two aspects, make us above all reflect on whether both theories equally well describe the language syntactically or whether they differ in the sets of their theses, i.e., whether **T1 = T2.**

As it is know, two axiomatic theories that do not differ from one another by the sets of their theses are equivalent, and to demonstrate that it suffices to show that every axiom and every definition in one theory is a thesis in the other theory, and conversely, every axiom and every definition in the latter is a thesis in the former.

Let us accordingly make a formal comparison of **T1** and **T2**.

Note first that all concepts at the *token level* are definable in **T2**(tk) in terms of concepts from the *type level* (definitions with the schemata DS, DCt_t, DR) or are such as in **T1**(tk) (Definitions D7 and D8(S)) and, what is more, they can be *characterized* as in **T1**(tk) (Fact 6): every axiom and every definition in **T1**(tk) is a thesis in **T2**(tk) (Fact 6a). Note also that Definitions D$\tilde{c}$ and DCt(S) in **T1**(tp) are such as in **T2**(**T2**(tp)). It can be demonstrated that the remaining definitions in **T1**(tp), that is definitions witch the schema DS, where **S** is a set from the system (**S**) other than Ct_T, Definition DCt_T, and definitions with the schema DR, where **R** is a relation in the system (**R**) other than $\tilde{c}$, are theorems in **T2**(**T2**(tk)). For Definition DU that fact was mentioned already in Sec. 5.2. We shall now prove the correctness of that statement only for expressions of the form DS.

DS. $X \in S \Leftrightarrow \exists x \in S(X = [x])$.

PROOF. Let $X \in S$. Since by Axiom A^{t}1 some $x_1 \in X$, by Definition DS of S, $x_1 \in S$. Note that $X = [x_1]$, because if $y \in X$, then in view of $x_1 \in X$ Definition D$\approx$ implies that $y \approx x_1$ and hence $y \in [x_1]$, and if $y \in [x_1]$, then $y \approx x_1$ and, by Lemma L1, $y \in X$. Thus the simple implication in DS is true. To prove the converse implication note that if $x_1 \in S$ and $X = [x_1]$, then by Definition DS $x_1 \in X_1$ and $X_1 \in S$. Now since $x_1 \in X$, it follows from Axiom A^{t}2 that $X_1 = X$ and $X \in S$. ∎

The foregoing considerations lead us to the conclusion that the theory **T1** can be grounded in the theory **T2**.

We shall prove that the converse also holds. Note first that the axioms and definitions adopted in **T2**(tp) either are dual analogues of the axioms and definitions of **T1**(tk) or are equivalent to the dual analogues of definitions of that theory (the expressions d(3a), (v), (i)–(iii)). As such they are, in agreement with Fact 1a, theorems or definitions in **T1** (**T1**(tp)). Thus all concepts at the *type level* can be *characterized* in **T1**(tp) in the same way as in **T2**(tp). This is possible owing to the fact that all concepts at that level are in **T1**(tp) definable in terms of concepts from the *token level* (definitions with the schemata DS, DCi_T, DR) or are the same as in **T2**(tp) (Definitions $DCt(S)$, $D\tilde{c}$). Both axioms of **T2'**(tp) joined to **T2**(tp) are also theorems in **T1**(**T1**(tp)). This follows directly from the convention that X, Y are variables which represent types, Definition DU, and the properties of equivalence classes. Further all the definitions of concepts from the *token level* adopted in **T2**(**T2**(tk)) are theorems or definitions in **T1**(tp). Definitions $D\tilde{c}$ and $DCt(S)$ are the same in both theories, and the expressions with the schemata DS, DCt_t and DR are provable in **T1**(tp). In their proofs in fact use is made of Theorems 2 and 3 (the formulas (∗) and (∗∗)). In this way every axiom and every definition of **T2** is a thesis in **T1**. Thus **T2** can be grounded in **T1**.

As a result of the above we may state

FACT 7.   *The theories* **T1** *and* **T2** *are equivalent.*   ∎

7. Final conclusions and remarks

From the point of view of the philosophy of language the theories **T1** and **T2** represent, respectively, two dual approaches to the syntax of language, the *nominalistic (concretistic)* and the *Platonic*. In the light of Fact 7 we may accordingly state that

(I)   *The two ontologically opposed approaches to the syntax of language represented by the theories* **T1** *and* **T2**, *are equivalent.*

The biaspectual formalizations of **T1** and **T2** at two different levels, that of *tokens* and that of *types*, as presented above, show clearly the analogies between the properties of the objects belonging to those two different levels. Dual expressions describe the analogous properties of dual concepts. The said analogies can be grasped in two ways. On the one hand, they can be perceived separately within both **T1** and **T2**. It suffices to compare any thesis of **T1**(tk), which describes the properties of concepts at the *token level* with the dual thesis of the dual theory **T1**(tp), which describes the properties of concepts at the *type level* (Fact 1), and to compare any thesis of **T2**(tp), which describes the properties of concepts at the *type level*, with either the dual thesis of the dual theory **T2**(tk) or its translation into the dual thesis of that theory — the theory which describes the properties of concepts at the *token level* (Fact 4, 5, and 6). On the other hand, we find these analogies when we compare the theories **T1** and **T2**, strictly

speaking when we compare the theses of **T1**(tk) with the theses of the dual theory **T2**(tp), and conversely. This is so because, in accordance with Fact 4, every property which is an attribute of an object at the *token level* is also an attribute of the dual object at the *type level*, while in accordance with Fact 5, every property which is an attribute of an object at the *type level* either is an attribute of the dual objects at the *token level* or can be transformed into such a property.

The above observations will be recorded as the following conclusion:

(II) *There is a formal mutual analogy between dual syntactic concepts at the token level and the type level.*

In view of the equivalence of the theories **T1** and **T2** it follows from the comments made above that whether elements of language are concrete or abstract entities is of no importance in theoretical enquiries concerned with the syntax of language. Hence note that

(III) *In purely theoretical syntactic considerations the philosophical aspects pertaining to the double ontological nature linguistic objects may be disregarded.*

But the conclusion (I) and (II) given above speak in favour of Słupecki ideology concerning the nature of linguistic objects. The possibility of constructing a theory of the syntax of language as the theory **T1**, which represents the *concretistic* approach and does not require, for the description of the basic syntactic concepts, the assumption of the existence of ideal objects (that is *types* understood as sets of equiform *tokens*) leads us in fact to the following essential conclusion of the present paper:

(IV) *In the syntactic analyses of language one may eliminate the assumption on the existence of ideal linguistic entities interpreted as classes of equiform tokens.*

* * *

Some final remarks. The studies presented in this paper cover only the categorial description of languages which do not include operators that bind variables. These studies can be generalized so as to cover such languages as well (see [24]). Further. This paper presents only a most essential fragment of syntactic problems. It discusses those syntactic concepts which are used for a general description of a language constructed in the spirit of Leśniewski and Ajdukiewicz. But it seems that the formulation of the fundamental philosophical conclusion present in this paper (Conclusion (IV)) can be affected neither by the expansion of the conceptual apparatus used and of the scope of syntactic problems, nor by the construction of the theoretical foundations of the syntax of language which would consider other formal models, such as Chomsky's transformational-generative models. The analyses pertaining to the two dual ontological approaches to the syntax of language can probably be easily adjusted to the construction of other theory of the syntax of language, in particular the theories of formal languages in Chomsky's spirit.

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