Complementary Sentential Logics

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We describe a simple axiomatic system by means of which exactly those sentences can be derived that are rated *non-tautologous* in classical sentential logic. Since the set of all *tautologies* is also specifiable by means of syntactic systems, the resulting picture would appear to give a fairly good account of that logic, let alone semantic considerations. The picture can then be completed by developing related systems adequate to specify the set of all *contradictions*, the set of all *non-contradictions*, the set of all *contingencies* and the set of all *non-contingencies* respectively: systems of this kind, which provide additional examples of paraconsistent calculi with a classical background, are also presented.

1. Preliminaries

We consider a sentential language L with '~' (negation) and ' ' (disjunction) as primitive connectives. 'p_0', 'p_1', 'p_2', … denote sentential variables (in the alphabetic order), while the letters '*A*', '*B*', '*C*,' (possibly with superscripts and/or subscripts) range over arbitrary sentences. Moreover, we use 'T' as an abbreviation for 'p_0 ~p_0' and ' ' abbreviates '~T'.

A *valuation* for L is any function mapping sentences into {0,1 }, subject to the classical conditions on '~' and ' '. If there is a valuation V such that V (*A*)=0, *A* is called a *non-tautology*, and V a *non-model* of *A.*

2. Axioms and Rules

We assume the usual notion of a *substitution instance*. In addition, we say that a sentence *A* is an *equivalent variant* of a sentence *B* iff *A* is obtained from *B* by replacement of equivalent sentences, counting the following pairs as equivalent:

{T, ~ }, { ,   }, {T,   T}, {T, T   }, {T, T T}.

Using this terminology, our system is defined by exactly one axiom and two rules of inference:

A.0
R.1 if *B* is an equivalent variant of *A*, $B \vdash A$
R.2 if *B* is a substitution instance of *A*, $B \vdash A$.

A *derivation* of a sentence *A* from a sentence *B* is a finite sequence $A_0 \ldots A_n$ ($n \geq 0$) such that $A_n = A$, $A_0 = B$, and $A_i \vdash A_{i+1}$ for each $i < n$. If there is a derivation *D* of *A* from , *A* is called a *non-theorem*, and *D* a *non-proof* (or a *disproof*) of *A*.

3. Examples

Note that (=~T) is a substitution instance of both p_i and $\sim p_i$ ($i \geq 0$). Hence all variables and negations thereof are immediately seen to be non-theorems by *R*.2. Also, disproofs of such basic non-tautologies as $\sim\sim p_i$, p_i p_j, $\sim p_i$ $\sim p_j$, $\sim(p_i$ $p_j)$, p_i $(p_j$ $p_k)$, $(p_i$ $p_j)$ p_k, etc. are all straightforward, using repeated applications of *R*.1 followed by one or more applications of *R*.2.

As an illustrative example, we give here a disproof of the sentence $\sim(p_0$ $\sim\sim p_1)$ $(\sim p_1$ $\sim(\sim(\sim p_2$ $\sim p_0)$ $\sim(p_0$ $p_2)))$:

1.		*A*.0
2.		*R*.1
3.	( )	*R*.1
4.	(~T)	*R*.1 + df
5.	(~(T))	*R*.1
6.	~T (~(~T ~))	*R*.1 + df
7.	~T (~(~T ~( )))	*R*.1
8.	~(T) (~(~(T T) ~( )))	*R*.1
9.	~(~~T) (~T ~(~(~ ~) ~( )))	*R*.1 + df
10.	~(~~T) (~T ~(~(~p_2 ~) ~(p_2)))	*R*.2
11.	~(~~p_1) (~p_1 ~(~(~p_2 ~) ~(p_2)))	*R*.2
12.	~(p_0 ~~p_1) (~p_1 ~(~(~p_2 ~p_0) ~(p_0 p_2)))	*R*.2

4. Soundness and Completeness

We now show that the semantic notion of a non-tautology and the syntactic notion of a non-theorem coincide, i.e. identify exactly the same set of sentences.

T.1 Every non-theorem is a non-tautology.

Suppose A is a non-theorem, let $A_0 \ldots A_n$ be a disproof of A=A_n, pick $k \leq n$ and assume each A_i ($i < k$) has a non-model. If A_k = (*A*.0), then clearly any valuation is a non-model of A_k. If some A_i is an equivalent variant of A_k (*R*.1), then $V(A_k)=V(A_i)$ for all valuations V, and therefore any non-model of A_i will be a non-model of A_k. Finally, if some A_i is a substitution instance of A_k (*R*.2), then for all valuations V there is a valuation V' so that $V(A_i)=V'(A_k)$, hence again the existence of a non-model of A_i implies the existence of a non-model of A_k. Thus, A_k is sure to be a non-tautology, and *T*.1 follows by mathematical induction and generalization.

T.2 Every non-tautology is a non-theorem.

Suppose A is a non-tautology and let V be any non-model of A. For each $i \geq 0$, define p_i^V =⊤ if $V(p_i)$=1, and p_i^V = if $V(p_i)$=0. Also, construct A^V from A by replacing each p_i ($i \geq 0$) with p_i^V. Clearly, $V(A^V)=V(A)$=0. And by standard techniques it is easy to form a sequence $A_0^V \ldots A_n^V$ so that $A_0^V = A^V$, A_n^V = , and each A_{i+1}^V is an equivalent variant of A_i^V. Hence the sequence $A_n^V \ldots A_0^V$ is a disproof of A^V (by repeated application of *R*.1). Since A^V can be traced back to a substitution instance of A (by repeated application of *R*.2), we can therefore infer that A is also a non-theorem. *T*.2 then follows by generalization.

5. Complementary Systems

We have seen that the syntactic system defined in Section 2, call it S , is adequate to specify the set of all *non-tautologies* of L. Since a valuation always assigns opposite values to a sentence and to its negation, it is clear that the set of all *non-contradictions* is also specifiable by means of a purely syntactic system: just take ⊤ as an axiom instead of and the resulting system, call it $S^⊤$, will do (or: just characterize $S^⊤$ indirectly, by defining a sentence A to be a theorem of $S^⊤$ iff ~A is a non-theorem of S). For the same reason, it is a fact that whenever a syntactic system S is given by means of which one can adequately specify the set of all classical *tautologies* of L, one can immediately define a perfectly symmetric system, S , which is adequate to specify the set of all classical *contradictions*: just introduce a conjunction connective ‘ ’ and replace each occurrence of ‘ ’ in the axioms and rules of inference of S with occurrences of ‘ ’ (or: just characterize S indirectly, by defining a sentence A to be a non-theorem of S iff ~A is a theorem of S). Accordingly, one would get a complete picture of the logic of L if one could define a third pair of complementary systems, say $S^{\top}$ and S , which prove adequate to specify the set of all *contingencies* (sentences that are both non-tautologous and non-contradictory) and the set of all *non-contingencies* (sentences that are either tautologous or contradictory) respectively. Indirectly, such systems can of course be characterized in terms of the corresponding pairs S , $S^{\top}$ and S , S . However, the more direct approach of Section 2 can also be exploited. For instance, counting the following pairs as equivalent

$$\{A, \sim\sim A\}, \{A, A \quad\}, \{A, \quad A\}, \{\top, A \;\; \top\}, \{\top, \top \;\; A\} \qquad A \quad \{\top, \sim\top, p_0, \sim p_0\}$$

$S^{\top}$ can be based on *R*.1-*R*.2 taking p_0 as the sole axiom (the proof parallels the arguments of Section 4; only, to show completeness, set $p_0^V = p_0$). By contrast, to provide an adequate basis for S it is sufficient to recall that a sentence with n distinct variables is a tautology iff its disjunctive Boolean expansion contains exactly 2^n disjuncts, while it is a contradiction iff its conjunctive Boolean expansion contains exactly 2^n conjuncts. Since such expansions can always be transformed into sentences of the form $p_0 \;\; \sim p_0$ or $\sim(p_0 \;\; \sim p_0)$ by means of Boolean equivalences, one can then take both $\top$ and as axioms, redefine *R*.1 in terms of such equivalences, and drop *R*.2.

All these results, of course, reflect the fact that the relevant sets of sentences (tautologies, contradictions, contingencies, etc.) are all decidable, and therefore come to no surprise. However, the point is that such sets are now seen to be *on a par* as far as their syntactical characterization is concerned. Which does not mean, of course, that they are all equally “interesting”. The fact remains that every substitution instance of a tautology of L is a valid sentence of, say, the language of quantification theory, whereas the substitution instances of a non-tautologous sentence of L do not have to be invalid in a language with quantifiers. They have to be non-tautologous – but that’s all.

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