EXPOSING SOME POINTS OF INTEREST ABOUT NON-EXPOSED POINTS OF DESIRABILITY

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ABSTRACT. We study the representation of sets of desirable gambles by sets of probability mass functions. Sets of desirable gambles are a very general uncertainty model, that may be non-Archimedean, and therefore not representable by a set of probability mass functions. Recently, Cozman (2018) has shown that imposing the additional requirement of even convexity on sets of desirable gambles guarantees that they are representable by a set of probability mass functions. Already more that 20 years earlier, Seidenfeld et al. (1995) gave an axiomatisation of binary preferences—on horse lotteries, rather than on gambles—that also leads to a unique representation in terms of sets of probability mass functions. To reach this goal, they use two devices, which we will call 'SSK–Archimedeanity' and 'SSK–extension'. In this paper, we will make the arguments of Seidenfeld et al. (1995) explicit in the language of gambles, and show how their ideas imply even convexity and allow for conservative reasoning with evenly convex sets of desirable gambles, by deriving an equivalence between the SSK–Archimedean natural extension, the SSK–extension, and the evenly convex natural extension.

keywords: sets of desirable gambles, representation, sets of probabilities, Archimedeanity, natural extension

1. INTRODUCTION

Over the last few decades, imprecise probability theory has taken an increasingly more prominent role in the field of reasoning and decision making under uncertainty. Imprecise-probabilistic uncertainty models extend classical probability theory by allowing for incomplete assessments, thereby creating a well-established theory that can deal with indecision and robustness. The decision theoretic foundations of such a theory were laid by C. A. B. Smith (1961) [24] and I. Levi (1974) [13]. Among more recent developments, [19–21, 25] focus on a theory of "desirability", which is the topic of this paper.

Sets of desirable gambles [17, 20, 25] generalise many of the other existing theories, such as closed and convex sets probabilities, coherent lower previsions, and belief functions. A set of desirable gambles D is a set of gambles—which are real-valued maps on the finite possibility space Ω—that the subject strictly prefers to the status quo indicated by 0. A gamble is commonly interpreted as an uncertain reward: if the subject has the gamble f on Ω, then after the actual outcome of the experiment turns out to be $\omega \in \Omega$, her capital is changed by the—possibly negative—payoff $f(\omega)$, described in a predetermined linear utility scale. Coherent sets of desirable gambles are more expressive than closed and convex sets of probabilities. When we surrender the closedness condition, the connection between convex sets of probabilities and coherent sets of desirable gambles becomes more intricate. But even when the set of probabilities is only convex—and not necessarily closed—there will be coherent sets of desirable gambles that cannot be expressed using such sets of probabilities: that is because sets of desirable gambles do not satisfy an Archimedean condition, and are therefore not always representable by real numbers.

Example 1. Consider the experiment of tossing a coin, so the possibility space Ω is equal to the binary set $\{\mathrm{h},\mathrm{t}\}$, where h stands for 'heads' and t for 'tails'. The set of gambles on Ω is a two-dimensional linear space under point-wise addition and point-wise scalar multiplication, so any gamble f can be depicted graphically as a point in the plane as in Figure 1.

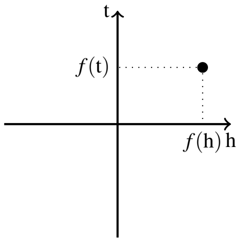


FIGURE 1. A gamble f on the binary possibility space $\Omega = \{\mathrm{h}, \mathrm{t}\}$

In this example we will contrast an Archimedean set of desirable gambles $D_A := \{\text{gambles } f : f(\mathrm{h}) + f(\mathrm{t}) > 0\}$ with a non-Archimedean one $D_{n-A} := D_A \cup \{\text{gambles } f : f(\mathrm{h}) + f(\mathrm{t}) = 0 \text{ and } f(\mathrm{h}) > f(\mathrm{t})\}$, as depicted in Figure 2.

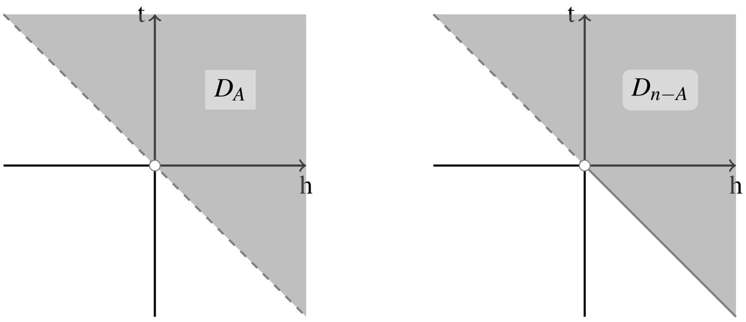


FIGURE 2. An Archimedean set of desirable gambles D_A and non-Archimedean set of desirable gambles D_{n-A}

If the subject's beliefs are described by the set of desirable gambles D_A then she finds desirable any gamble f whose uniform expected value $\frac{1}{2}(f(\mathrm{h}) + f(\mathrm{t}))$ is greater than 0: she believes that the coin is fair. On the other hand, if she uses the larger D_{n-A} as her set of desirable gambles, then in addition she finds desirable any gamble f with uniform expected value of 0 but whose payout in heads is greater than in tails: we take this to represent a belief that the coin is infinitesimally biased towards heads, but not by any real amount. While D_A is represented by the uniform probability on Ω, we will see that the set of desirable gambles D_{n-A} cannot be (two-way) represented by any real-valued probability. This will be evidenced by the fact that D_A satisfies a certain Archimedean condition, which D_{n-A} fails to satisfy, as we will point out informally in Example 4, and formally in Section 3. $\lozenge$

In this paper, we will be concerned with the representation of a set of desirable gambles by a set of probabilities. By being 'representable by a set of probabilities', we could mean two different things: a one-way and a two-way representation. A *one-way representation* of a set of desirable gambles D is a representation that does not completely determine D, but instead will determine a sub- or superset of D. In the subset-variant, a representation of D is a set of probability mass functions $\mathbf{R}(D)$ with the property that, if every element p of $\mathbf{R}(D)$ assigns a (strictly) positive p-expectation E_p to some gamble f—so $E_p(f) > 0$ for all p in $\mathbf{R}(D)$—, then f is desirable. A one-way representation that satisfies this property is called 'almost agreeing'. Blackwell & Girshick [2, Theorem 4.3.1] give necessary conditions on partial preference relations—which are equivalent to 'coherence' for sets of desirable gambles—for the existence of an almost agreeing representation. If $E_p(f) > 0$ for every p in $\mathbf{R}(D)$, then such a representation implies that $f \in D$, but the set $\{\text{gambles } f : (\forall p \in \mathbf{R}(D)) E_p(f) > 0\}$ may be a strict subset of D, as Blackwell & Girshick [2, P.119] show by means of an example. $\mathbf{R}(D)$ lets us determine all the gambles in the interior of D, but will be incapable of determining whether a gamble on the boundary of D is actually desirable or not. Such a representation may be useful if we are interested only in determining the interior of D.

If we want to retrieve the complete set D, including its possible non-Archimedean behaviour on its boundary, an almost agreeing representation will be insufficiently expressive. Because of this, we are often interested in a representation $\mathbf{R}(D)$ that also respects the converse relation, namely if $f \in D$, then $E_p(f) > 0$ for all p in $\mathbf{R}(D)$. This is the defining property of the superset-variant of a one-way representation of D, which is a set of probability mass functions $\mathbf{R}(D)$ whose elements p assign (strictly) positive p-expectations to every desirable gamble: for every f in D and every p in $\mathbf{R}(D)$, the p-expectation $E_p(f)$ is strictly positive. Not every coherent set of desirable gambles D will have such a representation; we will call the ones that have such a representation 'extendible to a probability'. A set of desirable gambles D being coherent and extendible to a probability, however, does not guarantee that we can retrieve D using its (one-way) representation $\mathbf{R}(D)$: as shown using an example by Cozman [4]—which we will repeat in Example 7—there are sets of desirable gambles D and $f \notin D$ for which $E_p(f) > 0$ for all p in $\mathbf{R}(D)$.

In this paper, we want to combine the two one-way representations mentioned above, and are interested in more stringent conditions on D that guarantee that $\mathbf{R}(D)$ is actually a *two-way representation* of D: we want to be able to retrieve D by $\mathbf{R}(D)$. This requires that $E_p(f) > 0$ for all p in $\mathbf{R}(D)$, if *and only if* $f \in D$.[1]

Example 2. The set of desirable gambles D_A from Example 1 is represented by the singleton $\mathbf{R}(D_A) = \{p_u\}$ consisting of the uniform probability p_u on $\Omega = \{\text{h}, \text{t}\}$. More specifically: $\mathbf{R}(D_A)$ is a two-way representation. Indeed, we have that $f \in D_A \Leftrightarrow f(\text{h}) + f(\text{t}) > 0 \Leftrightarrow E_{p_u}(f) > 0$, for any gamble f.

The set of desirable gambles D_{n-A} is also represented by the same $\mathbf{R}(D_{n-A}) = \{p_u\}$, but this is only a one-way almost agreeing representation. Indeed, if $E_{p_u}(f) > 0$ then f belongs to D, but not *vice versa*: For instance the gamble $f = (f(\text{h}), f(\text{t})) = (1, -1)$ has a zero uniform expectation $\frac{1}{2}(1-1)$, but belongs nonetheless to D_{n-A}. $\lozenge$

This picture becomes more complex once we allow for representing *sets* of probabilities $\mathbf{R}(D)$, instead of singletons. Such a set $\mathbf{R}(D)$ represents a set of desirable gambles D that is no longer a half-space. If a set of desirable gambles D is representable by a set of probabilities, there might be multiple representing sets of probabilities. However, there is one unique largest set of representing probabilities, which is always convex. If D is an open set, then its representing convex set of probabilities is a closed set, but more sophisticated situations are possible. Being two-way representable is important for computational aspects: general sets of desirable gambles are mostly difficult to handle from a computational point of view, but convex sets of probabilities, being Archimedean objects, are easier; see Quaeghebeur [16]. We are therefore looking for additional requirements on sets of desirable gambles to guarantee that they are two-way representable by a convex (but not necessarily closed) set of probabilities.

Example 3. To illustrate the fact stated above, consider the set of desirable gambles $D := \{\text{gambles } f : f(\text{h}) > 0 \text{ and } f(\text{t}) > 0\}$ on the possibility space $\Omega = \{\text{h}, \text{t}\}$, as indicated in Figure 3 below. This D is (two-way) represented by the set of probability mass functions $\{p_\text{h}, p_\text{t}\}$, which consists of the two degenerate probability mass functions p_h and p_t given

[1] We use this occasion to correct Seidenfeld et al. [20, Theorem 1.i and Theorem 2.i] which should read as one-way almost agreeing representations.

by $p_{\mathrm{h}}(\mathrm{h}) = 1 - p_{\mathrm{h}}(\mathrm{t}) = 1$ and $p_{\mathrm{t}}(\mathrm{t}) = 1 - p_{\mathrm{t}}(\mathrm{h}) = 1$. To see this, consider any gamble f on Ω, and infer that indeed $(\forall p \in \{p_{\mathrm{h}}, p_{\mathrm{t}}\}) E_p(f) > 0 \Leftrightarrow (E_{p_{\mathrm{h}}}(f) > 0 \text{ and } E_{p_{\mathrm{t}}}(f) > 0) \Leftrightarrow (f(\mathrm{h}) > 0 \text{ and } f(\mathrm{t}) > 0) \Leftrightarrow f \in D$. The set $\{p_{\mathrm{h}}, p_{\mathrm{t}}\}$ is not convex since it does not include any probability mass function $p_\alpha = \alpha p_{\mathrm{h}} + (1-\alpha) p_{\mathrm{t}}$ whose probability for heads is α, for any α in $(0,1)$.

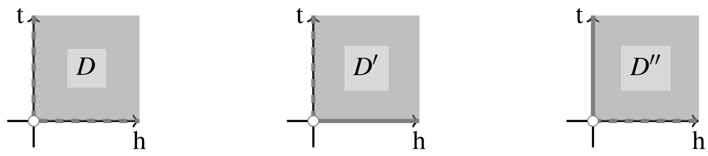


FIGURE 3. An open set of desirable gambles D two sets of desirable gambles D' and D'' that are not open

But D is also (two-way) represented by the simplex $\mathbf{R}(D) = \Sigma$ of all probability mass functions on Ω: indeed, for any gamble f in D and any probability p in Σ we have that $E(f) = p(\mathrm{h}) f(\mathrm{h}) + (1 - p(\mathrm{h})) f(\mathrm{t}) > 0$. So we have found two different two-way representations: $\{p_{\mathrm{h}}, p_{\mathrm{t}}\}$, and the unique largest Σ.

Contrast these conclusions with the sets of desirable gambles $D' := \{\text{gambles } f : f(\mathrm{h}) > 0 \text{ and } f(\mathrm{t}) \geq 0\}$ and $D'' := \{\text{gambles } f : f(\mathrm{h}) \geq 0 \text{ and } f(\mathrm{t}) > 0\}$, indicated in Figure 3. While D is an open set of desirable gambles, the sets D' and D'' are not open, and this is exemplified by the fact that their representing sets of probabilities are not closed: The set $\mathbf{R}(D') = \{p \in \Sigma : 0 < p(\mathrm{h}) \leq 1\}$ represents D' and the set $\mathbf{R}(D'') = \{p \in \Sigma : 0 \leq p(\mathrm{h}) < 1\}$ represents D''. As $\mathbf{R}(D')$ and $\mathbf{R}(D'')$ are convex but not closed sets of probabilities, they are not generated by a finitely many extreme points. $\Diamond$

We see that the boundary of a set of desirable gambles will play a crucial role in the question whether or not it is two-way representable by a set of probabilities. In order to regulate the boundary behaviour, Seidenfeld et al. [21] have introduced a continuity condition—which we will refer to as 'SSK–Archimedeanity'—for partial preference orders, in order to ensure representability by a set of probabilities. Translated to our setting, SSK–Archimedeanity requires that adding a desirable gamble to an *almost-desirable gamble*, which is the limit of a converging sequence of desirable gambles, should result in a desirable gamble. We refer to Section 3 for a more detailed discussion.

Example 4. Consider the sets of desirable gambles D_A and D_{n-A} defined in Example 1. We will show that D_A is an SSK–Archimedean set of desirable gambles, and D_{n-A} not, confirming our intuitions from Example 1. For D_A, any almost-desirable gamble g has a non-negative uniform expectation $E_{p_u}(g) \geq 0$, while any desirable gamble h has a positive uniform expectation $E_{p_u}(h) > 0$. Adding them results in a gamble $f := g + h$ whose uniform expectation $E_{p_u}(f) = E_{p_u}(g) + E_{p_u}(h) > 0$ is positive, whence f belongs to D_A. Since this is true for any choices of g and h, we conclude that indeed D_A is an SSK–Archimedean set of desirable gambles.

On the other hand, for D_{n-A}, note that the gamble $g = (g(\mathrm{h}), g(\mathrm{t})) = (-1, 1)$ is almost-desirable (as it is the limit of the converging sequence of gambles $((-1 + \frac{1}{n}, 1 + \frac{1}{n}) : n \in \mathbb{N})$ whose uniform expectation $\frac{1}{n}$ is positive), so adding it to the desirable gamble $h = (1, -1) \in D_{n-A}$ should result in a desirable gamble $f := g + h = 0$, which does not belong to D_{n-A}. This confirms that, indeed, D_{n-A} is not an SSK–Archimedean set of desirable gambles. $\Diamond$

Loosely speaking, the reason why there is no two-way representation for D_{n-A} in Example 4 is because D_{n-A} is "too large": we say that it is not 'extendible to a probability'. As we will discuss in Section 3, this is exactly what SSK–Archimedeanity can pick up.

Not being extendible to a probability is not the only reason why a set of desirable gambles can be non-representable by a (set of) probabilities. Indeed, it may happen that a set of desirable gambles D is extendible to a probability, but that it still is not 'regular' enough to have a two-way representation. It will turn out in Section 3.3 that this type of regularity is exactly 'even convexity'.

Example 5. In this example we will informally introduce a failure of even convexity, while being SSK–Archimedean. This example is due to Cozman [4], and we discuss it in more detail in Section 3.3, more specifically in Example 7. Consider a ternary possibility space, and the set of desirable gambles indicated in Figure 4.

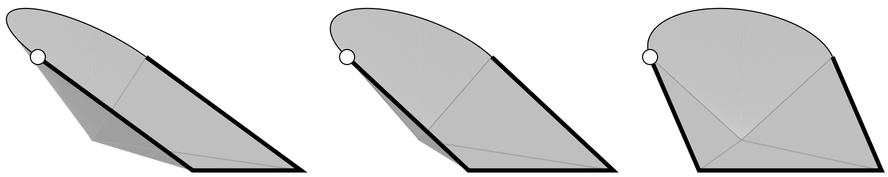


FIGURE 4. An SSK–Archimedean set of desirable gambles that is not evenly convex set—viewed from three different angles

While this set of desirable gambles is SSK–Archimedean—and therefore is extendible to a probability—it fails to be represented by a set of probabilities, as we will see in Section 3.3. The reason for this is that the ray through the gamble indicated by a white dot is a 'non-exposed extreme' ray of gambles. Loosely speaking, that this ray is extreme means that no gamble of this ray is a convex combination of gambles on other rays, and that this ray is non-exposed means that we cannot uniquely isolate it using a hyperplane. We will show in this paper that this is the only type of failure—having a non-exposed extreme ray—for an SSK–Archimedean set of desirable gambles to be represented by a set of probabilities. $\diamond$

The results of our paper add to the following literature. Quite recently, Cozman [4] has given an axiomatisation of sets of desirable gambles that guarantees a two-way representation. He shows that any *evenly convex* coherent set of desirable gambles—that is, a coherent set of desirable gambles that is an arbitrary intersection of affine open half-spaces—is uniquely represented by a convex set of probabilities, and he gives an elegant equivalent requirement in terms of gambles. Very recently, De Cooman [8] added to this with a discussion of Archimedeanity in general Banach spaces. Interestingly, he obtains results for arbitrary dimensions, and also for more than binary choices, modelled by imprecise choice functions introduced by Seidenfeld, Kadane and Schervish [12, 23]. In particular, De Cooman [8] obtains a very strong representation result for evenly convex sets of desirable gambles, and derives even a representation result for evenly convex choice functions from it. In this paper, we are after a similar result, but specialised to a finite dimension, which lays bare a tighter connection with SSK–Archimedeanity, and uses more familiar concepts from convex analysis [18] such as exposed rays.

Almost 25 years earlier, in 1995, Seidenfeld et al. [21] gave an axiomatisation of binary preferences and showed that it admits a unique representation in terms of convex sets of probabilities. Since binary preferences are closely related to sets of desirable gambles, Seidenfeld et al. [21]'s requirement must be similar to that of even convexity. There is however a difference: Seidenfeld et al. [21]'s options between which the subject must state her preferences, are horse lotteries, instead of gambles, but Cozman [4] has shown that their ideas can be straightforwardly used for gambles as well. Roughly speaking, and after translating results to sets of desirable gambles, here we show that any coherent set of desirable gambles that (i) satisfies SSK–Archimedeanity, in the same vein as Cozman [4], and (ii) is the result of a particular extension, which we will refer to as 'SSK–extension', is uniquely represented by a convex set of probabilities.

In this paper, we will make the arguments of Seidenfeld et al. [21] explicit in the language of gambles. One reason for doing so, is to be able to directly compare our results with Cozman [4]'s. There is another, maybe more compelling reason for doing so: doing conservative inference [see De Cooman [7] for an overview and connection with logical inference] is surprisingly easy with sets of desirable gambles. Given a consistent assessment A that consists of gambles that the subject finds desirable, there is always a unique smallest (least informative) coherent set of desirable gambles $\mathscr{E}(A)$ that includes A. This smallest coherent extension $\mathscr{E}(A)$ is called the *natural extension* [9, 17, 25]. We will add SSK–Archimedeanity as an optional rationality requirement, and find the unique smallest coherent and SSK–Archimedean set of desirable gambles $\mathscr{E}_{\mathrm{Arch}}(A)$—which we call the *SSK–Archimedean natural extension*, and characterise the conditions under which it is coherent. We will do the same for even convexity, leading to the *evenly convex natural extension* $\mathscr{E}_{\mathrm{e.c.}}(A)$. Quite interestingly, we will show that the evenly convex natural extension is equal to the SSK–extension of a set of desirable gambles that is extendible to a probability, which shows that the ideas of SSK–extension in Seidenfeld et al. [21], and of even convexity in Cozman [4], amount to the same result.

This paper is structured as follows. In Section 2, we review the rationality requirements of sets of desirable gambles, and show how they are connected with partial preferences. We use this connection in Section 3 to translate the requirement of SSK–Archimedeanity to our setting of sets of desirable gambles. In Section 3.1 we investigate some order-theoretic properties of SSK–Archimedean sets of desirable gambles, leading to the SSK–Archimedean natural extension in Section 3.2. We review Cozman [4]'s example of a coherent and SSK–Archimedean but not evenly convex set of desirable gambles in Section 3.3, showing that SSK–Archimedeanity is not sufficient to obtain a two-way representation. We add the new result that it is necessary, however. The missing link—namely, even convexity focused on convex cones—is the subject of Section 3.4. Our simpler way to obtain even convexity from an SSK–Archimedean set of desirable gambles will be explained in Section 4. Because we need the notion of indifference for the SSK–extension, Section 4.1 reviews how indifference can be reconciled with desirability, leading to the definition of the SSK–extension in Section 4.2. We connect this extension with even convexity in Section 4.3, leading to the evenly convex natural extension in Section 4.4.

2. Sets of desirable gambles

Consider a finite possibility space Ω. We will assume throughout that Ω contains $n \in \mathbb{N}$ distinct elements. A real-valued map on Ω is a *gamble*. The set of all gambles is denoted by $\mathscr{L}$, which is an n-dimensional linear space under point-wise addition of gambles, and point-wise scalar multiplication of a gamble with a real number. We attach the following interpretation to gambles. A gamble f is an uncertain reward: If the actual outcome turns out to be ω in Ω, then the subject's capital is changed by the—possibly negative—amount $f(\omega)$, described in a linear utility scale.

A set of desirable gambles is a subset of $\mathscr{L}$. It is meant to be the set of all the gambles that the subject (strictly) prefers to the status quo indicated by 0. This is the constant gamble that yields 0 in every outcome, so it leaves the subject's capital unchanged whatever happens. Sets of desirable gambles were used by Seidenfeld et al. [20], generalising the work of Blackwell & Girshick [2, P. 118], and later extensively, for instance by Walley [25, 26], De Cooman and Quaeghebeur [9], and Quaeghebeur [17].

2.1. Coherence.

Not every set of desirable gambles reflects a rational belief. For example, any gamble $f \lessdot 0$—by which we mean $f(\omega) < 0$ for all ω—can never be desirable, since it makes the subject lose capital. We collect these gambles in $\mathscr{L}_{\lessdot 0}$. On the other hand, any gamble $f \gtrdot 0$—by which we mean $f(\omega) > 0$ for all ω in Ω—will be desirable, since the subject's capital will increase certainly. These gambles are collected in $\mathscr{L}_{\gtrdot 0}$.

Only *coherent* sets of desirable gambles will be used to describe rational beliefs:

Definition 1 (Coherent set of desirable gambles)**.** Let $\mathbb{R}_{>0}$ be the set of all (strictly) positive real numbers. A set of desirable gambles D is called *coherent* if for all f and g in $\mathscr{L}$, and λ in $\mathbb{R}_{>0}$:

D_1. $0 \notin D$;
D_2. $\mathscr{L}_{\gtrdot 0} \subseteq D$;
D_3. if $f \in D$ then $\lambda f \in D$;
D_4. if $f, g \in D$ then $f + g \in D$.

We collect all the coherent sets of desirable gambles in $\overline{\mathbf{D}}$.

Axioms D_3 and D_4 make a coherent set of desirable gambles D a *convex cone*: $D = \operatorname{posi}(D)$, where posi is the 'positive hull operator' defined as[2]

$$\operatorname{posi}(A) := \left\{ \sum_{k=1}^{m} \lambda_k f_k : m \in \mathbb{N}, f_1, \ldots, f_m \in A, \lambda_1, \ldots, f_m \in \mathbb{R}_{>0} \right\} \text{ for any } A \subseteq \mathscr{L}.$$

So any convex cone D that includes the (interior of the) positive orthant $\mathscr{L}_{\gtrdot 0}$, and does not include 0 is a coherent set of desirable gambles. This implies that D has nothing in common with the point-wise negative orthant $\mathscr{L}_{\lessdot 0}$: otherwise, if some $f \lessdot 0$ would belong to D, then $-f \gtrdot 0$ so $-f \in D$ by Axiom D_2, whence $0 = f + (-f) \in D$ by Axiom D_4, which would contradict Axiom D_1. This is the notion of coherence that Cozman [4] uses, and is a particular instance of the coherence discussed by De Cooman and Quaeghebeur [9].

The collection $\overline{\mathbf{D}}$ of all coherent sets of desirable gambles is therefore equal to the collection of all the convex cones that include $\mathscr{L}_{\gtrdot 0}$ and does not include 0. This is an *intersection structure*: for any collection of coherent sets of desirable gambles $\mathbf{D} \subseteq \overline{\mathbf{D}}$, their intersection $\bigcap \mathbf{D}$ is a coherent set of desirable gambles: $\bigcap \mathbf{D} \in \overline{\mathbf{D}}$. This is what allows for *conservative reasoning* with sets of desirable gambles. Consider a subset $A \subseteq \mathscr{L}$ of gambles that the subject assesses to be desirable, i.e., prefers to 0. Such a set A is called an *assessment*. If we do conservative reasoning, we will look for the implications of this assessment *using coherence only*: we will look for $\mathscr{E}(A) := \bigcap\{D \in \overline{\mathbf{D}} : A \subseteq D\}$. If $A \subseteq D$ for at least one coherent D, then $\mathscr{E}(A)$ is again a coherent set of desirable gambles because $\overline{\mathbf{D}}$ is an intersection structure. Since $\mathscr{E}(A)$ is the least informative (smallest) coherent set of desirable gambles that includes A, it is called the *natural extension* of A. The unique smallest coherent set of desirable gambles is given by $\mathscr{L}_{\gtrdot 0}$; it is equal to the natural extension $\mathscr{E}(\emptyset)$ of no assessment. $\mathscr{L}_{\gtrdot 0}$ is therefore called the *vacuous* set of desirable gambles, indicated by D_{v}. For more information, we refer to [7, 17, 20, 25].

It is easy to make conservative inferences using sets of desirable gambles. It turns out that, when a given assessment A can be extended to a coherent set of desirable gambles, then the smallest such extension—its natural extension—is given by

$$\mathscr{E}(A) = \operatorname{posi}(A \cup \mathscr{L}_{\gtrdot 0}). \tag{1}$$

Such an assessment A can be coherently extended precisely when

$$(\{0\} \cup \mathscr{L}_{\lessdot 0}) \cap \operatorname{posi}(A) = \emptyset, \tag{ANP}$$

and this requirement is called *avoiding non-positivity*. We refer to De Cooman & Quaeghebeur [9] for a proof of both statements.

We collect the maximal elements of $\overline{\mathbf{D}}$—that is, the undominated elements of the partial order of coherent set of desirable gambles $\overline{\mathbf{D}}$ ordered by $\subseteq$—in $\widehat{\mathbf{D}}$:

$$\widehat{\mathbf{D}} := \{D \in \overline{\mathbf{D}} : (\forall D' \in \overline{\mathbf{D}}) D \not\subset D'\}.$$

[2] posi describes the defining property of convex cones: a set A is a convex cone if and only if $\operatorname{posi}(A) = A$.

As shown by Couso & Moral [3] and De Cooman & Quaeghebeur [9], any coherent set of desirable gambles D is maximal if and only if

$$(\forall f \in \mathscr{L} \setminus \{0\})(f \in D \text{ or } -f \in D). \tag{2}$$

Interestingly, any coherent set of desirable gambles D is dominated by a maximal one; see [9] for a proof. This yields [9, Corollary 4] a (two-way) representation of coherent sets of desirable gambles: for any coherent set of desirable gambles D, it holds that

$$D = \bigcap \{D' \in \widehat{\mathbf{D}} : D \subseteq D'\}, \tag{3}$$

so that the collection $\{D' \in \widehat{\mathbf{D}} : D \subseteq D'\}$ of maximal sets of desirable gambles, is a representation of D.

2.2. **Partial preferences.** Every set of desirable gambles corresponds uniquely to a preference relation $\prec$ on $\mathscr{L}$, which is a vector ordering on $\mathscr{L}$, meaning that it satisfies $f \prec g \Leftrightarrow \lambda f + h \prec \lambda g + h$, for all gambles f, g and h, and all positive real numbers λ. Given a set of desirable gambles D, the corresponding preference relation is given by $f \prec g \Leftrightarrow g - f \in D$ for all gambles f and g, and *vice versa*, given a preference relation $\prec$, the corresponding set of desirable gambles is given by $D = \{f \in \mathscr{L} : 0 \prec f\}$. The two operations commute.

A set of desirable gambles is coherent if and only if its corresponding preference relation $\prec$ is a strict partial order—meaning that it is irreflexive and transitive—that includes $\lessdot$—meaning that $f \lessdot g \Rightarrow f \prec g$, for all gambles f and g.

2.3. **A slightly stronger notion of coherence.** Mostly, for instance in [6, 9, 17], a slightly stronger notion of coherence is used. They consider any gamble $f > 0$—by which we mean $f(\omega) \geq 0$ for any ω in Ω and $f(\omega) > 0$ for some ω in Ω—automatically desirable, since the subject's capital can never decrease and might increase. These gambles are collected in $\mathscr{L}_{>0}$. Therefore, they use a stronger requirement and replaced Axiom D_2 by

D_2'. $\mathscr{L}_{>0} \subseteq D$.

This norm of coherence is less obvious to connect with the work of Cozman [4] and Seidenfeld et al. [21], which is why we will refer to Axioms D_1–D_4 as our notion of 'coherence'.

Interestingly, using the stronger variant Axiom D_2' implies

D_5. If $f \in D$ and $g > f$ then $g \in D$, for all f and g in $\mathscr{L}$.

Indeed, if $g > f$ then $g - f \in \mathscr{L}_{>0}$, which by Axiom D_2' belongs to D. Assuming that $f \in D$, Axiom D_4 implies then that $g = f + (g - f)$ indeed belongs to D.

We could opt to add Axiom D_5 as a rationality requirement, and use Axioms D_1–D_5 as our notion of 'coherence'. Adding it avoids for instance the classification of the set $D = \{f \in \mathscr{L} : f(\mathrm{h}) > 0 \text{ or } (f(\mathrm{h}) = 0 \text{ and } f(\mathrm{t}) < 0)\}$ as coherent, which may be natural thing to avoid as D contains for instance the gamble $f = (f(\mathrm{h}), f(\mathrm{t})) = (0, -1)$, which is dominated by 0. Axiom D_5 is closed under arbitrary intersections, so we would still yield an intersection structure, which therefore allows for conservative reasoning, too. The expression (1) for the natural extension would then become $\mathscr{E}(A) = \mathrm{posi}(\{g : (\exists f \in A) f \leq g\} \cup \mathscr{L}_{>0})$, and the requirement (ANP) would become $\mathscr{L}_{\leq 0} \cap \mathrm{posi}(A) = \emptyset$, where we used the notation $\mathscr{L}_{\leq 0} := \{f \in \mathscr{L} : (\forall \omega \in \Omega) f(\omega) \leq 0\}$ to indicate the set of (pointwise) non-positive gambles, but other than that, the results in our paper would remain valid *mutatis mutandis*. Since omitting Axiom D_5 is in line with the choice of Cozman [4] and also of De Cooman and Quaeghebeur [9], and yields a more general theory, we opt here to not require Axiom D_5 as a standard of coherence, but instead stick with Axioms D_1–D_4 as our notion of 'coherence'.

3. SSK–ARCHIMEDEANITY

Seidenfeld et al. [21] introduce a type of Archimedeanity that we will call 'SSK–Archimedeanity'. They show that SSK–Archimedeanity is necessary for even convexity in their setting, which is preference relations over horse lotteries. In order to use their ideas in the present context, let us spell out here what horse lotteries are. Consider a countable set $\mathscr{R}$ of rewards. Any real-valued map H on $\Omega \times \mathscr{R}$ such that

$$H(\omega, r) \geq 0 \text{ for all } r \text{ in } \mathscr{R}, \text{ and } \sum_{r \in \mathscr{R}} H(\omega, r) = 1$$

for any ω in Ω, is called a horse lottery. The idea behind a horse lottery H is that it returns, for every outcome ω in Ω, a probability mass function $H(\omega, \bullet)$ over $\mathscr{R}$, and a subject may describe her beliefs about which element ω of Ω is true by specifying a preference relation $\prec$ over the set of horse lotteries.

In Seidenfeld et al. [21]'s setting SSK–Archimedeanity is expressed as follows. A preference relation $\prec$ (on horse lotteries) is SSK–Archimedean if for every sequence of horse lotteries $H_1, H_2, \ldots,$ that converges to a horse lottery H,[3] every sequence $M_1, M_2, \ldots,$ that converges to M, and all horse lotteries J and N:

$$\text{if } (\forall k) H_k \prec M_k \text{ and } M \prec N \text{ then } H \prec N; \tag{Arch-a}$$

$$\text{if } (\forall k) H_k \prec M_k \text{ and } J \prec H \text{ then } J \prec M. \tag{Arch-b}$$

We will now interpret $\prec$ as a preference relation on gambles, with corresponding set of desirable gambles D. We will call a set of desirable gambles SSK–Archimedean if its corresponding preference relation $\prec$ satisfies the Equations (Arch-a) and (Arch-b) above.[4]

In the context of horse lottery, requirements (Arch-a) and (Arch-b) are not equivalent. However, if H_k, M_k, J and N are gambles, it turns out that both expressions are equivalent to each other, and also to the following convenient expression:

Proposition 2. *Consider any set of desirable gambles D. Then D is SSK–Archimedean if and only if*

$$\operatorname{cl}(D) + D \subseteq D. \tag{Arch}$$

Here, the addition $A + B$ of two sets A and B is taken to be the *Minkowski addition*, defined as

$$A + B = \{a + b : a \in A \text{ and } b \in B\},$$

and $\operatorname{cl}(D)$ is the (topological) closure of D. Because $\mathscr{L}$ is a metric space, we have that a gamble f belongs to $\operatorname{cl}(D)$ if and only if there is a sequence $f_1, f_2, \ldots$ of gambles in D that converges to f. Note that $0 \in \operatorname{cl}(D)$ for any set of desirable gambles D that satisfies Axiom D_2, whence $D \subseteq \operatorname{cl}(D) + D$, so the requirement (Arch) specialises to $\operatorname{cl}(D) + D = D$, for sets of desirable gambles D that satisfy Axiom D_2.

Proof of Proposition 2. We first prove necessity. Assume that D is SSK–Archimedean, meaning that its corresponding preference relation $\prec$ on gambles satisfies the requirements (Arch-a) and (Arch-b). We need to show that $\operatorname{cl}(D) + D \subseteq D$, or, in other words, that $f + g \in D$ for every f in $\operatorname{cl}(D)$ and g in D. So consider any f in $\operatorname{cl}(D)$ and g in D. Then there is a sequence $f_1, f_2, \ldots$ of gambles in D that converges to f. Define the sequences of gambles $H_k := 0 \to H := 0$, $M_k := f_k \to M := f$, and the gamble $N := g + f$. Since, for every k, $f_k = M_k$ is desirable, we have that $H_k = 0 \prec M_k$. Similarly, since $g = N - M$ is desirable, we have that $0 \prec N - M$, or in other words, $M \prec N$. From the requirement (Arch-a), we infer that $H \prec N$, or in other words, that $0 \prec g + f$, so that indeed $g + f \in D$.

[3] We take this convergence to be *point-wise* convergence: $\lim_{k\to\infty} H_k = H$—which we indicate by $H_k \to H$—if and only if $\lim_{k\to\infty} H_k(\omega, r) = H(\omega, r)$ for every ω in Ω and r in $\mathscr{R}$.

[4] The embedding of the horse lotteries in a linear space can be done using the embedding by Hausner [11, Chapter 7]. In this paper, we do not focus on this embedding, but we rather use it to translate the idea of SSK–Archimedeanity of Seidenfeld et al. [21] expressed in terms of horse lotteries.

We now turn to sufficiency. Assume that D satisfies $\mathrm{cl}(D)+D \subseteq D$, meaning that $f+g \in D$ for every sequence $f_1, f_2, \ldots$ in D that converges to f, and every g in D. We need to show that the corresponding preference relation satisfies the requirements (Arch-a) and (Arch-b).

For requirement (Arch-a), consider any sequences of gambles $H_k \to H$ and $M_k \to M$ and any gamble H. Assume that $H_k \prec M_k$ for all k, and $M \prec N$. We need to show that then $H \prec N$. To this end, define the sequence of gambles $f_k := M_k - H_k \to f := M - H$, and the gamble $g := N - M$. For every k, we have $0 \prec M_k - H_k = f_k$, so $f_k \in D$. Similarly, we have $0 \prec N - M = g$, so $g \in D$. By the assumption, we have that $f+g \in D$, or in other words, $0 \prec f+g$, whence $0 \prec N - H$, so indeed $H \prec N$.

For requirement (Arch-b), consider any sequences of gambles $H_k \to H$ and $M_k \to M$ and any gamble J. Assume that $H_k \prec M_k$ for all k, and $J \prec H$. By considering the gambles $f_k := M_k - H_k \to f := M - H$, and the gamble $g := H - J$, we find that indeed $J \prec N$ using a very similar argument as above. $\square$

If D is coherent, then its (topological) closure is given by $\mathrm{cl}(D) = \{f \in \mathscr{L} : (\forall \varepsilon \in \mathbb{R}_{>0}) f + \varepsilon \in D\}$, so that $f \in \mathrm{cl}(D) \Leftrightarrow (\forall \varepsilon \in \mathbb{R}_{>0}) f + \varepsilon \in D$, for any gamble f. We collect all the SSK–Archimedean sets of desirable gambles in the set $\mathbf{D}_{\mathrm{Arch}} := \{D \subseteq \mathscr{L} : D \text{ satisfies (Arch)}\}$, and the coherent SSK–Archimedean sets of desirable gambles in $\overline{\mathbf{D}}_{\mathrm{Arch}} := \overline{\mathbf{D}} \cap \mathbf{D}_{\mathrm{Arch}}$.

Example 6. The goal of this example is to provide insight in SSK–Archimedeanity by giving explicit examples of coherent SSK–Archimedean and non-SSK–Archimedean sets of desirable gambles. It formally confirms the ideas of Example 4. Consider a binary possibility space $\Omega = \{\mathrm{h}, \mathrm{t}\}$.

Let us start with the vacuous set of desirable gambles $D_{\mathrm{v}} := \mathscr{L}_{>0}$; see Figure 5 for a graphical representation.

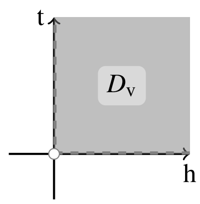


FIGURE 5. The vacuous set of desirable gambles D_{v}

A dotted line indicates a boundary that is not included in D_{v}, and a white dot is a gamble that is not included in D_{v}. D_{v} is SSK–Archimedean. To show this, note that $\mathrm{cl}(D_{\mathrm{v}}) = \mathscr{L}_{\geq 0}$, which collects the gambles f for which $f(\omega) \geq 0$ for all ω in Ω. Then, for any gambles f in $\mathrm{cl}(D_{\mathrm{v}})$ and g in D_{v}, indeed $f + g \in \mathscr{L}_{>0} = D_{\mathrm{v}}$.

Let us now move to a more general (but finite) possibility space $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$. We will give (i) a coherent set of desirable gambles that is SSK–Archimedean, and (ii) one that is not.

For (i), given a probability mass function p on Ω (so $p(\omega_k) \geq 0$ for every ω_k in Ω, and $\sum_{k=1}^{n} p(\omega_k) = 1$), we define the corresponding set of desirable gambles

$$D_p := \{f \in \mathscr{L} : E_p(f) > 0\} \tag{4}$$

as the set of gambles f with positive p-expectation $E_p(f) := \sum_{k=1}^{n} p(\omega_k) f(\omega_k)$. We can think of D_p as a linear half-space, so it is clearly a coherent set of desirable gambles: it is a convex cone that includes $\mathscr{L}_{>0}$ and does not include 0. Because it will turn out useful for later reference, we will collect all the probability mass functions in the set Σ. Figure 6 shows a graphical representation of D_p in the binary possibility space $\{\mathrm{h},\mathrm{t}\}$, where p is the uniform probability mass function on $\{\mathrm{h},\mathrm{t}\}$.

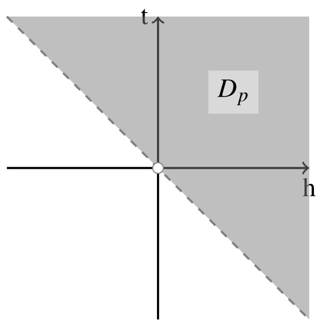


FIGURE 6. A set of desirable gambles D_p corresponding to the uniform probability mass function $p = (1/2, 1/2)$

We claim that, for any p in Σ, the coherent set of desirable gambles D_p is SSK–Archimedean. To see this, note that $\mathrm{cl}(D_p) = \{f \in \mathscr{L} : E_p(f) \geq 0\}$. Consider any f in $\mathrm{cl}(D_p)$ and g in D_p, then $E_p(f) \geq 0$ and $E_p(g) > 0$, whence $E_p(f+g) > 0$ so indeed $f + g \in D_p$.

As a more general example, consider any coherent set of desirable gambles D that is *open*, in the sense that D coincides with its (topological) interior $\mathrm{int}(D)$. Such a D is SSK–Archimedean. To this end, consider any f in D and g in $\mathrm{cl}(D)$, and we will see that $f+g$ belongs to D. Since D coincides with its interior $\mathrm{int}(D) = \{h \in \mathscr{L} : (\exists \delta \in \mathbb{R}_{>0}) h - \delta \in D\}$, this means that $f - \delta \in D \subseteq \mathrm{cl}(D)$ for some real $\delta > 0$. Because D is coherent, it is a convex cone, and therefore so is its closure $\mathrm{cl}(D)$, meaning that $h := f + g - \delta \in \mathrm{cl}(D)$. That h belongs to $\mathrm{cl}(D)$ means that $h + \varepsilon \in D$ for every real $\varepsilon > 0$. By considering $\varepsilon = \delta$, we find that indeed $f + g = h + \delta \in D$.

For (ii), to give a coherent set of desirable gambles that is not SSK–Archimedean, consider a *maximal* set of desirable gambles $\widehat{D}$, as described by Equation (2). We can think of a coherent maximal set of desirable gambles as a convex superset of a D_p that includes a maximally large part of its boundary, without including 0. An example of a coherent maximal set of desirable gambles on a binary possibility space is depicted in Figure 7.

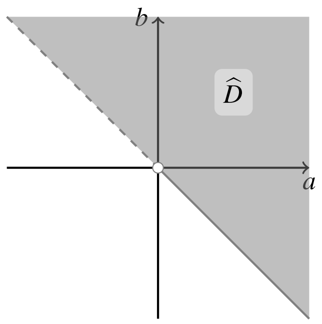


FIGURE 7. A coherent maximal set of desirable gambles $\widehat{D}$

We now prove our claim that any coherent maximal set of desirable gambles $\widehat{D}$ is not SSK–Archimedean. Consider a gamble g in $\widehat{D}$ on its boundary, so that $(\forall \varepsilon > 0) g - \varepsilon \notin \widehat{D}$.

Then $g \in \widehat{D}$ and, since $g-\varepsilon \notin \widehat{D}$, by the characteristic property of coherent maximal sets of desirable gambles in Equation (2), we have $-g+\varepsilon \in \widehat{D}$ for every ε in $\mathbb{R}_{>0}$, so $-g \in \mathrm{cl}(\widehat{D})$. But $g-g=0$ does not belong to $\widehat{D}$, by coherence, and therefore indeed $\widehat{D}+\mathrm{cl}(\widehat{D}) \nsubseteq \widehat{D}$. $\Diamond$

3.1. **SSK–Archimedeanity and probabilities.** Let us make a digression towards probabilities, and state some well-known facts about the connection between coherent sets of desirable gambles, lower previsions and sets of probability mass functions. Given any coherent set of desirable gambles D, we let its corresponding lower prevision $\underline{P}_D$ be the functional on $\mathscr{L}$ that maps every gamble to its supremum acceptable buying price:

$$\underline{P}_D(f) := \sup\{\mu \in \mathbb{R} : f-\mu \in D\} \text{ for every gamble } f.$$

So a lower prevision is a real-valued functional. Similarly, its corresponding upper prevision $\overline{P}_D$ is the infimum acceptable selling price:

$$\overline{P}_D(f) := \inf\{\mu \in \mathbb{R} : \mu-f \in D\} \text{ for every gamble } f.$$

By the coherence of D, we have that $\underline{P}_D(f) \leq \overline{P}_D(f)$, for every f in $\mathscr{L}$ (see Walley [25, Section 2.3.5]). Moreover, $\underline{P}_D$ and $\overline{P}_D$ are related by *conjugacy*: $\underline{P}_D(f) = -\overline{P}_D(-f)$ for every f in $\mathscr{L}$ (see Walley [25, Section 2.3.5] for more information), so we may focus on either one of them. One may use $\underline{P}_D$ to retrieve the closure $\mathrm{cl}(D)$ of D: it follows that $\mathrm{cl}(D) = \{f \in \mathscr{L} : \underline{P}_D(f) \geq 0\}$. Also D's (topological) interior $\mathrm{int}(D) = \{f \in \mathscr{L} : (\exists \delta \in \mathbb{R}_{>0}) f-\delta \in D\}$ can be retrieved using $\underline{P}_D$: it turns out that $\mathrm{int}(D) = \{f \in \mathscr{L} : \underline{P}_D(f) > 0\}$. We refer to De Bock [6, Section 2.3] for more information about this.

We call a lower prevision $\underline{P}$ *coherent* if there is a coherent set of desirable gambles D such that $\underline{P} = \underline{P}_D$. Coherence has an equivalent definition directly in terms of $\underline{P}$ (see, for instance, Walley [25, Section 2.3.3] or Miranda [14, Section 2.1]), but this is of no importance in this paper. Given a coherent lower prevision $\underline{P}$, we collect in $\mathbf{P}(\underline{P})$ the set of probability mass functions whose expectation dominates $\underline{P}$:

$$\mathbf{P}(\underline{P}) = \{p \in \Sigma : (\forall f \in \mathscr{L}) \underline{P}(f) \leq E_p(f)\} = \{p \in \Sigma : \underline{P} \leq E_p\}, \tag{5}$$

where we used the notation $\underline{P} \leq E_p$ to indicate $\underline{P}(f) \leq E_p(f)$ for all f in $\mathscr{L}$. The set $\mathbf{P}(\underline{P})$ is called the *credal set*. It is a closed and convex non-empty set of probability mass functions, and in a one-to-one correspondence with a lower prevision: given an arbitrary credal (closed and convex) set of probabilities $\mathbf{P}$, its corresponding lower prevision $\underline{P}_{\mathbf{P}}$ is

$$\underline{P}_{\mathbf{P}}(f) := \min\{E_p(f) : p \in \mathbf{P}\} \text{ for every gamble } f.$$

Both operations (of going from $\underline{P}$ to $\mathbf{P}(\underline{P})$ and going from $\mathbf{P}$ to $\underline{P}_{\mathbf{P}}$) commute; see Walley [25, Section 3.3.3] for a proof and Miranda & de Cooman [15, Section 2.2.2] for more information.

Note that the interior $\mathrm{int}(\mathbf{P})$ of a credal set $\mathbf{P}$ may be empty. Indeed, in a ternary possibility space $\Omega = \{a,b,c\}$, the set $\{p \in \Sigma : \frac{1}{3} \leq p(a) \leq \frac{2}{3} \text{ and } p(b) = p(c)\}$ as indicated in Figure 8—which is non-empty, closed and convex, and therefore a valid credal set—has empty interior. This is because $\mathbf{P}$ is a subset of an affine space of lower dimension—of dimension 2 in this case. The smallest affine space that includes any set C is its *affine hull* $\mathrm{aff}(C)$. It is given by

$$\mathrm{aff}(C) := \Big\{\sum_{k=1}^{m} \lambda_k f_k : m \in \mathbb{N}, f_1, \dots, f_m \in C, \lambda_1, \dots, \lambda_m \in \mathbb{R}, \sum_{k=1}^{m} \lambda_k = 1\Big\}.$$

It can always be written as $\mathrm{aff}(C) = \{f\} + K$, where K is a linear space, and f a point in C; see Rockafellar [18, Theorem 1.2]. When $0 \in \mathrm{aff}(C)$, this affine hull is actually its *linear hull* $\mathrm{span}(C)$, so:

$$\mathrm{aff}(C) = \mathrm{span}(C) := \Big\{\sum_{k=1}^{m} \lambda_k f_k : m \in \mathbb{N}, f_1, \dots, f_m \in C, \lambda_1, \dots, \lambda_m \in \mathbb{R}\Big\}.$$

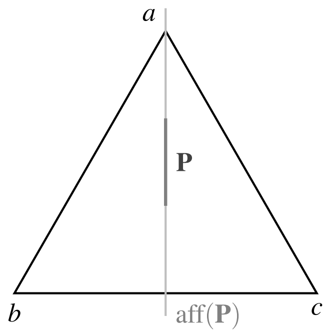


FIGURE 8. The credal set $\mathbf{P} = \{p \in \Sigma : \frac{1}{3} \leq p(a) \leq \frac{2}{3} \text{ and } p(b) = p(c)\}$ and its two-dimensional affine hull $\mathrm{aff}(\mathbf{P})$

On the other hand, the *relative interior* [18, Section 6] of a non-empty convex set in a finite-dimensional space is never empty. For any convex set $C \subseteq \mathbb{R}^n$, its relative interior $\mathrm{ri}(C)$ is defined as the interior of C when C is regarded as a subset of its affine hull $\mathrm{aff}(C)$, rather than of the complete n-dimensional space:

$$\mathrm{ri}(C) := \{x \in \mathrm{aff}(C) : (\exists \varepsilon \in \mathbb{R}_{>0})(\{x\} + \varepsilon B^n) \cap \mathrm{aff}(C) \subseteq C\}, \tag{6}$$

where $B^n := \{x \in \mathbb{R}^n : |x| < 1\}$ is the Euclidean unit ball in $\mathbb{R}^n$.

If C is non-empty, then so is its relative interior $\mathrm{ri}(C)$, whereas the interior $\mathrm{int}(C)$ may be empty. When $\mathrm{int}(C)$ is non-empty, then it is equal to its relative interior $\mathrm{ri}(C)$. Going back to the example above, we have $\mathrm{ri}(\mathbf{P}) = \{p \in \mathbf{P} : \frac{1}{3} < p(a) < \frac{2}{3} \text{ and } p(b) = p(c)\}$. Because we will need them later on, we will mention the following two useful facts about the relative interior, proved by Rockafellar [18]:

Theorem 3 ([18, Theorem 6.1])**.** *Let C be a non-empty convex set in a finite-dimensional space, and $x \in \mathrm{ri}(C)$ and $y \in \mathrm{cl}(C)$. Then $(1-\lambda)x + \lambda y \in \mathrm{ri}(C)$ for every real λ in $[0,1)$.*

Theorem 4 ([18, Theorem 6.4])**.** *Let C be a non-empty convex set in a finite-dimensional space. Then $z \in \mathrm{ri}(C)$ if and only if*

$$(\forall x \in C)(\exists \mu > 1)((1-\mu)x + \mu z \in C).$$

Lower previsions—and credal sets—are less expressive than sets of desirable gambles. Given a coherent lower prevision $\underline{P}$, there may be multiple coherent sets of desirable gambles D such that $\underline{P}_D = \underline{P}$, the smallest of which is $\{f \in \mathscr{L} : \underline{P}(f) > 0\}$. Moreover, it follows from De Bock [6, Equation (2.8)] that any set of desirable gambles D' between $\mathrm{int}(D)$ and $\mathrm{cl}(D)$ has the same lower prevision: $\underline{P}_{D'} = \underline{P}_D$. This means that lower previsions cannot distinguish between sets of desirable gambles with the same interior but different boundaries.

This implies that the credal set $\mathbf{P}(D) := \mathbf{P}(\underline{P}_D)$ based on a coherent set of desirable gambles D (through its lower prevision $\underline{P}_D$) is given by

$$\mathbf{P}(D) = \{p \in \Sigma : (\forall f \in D) E_p(f) \geq 0\} = \{p \in \Sigma : \mathrm{int}(D) \subseteq D_p\}, \tag{7}$$

which is non-empty. This follows from Seidenfeld et al. [20, Theorem 1], and the core of the result was already present in Blackwell & Girshick [2, Theorem 4.3.1]. Here we give a short argument focused on our current setting, based on the same ideas. Note that $\mathbf{P}(D) = \{p \in \Sigma : \underline{P}_D \leq E_p\}$, using Equation (5). We will first show that $\underline{P}_D \leq E_p \Leftrightarrow (\forall f \in D) E_p(f) \geq 0$, establishing the first identity. To this end, infer the following chain of equivalence:

$$(\forall f \in \mathscr{L}) \underline{P}_D(f) \leq E_p(f) \Leftrightarrow (\forall f \in \mathscr{L}) 0 \leq E_p(\underbrace{f - \underline{P}_D(f)}_{\in \mathrm{cl}(D)}) \Leftrightarrow (\forall g \in \mathrm{cl}(D)) 0 \leq E_p(g).$$

This last equivalent statement implies $(\forall g \in D) 0 \leq E_p(g)$. To show that it is also implied by this, consider any g in $\mathrm{cl}(D)$. Then $g+\varepsilon \in D$ for all $\varepsilon > 0$, so that $0 \leq E_p(g+\varepsilon) = E_p(g)+\varepsilon$ for all $\varepsilon > 0$ by the assumption. But then indeed necessarily $0 \leq E_p(g)$.

To prove the second identity, note that indeed $(\forall f \in D) E_p(f) \geq 0 \Leftrightarrow D \subseteq \mathrm{cl}(D_p) \Leftrightarrow \mathrm{int}(D) \subseteq \mathrm{int}(D_p) = D_p$, where the first equivalence follows from the definition of D_p in Equation (4).

To summarise: given a coherent set of desirable gambles D and any gamble f, we have

$$f \in D \Rightarrow \underline{P}_D(f) \geq 0 \Leftrightarrow (\forall p \in \mathbf{P}(D)) E_p(f) \geq 0 \Leftrightarrow f \in \mathrm{cl}(D), \tag{8}$$

and

$$f \in \mathrm{int}(D) \Leftrightarrow \underline{P}_D(f) > 0 \Leftrightarrow (\forall p \in \mathbf{P}(D)) E_p(f) > 0 \Rightarrow f \in D, \tag{9}$$

but $\mathbf{P}(D)$ cannot generally reconstruct D: it is no two-way representation of D.

We can use this, and the observation in Example 6 that any D_p is SSK–Archimedean, to show that the largest SSK–Archimedean coherent sets of desirable gambles are actually the D_p:

Proposition 5. *The maximal elements of $\overline{\mathbf{D}}_{\mathrm{Arch}}$—that is, the elements of $\overline{\mathbf{D}}_{\mathrm{Arch}}$ that are no strict subset of other elements of $\overline{\mathbf{D}}_{\mathrm{Arch}}$—are precisely $\{D_p : p \in \Sigma\}$. In other words, if we define*

$$\widehat{\mathbf{D}}_{\mathrm{Arch}} := \{D \in \overline{\mathbf{D}}_{\mathrm{Arch}} : (\forall D' \in \overline{\mathbf{D}}_{\mathrm{Arch}}) D \not\subset D'\}$$

as the maximal elements of $\overline{\mathbf{D}}_{\mathrm{Arch}}$, we have

$$\widehat{\mathbf{D}}_{\mathrm{Arch}} = \{D_p : p \in \Sigma\}. \tag{10}$$

Proof. We will show that Equation (10) holds: we will show (i) that $\widehat{\mathbf{D}}_{\mathrm{Arch}} \supseteq \{D_p : p \in \Sigma\}$ and (ii) that $\widehat{\mathbf{D}}_{\mathrm{Arch}} \subseteq \{D_p : p \in \Sigma\}$.

For (i), to show that any D_p belongs to $\widehat{\mathbf{D}}_{\mathrm{Arch}}$, infer from Example 6 that $D_p \in \overline{\mathbf{D}}_{\mathrm{Arch}}$. We show that it is no strict subset of another set of desirable gambles in $\overline{\mathbf{D}}_{\mathrm{Arch}}$. To this end, assume *ex absurdo* that there is some D' in $\overline{\mathbf{D}}_{\mathrm{Arch}}$ such that $D_p \subset D'$. Since D' is a coherent set of desirable gambles, it is dominated by a maximal one, say $D'' \in \widehat{\mathbf{D}}$, so $D_p \subset D' \subseteq D''$, and therefore $\mathrm{int}(D_p) \subseteq \mathrm{int}(D'')$. But then $\mathbf{P}(D'') \subseteq \mathbf{P}(D_p)$, using the second identity in Equation (7). Furthermore, infer from Equation (4) that $\mathrm{int}(D_p) = D_p$, whence $\mathbf{P}(D_p) = \{p\}$, again using the second identity in Equation (7). But D'' is a coherent set of desirable gambles, so $\mathbf{P}(D'')$ is non-empty, and therefore $\mathbf{P}(D'') = \{p\}$. Using Equation (9), this would imply that $\mathrm{int}(D'') = \{f \in \mathscr{L} : E_p(f) > 0\} = D_p$. In turn, this would imply that $\mathrm{int}(D') = D_p$, and hence that D' contains a part of the boundary of D_p. To show that this is impossible, consider any g in D' on the boundary of D_p. Then $E_p(g) = 0$, so $E_p(-g) = 0$, whence $-g \in \mathrm{cl}(D_p) \subseteq \mathrm{cl}(D')$, so by the SSK–Archimedeanity of D', we would infer that $g - g = 0 \in D'$, contradicting the coherence of D'. This proves that D_p is no strict subset of another set of desirable gambles in $\overline{\mathbf{D}}_{\mathrm{Arch}}$, as desired.

For (ii), to show that *all* the maximal SSK–Archimedean coherent sets of desirable gambles are of this form, assume *ex absurdo* that $\widehat{\mathbf{D}}_{\mathrm{Arch}} \not\subseteq \{D_p : p \in \Sigma\}$. Since we have already established that $\widehat{\mathbf{D}}_{\mathrm{Arch}} \supseteq \{D_p : p \in \Sigma\}$, this would mean that $\widehat{\mathbf{D}}_{\mathrm{Arch}} \supset \{D_p : p \in \Sigma\}$. We will show that this is impossible. To this end, consider any D in $\widehat{\mathbf{D}}_{\mathrm{Arch}} \setminus \{D_p : p \in \Sigma\}$. Let $\mathbf{P}(D)$ be the credal (closed and convex) set of probability mass functions corresponding with D, given by Equation (7), which is non-empty by the coherence of D, and therefore its relative interior $\mathrm{ri}(D)$ is also non-empty. Use Proposition 7 below to infer that then $D \subseteq D_p$ for all p in $\mathrm{ri}(D)$, so D is dominated by an element of $\overline{\mathbf{D}}_{\mathrm{Arch}}$, a contradiction with the fact that D is maximal. □

Lemma 6. *Consider any coherent set of desirable gambles D that is SSK–Archimedean. Then $\overline{P}_D(f) > 0$ for all f in D.*

Proof. Consider any f in D. Assume *ex absurdo* that $\overline{P}_D(f) = 0$, then necessarily $\underline{P}_D(f) = 0$. Using the conjugacy of $\underline{P}_D$ and $\overline{P}_D$, this would imply that $\underline{P}_D(-f) = -\overline{P}_D(f) = 0$, so $-f \in \mathrm{cl}(D)$ by Equation (8). By the SSK–Archimedeanity of D, we would infer that $f - f = 0 \in D$, which contradicts the coherence of D. Therefore indeed $\overline{P}_D(f) > 0$. □

Proposition 7. *Every element of* $\overline{\mathbf{D}}_{\mathrm{Arch}}$ *is dominated by a maximal element. More precisely: Consider any coherent set of desirable gambles D that is SSK–Archimedean. Then $D \subseteq D_p$ for every p in the non-empty relative interior* $\mathrm{ri}(\mathbf{P}(D))$ *of* $\mathbf{P}(D)$ *given by Equation* (7).

Proof. Consider any p^* in $\mathrm{ri}(\mathbf{P}(D))$ and assume *ex absurdo* that $D \not\subseteq D_{p^*}$. This would imply that there is some f in D such that $E_{p^*}(f) \leq 0$. We now use Theorem 4 above to infer that, for any p' in $\mathbf{P}(D)$, there is a real $\mu > 1$ such that the probability mass function $p'' := (1-\mu)p' + \mu p^*$ belongs to $\mathbf{P}(D)$. By Equation (8) then $E_{p''}(f) \geq 0$ and $E_{p'}(f) \geq 0$, so using the linearity of the expectation operators, we would infer that

$$\underbrace{E_{p''}(f)}_{\geq 0} = \underbrace{(1-\mu)E_{p'}(f)}_{\leq 0} + \underbrace{\mu E_{p^*}(f)}_{\leq 0},$$

whence $E_{p'}(f) = 0$. Since the choice of p' in $\mathbf{P}(D)$ was arbitrary, this implies that $\underline{P}_D(f) = \overline{P}_D(f) = 0$. This is a contradiction with the fact that $\overline{P}_D(f) > 0$ for all f in D, established in Lemma 6 above. □

This brings us to one of the general claims of this paper that for any coherent set of desirable gambles D, the requirements of SSK–Archimedeanity and SSK–extension are sufficient and necessary for a representation of D in terms of probabilities. More precisely, we will show that any coherent set of desirable gambles D that is extendible to a probability and is the result of an SSK–extension, is an intersection of D_p, with p in D's representing set $\mathbf{R}(D)$ of probability mass functions. By Proposition 7 we already know that any SSK–Archimedean coherent set of desirable gambles will be a subset of intersections of D_p, but we will show that, using SSK–extension, we have an equality, which is therefore a two-way representation of such sets of desirable gambles.

3.2. **SSK–Archimedean natural extension.** In this section we want to connect SSK–Archimedeanity to the concept of natural extension, which we discussed in Section 2.1. The questions we ask, are "When can a set of desirable gambles be extended to a coherent one that satisfies SSK–Archimedeanity?", and "What does the smallest of these extensions look like?" To answer these, note first that the property of being SSK–Archimedean is closed under arbitrary intersections:

Proposition 8. *Consider an arbitrary non-empty collection* $\mathbf{D} \subseteq \mathbf{D}_{\mathrm{Arch}}$ *of sets of desirable gambles that are SSK–Archimedean. Then their intersection* $\bigcap \mathbf{D}$ *is a set of desirable gambles that is SSK–Archimedean.*

Proof. We need to prove that $\mathrm{cl}(\bigcap \mathbf{D}) + \bigcap \mathbf{D} \subseteq \bigcap \mathbf{D}$. For any arbitrary intersection, we have [see, for instance Dugundji [10, Chapter 3, Section 4.5]] that $\mathrm{cl}(\bigcap \mathbf{D}) \subseteq \bigcap_{D \in \mathbf{D}} \mathrm{cl}(D)$.[5] So we infer that $\mathrm{cl}(\bigcap \mathbf{D}) + \bigcap \mathbf{D} \subseteq \bigcap_{D \in \mathbf{D}} \mathrm{cl}(D) + \bigcap \mathbf{D}$.

We will now show the intermediate result that $\bigcap_{D \in \mathbf{D}} \mathrm{cl}(D) + \bigcap \mathbf{D} \subseteq \bigcap_{D \in \mathbf{D}} (\mathrm{cl}(D) + D)$. To this end, consider any gamble f in $\bigcap_{D \in \mathbf{D}} \mathrm{cl}(D) + \bigcap \mathbf{D}$. This means that

$$(\exists g, h \in \mathscr{L})(\forall D \in \mathbf{D})(g \in \mathrm{cl}(D) \text{ and } h \in D \text{ and } f = g + h).$$

Therefore, in particular

$$(\forall D \in \mathbf{D})(\exists g \in \mathrm{cl}(D), \exists h \in D) f = g + h,$$

[5] If $\mathbf{D}$ is a collection of *coherent* sets of desirable gambles, we actually have an equality: Since there is a gamble that belongs to the relative interior of every coherent set of desirable gambles (for instance, the constant gamble 1), we can use Rockafellar [18, Theorem 6.5], which yields this equality. But if the collection $\mathbf{D}$ consists of sets of desirable gambles that are not necessarily coherent, we only have inclusion.

whence $f \in \mathrm{cl}(D) + D$ for every D in $\mathbf{D}$, so indeed $f \in \bigcap_{D \in \mathbf{D}}(\mathrm{cl}(D) + D)$.

We conclude that indeed $\mathrm{cl}(\bigcap \mathbf{D}) + \bigcap \mathbf{D} \subseteq \bigcap_{D \in \mathbf{D}} \mathrm{cl}(D) + \bigcap \mathbf{D} \subseteq \bigcap_{D \in \mathbf{D}}(\mathrm{cl}(D) + D) \subseteq \bigcap_{D \in \mathbf{D}} D = \bigcap \mathbf{D}$, where the last inclusion follows from our assumption that every D in $\mathbf{D}$ is SSK–Archimedean. □

We have already seen in Section 2.1 that $\overline{\mathbf{D}}$ is an intersection structure—meaning that coherence is closed under (arbitrary) intersections. Therefore, the set $\overline{\mathbf{D}}_{\mathrm{Arch}}$ is an intersection structure as well: for any collection of coherent and SSK–Archimedean sets of desirable gambles $\mathbf{D} \subseteq \overline{\mathbf{D}}_{\mathrm{Arch}}$, their intersection $\bigcap \mathbf{D}$ is a coherent and SSK–Archimedean set of desirable gambles, so $\bigcap \mathbf{D} \in \overline{\mathbf{D}}_{\mathrm{Arch}}$.

We already know from Proposition 5 that D_p is SSK–Archimedean and coherent, for every p in Σ. Therefore, Proposition 8 implies that $\mathscr{L}_{\gtrdot 0} = \bigcap\{D_p : p \in \Sigma\}$ is a coherent and SSK–Archimedean set of desirable gambles. Since this is the unique smallest coherent one, it is also the unique smallest coherent and SSK–Archimedean one.

As we will see, the notion of "*extendibility to a probability*" will characterise the coherent sets of desirable gambles that can be coherently extended to one that satisfies SSK–Archimedeanity:

Definition 9 (Extendibility to a probability)**.** Consider any assessment $A \subseteq \mathscr{L}$. We say that A is *extendible to a probability* when

$$(\exists p \in \Sigma)\ A \subseteq D_p. \qquad \text{(extendibility)}$$

Lemma 10. *Consider any assessment $A \subseteq \mathscr{L}$. If A is extendible to a probability then*

(i) A avoids non-positivity;
(ii) $\mathscr{E}(A)$ is extendible to a probability.

Proof. Since A is extendible to a probability, we have that $A \subseteq D_p$ for some p in Σ. To show (i) that A avoids non-positivity (ANP), we need to show that $(\{0\} \cup \mathscr{L}_{\lessdot 0}) \cap \mathrm{posi}(A) = \emptyset$. Note that D_p is a convex cone, so $D_p = \mathrm{posi}(D_p)$, and therefore $\mathrm{posi}(A) \subseteq D_p$. But 0 nor any gamble f in $\mathscr{L}_{\lessdot 0}$ has a positive p-expectation: $E_p(0) = 0$ and $E_p(f) \leq 0$, so 0 nor f can indeed not belong to $\mathrm{posi}(A)$.

To show (ii) that $\mathscr{E}(A)$ is extendible to a probability, note that (i) already implies that A's natural extension $\mathscr{E}(A)$ is coherent, which is then the smallest coherent set of desirable gambles that includes A. Since we already know that D_p is a coherent set of desirable gambles that includes A, we infer that indeed $\mathscr{E}(A) \subseteq D_p$. □

Proposition 7 tells us that any coherent and SSK–Archimedean set of desirable gambles is extendible to a probability. But there is a closer connection between this property and SSK–Archimedeanity. Indeed, 'extendibility to a probability' plays a role similar to that of 'avoiding non-positivity' (ANP): as it turns out, it expresses the condition under which it is possible to extend an assessment to an SSK–Archimedean coherent set of desirable gambles:

Theorem 11. *For any set of gambles $A \subseteq \mathscr{L}$, we define the* SSK–Archimedean natural extension $\mathscr{E}_{\mathrm{Arch}}$ *as*[6]

$$\mathscr{E}_{\mathrm{Arch}}(A) := \bigcap\{D \in \overline{\mathbf{D}}_{\mathrm{Arch}} : A \subseteq D\}.$$

Consider any assessment $A \subseteq \mathscr{L}$, then the following statements are equivalent:

(i) A is extendible to a probability [i.e., satisfies (extendibility)*];*
(ii) A is included in a coherent and SSK–Archimedean set of desirable gambles;
(iii) $\mathscr{E}_{\mathrm{Arch}}(A) \neq \mathscr{L}$;
(iv) The set of desirable gambles $\mathscr{E}_{\mathrm{Arch}}(A)$ is coherent and SSK–Archimedean;
(v) $\mathscr{E}_{\mathrm{Arch}}(A)$ is the smallest coherent and SSK–Archimedean set of desirable gambles that includes A.

[6] As usual, we let $\bigcap \emptyset = \mathscr{L}$.

When any, and hence all, of these equivalent statements hold, then

$$\mathscr{E}_{\mathrm{Arch}}(A) = \mathrm{cl}(\mathscr{E}(A)) + \mathscr{E}(A),$$

where $\mathscr{E}$ is the natural extension defined in Equation (1)*.*

Proof. It follows from the fact that $\overline{\mathbf{D}}_{\mathrm{Arch}}$ is an intersection structure (as a consequence of Proposition 8), the definition of $\mathscr{E}_{\mathrm{Arch}}(A)$, and the fact that $\mathscr{L}$ is not a coherent set of desirable gambles, that the four statements (ii)–(v) are equivalent. Next, we prove that (i)$\Leftrightarrow$(ii).

To show that (i)$\Rightarrow$(ii), it suffices to note that D_p is a coherent and SSK–Archimedean set of desirable gambles, as shown in Proposition 5.

To show that (ii)$\Rightarrow$(i), assume that (ii) holds: let $A \subseteq D$ for some D in $\overline{\mathbf{D}}_{\mathrm{Arch}}$. Then, by Proposition 7, $D \subseteq D_p$ for all p in $\mathrm{ri}(\mathbf{P}(D))$, which is non-empty by the coherence of D. This implies that indeed $A \subseteq D_p$ for some p in Σ.

Finally, we prove that $\mathscr{E}_{\mathrm{Arch}}(A) = \mathrm{cl}(\mathscr{E}(A)) + \mathscr{E}(A)$ whenever any (and hence all) of the equivalent statements (i)–(v) hold. To this end, we will show that $D^* := \mathrm{cl}(\mathscr{E}(A)) + \mathscr{E}(A)$ is the smallest coherent and SSK–Archimedean set of desirable gambles that includes A. Using statement (v) it then follows that D^* equals $\mathscr{E}_{\mathrm{Arch}}(A)$.

First, we show that D^* is coherent.

For Axiom D_1, consider any h in D^*. Then $h = f + g$ for some f in $\mathscr{E}(A)$ and g in $\mathrm{cl}(\mathscr{E}(A))$. By statement (i), we have that A is extendible to a probability, and therefore, using Lemma 10 above, so is $\mathscr{E}(A)$. This implies that $E_p(f) > 0$ and $E_p(g) \geq 0$ for some p in Σ, so, by the linearity of the expectation operators, we have that $E_p(h) = E_p(f+g) > 0$. Therefore indeed $h \neq 0$.

For Axiom D_2, consider any f in $\mathscr{L}_{>0}$. Then $f \in \mathscr{E}(A)$ by the coherence of $\mathscr{E}(A)$. Since $0 \in \mathrm{cl}(\mathscr{E}(A))$ [to see this, infer that the sequence of constant gambles $\frac{1}{k} \to 0$ belongs to $\mathscr{E}(A)$ by the coherence of $\mathscr{E}(A)$], this implies that indeed $f = f + 0 \in D^*$.

D^* satisfies Axioms D_3 and D_4—requiring that D^* be a convex cone—because it is the Minkowski addition of two convex cones.

So D^* is a coherent set of desirable gambles. We now show that it is also SSK–Archimedean. We use the fact that $\mathrm{cl}(C_1) + \mathrm{cl}(C_2) \subseteq \mathrm{cl}(C_1 + C_2)$ for any convex sets C_1 and C_2 to show, indeed:

$$\begin{aligned}
\mathrm{cl}(D^*) + D^* &= \mathrm{cl}(\mathrm{cl}(\mathscr{E}(A)) + \mathscr{E}(A)) + \mathrm{cl}(\mathscr{E}(A)) + \mathscr{E}(A) \\
&\subseteq \mathrm{cl}(\mathrm{cl}(\mathscr{E}(A)) + \mathscr{E}(A) + \mathscr{E}(A)) + \mathscr{E}(A) \\
&= \mathrm{cl}(\mathrm{cl}(\mathscr{E}(A)) + \mathscr{E}(A)) + \mathscr{E}(A) \\
&\subseteq \mathrm{cl}(\mathrm{cl}(\mathscr{E}(A)) + \mathrm{cl}(\mathscr{E}(A))) + \mathscr{E}(A) \\
&= \mathrm{cl}(\mathrm{cl}(\mathscr{E}(A))) + \mathscr{E}(A) = \mathrm{cl}(\mathscr{E}(A)) + \mathscr{E}(A) = D^*
\end{aligned}$$

where we used $D + D = D$ in the third line, and $\mathrm{cl}(D) + \mathrm{cl}(D) = \mathrm{cl}(D)$ in the fifth line, for any coherent set of desirable gambles D.

So now we know that D^* is coherent and SSK–Archimedean, and it clearly includes A: to see this, note that $A \subseteq \mathscr{E}(A)$ and $0 \in \mathrm{cl}(\mathscr{E}(A))$, whence $A \subseteq \mathscr{E}(A) \subseteq \mathscr{E}(A) + \mathrm{cl}(\mathscr{E}(A))$. But is it also the smallest such set? To show that this is indeed the case, note that any coherent set of desirable gambles that includes A must include $\mathscr{E}(A)$, its natural extension, so $\mathscr{E}(A) \subseteq D^*$, and therefore $\mathscr{E}(A) + \mathrm{cl}(\mathscr{E}(A)) \subseteq D^* + \mathrm{cl}(\mathscr{E}(A))$. But $\mathrm{cl}(\mathscr{E}(A)) \subseteq \mathrm{cl}(D^*)$, whence, by the SSK–Archimedeanity of D^*, indeed $\mathscr{E}(A) + \mathrm{cl}(\mathscr{E}(A)) \subseteq D^*$. This means that D^* is indeed the smallest coherent and SSK–Archimedean set of desirable gambles that includes A, and is therefore, by statement (v), equal to $\mathscr{E}_{\mathrm{Arch}}(A)$. □

Theorem 11 above implies in particular that the SSK–Archimedean natural extension $\mathscr{E}_{\mathrm{Arch}}(D)$ of any coherent set of desirable gambles D that is extendible to a probability, is given by $D + \mathrm{cl}(D)$. The set of desirable gambles D and its SSK–Archimedean natural extension $D + \mathrm{cl}(D)$ are closely related: we have $D \subseteq D + \mathrm{cl}(D) \subseteq \mathrm{cl}(D)$. Indeed, that

$D \subseteq D + \mathrm{cl}(D)$ follows from the fact that $0 \in \mathrm{cl}(D)$, as already noted immediately after Proposition 2. To see the second inclusion, note that indeed $D + \mathrm{cl}(D) \subseteq \mathrm{cl}(D) + \mathrm{cl}(D) = \mathrm{cl}(D)$. This argument shows that D and $D + \mathrm{cl}(D)$ coincide except for their boundary. More specifically, $D + \mathrm{cl}(D)$ contains D and may contain additional gambles on D's boundary. This implies that the interiors of D and its SSK–Archimedean natural extension are equal: $\mathrm{int}(D) = \mathrm{int}(D + \mathrm{cl}(D))$, and hence D and $D + \mathrm{cl}(D)$ have the same credal set $\mathbf{P}(D) = \mathbf{P}(D + \mathrm{cl}(D))$ by Equation (7). Lemma 26 later on establishes a strengthening of this idea.

3.3. **No two-way representation & even convexity.** By collecting in

$$\mathbf{R}(A) \coloneqq \{p \in \Sigma : A \subseteq D_p\} \tag{11}$$

all probability mass functions p for which D_p dominates A, then $A \subseteq \bigcap\{D_p : p \in \mathbf{R}(A)\}$. For a coherent set of desirable gambles D this implies a one-way representation:[7]

$$D \subseteq \bigcap\{D_p : p \in \mathbf{R}(D)\} \tag{12}$$

and $\mathbf{R}(D) \neq \emptyset$ if and only if D is extendible to a probability.

The one-way representation is done using the set $\{D_p : p \in \mathbf{R}(D)\}$, which contains, by Proposition 5, the maximal elements of $\overline{\mathbf{D}}_{\mathrm{Arch}}$ that dominate D. Moreover, if D is SSK–Archimedean, by Proposition 7 we have that $\mathrm{ri}(\mathbf{P}(D)) \subseteq \mathbf{R}(D) \subseteq \mathbf{P}(D)$, which is by Rockafellar [18, Corollary 6.3.1] equivalent to

$$\mathrm{cl}(\mathbf{R}(D)) = \mathbf{P}(D). \tag{13}$$

In other words: $\mathbf{R}(D)$ consists of $\mathrm{ri}(\mathbf{P}(D))$, possibly together with elements of $\mathbf{P}(D)$ on its boundary.

The set of probability mass functions $\mathbf{R}(D)$ would be a two-way representation if we can retrieve the original set of desirable gambles D with it—in other words, if

$$D = \bigcap\{D_p : p \in \mathbf{R}(D)\}.$$

However, SSK–Archimedeanity is not strong enough to guarantee this, as Cozman [4, Example 17] showed by means of an example. Here, we will present the idea of his example.

Example 7. Consider a ternary possibility space $\Omega = \{a, b, c\}$. A generic coherent set of desirable gambles is a convex cone in the 3-dimensional space $\mathscr{L}$ of gambles on Ω, as indicated in Figure 9, with a non-orthogonal coordinate system, so that the positive orthant $\mathscr{L}_{\geqslant 0}$ is included in D.

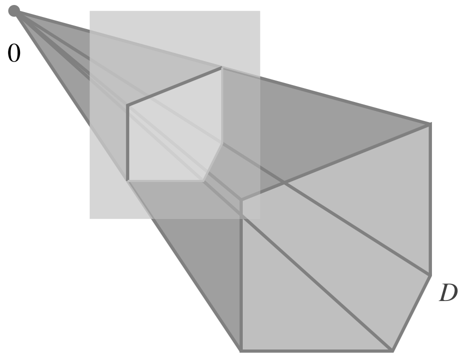
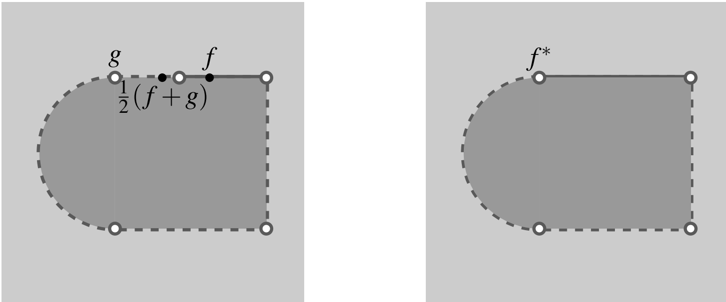


FIGURE 9. A generic set of desirable gambles in a ternary space

[7] As usual, we let $\bigcap\emptyset = \mathscr{L}$

We intersect this set of desirable gambles with a plane, indicated in light grey. We can think of this plane as the plane with normal in the direction of $(1,1,1)$, through the point $(1,1,1)$. The following two pictures are the result of such an intersection with two different sets of desirable gambles.



not SSK–Archimedean SSK–Archimedean

FIGURE 10. A graphical representation of a non-SSK–Archimedean and an SSK–Archimedean coherent set of desirable gambles

Figure 4 serves as an illustration of this sets of desirable gambles D.

It turns out that the set of desirable gambles D depicted in the left of Figure 10 is not SSK–Archimedean. To see this, consider the gambles f in D and g in $\mathrm{cl}(D)$. Then their addition—scaled with a factor $\frac{1}{2}$ so that the result lays within the intersecting plane—$\frac{1}{2}(f+g)$ will not be an element of D, contradicting its SSK–Archimedeanity. We will later—in Section 3.4—see, that for any linear part of the boundary of D, SSK–Archimedeanity requires that it either has nothing in common with D, or its relative interior is included in D.

The set of desirable gambles D depicted in the right of Figure 10 is SSK–Archimedean, as Cozman [4, Example 17] shows. It is, however, no intersection of D_p's. As we will see in Theorem 31, this is because the gamble f^*, which lies on a non-exposed ray of D, does not belong to D. $\lozenge$

This observation is in line with the findings of Seidenfeld et al. [21] for partial preferences on horse lotteries: they showed that a coherent partial preference $\prec$ (on horse lotteries) is two-way represented by a convex (not necessarily closed) set of probability mass functions if and only if $\prec$ is SSK–Achimedean and the result of an extension (what we will call SSK–extension later on in Section 4). Cozman [4] acknowledges that SSK–Archimedeanity is not sufficient to have a two-way representation, and shows that the requirement of *even convexity* is sufficient.

Definition 12 (Even convexity; see Daniilidis & Martinez–Legaz [5, Section 2][8])**.** A set C in a finite-dimensional space is called *evenly convex* if it is an (arbitrary) intersection of affine open half-spaces.

Clearly, intersections of evenly convex sets are evenly convex: they form an intersection structure. This allows us to introduce an *evenly convex closure* operator:

$$\mathrm{eco}(C) = \bigcap\{S : S \text{ is evenly convex and } C \subseteq S\}$$

[8]Actually, Daniilidis and Martinez–Legaz define [5, Definition 1] even convexity differently, but they acknowledge immediately after their definition that it coincides with the definition we use. A second difference is that they work in the more general context of possibly infinite-dimensional separable Banach spaces. Because our linear space $\mathscr{L}$ of gambles has a finite dimension n, $\mathscr{L}$ is automatically a separable Banach space, and therefore all the results in Daniilidis and Martinez–Legaz [5] apply in the present context.

$$= \bigcap\{S : S \text{ is an affine open half-space and } C \subseteq S\}$$

for any set C in a finite-dimensional space. $\mathrm{eco}(C)$ is the smallest evenly convex set that includes C. Using the second identity of this equation, a set C is evenly convex if and only if it is equal to its evenly convex closure: C is evenly convex if and only if $C = \mathrm{eco}(C)$.

For any p in Σ, its associated set of desirable gambles D_p is an affine open half-space, and is therefore evenly convex. More generally, as Daniilidis and Martinez–Legaz [5, Section 2] note, any closed convex set and any open convex set is evenly convex. We collect all the evenly convex coherent sets of desirable gambles in the set $\overline{\mathbf{D}}_{\text{e.c.}}$, which is non-empty because $D_p \in \overline{\mathbf{D}}_{\text{e.c.}}$ for every p in Σ, as we just have seen.

It turns out that even convexity guarantees a two-way representation:

Proposition 13. *Consider any coherent set of desirable gambles D. Then $\mathrm{eco}(D)$ is coherent if and only if D is extendible to a probability, and in that case*

$$\mathrm{eco}(D) = \bigcap\{D_p : p \in \mathbf{R}(D)\}.$$

As a consequence

$$D \in \overline{\mathbf{D}}_{\text{e.c.}} \Leftrightarrow D = \bigcap\{D_p : p \in \mathbf{R}(D)\}, \tag{14}$$

for any set of desirable gambles D, so any set of desirable gambles is evenly convex if and only if it is two-way represented by $\mathbf{R}(D)$. Finally, SSK–Archimedeanity is necessary for even convexity: any D in $\overline{\mathbf{D}}_{\text{e.c.}}$ is SSK–Archimedean.

Proof. Before we start the actual proof, note already that, for any coherent set of desirable gambles D,

$$\begin{aligned}
\mathrm{eco}(D) &= \bigcap\{S \subseteq \mathscr{L} : S \text{ is evenly convex and } D \subseteq S\} \\
&\subseteq \bigcap\{D_p : p \in \Sigma \text{ and } D \subseteq D_p\} = \bigcap\{D_p : p \in \mathbf{R}(D)\},
\end{aligned} \tag{15}$$

where the set inclusion follows from our observation earlier that every D_p is evenly convex. Since $\mathbf{R}(D) \neq \emptyset$ whenever D is extendible to a probability, and since we have already seen in Example 6 that every D_p is a coherent set of desirable gambles, this also implies that $\mathrm{eco}(D)$ is included in a coherent set of desirable gambles $\bigcap\{D_p : p \in \mathbf{R}(D)\}$ when D is extendible to a probability.

The proof is structured as follows. We will prove the following two statements in one fell swoop: 'if D is not extendible to a probability, then $\mathrm{eco}(D)$ is not coherent', and 'if D is extendible to a probability, then $\mathrm{eco}(D) = \bigcap\{D_p : p \in \mathbf{R}(D)\}$', which considered together, proves the equivalence $\mathrm{eco}(D)$ is coherent $\Leftrightarrow$ D is extendible to a probability. Our strategy for this will be to consider any $f \notin \mathrm{eco}(D)$, and infer from this that $f \neq 0$ when D is not extendible to a probability—meaning that $0 \in \mathrm{eco}(D)$ which is therefore not coherent since it violates Axiom D_1—and that $f \notin \bigcap\{D_p : p \in \mathbf{R}(D)\}$ when D is extendible to a probability, which implies, together with the initial observation in Equation (15), that $\mathrm{eco}(D) = \bigcap\{D_p : p \in \mathbf{R}(D)\}$. Once we have done so, our proof will be complete: the consequence in Equation (14) follows readily from our observation earlier that D is evenly convex if and only if it is equal to $\mathrm{eco}(D)$, and that every evenly convex set of desirable gambles is SSK–Archimedean by the fact that every D_p is SSK–Archimedean, established in Example 6, and that SSK–Archimedeanity is closed under arbitrary intersections, established in Proposition 8.

So consider any coherent set of desirable gambles D and any gamble $f \notin \mathrm{eco}(D)$, implying that $f \notin S$ for some affine open half-space S such that $D \subseteq S$. Since S is an affine open half-space, there is a real-valued non-zero linear functional Λ on $\mathscr{L}$ and h in $\mathscr{L}$ such that

$$\begin{aligned}
S &= \{h' \in \mathscr{L} : \Lambda(h') > 0\} + \{h\} \\
&= \{h' + h : h' \in \mathscr{L} \text{ and } \Lambda(h') > 0\} = \{g \in \mathscr{L} : \Lambda(g - h) > 0\} \\
&= \{g \in \mathscr{L} : \Lambda(g) > \Lambda(h)\} = \{g \in \mathscr{L} : \Lambda(g) > \lambda\} = \{g \in \mathscr{L} : \Lambda(g) > 0\} + \{\lambda\},
\end{aligned}$$

where $\lambda := \Lambda(h) \in \mathbb{R}$. In other words, $g \in S \Leftrightarrow \Lambda(g) > \lambda$, for any gamble g. Since Λ is a linear functional, it is completely determined by the values it takes on elementary indicators, that is on gambles $\mathbb{I}_{\{\omega_k\}}$ with k in $\{1,\dots,n\}$, defined as

$$\mathbb{I}_{\{\omega_k\}}(\omega) = \begin{cases} 1 & \text{if } \omega = \omega_k \\ 0 & \text{else} \end{cases} \qquad \text{for all } \omega \text{ in } \Omega.$$

Indeed, since any gamble g can be written as $g = \sum_{k=1}^{n} g(\omega_k)\mathbb{I}_{\{\omega_k\}}$, we have that

$$\Lambda(g) = \Lambda\Big(\sum_{k=1}^{n} g(\omega_k)\mathbb{I}_{\{\omega_k\}}\Big) = \sum_{k=1}^{n} g(\omega_k)\Lambda(\mathbb{I}_{\{\omega_k\}})$$

by the linearity of Λ. We can think of $(\Lambda(\mathbb{I}_{\{\omega_1\}}),\dots,\Lambda(\mathbb{I}_{\{\omega_n\}}))$ as an n-dimensional vector and therefore as an element of $\mathscr{L}$. In fact, $(\Lambda(\mathbb{I}_{\{\omega_1\}}),\dots,\Lambda(\mathbb{I}_{\{\omega_n\}}))$ can be seen as a normal vector on $\ker\Lambda := \{h' \in \mathscr{L} : \Lambda(h') = 0\}$, the kernel of Λ, which is a linear subspace of dimension $n-1$ by the rank–nullity theorem.

We show the first intermediate result that, as a consequence of $D \subseteq S$, the normal vector $(\Lambda(\mathbb{I}_{\{\omega_1\}}),\dots,\Lambda(\mathbb{I}_{\{\omega_n\}}))$ belongs to $\mathscr{L}_{\geq 0}$ and $\lambda \leq 0$. To see this, consider for any k in $\{1,\dots,n\}$, and α and β in $\mathbb{R}_{>0}$, the gamble $g_{\alpha,\beta} := \alpha\mathbb{I}_{\{\omega_k\}} + \beta \in \mathscr{L}_{>0}$, which belongs to D by its coherence [more specifically, Axiom D_2]. Because $D \subseteq S$, we find that $g_{\alpha,\beta} \in S$, which implies that $\Lambda(g_{\alpha,\beta}) = \alpha\Lambda(\mathbb{I}_{\{\omega_k\}}) + \beta\Lambda(1) > \lambda$. Since the choice of α in $\mathbb{R}_{>0}$ was arbitrarily large, this implies that indeed $\Lambda(\mathbb{I}_{\{\omega_k\}}) \geq 0$, for any k in $\{1,\dots,n\}$. Furthermore, since the choices of α and β in $\mathbb{R}_{>0}$ were arbitrarily small, this implies that indeed $\lambda \leq 0$.

So we find that $S = \{g \in \mathscr{L} : \Lambda(g) > \lambda\}$ for some real $\lambda \leq 0$ and some non-zero linear functional Λ such that $(\Lambda(\mathbb{I}_{\{\omega_1\}}),\dots,\Lambda(\mathbb{I}_{\{\omega_n\}})) \in \mathscr{L}_{\geq 0}$. Clearly $(\Lambda(\mathbb{I}_{\{\omega_1\}}),\dots,\Lambda(\mathbb{I}_{\{\omega_n\}})) \neq 0$ since otherwise $\Lambda(g) = 0$ for all gambles g, contradicting that Λ is non-zero. Therefore $\Lambda(1) = \sum_{\ell=1}^{n} \Lambda(\mathbb{I}_{\{\omega_\ell\}}) > 0$, and hence the real-valued function q on Ω defined as $q(\omega_k) := \Lambda(\mathbb{I}_{\{\omega_k\}})/\Lambda(1)$ for every k in $\{1,\dots,n\}$ satisfies $q(\omega) \geq 0$ for all ω in Ω, and $\sum_{k=1}^{n} q(\omega_k) = 1$, so q is a probability mass function. Since q is a scaled variant of Λ, it satisfies $\Lambda(g) > 0 \Leftrightarrow E_q(g) > 0$, for all gambles g. Therefore

$$S = \{g \in \mathscr{L} : \Lambda(g) > 0\} + \{\lambda\} = \{g \in \mathscr{L} : E_q(g) > 0\} + \{\lambda\} = D_q + \{\lambda\}.$$

Let us now, as a second and final intermediate result, show that q belongs to $\mathbf{P}(D)$. Note already that if $\lambda = 0$, then $D \subseteq S = D_q + \{0\} = D_q$, so that by Equation (11) $q \in \mathbf{R}(D) \subseteq \mathbf{P}(D)$ and we are done. In the light of our first intermediate result, we may therefore assume that $\lambda < 0$. To show that $q \in \mathbf{P}(D)$, it suffices by Equation (7) to show that $\operatorname{int}(D) \subseteq D_q$. To this end, consider any g in $\operatorname{int}(D)$, meaning that $g - \delta \in D$ for some δ in $\mathbb{R}_{>0}$. Therefore $\mu g - \mu\delta \in D$ for any μ in $\mathbb{R}_{>0}$, using the coherence of D [more specifically, Axiom D_3]. Since $D \subseteq S = D_q + \{\lambda\}$ we find that $\mu g - \mu\delta \in D_q + \{\lambda\}$, or, in other words, that $E_q(\mu g) > \lambda + \mu\delta$. Since the choice of μ in $\mathbb{R}_{>0}$ was arbitrary, we may consider the specific value $\mu = -\frac{\lambda}{\delta} > 0$, which leads us to conclude that $E_q(\lambda g) > 0$, and hence $E_q(g) > 0$ by the linearity of the expectation operator E_q. Therefore $g \in D_q$, whence indeed $q \in \mathbf{P}(D)$.

To finish the proof, we will show (i) that $f \neq 0$ if D is not extendible to a probability, and (ii) that $f \notin \bigcap\{D_p : p \in \mathbf{R}(D)\}$ if D is extendible to a probability.

For (i), let us assume that D is not extendible to a probability, which implies that $D \nsubseteq D_q$. Since $D_q + \{\lambda\}$ contains D, this means that necessarily $\lambda < 0$. But $f \notin D_q + \{\lambda\}$ implies that $E_q(f) \leq \lambda < 0$, whence indeed $f \neq 0$. Therefore we have shown $f \notin \operatorname{eco}(D) \Rightarrow f \neq 0$, or equivalently, $f = 0 \Rightarrow f \in \operatorname{eco}(D)$, for every gamble f, so 0 belongs to $\operatorname{eco}(D)$, which therefore violates Axiom D_1 so it is indeed not coherent.

For (ii), let us assume that D is extendible to a probability. Then by Theorem 11 D's SSK–Archimedean natural extension $D^* := \mathscr{E}_{\mathrm{Arch}}(D) = D + \operatorname{cl}(D)$ is coherent and SSK–Archimedean, and therefore by Proposition 7 we have $\operatorname{ri}(\mathbf{P}(D^*)) \subseteq \mathbf{R}(D^*) \subseteq \mathbf{P}(D^*)$. We will prove the intermediate result that $\mathbf{R}(D) = \mathbf{R}(D^*)$. Since $D \subseteq D^*$, using Equation (11) we find that automatically $\mathbf{R}(D) \supseteq \mathbf{R}(D^*)$, so it remains to show that $\mathbf{R}(D) \subseteq \mathbf{R}(D^*)$. To this end, consider any p in $\mathbf{R}(D)$, meaning that $D \subseteq D_p$. We need to show that $D^* \subseteq D_p$. So consider any h in D^*, meaning that $h = g_1 + g_2$ for some g_1 in D and g_2 in $\mathrm{cl}(D)$. Since $D \subseteq D_p$, we have that $E_p(g_1) > 0$ and $E_p(g_2) \geq 0$, whence $E_p(h) = E_p(g_1) + E_p(g_2) > 0$, and therefore indeed $h \in D_p$. So we have found that $\mathbf{R}(D) = \mathbf{R}(D^*)$, and therefore, using Equation (13), also $\mathbf{P}(D) = \mathrm{cl}(\mathbf{R}(D)) = \mathrm{cl}(\mathbf{R}(D^*)) = \mathbf{P}(D^*)$. This means that $\mathrm{ri}(\mathbf{P}(D)) \subseteq \mathbf{R}(D) \subseteq \mathbf{P}(D)$.

We finish by distinguishing two cases: $q \in \mathbf{R}(D)$, or $q \notin \mathbf{R}(D)$. If $q \in \mathbf{R}(D)$ then $D \subseteq D_q$ by Equation (11), so D_q is an affine open half-space that includes D. Furthermore, since $f \notin D_q + \{\lambda\}$ we find that $f \notin D_q$, because we already established above that $\lambda \leq 0$. This implies that indeed $f \notin \bigcap\{D_p : p \in \mathbf{R}(D)\}$.

If, on the other hand, $q \notin \mathbf{R}(D)$, then $D \nsubseteq D_q$, and hence $\lambda < 0$ since $D \subseteq D_q + \{\lambda\}$. This implies that $E_q(f) < 0$. We will show that this implies that $f \notin D_{p^*}$ for some p^* in $\mathbf{R}(D)$. To this end, consider any p in $\mathrm{ri}(\mathbf{P}(D))$. If $E_p(f) \leq 0$ then $f \notin D_p$ and we are done, so assume that $E_p(f) > 0$. Let $\alpha := \frac{E_p(f)}{E_p(f) - E_q(f)} \in (0,1)$, then $1 - \alpha = \frac{-E_q(f)}{E_p(f) - E_q(f)}$, and consider the convex combination $p^* := \alpha q + (1-\alpha)p$, which belongs to $\mathrm{ri}(\mathbf{P}(D))$ by Theorem 3, and therefore also to $\mathbf{R}(D)$. But $E_{p^*}(f) = \frac{E_p(f)}{E_p(f) - E_q(f)} E_q(f) - \frac{E_q(f)}{E_p(f) - E_q(f)} E_p(f) = 0$, so that $f \notin D_{p^*}$ and therefore indeed $f \notin \bigcap\{D_p : p \in \mathbf{R}(D)\}$. So we have shown $f \notin \mathrm{eco}(D) \Rightarrow f \notin \bigcap\{D_p : p \in \mathbf{R}(D)\}$, or equivalently, $f \in \bigcap\{D_p : p \in \mathbf{R}(D)\} \Rightarrow f \in \mathrm{eco}(D)$, for every gamble f, which, together with Equation (15) shows that indeed $\mathrm{eco}(D) = \bigcap\{D_p : p \in \mathbf{R}(D)\}$. □

Thus a coherent set of desirable gambles D is evenly convex if and only if it is (two-way) represented by a set $\mathbf{R}(D)$ of probability mass functions, which as Cozman [4, Theorem 9] shows, is evenly convex itself, and is the unique largest representing set of probabilities. This clearly shows the importance of even convexity for our purpose. Cozman [4] investigates the notion of evenly convex sets of desirable gambles in greater detail, and provides a useful equivalent condition.

Because we will also be interested in conservative reasoning, gather from the discussion above that the set $\overline{\mathbf{D}}_{\text{e.c.}}$ is an intersection structure, just as the set $\overline{\mathbf{D}}_{\text{Arch}}$ of all SSK–Archimedean coherent sets of desirable gambles as shown in Proposition 8. Interestingly, $\overline{\mathbf{D}}_{\text{e.c.}}$ has the same set $\widehat{\mathbf{D}}_{\text{e.c.}}$ of maximal elements as $\overline{\mathbf{D}}_{\text{Arch}}$:

Proposition 14. *The maximal elements of $\overline{\mathbf{D}}_{\text{e.c.}}$ are precisely the maximal elements of $\overline{\mathbf{D}}_{\text{Arch}}$:*

$$\widehat{\mathbf{D}}_{\text{e.c.}} = \widehat{\mathbf{D}}_{\text{Arch}} = \{D_p : p \in \Sigma\}.$$

Moreover, every element of $\overline{\mathbf{D}}_{\text{e.c.}}$ is dominated by an element of $\widehat{\mathbf{D}}_{\text{e.c.}}$. Finally, the unique smallest evenly convex coherent set of desirable gambles is equal to $\mathscr{L}_{>0}$.

Proof. For the first statement, we will first show that any D_p, with p in Σ, is a maximal element of $\overline{\mathbf{D}}_{\text{e.c.}}$. To this end, assume *ex absurdo* that $D_p \subset D$ for some D in $\overline{\mathbf{D}}_{\text{e.c.}}$ and p in Σ. But $D = \bigcap\{D_q : q \in \mathbf{R}(D)\}$, so $D_p \subset \bigcap\{D_q : q \in \mathbf{R}(D)\}$, which is absurd.

Conversely, to show that any maximal element D of $\overline{\mathbf{D}}_{\text{e.c.}}$ is equal to D_p for some p in Σ, assume *ex absurdo* that $\widehat{\mathbf{D}}_{\text{e.c.}} \nsubseteq \{D_p : p \in \Sigma\}$. Since we have already established that $\widehat{\mathbf{D}}_{\text{e.c.}} \supseteq \{D_p : p \in \Sigma\}$, this would mean that $\widehat{\mathbf{D}}_{\text{e.c.}} \supset \{D_p : p \in \Sigma\}$. We will show that this is impossible. To this end, consider any D in $\widehat{\mathbf{D}}_{\text{e.c.}} \setminus \{D_p : p \in \Sigma\}$. Since D is evenly convex, by Equation (14) we would infer that $D \subseteq D_p$ for every p in the non-empty set $\mathbf{R}(D)$. But we have already established that D_p is an element of $\overline{\mathbf{D}}_{\text{e.c.}}$, so this would mean that D is dominated in $\overline{\mathbf{D}}_{\text{e.c.}}$, a contradiction with the fact that D is maximal.

For the second statement, consider any D in $\overline{\mathbf{D}}_{\text{e.c.}}$. Then by Equation (14) we infer that, indeed, $D \subseteq D_p$ for every p in the non-empty set $\mathbf{R}(D)$.

Finally, for the third statement, we have established above that D_p is an element of $\overline{\mathbf{D}}_{\text{e.c.}}$ for every p in Σ. Since $\overline{\mathbf{D}}_{\text{e.c.}}$ is an intersection structure, this implies that $\bigcap\{D_p : p \in \Sigma\} =$

$\mathscr{L}_{>0}$ belongs to $\overline{\mathbf{D}}_{\text{e.c.}}$. But $\mathscr{L}_{>0}$ is the unique smallest coherent set of desirable gambles, which implies that it is indeed the unique smallest evenly convex coherent set of desirable gambles. □

3.4. **Bouligand tangent cones.** There is a useful characterisation of even convexity by Daniilidis and Martinez–Legaz [5], in terms of *Bouligand tangent cones*, that will help us connect with the SSK–extension in Section 4. Given any convex set C in a finite-dimensional linear space, we call $\ell(C)$ the largest linear space included in C:[9]

$$\ell(C) := C \cap (-C) = \{f : f \in C \text{ and } -f \in C\}.$$

For any convex set C and any point f in $\mathrm{cl}(C)$, the *Bouligand tangent cone* of C in f is the set[10]

$$T_C(f) := \mathrm{cl}\big(\bigcup\{\lambda(C - \{f\}) : \lambda \in \mathbb{R}_{>0}\}\big),$$

which is a closed convex cone [5].

Example 8. Consider the binary possibility space $\Omega := \{a, b\}$ and the coherent set of desirable gambles D as depicted in Figure 11.

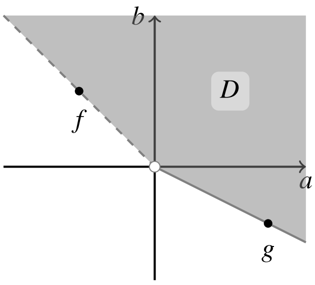


FIGURE 11. The set of desirable gambles D used in Example 8

We depict for the four different gambles f, g, 0 and 1 in $\mathrm{cl}(D)$ their Bouligand tangent cones $T_D(f)$, $T_D(g)$, $T_D(0)$ and $T_D(1)$ graphically in Figure 12.

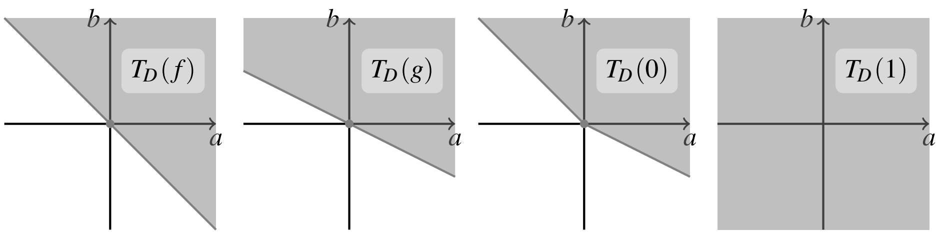


FIGURE 12. The Bouligand tangent cones $T_D(f)$, $T_D(g)$, $T_D(0)$ and $T_D(1)$

We see that $\ell(T_D(f)) = \{\lambda f : \lambda \in \mathbb{R}\}$, $\ell(T_D(g)) = \{\lambda g : \lambda \in \mathbb{R}\}$, $\ell(T_D(0)) = \{0\}$ and $\ell(T_D(0)) = \mathscr{L}$. ◊

[9] We define $-C := \{-x : x \in C\}$.

[10] We define $\lambda C := \{\lambda x : x \in C\}$ as the set of multiplications of elements of C with a given real λ.

A useful fact about Bouligand tangent cones is the following property:

$$f \in \operatorname{int}(D) \Leftrightarrow T_D(f) = \mathscr{L}, \tag{16}$$

for any coherent set of desirable gambles D and any gamble f in $\operatorname{cl}(D)$; see [1, Sections 4.1.3 and 4.2.1]. It will be convenient to first prove some other facts about $T_D(f)$:

Lemma 15. *Consider any coherent set of desirable gambles D and any gamble f in $\operatorname{cl}(D) \setminus D$. Then*

(i) $\{\lambda f : \lambda \in \mathbb{R}\} \subseteq T_D(f)$*;*
(ii) $\operatorname{cl}(D) + T_D(f) \subseteq T_D(f)$*;*
(iii) $\operatorname{cl}(D) \subseteq T_D(f)$*;*
(iv) $\operatorname{int}(D) \cap \ell(T_D(f)) = \emptyset$*;*
(v) $\{f\} + \ell(T_D(f)) = \ell(T_D(f))$.

Proof. For Property (i), since the closure operator cl preserves set inclusion (is monotone) and $0, 2f \in \operatorname{cl}(D)$, note that $f, -f \in \operatorname{cl}(D - \{f\}) \subseteq \operatorname{cl}\big(\bigcup\{\lambda(D - \{f\}) : \lambda \in \mathbb{R}_{>0}\}\big) = T_D(f)$. Using the fact that $T_D(f)$ is a convex cone, we infer that indeed $\{\lambda f : \lambda \in \mathbb{R}\} = \operatorname{posi}\{-f, f\} \subseteq T_D(f)$.

For Property (ii), consider any g in $\operatorname{cl}(D)$ and h in $T_D(f)$, which implies that $g + \gamma \in D$ and $h + \varepsilon \in \bigcup_{\lambda \in \mathbb{R}_{>0}} \lambda(D - \{f\})$ for all ε, γ in $\mathbb{R}_{>0}$, whence $g + h + \varepsilon + \gamma \in \{g + \gamma\} + \bigcup_{\lambda \in \mathbb{R}_{>0}} \lambda(D - \{f\}) = \bigcup_{\lambda \in \mathbb{R}_{>0}} \lambda(D + \frac{1}{\lambda}\{g + \gamma\} - \{f\}) \subseteq \bigcup_{\lambda \in \mathbb{R}_{>0}} \lambda(D - \{f\})$, for all ε, γ in $\mathbb{R}_{>0}$, where the set inclusion follows from the coherence [more specifically, Axiom D_4] of D. Therefore indeed $g + h \in \operatorname{cl}\big(\bigcup_{\lambda \in \mathbb{R}_{>0}} \lambda(D - \{f\})\big) = T_D(f)$.

For Property (iii), infer from Property (i) that $0 \in T_D(f)$, whence $\operatorname{cl}(D) \subseteq \operatorname{cl}(D) + T_D(f)$ by the definition of the Minkowski addition. Using Property (ii), this implies that indeed $\operatorname{cl}(D) \subseteq T_D(f)$.

For Property (iv), assume *ex absurdo* that $\operatorname{int}(D) \cap \ell(T_D(f)) \neq \emptyset$, say $g \in \operatorname{int}(D) \cap \ell(T_D(f))$. Then $-g \in T_D(f)$ and $g \in \operatorname{int}(D)$, whence $g - \delta \in D$ for some $\delta \in \mathbb{R}_{>0}$. By Property (iii), this would imply that $g - \delta \in T_D(f)$. Since $T_D(f)$ is a convex cone—so $\operatorname{posi}(T_D(f)) = T_D(f)$—we would infer that $\frac{1}{\delta}(-g + g - \delta) = -1 \in T_D(f)$. But also $D \subseteq T_D(f)$ by Property (iii), which would imply that $\operatorname{posi}(D \cup \{-1\}) = \mathscr{L} \subseteq T_D(f)$. By Equation (16) this would imply that $f \in \operatorname{int}(D)$, contradicting the fact that f belongs to $\operatorname{cl}(D) \setminus D$.

For Property (v), note that Property (i) implies that $T_D(f)$ includes the linear space $\{\lambda f : \lambda \in \mathbb{R}\}$, and therefore so does the largest linear space $\ell(T_D(f))$ included in $T_D(f)$. This implies that $f \in \ell(T_D(f))$, whence indeed $\{f\} + \ell(T_D(f)) = \ell(T_D(f))$. □

Bouligand tangent cones are important to our purpose mainly due to the following representation result by Daniilidis and Martinez–Legaz [5]:

Theorem 16 (See Daniilidis and Martinez–Legaz [5, Theorem 5])**.** *Consider any convex subset C of a finite-dimensional space. Then C is evenly convex if and only if*

$$(\forall f \in \operatorname{cl}(C) \setminus C)(\{f\} + \ell(T_C(f))) \cap C = \emptyset.$$

Using Lemma 15(v), this immediately yields:

Corollary 17. *Consider any coherent set of desirable gambles D. Then D is evenly convex if and only if*

$$\ell(T_D(f)) \cap D = \emptyset \text{ for all } f \text{ in } \operatorname{cl}(D) \setminus D.$$

We know by the example in Cozman [4, Example 17] and Proposition 13 that that there is a gap between SSK–Archimedeanity and even convexity, which we are looking to bridge. More specifically, we are looking for an additional requirement on coherent sets of desirable gambles D that ensures that D is evenly convex—in other words, that $\operatorname{eco}(D) = D$—which by Corollary 17 above is equivalent to the requirement that $\ell(T_D(f)) \cap D = \emptyset$ for all f in

$\mathrm{cl}(D)\setminus D$. The following proposition shows the extent to which SSK–Archimedeanity takes care of the linear part $\ell(T_D(f))\cap\mathrm{cl}(D)$ of D's boundary.

Proposition 18. *Consider any coherent set of desirable gambles D. Then D is SSK–Archimedean if and only if*

$$K\cap D=\emptyset \text{ or } \mathrm{ri}(K\cap\mathrm{cl}(D))\subseteq D, \text{ for any linear space } K\subseteq\mathscr{L}.$$

As a consequence, if D is SSK–Archimedean then

$$\ell(T_D(f))\cap D=\emptyset \text{ or } \mathrm{ri}(\ell(T_D(f))\cap\mathrm{cl}(D))\subseteq D, \text{ for all } f \text{ in } \mathrm{cl}(D)\setminus D.$$

Proof. We start with the first statement. For necessity, assume that D is SSK–Archimedean, and consider any linear space $K\subseteq\mathscr{L}$. If $K\cap D=\emptyset$, then the proof is done, so assume that $K\cap D\neq\emptyset$, say $g\in K\cap D\subseteq K\cap\mathrm{cl}(D)$. Consider any h in $\mathrm{ri}(K\cap\mathrm{cl}(D))$; we need to show that then $h\in D$. If $h=g$ then the proof is done, so assume that $h\neq g$. Using Theorem 4, we infer that $f:=(1-\mu)g+\mu h\in K\cap\mathrm{cl}(D)$ for some real $\mu>1$. Then indeed

$$h=\underbrace{\frac{1}{\mu}f}_{\in\mathrm{cl}(D)}+\underbrace{\frac{\mu-1}{\mu}g}_{\in D}\in D,$$

by the SSK–Archimedeanity of D.

For sufficiency, assume that $K\cap D=\emptyset$ or $\mathrm{ri}(K\cap\mathrm{cl}(D))\subseteq D$ for any linear space $K\subseteq\mathscr{L}$. Consider any f in D and g in $\mathrm{cl}(D)$; we need to show that then $f+g\in D$. If $g=0$ or $g\in D$ then the proof is done, so assume that $g\neq 0$ and $g\notin D$. Let $K:=\mathrm{span}\{f,g\}=\{\lambda_1 f+\lambda_2 g:\lambda_1,\lambda_2\in\mathbb{R}\}$. Then $f\in K$, so $K\cap D\neq\emptyset$. By the assumption, therefore $\mathrm{ri}(K\cap\mathrm{cl}(D))\subseteq D$. Note that $K\cap\mathrm{cl}(D)$ contains 0, so its affine hull $\mathrm{aff}(K\cap\mathrm{cl}(D))$ is actually equal to its linear hull $\mathrm{span}(K\cap\mathrm{cl}(D))=K$, which is two-dimensional. To see this, note that $K=\mathrm{span}\{f,g\}$, so K has dimension 0, 1 or 2. Since $g\neq 0$ it cannot have dimension 0. It can only have dimension 1 if $\lambda f=g$ for some λ in $\mathbb{R}$. $\lambda>0$ is impossible by the coherence of D [since $f\in D$ but $g\notin D$]; $\lambda=0$ is impossible since $g\neq 0$, and $\lambda<0$ is impossible because it would imply that $0\in\mathrm{ri}(K\cap\mathrm{cl}(D))$ [since 0 would belong to the relative interior of $\{\alpha f+(1-\alpha)g:\alpha\in[0,1]\}$ in the one-dimensional linear space $\mathrm{aff}(K\cap\mathrm{cl}(D))$], which is a subset of D, contradicting the coherence of D.

Note that by Equation (6), $(f+g)/2$ belongs to the relative interior of the convex hull of $\{0,f,g,f+g\}\subseteq K\cap\mathrm{cl}(D)$, which has K as its affine hull: indeed $(\{(f+g)/2\}+\varepsilon B^n)\cap\mathrm{aff}(K)\subseteq\{0,f,g,f+g\}$. But since $\{0,f,g,f+g\}\subseteq K\cap\mathrm{cl}(D)$, we have that $(f+g)/2$ belongs to $\mathrm{ri}(K\cap\mathrm{cl}(D))$. Since $\mathrm{ri}(K\cap\mathrm{cl}(D))\subseteq D$, we infer that $(f+g)/2\in D$, whence by the coherence of D, indeed $f+g\in D$.

The second statement follows immediately once we realise that $\ell(T_D(f))$ is a linear space. □

By combining Corollary 17 with Proposition 18, we find that, if a coherent set of desirable gambles D is SSK–Archimedean but not evenly convex, then

$$\mathrm{ri}(\ell(T_D(f))\cap\mathrm{cl}(D))\subseteq D \text{ for some } f \text{ in } \mathrm{cl}(D)\setminus D.$$

This means that there is a linear part $\mathrm{ri}(\ell(T_D(f))\cap\mathrm{cl}(D))$ of the boundary of D that is contained in D, but not its endpoint $f\in\mathrm{cl}(D)\setminus D$. As we will see in Section 4, this endpoint will necessarily lie on a non-exposed ray of D. So we see that SSK–Archimedeanity takes care of the relative interior of the linear parts of the boundary, but not of its endpoints. This implies that the coherent set of desirable gambles of Example 7 is the only 'type' of SSK–Archimedean but not evenly convex coherent set of desirable gambles.

4. SSK–EXTENSION

So far, we have seen that SSK–Archimedeanity is insufficient to guarantee a two-way representation. This observation is in line with Cozman [4, Example 17] example, and with the results in Seidenfeld et al. [21], who introduce an additional operation called 'SSK–extension' that yields a representation in terms of probabilities, in their setting of partial preferences on horse lotteries. This section investigates whether the same ideas can be used to obtain representation in our current setting, by imposing a suitable adaptation of SSK–extension.

The idea is to add to a coherent and SSK–Archimedean set of desirable gambles D those gambles f whose negation $-f$ are "precluded from being desirable" (precluded from being preferred to 0), and that are "precluded from being indifferent to 0".

4.1. **Indifference and desirability.** Let us first review how to combine an indifference and desirability statement. Besides her set of gambles D that a subject prefers to 0, she might specify some gambles I that she finds indifferent—equivalent—to 0.

Definition 19 (Coherent set of indifferent gambles)**.** A set of indifferent gambles I is called *coherent* if for all f and g in $\mathscr{L}$ and λ in $\mathbb{R}$:

I_1. $0 \in I$;
I_2. $\mathscr{L}_{>0} \cap I = \emptyset$;
I_3. if $f \in I$ then $\lambda f \in I$;
I_4. if $f, g \in I$ then $f + g \in I$.

We collect all the coherent sets of indifferent gambles in $\overline{\mathbf{I}}$.

Axioms I_3 and I_4 make a coherent set of indifferent gambles I a linear space: $I = \operatorname{span} I$, that is non-empty by Axiom I_1. So a coherent set of indifferent gambles I is a linear space that has nothing in common with $\mathscr{L}_{>0}$ nor $\mathscr{L}_{<0}$. The smallest coherent set of indifferent gambles is $I = \{0\}$: it is the smallest linear space that includes 0. This set specifies that the only gamble that the subject finds indifferent to 0 is 0 itself. This means that any gamble is only indifferent to itself, so it stipulates only trivial indifferences, and is therefore called the *trivial* set of indifferences. A coherent set of indifferent gambles I is non-trivial if and only if $I \supset \{0\}$, or, in other words, if and only if $\dim I \geq 1$.

The interaction between indifferent and desirable gambles is subject to rationality criteria as well: they should be compatible with one another.

Definition 20 (Compatibility between indifference and desirability)**.** Given a set of desirable gambles D and a coherent set of indifferent gambles I, we call D *compatible* with I if $D + I \subseteq D$.

The idea behind this is that adding an indifferent gamble to a desirable one should result in a desirable gamble. Since $0 \in I$ by Axiom I_1 so $D \subseteq D + I$, we see that D and I are compatible if and only if $D = D + I$. Note that the trivial set of indifferent gambles $\{0\}$ is compatible with any set of desirable gambles D: indeed $D + \{0\} = D$.

We can combine a desirability assessment $A \subseteq \mathscr{L}$ with a coherent set of indifferent gambles I. We call any coherent set of desirable gambles D a *coherent extension of A compatible with indifference described by I*, or shorter, a *coherent extension of A under I*, when $D \supseteq A$ and $D + I \subseteq D$. It turns out that an assessment $A \subseteq \mathscr{L}$ can be coherently extended under I, precisely when [see [9, Proposition 12]]

$$0 \notin \operatorname{posi}(I + (\mathscr{L}_{>0} \cup A)). \tag{ANP-I}$$

If this is the case, then the smallest coherent extension $D = \bigcap\{D' \in \overline{\mathbf{D}} : D' \supseteq A \text{ and } D' + I \subseteq D'\}$ of A under I is given by [see [9, Theorem 13]] $D = \operatorname{posi}(I + (\mathscr{L}_{>0} \cup A))$.

This result implies that a coherent set of desirable gambles D can be coherently extended under indifference described by I, precisely when $0 \notin D + I$, or, equivalently, when

$$D \cap I = \emptyset. \tag{17}$$

The smallest coherent extension of D under I is given by

$$D+I. \tag{18}$$

4.2. **Defining SSK–Extension.** The idea will be to extend a coherent set of desirable gambles D that is extendible to a probability to an extension $\operatorname{ext}(D)$ called 'SSK–extension' by adding gambles whose negation are precluded from being desirable and that are precluded from being (non-trivially) indifferent to 0:

$$\begin{aligned}\operatorname{ext}(D) = D \cup \{f \in \mathscr{L} : -f &\text{ is precluded from being desirable and}\\ f &\text{ is precluded from being (non-trivially) indifferent to } 0\}.\end{aligned} \tag{19}$$

What do we mean by this?

Definition 21 (Precluded desirability; see [21, Definition 21])**.** Given a set of desirable gambles D, a gamble f is called *precluded from being desirable* if there is no coherent SSK–Archimedean extension of D that contains f.

Lemma 22. *Consider any coherent set of desirable gambles D that is extendible to a probability, and any gamble f. Then f is precluded from being desirable if and only if $-f \in \operatorname{cl}(D)$.*

Proof. We begin our proof by noting that 'f is not precluded from being desirable' is by definition equivalent to '$D \cup \{f\}$ is included in a coherent and SSK–Archimedean set of desirable gambles', which in turn is equivalent to $D \cup \{f\} \subseteq D_p$ for some p in Σ, using Theorem 11. It therefore suffices to show that

$$D \cup \{f\} \subseteq D_p \text{ for some } p \text{ in } \Sigma \;\Leftrightarrow\; -f \notin \operatorname{cl}(D).$$

For necessity, assume that $D \cup \{f\} \subseteq D_p$ for some p in Σ, which then automatically belongs to $\mathbf{R}(D \cup \{f\})$ by Equation (11). Since $\mathbf{R}(D \cup \{f\}) \subseteq \mathbf{R}(D) \subseteq \mathbf{P}(D)$, we have found that $E_p(f) > 0$—and therefore $E_p(-f) < 0$—for some p in $\mathbf{P}(D)$, which by Equation (8) indeed is equivalent to $-f \notin \operatorname{cl}(D)$.

For sufficiency, assume that $-f \notin \operatorname{cl}(D)$, which is, as we just have seen, by Equation (8) equivalent to $E_p(f) > 0$—or in other words $f \in D_p$—for some p in $\mathbf{P}(D)$. Recall from our discussion after Theorem 11 that, because D is extendible to a probability, D's credal set $\mathbf{P}(D)$ equals the credal set $\mathbf{P}(D + \operatorname{cl}(D))$ of its SSK–Archimedean natural extension, so consider any q in $\operatorname{ri}(\mathbf{P}(D))$. Since $D + \operatorname{cl}(D)$ is SSK–Archimedean, by Proposition 7 we have that $D \subseteq D + \operatorname{cl}(D) \subseteq D_q$. If $E_q(f) > 0$ then $f \in D_q$ so $D \cup \{f\} \subseteq D_q$, and the proof is complete, so assume that $E_q(f) \leq 0$. Let $\alpha := \frac{-E_q(f)}{E_p(f) - E_q(f)} \in [0,1)$, and consider any λ in $(\alpha, 1)$. Using Theorem 3 we find that the convex combination $p^* := \lambda p + (1-\lambda)q$ belongs to $\operatorname{ri}(\mathbf{P}(D))$, whence $D \subseteq D_{p^*}$. Note that

$$\begin{aligned} E_{p^*}(f) &= \underbrace{\lambda}_{>\alpha} \underbrace{E_p(f)}_{>0} + \underbrace{(1-\lambda)}_{<1-\alpha} \underbrace{E_q(f)}_{\leq 0} \\ &> \alpha E_p(f) + (1-\alpha) E_q(f) = \frac{-E_q(f)}{E_p(f) - E_q(f)} E_p(f) + \frac{E_p(f)}{E_p(f) - E_q(f)} E_q(f) = 0 \end{aligned}$$

whence $f \in D_{p^*}$. So we infer that indeed $D \cup \{f\} \subseteq D_{p^*}$. □

Definition 23 (Precluded indifference; see [21, Definition 22])**.** Given a set of desirable gambles D, a gamble f is called *precluded from being (non-trivially) indifferent to* 0 if there is no non-trivial coherent set of indifferent gambles I that contains f and results in a coherent extension of D under I that is SSK–Archimedean.

Lemma 24. *Consider any coherent set of desirable gambles D and any gamble f. Then f is precluded from being (non-trivially) indifferent to* 0 *if and only if*

$$(\forall I \in \bar{\mathbf{I}})\big((\dim I \geq 1 \text{ and } f \in I) \Rightarrow (D \cap I \neq \emptyset \text{ or } (\forall p \in \Sigma) D + I \not\subseteq D_p)\big)$$

Proof. We need to show that there is a coherent extension of D under I that is SSK–Archimedean, if and only if $D\cap I=\emptyset$ and $(\exists p\in\Sigma)D+I\subseteq D_p$. By Equations (17) and (18) we have that D is coherently extendible under I precisely when $D\cap I=\emptyset$, and moreover that the smallest such extension is $D+I$. By Theorem 11, this smallest coherent extension $D+I$ is extendible to an SSK–Archimedean set of desirable gambles precisely when $(\exists p\in\Sigma)D+I\subseteq D_p$. □

This means that we can rewrite Equation (19) to a more convenient equivalent variant:

$$\begin{aligned}\operatorname{ext}(D)=D\cup\{&f\in\mathscr{L}:f\in\operatorname{cl}(D)\text{ and}\\&(\forall I\in\bar{\mathbf{I}})\big((\dim I\geq 1\text{ and }f\in I)\Rightarrow(D\cap I\neq\emptyset\text{ or }(\forall p\in\Sigma)D+I\not\subseteq D_p)\big)\}\\=D\cup\{&f\in\operatorname{cl}(D)\setminus D:\\&(\forall I\in\bar{\mathbf{I}})\big((\dim I\geq 1\text{ and }f\in I)\Rightarrow(D\cap I\neq\emptyset\text{ or }(\forall p\in\Sigma)D+I\not\subseteq D_p)\big)\},\end{aligned}\tag{20}$$

so clearly $\operatorname{ext}(D)\subseteq\operatorname{cl}(D)$.

It turns out that there is another, yet more useful, equivalent expression, that uses only coherent sets of indifferent gambles of maximal dimension $n-1$:

Proposition 25. *Consider any coherent set of desirable gambles D that is extendible to a probability. Then its SSK–extension* $\operatorname{ext}(D)$ *is given by*

$$\operatorname{ext}(D)=D\cup\{f\in\operatorname{cl}(D)\setminus D:(\forall p\in\Sigma)(E_p(f)=0\Rightarrow D\not\subseteq D_p)\}.$$

Furthermore, ext *is a* closure operator*: it satisfies*

(i) $D\subseteq\operatorname{ext}(D)$ — ext *is extensive;*

(ii) $D\subseteq D'\Rightarrow\operatorname{ext}(D)\subseteq\operatorname{ext}(D')$ — ext *is monotone;*

(iii) $\operatorname{ext}(\operatorname{ext}(D))=\operatorname{ext}(D)$ — ext *is idempotent,*

for any coherent sets of desirable gambles D and D' that are extendible to a probability.

Proof. We start with the first statement. It suffices to show that

$$\begin{aligned}A_1:=\{&f\in\operatorname{cl}(D)\setminus D:\\&(\forall I\in\bar{\mathbf{I}})\big((\dim I\geq 1\text{ and }f\in I)\Rightarrow(D\cap I\neq\emptyset\text{ or }(\forall p\in\Sigma)D+I\not\subseteq D_p)\big)\}\\=\{&f\in\operatorname{cl}(D)\setminus D:(\forall p\in\Sigma)(E_p(f)=0\Rightarrow D\not\subseteq D_p)\}=:A_2.\end{aligned}$$

To show that $A_1\subseteq A_2$, consider any f in $\operatorname{cl}(D)\setminus D$ such that $f\notin A_2$. We will show that then $f\notin A_1$. $f\notin A_2$ implies that there is some probability mass function p such that $E_p(f)=0$ and $D\subseteq D_p$. Let $I:=\ker E_p=\{h\in\mathscr{L}:E_p(h)=0\}$, the kernel of the linear transformation E_p on $\mathscr{L}$. We will show that then $D+I\subseteq D_p$. To this end, consider any f_1 in D and f_2 in I. Since $D\subseteq D_p$, we have $E_p(f_1)>0$, and $E_p(f_2)=0$ by definition. Therefore $E_p(f_1+f_2)>0$, whence indeed $f_1+f_2\in D_p$. Also, $D\cap I=\emptyset$: indeed, $f_1\in D$ implies $E_p(f_1)>0$ while $f_2\in I$ is equivalent to $E_p(f_2)=0$. So $f\in I$ and I has dimension $n-1\geq 1$, whence indeed $f\notin A_1$.

To show that $A_2\subseteq A_1$, consider any f in $\operatorname{cl}(D)\setminus D$ such that $f\notin A_1$. We will show that then $f\notin A_2$. $f\notin A_1$ implies that $f\in I$, $D\cap I=\emptyset$ and $D+I\subseteq D_p$ for some p in Σ and I in $\bar{\mathbf{I}}$ such that $\dim I\geq 1$. Since $0\in I$, we have that $D\subseteq D+I\subseteq D_p$. We will show that then $I\subseteq\ker E_p$. To see this, assume *ex absurdo* that $I\not\subseteq\ker E_p$, so $E_p(g)\neq 0$ for some g in I. Since I is a linear space, we may assume without loss of generality that $E_p(g)<0$. Consider the constant gamble $h:=-E_p(g)>0$, which belongs to D by its coherence. But this would imply that $g+h\in D+I\subseteq D_p$, while $E_p(g+h)=0$, a contradiction. Therefore indeed $I\subseteq\ker E_p$, whence $f\in\ker E_p$ so $E_p(f)=0$. This implies that indeed $f\notin A_2$.

We now turn to the second statement. For (i), that ext is extensive follows from its definition since $\operatorname{ext}(D)$ is the union of D with another set.

For (ii), to show that ext is monotone, we must show that $\operatorname{ext}(D)\subseteq\operatorname{ext}(D')$. So consider any gamble f in $\operatorname{ext}(D)$, and we will show that f belongs to $\operatorname{ext}(D')$. Since $f\in\operatorname{ext}(D)$, we find that $f \in D$ or $D \nsubseteq D_p$ for every p in Σ such that $E_p(f) = 0$. Since $D \subseteq D'$, infer that $f \in D$ implies $f \in D'$, and that $D \nsubseteq D_p$ implies $D' \nsubseteq D_p$. Therefore indeed $f \in \operatorname{ext}(D')$.

For (iii), to show that ext is idempotent, note that $\operatorname{ext}(D) \subseteq \operatorname{ext}(\operatorname{ext}(D))$ using that ext is extensive and monotone, which we just have established. Assume *ex absurdo* that $\operatorname{ext}(D) \subset \operatorname{ext}(\operatorname{ext}(D))$, then

$$(\exists f \in \underbrace{\operatorname{cl}(\operatorname{ext}(D))}_{=\operatorname{cl}(D)} \setminus \underbrace{\operatorname{ext}(D)}_{\supseteq D})(\forall p \in \Sigma)(E_p(f) = 0 \Rightarrow \operatorname{ext}(D) \nsubseteq D_p).$$

But then $f \in \operatorname{cl}(D) \setminus D$ and $f \notin \operatorname{ext}(D)$, so $E_q(f) = 0$ and $D \subseteq D_q$ for some probability mass function q. Since in particular $\operatorname{ext}(D) \nsubseteq D_q$, we would infer that there is some $g \in \operatorname{ext}(D) \setminus D$ such that $E_q(g) \leq 0$, and therefore $E_q(g) = 0$ because $g \in \operatorname{ext}(D) \subseteq \operatorname{cl}(D)$. But since g belongs to $\operatorname{ext}(D) \setminus D$, this would imply that $E_p(g) = 0 \Rightarrow D \nsubseteq D_p$ for every probability mass function p, so in particular for the probability mass function q we have that $D \nsubseteq D_q$, a contradiction with the earlier established fact that $D \subseteq D_q$. This shows that it is impossible that $\operatorname{ext}(D) \subset \operatorname{ext}(\operatorname{ext}(D))$, whence indeed $\operatorname{ext}(D) = \operatorname{ext}(\operatorname{ext}(D))$. □

The following fact will be useful on multiple occasions.

Lemma 26. *Consider any assessment $A \subseteq \mathscr{L}$ that is extendible to a probability, and any $A \subseteq B \subseteq \operatorname{ext}(\mathscr{E}(A))$. Then A, B and $\operatorname{ext}(\mathscr{E}(A))$ all have the same (one-way) representing set of probabilities. In other words, $\mathbf{R}(A) = \mathbf{R}(B) = \mathbf{R}(\operatorname{ext}(\mathscr{E}(A)))$. As a consequence, $\operatorname{ext}(\mathscr{E}(A))$ is extendible to a probability.*

Proof. Since $A \subseteq B \subseteq \operatorname{ext}(\mathscr{E}(A))$, we have by the definition of $\mathbf{R}$ in Equation (11) that $\mathbf{R}(A) \supseteq \mathbf{R}(B) \supseteq \mathbf{R}(\operatorname{ext}(\mathscr{E}(A)))$, so it suffices to show that $\mathbf{R}(A) \subseteq \mathbf{R}(\operatorname{ext}(\mathscr{E}(A)))$. To this end, consider any p in $\mathbf{R}(A)$, so $A \subseteq D_p$. Assume *ex absurdo* that $\operatorname{ext}(\mathscr{E}(A)) \nsubseteq D_p$, meaning that $E_p(f) \leq 0$ for some f in $\operatorname{ext}(\mathscr{E}(A))$. Recall using Lemma 10 that $\mathscr{E}(A) \subseteq D_p$, so we would infer that $f \notin \mathscr{E}(A)$. Use Proposition 25 to infer that this would imply that f belongs to

$$\begin{aligned}\{g \in \operatorname{cl}(\mathscr{E}(A)) \setminus \mathscr{E}(A) : (\forall q \in \Sigma)(E_q(g) = 0 \Rightarrow \mathscr{E}(A) \nsubseteq D_q)\}\\ = \{g \in \operatorname{cl}(\mathscr{E}(A)) \setminus \mathscr{E}(A) : (\forall q \in \Sigma)(\mathscr{E}(A) \subseteq D_q \Rightarrow E_q(g) \neq 0)\}\end{aligned}$$

so in particular $E_p(f) \neq 0$. Since f belongs to $\operatorname{cl}(\mathscr{E}(A))$, we have that $E_p(f) \geq 0$, so this would imply that $E_p(f) > 0$, a contradiction.

To show that $\operatorname{ext}(\mathscr{E}(A))$ is extendible to a probability, note that $\mathbf{R}(A) \neq \emptyset$ because A is extendible to a probability, and therefore also $\mathbf{R}(\operatorname{ext}(\mathscr{E}(A))) \neq \emptyset$, which means that $\operatorname{ext}(\mathscr{E}(A))$ is indeed extendible to a probability. □

Lemma 26 implies that $D \subseteq D_p \Leftrightarrow \operatorname{ext}(D) \subseteq D_p$, for any coherent set of desirable gambles D that is extendible to a probability, and any probability mass function p. In other words, in the light of Equation (12), for any coherent set of desirable gambles D that is extendible to a probability, we have $D \subseteq \operatorname{ext}(D) \subseteq \bigcap\{D_p : p \in \mathbf{R}(D)\}$, so $\mathbf{R}(D)$ is a one-way representation not only for D, but also for $\operatorname{ext}(D)$.

4.3. **SSK–extension, even convexity, and exposed rays.** In this section we will lay bare a connection between the SSK–extension, even convexity, and exposed rays, which will allow us to establish that $\mathbf{R}(D)$ is not only a one-way representation of $\operatorname{ext}(D)$, but in fact a two-way representation. The main result in this section is the equivalence in Theorem 31 between the property for a coherent set of desirable gambles that is extendible to a probability of being two-way represented by a set of probability mass functions, and five other conditions.

Definition 27 (Exposed ray; see [18, Section 18])**.** Consider any coherent set of desirable gambles D. We call any set $\operatorname{posi}\{f\}$ with f in $\operatorname{cl}(D)$ a *ray of D*. A *supporting half-space to D* is an affine closed half-space that contains D and has a point of $\operatorname{cl}(D)$ on its boundary. A *supporting hyperplane to D* is the boundary of a supporting half-space to D, thus it includes a point of $\operatorname{cl}(D)$. A ray $\operatorname{posi}\{f\}$ of D is an *exposed ray of D* if there is a supporting hyperplane H to D such that $f \in H$ and $D \cap H = \emptyset$.

Lemma 28. *Consider any coherent set of desirable gambles D. A ray $\operatorname{posi}\{f\}$ of D is exposed if and only if there is a probability mass function p such that $E_p(f) = 0$ and $D \subseteq D_p$.*

Proof. For necessity, since $\operatorname{posi}\{f\}$ is an exposed ray of D, we have that $f \in H$ for some supporting hyperplane H to D, for which moreover $D \cap H = \emptyset$. Note that $0 \in H$ since 0 belongs to every supporting half-space to D. Thus H is a hyperplane that contains 0, so it is a linear space. Consider the set of desirable gambles $D' := H + \mathscr{L}_{>0}$, which is coherent: It satisfies Axiom D_1 since otherwise, were $0 \in D'$, then $h < 0$ for some h in H, and since H is a linear space also $-h \in H$. But $-h > 0$ also belongs to D by the coherence of D, a contradiction with $D \cap H = \emptyset$. It satisfies Axiom D_2 since $0 \in H$ so it includes $\mathscr{L}_{>0}$. Moreover, D' is a convex cone because it is the Minkowski addition of two convex cones, so D' satisfies Axioms D_3 and D_4. Then $D' = D_p$ for some p in Σ, because D' is coherent and an open half-space, whence $H = \ker E_p$. Then indeed $D \cap \ker E_p = \emptyset$—or equivalently $D \subseteq D_p$—and $f \in \ker E_p$—or equivalently $E_p(f) = 0$.

For sufficiency, note that $D \subseteq D_p$ is equivalent to $D \cap \ker E_p = \emptyset$. This means that $\ker E_p$ is a supporting hyperplane to D since $E_p(f) = 0$, so $f \in \ker E_p$. This shows that $\operatorname{posi}\{f\}$ is indeed an exposed ray of D. □

Lemma 28 gives an easy categorisation of the rays $\operatorname{posi}\{f\}$ of a coherent set of desirable gambles D: for any f in $\operatorname{cl}(D)$, the ray $\operatorname{posi}\{f\}$

- is exposed if $(\exists p \in \Sigma)(E_p(f) = 0 \text{ and } D \subseteq D_p)$;
- is non-exposed if $(\forall p \in \Sigma)(E_p(f) = 0 \Rightarrow D \nsubseteq D_p)$, or, $(\forall p \in \Sigma)(D \subseteq D_p \Rightarrow E_p(f) \neq 0)$.

The Bouligand tangent cone $T_D(f)$ of an exposed ray $\operatorname{posi}\{f\}$ of D has the following useful property:

Lemma 29. *Consider any coherent set of desirable gambles D, with exposed ray $\operatorname{posi}\{f\}$. Then $\ell(T_D(f)) \subseteq \ker E_p$ for every p in Σ such that $E_p(f) = 0$ and $D \subseteq D_p$.*

Proof. Because $D \subseteq D_p$, we have that

$$T_D(f) = \operatorname{cl}\Bigg(\bigcup_{\lambda \in \mathbb{R}_{>0}} \lambda(D - \{f\})\Bigg) \subseteq \operatorname{cl}\Bigg(\bigcup_{\lambda \in \mathbb{R}_{>0}} \lambda(D_p - \{f\})\Bigg) = T_{D_p}(f).$$

Since $E_p(f) = 0$, we know that f belongs to the boundary of the open half-space D_p, and hence $D_p - \{f\} = D_p$, so $\lambda(D_p - \{f\}) = D_p$. Therefore we have, for any gamble g,

$$g \in T_{D_p}(f) \Leftrightarrow g \in \operatorname{cl}\Bigg(\bigcup_{\lambda \in \mathbb{R}_{>0}} D_p\Bigg) \Leftrightarrow g \in \operatorname{cl}(D_p) \Leftrightarrow E_p(g) \geq 0,$$

whence $T_{D_p}(f) = \operatorname{cl}(D_p)$. Note that therefore $-T_{D_p}(f) = -\operatorname{cl}(D_p)$, and hence indeed $\ell(T_D(f)) \subseteq \operatorname{cl}(D_p) \cap -\operatorname{cl}(D_p) = \ker E_p$. □

The observation in Lemma 29 allows us to show that the SSK-extension produces SSK–Archimedean and evenly convex sets of desirable gambles, even if its input is not SSK–Archimedean:

Proposition 30. *Consider any coherent set of desirable gambles D that is extendible to a probability. Then $\operatorname{ext}(D)$ is a coherent, SSK–Archimedean and evenly convex set of desirable gambles.*

Proof. We first show that $\operatorname{ext}(D)$ is coherent. For Axiom D_1, we need to show that $0 \notin \operatorname{ext}(D)$. Since $0 \in \operatorname{cl}(D) \setminus D$ by the coherence of D, it suffices by Proposition 25 to show that $D \subseteq D_p$ for some p in Σ. Using Proposition 7, this is indeed guaranteed by the SSK–Archimedeanity of D.

Since $D \subseteq \mathrm{ext}(D)$, Axiom D_2 is satisfied by the coherence of D.

For Axiom D_3, consider any f in $\mathscr{L}$, λ in $\mathbb{R}_{>0}$ and any p in Σ. Note that $f \in D \Leftrightarrow \lambda f \in D$, $f \in \mathrm{cl}(D) \Leftrightarrow \lambda f \in \mathrm{cl}(D)$, and $E_p(f) = 0 \Leftrightarrow E_p(\lambda f) = 0$, which establishes that $\mathrm{ext}(D)$ satisfies Axiom D_3.

For Axiom D_4, consider any f and g in $\mathrm{ext}(D)$. If $f+g \in D$ then the proof is done, so assume that $f+g \notin D$, whence $f \notin D$ or $g \notin D$—say, $f \notin D$ without loss of generality. Since $f, g \in \mathrm{ext}(D) \subseteq \mathrm{cl}(D)$, this also implies that $f+g \in \mathrm{cl}(D) \setminus D$ because $\mathrm{cl}(D)$ is a convex cone. Assume *ex absurdo* that $f+g \notin \mathrm{ext}(D)$. Using Proposition 25, this would imply that $E_q(f+g) = 0$ and $D \subseteq D_q$ for some q in Σ. But f and g belong to $\mathrm{cl}(D)$, which implies by Equation (8) that $E_q(f) \geq 0$ and $E_q(g) \geq 0$, whence $E_q(f) = E_q(g) = 0$. We have already established that f belongs to $\mathrm{ext}(D) \setminus D$, so $E_p(f) = 0 \Rightarrow D \nsubseteq D_p$ for all p in Σ, a contradiction with the earlier established fact that $E_q(f) = 0$ and $D \subseteq D_q$. Therefore indeed $f+g \in \mathrm{ext}(D)$.

To show that $\mathrm{ext}(D)$ is SSK–Archimedean and evenly convex, it suffices to show that $\mathrm{ext}(D)$ is evenly convex. Indeed, by Proposition 13 we know that SSK–Archimedeanity is necessary for even convexity.

So we will show that $\mathrm{ext}(D)$ is evenly convex. By Corollary 17 it suffices to show that $\ell(T_{\mathrm{ext}(D)}(f)) \cap \mathrm{ext}(D) = \emptyset$ for every f in $\mathrm{cl}(\mathrm{ext}(D)) \setminus \mathrm{ext}(D) = \mathrm{cl}(D) \setminus \mathrm{ext}(D)$. So consider any f in $\mathrm{cl}(D) \setminus \mathrm{ext}(D)$. By Proposition 25, this implies that $E_p(f) = 0$ and $D \subseteq D_p$ for some p in Σ, and therefore by Lemma 26, $\mathrm{ext}(D) \subseteq D_p$. We infer that $\mathrm{posi}\{f\}$ is an exposed ray of $\mathrm{ext}(D)$, so we can apply Lemma 29 above to infer that then $\ell(T_{\mathrm{ext}(D)}(f)) \subseteq \ker E_p$, whence $\ell(T_{\mathrm{ext}(D)}(f)) \cap D_p = \emptyset$. Since $\mathrm{ext}(D) \subseteq D_p$, this implies that indeed $\ell(T_{\mathrm{ext}(D)}(f)) \cap \mathrm{ext}(D) = \emptyset$. □

Now we are ready to establish one of the main results or our paper, namely the establishment of equivalent conditions for a coherent set of desirable gambles to be representable by a set of probability mass functions.

Theorem 31. *Consider any coherent set of desirable gambles D that is extendible to a probability. Then the following statements are equivalent:*

(i) D is closed under SSK–extension: $D = \mathrm{ext}(D)$;
(ii) D contains all its non-exposed rays;
(iii) D is evenly convex: $D = \mathrm{eco}(D)$;
(iv) D satisfies the following requirement of non-exposedness:

$$(\forall f \notin D)(\exists p \in \Sigma)(f \notin D_p \textit{ and } D \subseteq D_p). \qquad \text{(non-exposedness)}$$

(v) There is a non-empty (not necessarily convex) set of probability mass functions $\mathbf{P} \subseteq \Sigma$ that two-way represents D: in other words, $D = \bigcap\{D_p : p \in \mathbf{P}\}$;
(vi) D is two-way represented by $\mathbf{R}(D)$: $D = \bigcap\{D_p : p \in \mathbf{R}(D)\}$.

Proof. Proposition 13 already implies that (iii)⇔(vi), so it suffices to show the following two chains of implications: (vi)⇒(iv)⇒(ii)⇒(i)⇒(iii) and (vi)⇒(v)⇒(iii).

To show that (vi)⇒(iv), consider any $f \notin D$. Then $f \notin D_p$ for some p in $\mathbf{R}(D)$, for which indeed $D \subseteq D_p$ by definition.

To show that (iv)⇒(ii), note that requirement (non-exposedness) implies in particular that

$$(\forall f \in \mathrm{cl}(D) \setminus D)(\exists p \in \Sigma)(f \notin D_p \text{ and } D \subseteq D_p),$$

and hence, since $\mathrm{cl}(D) \subseteq \mathrm{cl}(D_p)$, we have $E_p(f) \geq 0$ for any f in $\mathrm{cl}(D)$. This implies that

$$(\forall f \in \mathrm{cl}(D) \setminus D)(\exists p \in \Sigma)(E_p(f) = 0 \text{ and } D \subseteq D_p),$$

meaning that all D's rays that do not belong to D are exposed, using Lemma 28. In other words, D indeed contains all its non-exposed rays.

To show that (ii)⇒(i), since D contains all its non-exposed rays, by Lemma 28 this implies that $D = \{f \in \mathrm{cl}(D) : (\forall p \in \Sigma)(E_p(f) = 0 \Rightarrow D \not\subseteq D_p)\}$. By Proposition 25 therefore indeed $D = \mathrm{ext}(D)$.

To show that (i)⇒(iii), note that $\mathrm{ext}(D)$ is evenly convex by Proposition 30. Therefore indeed, since $D = \mathrm{ext}(D)$, so is D.

The second chain of implications is immediate: to see that (vi)⇒(v) use $\mathbf{P} := \mathbf{R}$ as set of probability mass functions that two-way represents D, and to see that (v)⇒(iii) note that $\bigcap\{D_p : p \in \mathbf{P}\}$ is an intersection of affine open semispaces, so that D is evenly convex by Definition 12. □

Theorem 31 extends Cozman's [4, Theorem 16], which gives only necessary conditions for a coherent and SSK–Archimedean set of desirable gambles D to be evenly convex, or in other words, be two-way represented by $\mathbf{R}(D)$. Cozman's [4, Theorem 9] furthermore implies that if any (and hence all) of our six equivalent conditions hold, then $\mathbf{R}(D)$ is the unique largest representing set of D.

We want to highlight that for any coherent set of desirable gambles D, the requirement (non-exposedness) implies extendibility to a probability: indeed, if D is coherent, then it does not contain 0, so that (non-exposedness) implies in particular that $D \subseteq D_p$ for some p in Σ. We therefore immediately obtain the following characterisation:

Corollary 32. *A coherent set of desirable gambles D is two-way represented by a set of probability mass functions if and only if D satisfies* (non-exposedness).

Our Corollary 32 has the same content as De Cooman's [8, Corollary 22] specialised to a finite-dimensional context, but the conclusions are obtained in a very different way. While De Cooman's [8, Corollary 22] hold in more general Banach spaces of arbitrary dimension, it uses the Hahn–Banach theorem version for superlinear bounded real functionals. Our result is obtained using more basic concepts only, which might be valuable to some readers.

4.4. **Even convexity and natural extension.** Let us show how our findings above can help us extend an assessment $A \subseteq \mathscr{L}$ to the smallest coherent and evenly convex set of desirable gambles that includes A. We have seen in Theorem 11 that the SSK–Archimedean natural extension $\mathscr{E}_{\mathrm{Arch}}(A)$ is the smallest coherent and SSK–Archimedean set of desirable gambles that includes A, but it is not necessarily evenly convex. It turns out that the SSK–extension plays a role similar to that of $\mathscr{E}_{\mathrm{Arch}}(A)$:

Theorem 33. *Consider any assessment $A \subseteq \mathscr{L}$. Then the following statements are equivalent:*

(i) A is extendible to a probability;
(ii) A is included in a coherent and evenly convex set of desirable gambles;
(iii) $\mathrm{ext}(\mathscr{E}(A)) \neq \mathscr{L}$;
(iv) The set of desirable gambles $\mathrm{ext}(\mathscr{E}(A))$ is coherent and evenly convex;
(v) $\mathrm{ext}(\mathscr{E}(A))$ is the smallest coherent and evenly convex set of desirable gambles that includes A.

When any, and hence all, of these equivalent statements hold, then $\mathrm{ext}(\mathscr{E}(A)) = \bigcap\{D_p : p \in \mathbf{R}(A)\}$. *As a consequence,* $\mathrm{ext}(\mathscr{E}(A))$ *coincides with A's* evenly convex natural extension:[11]

$$\mathrm{ext}(\mathscr{E}(A)) = \bigcap\{D \in \overline{\mathbf{D}}_{\mathrm{e.c.}} : A \subseteq D\},$$

for any $A \subseteq \mathscr{L}$.

Proof. We will show the following implications, guaranteeing the equivalence of the five statements (i)–(v):

[11]As usual, we let $\bigcap\emptyset = \mathscr{L}$.

$$
\begin{tikzcd}
\text{(ii)} \arrow[r, Leftrightarrow] & \text{(i)} \arrow[dd, Rightarrow] & \\
& & \text{(iii)} \arrow[ul, Rightarrow] \\
\text{(v)} \arrow[r, Leftrightarrow] & \text{(iv)} \arrow[ur, Rightarrow] &
\end{tikzcd}
$$

To show that that (i)⇔(ii), note using Theorem 11 that (i) is equivalent to the statement '$A \subseteq D$ for some D in $\overline{\mathbf{D}}_{\mathrm{Arch}}$'. We know [see Proposition 7] that every coherent and SSK–Archimedean set of desirable gambles is included in an element of $\widehat{\mathbf{D}}_{\mathrm{Arch}}$, the maximal coherent and SSK–Archimedean sets of desirable gambles, so the statement is in turn equivalent to '$A \subseteq \widehat{D}$ for some $\widehat{D}$ in $\widehat{\mathbf{D}}_{\mathrm{Arch}}$'. But Proposition 14 guarantees that $\widehat{\mathbf{D}}_{\mathrm{Arch}}$ is identical to $\widehat{\mathbf{D}}_{\mathrm{e.c.}}$, the maximal coherent evenly convex sets of desirable gambles. Therefore, we have the equivalence with '$A \subseteq \widehat{D}$ for some $\widehat{D}$ in $\widehat{\mathbf{D}}_{\mathrm{e.c.}}$', which in turn is equivalent to (ii) once we realise using Proposition 14 that every coherent evenly convex set of desirable gambles is dominated by an element of $\widehat{\mathbf{D}}_{\mathrm{e.c.}}$.

To show that (i)⇒(iv), note using Lemma 10 that A avoids non-positivity—so that $\mathscr{E}(A)$ is coherent—and that $\mathscr{E}(A)$ is extendible to a probability. Proposition 30 tells us that then indeed $\mathrm{ext}(\mathscr{E}(A))$ is coherent and evenly convex.

To show that (iv)⇒(iii), simply use the fact that $\mathscr{L}$ is not a coherent set of desirable gambles: it violates Axiom D_1.

To show that (iii)⇒(i), we will show the contrapositive statement that $\mathrm{ext}(\mathscr{E}(A))$ equals $\mathscr{L}$ when A is not extendible to a probability. Recall using Theorem 11 that A is not extendible to a probability implies that A—and hence also $\mathscr{E}(A) \supseteq A$—is not included in a coherent and SSK–Archimedean set of desirable gambles. This means that there is no coherent SSK–Archimedean extension of $\mathscr{E}(A)$, whence by Definition 21, any gamble is precluded from being desirable, and by Definition 23 that any gamble is precluded from being (non-trivially) indifferent to 0. Therefore, by Equation (19), $\mathscr{E}(A)$'s SSK–extension $\mathrm{ext}(\mathscr{E}(A))$ is equal to $\mathscr{L}$.

Finally, to show that (iv)⇔(v), note that (v) is a strengthening of (iv). It therefore suffices to show that (iv)⇒(v). To this end, assume that $\mathrm{ext}(\mathscr{E}(A))$ is an evenly convex coherent set of desirable gambles, and we need to show that it is equal to the smallest evenly convex coherent set of desirable gambles D that includes A. To this end, since $A \subseteq D \subseteq \mathrm{ext}(\mathscr{E}(A))$, use Lemma 26 to infer that $\mathbf{R}(D) = \mathbf{R}(\mathrm{ext}(\mathscr{E}(A)))$. But since both D and $\mathrm{ext}(\mathscr{E}(A))$ are coherent and evenly convex, we infer using Proposition 13 that indeed $D = \bigcap\{D_p : p \in \mathbf{R}(D)\} = \bigcap\{D_p : p \in \mathbf{R}(\mathrm{ext}(\mathscr{E}(A)))\} = \mathrm{ext}(\mathscr{E}(A))$.

For the second statement, to show that then $\mathrm{ext}(\mathscr{E}(A)) = \bigcap\{D_p : p \in \mathbf{R}(A)\}$, it suffices to note, using Lemma 26 again, that $\mathbf{R}(A) = \mathbf{R}(\mathrm{ext}(\mathscr{E}(A)))$.

To finish the proof we will show that $\mathrm{ext}(\mathscr{E}(A))$ is equal to A's evenly convex natural extension $\bigcap\{D \in \overline{\mathbf{D}}_{\mathrm{e.c.}} : A \subseteq D\}$. To this end, note that if A is not included in a coherent and evenly convex set of desirable gambles then $\bigcap\{D \in \overline{\mathbf{D}}_{\mathrm{e.c.}} : A \subseteq D\} = \bigcap\emptyset = \mathscr{L}$. But using the equivalence (ii)⇔(iii) we also have that $\mathrm{ext}(\mathscr{E}(A)) = \mathscr{L}$, so in this case indeed $\bigcap\{D \in \overline{\mathbf{D}}_{\mathrm{e.c.}} : A \subseteq D\} = \mathrm{ext}(\mathscr{E}(A))$.

If, on the other hand, A is included in a coherent and evenly convex set of desirable gambles, then $\bigcap\{D \in \overline{\mathbf{D}}_{\mathrm{e.c.}} : A \subseteq D\}$ is the smallest coherent and evenly convex set of desirable gambles that includes A, which indeed coincides with $\mathrm{ext}(\mathscr{E}(A))$ using the equivalence (ii)⇔(v). □

De Cooman [8, Theorem 21] obtained a similar conclusion, in a more general context of Banach spaces of arbitrary dimension, but without reference to the SSK–extension.

Our Theorem 33 above implies that $\mathrm{ext}(D)$ is the smallest evenly convex and coherent extension of any coherent set of desirable gambles D that is extendible to a probability. In other words, if we start with a given coherent set of desirable gambles D that is extendible to a probability, and impose additionally the requirement (non-exposedness)—or equivalently, as we have seen in Theorem 31, even convexity—on it *but nothing else*, then we end up with $\operatorname{ext}(D)$.

Note that, quite interestingly, the SSK–extension ext spells out the condition under which a set of desirable gambles D can be coherently extended to an evenly convex one: as we have seen in Theorem 33, this can be done precisely when D is extendible to a probability. This is expressed by the fact that ext coincides with the evenly convex natural extension on the entire domain of subsets of gambles.

One might wonder what the SSK–extension $\operatorname{ext}(\mathscr{E}_{\mathrm{Arch}}(D))$—which, as we have seen, is the smallest evenly convex extension—of the SSK–Archimedean natural extension $\mathscr{E}_{\mathrm{Arch}}(D)$ of a coherent set of desirable gambles D that is extendible to a probability, may be. Since we have seen that SSK–Archimedeanity is necessary for even convexity, this suggests that the SSK–Archimedean natural extension is only an intermediate step to the SSK–extension. Is this the case? More specifically, is $\operatorname{ext}(\mathscr{E}_{\mathrm{Arch}}(D)) = \operatorname{ext}(D)$?

Since $\operatorname{ext}(D)$ is a coherent and SSK–Archimedean extension of D by Proposition 30, and $\mathscr{E}_{\mathrm{Arch}}(D)$ is the smallest such extension of D by Theorem 11, this means that $\mathscr{E}_{\mathrm{Arch}}(D) \subseteq \operatorname{ext}(D)$. But since ext is a closure operator [see Proposition 25], we find that $\operatorname{ext}(\mathscr{E}_{\mathrm{Arch}}(D)) \subseteq \operatorname{ext}(\operatorname{ext}(D)) = \operatorname{ext}(D) \subseteq \operatorname{ext}(\mathscr{E}_{\mathrm{Arch}}(D))$, where the first inclusion holds because ext is monotone, the equality because ext is idempotent, and the second inclusion because ext is extensive. This means that indeed $\operatorname{ext}(D) = \operatorname{ext}(\mathscr{E}_{\mathrm{Arch}}(D))$, so that SSK–Archimedeanity is an intermediate step towards even convexity which can be omitted, but nevertheless is crucial to define the SSK–extension. This idea, and a similar one for not necessarily coherent assessments $A \subseteq \mathscr{L}$ that are extendible to a probability, is shown in the commuting diagram of Figure 13, and illustrated in Example 9 below.

$$
\begin{tikzcd}
\begin{array}{l} D\text{: coherent} \\ \quad\text{extendible to a probability} \end{array} \arrow[rr, "\mathscr{E}_{\mathrm{Arch}}"] \arrow[dr, "\operatorname{ext}"] & & \begin{array}{l} \mathscr{E}_{\mathrm{Arch}}(D)\text{: coherent} \\ \quad\text{SSK–Archimedean} \end{array} \arrow[dl, "\operatorname{ext}"] \\
& \begin{array}{l} \operatorname{ext}(D) = \operatorname{ext}(\mathscr{E}_{\mathrm{Arch}}(D))\text{: coherent} \\ \quad\text{evenly convex} \end{array} &
\end{tikzcd}
$$

FIGURE 13a

$$
\begin{tikzcd}
A\text{: extendible to a probability} \arrow[rr, "\mathscr{E}_{\mathrm{Arch}}"] \arrow[dr, "\operatorname{ext}\circ\mathscr{E}"] & & \begin{array}{l} \mathscr{E}_{\mathrm{Arch}}(A)\text{: coherent} \\ \quad\text{SSK–Archimedean} \end{array} \arrow[dl, "\operatorname{ext}"] \\
& \begin{array}{l} \operatorname{ext}(\mathscr{E}(A)) = \operatorname{ext}(\mathscr{E}_{\mathrm{Arch}}(A))\text{: coherent} \\ \quad\text{evenly convex} \end{array} &
\end{tikzcd}
$$

FIGURE 13b

FIGURE 13. Commuting diagrams involving the natural extension $\mathscr{E}$, the SSK–Archimedean natural extension $\mathscr{E}_{\mathrm{Arch}}$ and the SSK–extension ext. FIGURE 13a. starting with a coherent set of desirable gambles that is extendible to a probability. FIGURE 13b. starting with an arbitrary set of desirable gambles that is extendible to a probability.

Example 9. Let us revisit Cozman's [4, Example 17] example, which we used in our Example 7, to gain more intuition about the difference between the SSK–Archimedean natural extension $\mathscr{E}_{\text{Arch}}$ and the SSK–extension ext. We consider a ternary possibility space and two different coherent sets of desirable gambles D_1 and D_2 that are extendible to a probability, and will calculate their SSK–extensions in two different ways: once using the SSK–Archimedean natural extension, and once directly.

We show in Figure 14 the two intersections of the two sets of desirable gambles with a plane. This figure is meant to be understood in the same way as Figure 10 in Example 7.

FIGURE 14. The two sets of desirable gambles D_1 and D_2

D_1 and D_2 differ only by the ray $\text{posi}\{f\}$: $D_2 = D_1 \cup \text{posi}\{f\}$. This ray $\text{posi}\{f\}$ is an element of the linear space $\ell(T_{D_2}(f))$, indicated in Figure 15.

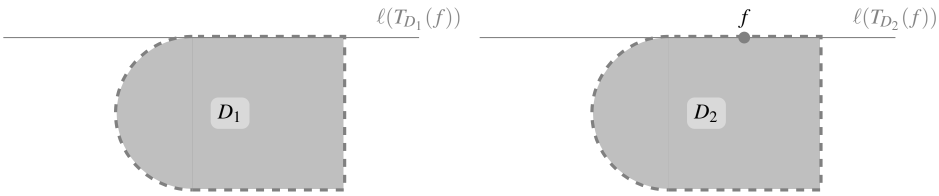


FIGURE 15. The largest linear spaces $\ell(T_{D_1}(f))$ and $\ell(T_{D_2}(f))$ included in the Bouligand tangent cones of D_1 and D_2

We see that $\ell(T_{D_1}(f))$ has nothing in common with D_1: $\ell(T_{D_1}(f)) \cap D_1 = \emptyset$, so the entire linear part of D_1's boundary $\ell(T_{D_1}(f)) \cap \text{cl}(D_1)$ does not belong to D_1. However, $\ell(T_{D_2}(f))$ does have something in common with D_2, namely the ray $\text{posi}\{f\}$.

We are first looking for the SSK–Archimedean natural extension of each of these two sets of desirable gambles. Note that D_1 is open and therefore SSK–Archimedean, as we have observed in Example 6, so that $\mathscr{E}_{\text{Arch}}(D_1) = D_1$. However, D_2 is not SSK–Archimedean, so we will have $\mathscr{E}_{\text{Arch}}(D_2) \supset D_2$. To see this, use Proposition 18, which establishes that any SSK–Archimedean set of desirable gambles D either has nothing in common with the linear space $\ell(T_D(f))$—as is the case for D_1—or contains the entire relative interior $\text{ri}(\ell(T_D(f)) \cap \text{cl}(D))$ of the linear part of the boundary it has something in common with. This implies that we should extend D_2 with at least $\text{ri}(\ell(T_{D_2}(f)) \cap \text{cl}(D_2))$. The same argument used in Example 7, which uses requirement (Arch) directly, shows that D_2's SSK–Archimedean natural extension $\mathscr{E}_{\text{Arch}}(D_2) = D_2 \cup \text{ri}(\ell(T_{D_2}(f)) \cap \text{cl}(D_2))$ is indeed precisely this extension. Figure 16 depicts the SSK–Archimedean natural extensions of D_1 and D_2 graphically.

Note that D_1 is open, and therefore evenly convex. Using Theorem 33 we infer that $\text{ext}(D_1) = \text{ext}(\mathscr{E}_{\text{Arch}}(D_1)) = D_1$. D_2 nor $\mathscr{E}_{\text{Arch}}(D_2)$, however, are evenly convex. To see that D_2 is not evenly convex, use Proposition 13 and the fact that D_2 is not SSK–Archimedean, established above. To see that also $\mathscr{E}_{\text{Arch}}(D_2)$ is not evenly convex, note that it does not contain the gamble g—or the ray $\text{posi}\{g\}$—indicated in Figure 16, which is

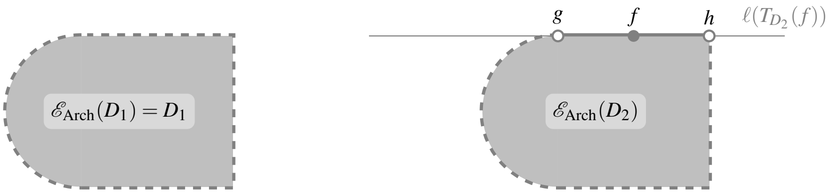


FIGURE 16. The SSK–Archimedean natural extensions of D_1 and D_2

an "endpoint" of the linear part $\ell(T_{D_2}(f)) \cap \text{cl}(D_2)$ of D_2's boundary, and therefore, as we have seen in Example 7, not required by SSK–Archimedeanity to belong to D_2. But this "endpoint" g is non-exposed, as we will see, in contradistinction to the other "endpoint" h. To see that g is non-exposed, consider any p_k in Σ such that $E_{p_k}(f) = 0$. Then $\ker E_{p_k}$ intersects $\mathscr{E}_{\text{Arch}}(D_2)$, as is depicted graphically in Figure 17, so that $\text{posi}\{g\}$ is a non-exposed ray by Lemma 28.

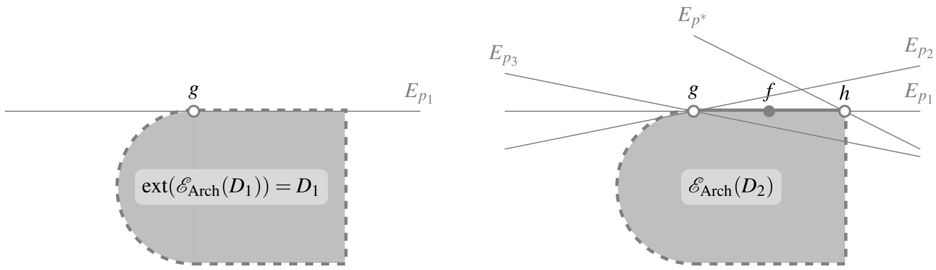


FIGURE 17. g is exposed for D_1 but not for $\mathscr{E}_{\text{Arch}}(D_2)$, and h is exposed for $\mathscr{E}_{\text{Arch}}(D_2)$

Note that $\text{posi}\{g\}$ is an exposed ray of D_1, since $\ker E_{p_1}$ has nothing in common with D_1. We see that $\text{posi}\{g\}$ is $\mathscr{E}_{\text{Arch}}(D_2)$'s only non-exposed ray in $\ell(T_{D_2}(f)) \cap \text{cl}(D_2)$ that does not belong to $\mathscr{E}_{\text{Arch}}(D_2)$: the other ray $\text{posi}\{h\}$ that does not belong to $\mathscr{E}_{\text{Arch}}(D_2)$ is exposed since $E_{p^*} = 0$ and $\ker E_{p^*}$ does not intersect $\mathscr{E}_{\text{Arch}}(D_2)$. Therefore by Theorem 31 we have that $\text{posi}\{g\} \subseteq \text{ext}(\mathscr{E}_{\text{Arch}}(D_2))$. In fact, using Proposition 25 we see that $\text{ext}(\mathscr{E}_{\text{Arch}}(D_2)) = \mathscr{E}_{\text{Arch}}(D_2) \cup \text{posi}\{g\}$, which is the smallest coherent and evenly convex set of desirable gambles that includes $\mathscr{E}_{\text{Arch}}(D_2)$, as depicted in Figure 18.

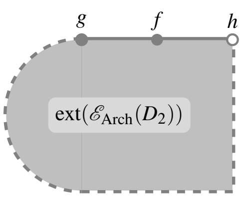


FIGURE 18. The SSK–extension of $\mathscr{E}_{\text{Arch}}(D_2)$

Note that we would obtain the same smallest coherent and evenly convex set of desirable gambles if we were to skip the SSK–Archimedean extension, and directly look for D_2's SSK–extension $\text{ext}(D_2)$, as predicted by Theorem 33. To see this, note that the ray $\text{posi}\{g\}$ is a non-exposed ray of D_2: for every $p_k \neq p_1$ the intersection $\ker E_{p_k} \cap D_2$ is the same non-empty set, and for p_1 we find that $\text{posi}\{f\}$ belongs to $\ker E_{p_1}$ as indicated in Figure 19.

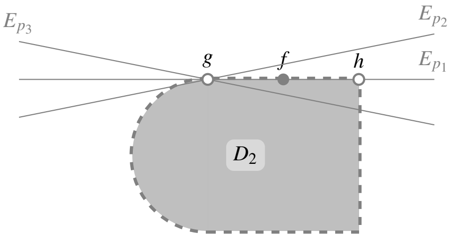


FIGURE 19. g is non exposed for D_2

In fact, any other gamble on the line between g and h—in other words, any gamble in $\mathrm{ri}(\ell(T_D(f)) \cap \mathrm{cl}(D))$—is non-exposed, for very similar reasons. Using Theorem 31, this means that $\mathrm{ri}(\ell(T_{D_2}(f)) \cap \mathrm{cl}(D_2)) \subseteq \mathrm{ext}(D_2)$. Proposition 25 tells us, here again, that D_2's SSK–extension is precisely $\mathrm{ext}(D_2) = D_2 \cup \mathrm{posi}\{g\} \cup \mathrm{ri}(\ell(T_{D_2}(f)) \cap \mathrm{cl}(D_2))$, so it is equal to $\mathrm{ext}(\mathscr{E}_{\mathrm{Arch}}(D_2))$, which is depicted in Figure 18. $\Diamond$

5. CONCLUSIONS

We have shown how SSK–Archimedeanity and SSK–extension can be used to obtain a two-way representation of sets of desirable gambles in terms of sets of probabilities, by mimicking the arguments of Seidenfeld et al. [21] in terms of preference relations on horse lotteries. Quite interestingly, but not entirely surprising, the combination of SSK–extension with SSK–Archimedeanity leads to evenly convex sets of desirable gambles. But this relation is deeper than could be thought based on the works of Seidenfeld et al. [21] and Cozman [4] alone: we show that the SSK–extension is the *smallest* evenly convex extension of a coherent set of desirable gambles that is extendible to a probability. Furthermore, we extended Cozman's [4, Theorem 16] and gave equivalent conditions for a coherent set of desirable gambles that is extendible to a probability to be represented by a probability. One of these equivalent conditions is given in Theorem 31(iv), which is a condition on coherent sets of desirable gambles that turns out to be equivalent to representability by a set of probability mass functions.

We remind the reader that we use the framework of a finite state space Ω. One challenge in generalising to an infinite state space is coherence for finitely-additive probabilities. For example, if $\Omega = \{\omega_k : k \in \mathbb{N}\}$ and f the bounded gamble defined as $f(\omega_k) = -\frac{1}{k}$ for each k in $\mathbb{N}$, then f is point-wise negative. But if p is a merely finitely-additive probability with $p(\omega_k) = 0$ for each k in $\mathbb{N}$, then $E_p(f) = 0$, and f is almost-desirable. This conflicts with the fact that $-f$ is point-wise positive and hence required to be desirable by clause D_2 of Definition 1. We leave to future work extending these results to the infinite state space case with finitely additive probabilities.

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