C. S. Peirce and the square root of minus one:

Quaternions and a complex approach to classes of signs and categorical degeneration

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Abstract

The beginning for C. S. Peirce was the reduction of the traditional categories in a list composed of a fundamental triad: quality, respect and representation. Thus, these three would be named as Firstness, Secondness and Thirdness, as well given the ability to degeneration. Here we show how this degeneration categorical is related to mathematical revolution which Peirce family, especially his father Benjamin Peirce, took part: the advent of quaternions by William Rowan Hamilton, a number system that extends the complex numbers, i.e. those numbers which consists of an imaginary unit built by the square root of minus one. This is a debate that can, and should, have contributions that take into account the role that mathematical analysis and linear algebra had in C. S. Peirce's past.

Keywords: Philosophy of Mathematics, Semiotics, Quaternions

Starting point for the study of signs, the semiotic categories are the Peircean response to the great metaphysical models. The beginning for C. S. Peirce was the reduction of the traditional categories in a list composed of a fundamental triad: quality, respect and representation. Thus, these three would be named as Firstness, Secondness and Thirdness, as well given the ability to degeneration. However, this degeneration, which allows the classes of signs' extension, does not receive proper attention in language studies. Here we show how this degeneration categorical is related to mathematical revolution which Peirce family, especially his father Benjamin Peirce, took part: the advent of quaternions by William Rowan Hamilton, a number system that extends the complex numbers, i.e. those numbers which consists of an imaginary unit built by the square root of minus one. Given that the imaginary part of a quaternion is characterized, in their algebraic and geometrical properties, while a versor, this number

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system makes itself ideal for application in the mechanics of a 3-dimensional and/or 4-dimensional space, linking to the very question of analytic geometry, or even in algebraic reflections. Here we see that the engagement of C. S. Peirce in the design of his father's linear algebra made him build his semiotic systems in the manner of mathematical imagination. Such consideration, developed by this article, is consolidated as a further contribution to the ongoing debate about the Peircean trichotomies and the development classes of signs, represented here by the development of a spherical representation of classes of signs. This is a debate that can, and should, have contributions that take into account the role that mathematical analysis and linear algebra had in C. S. Peirce's past.

Peirce family and quaternions

Known for his letters and to leave in his estate, a series of impressionist writings, C. S. Peirce has a text where he paints a significant portrait about the relationship and the theoretical disagreements he had with the mathematical thought of his father, Benjamin Peirce. This does not mean an ordinary family disagreement. One of the first American mathematician with some worldwide projection, Benjamin Peirce is well remembered for his work in statistical theory, in analytic geometry, and linear algebra, as well for his phrase "Mathematics is the science which draws necessary conclusions" (PEIRCE, 1881, p. 97).

In fact, this is the opening sentence of his treatise *Linear Associative Algebra*, from 1870 and republished posthumously in 1881, with notes and additions, by C. S. Peirce. And it is precisely in *Linear Associative Algebra* which deals with mentioned Peirce's text. This text, dated June 28th, 1910, was in a copy of the *Traité des Substitutions et des Equations Algébriques*, by Camille Jordan, and was published by the initiative of Raymond Clare Archibald in 1927. I highlight below some excerpts from the text:

> I will record a reminiscence about this book. It was published in 1870, the same year as the date of the original edition of my father's *Linear Associative Algebra* (though I am sure this was not lithographed for a year or two after the general theory was complete). I had first put my father up to that investigation by persistent hammering upon the desirability of it. There was one feature of this work, however, which I never could approve of, and in vain endeavoured to get him to change. It was his making his coefficients, or scalars, to be susceptible of

> taking imaginary values. In vain I represented to him that the system of imaginary quantity has two dimensions, and is consequently a double algebra. But it was always next to impossible to induce him to take a logical view of any subject. He did me the honor to reply to my arguments in a footnote on p. 19 of the Ed. of 1870 (p. 9 of that of 1882). The reply is pure bosh. His "broad philosophy" which could not be definitely expressed, was a mere habit of feeling. He was a creature of feeling, and had a superstitious reverence for "the square root of minus one"; and as to the absence of it "trammeling" research, that only means that he was not in possession of any machinery for dealing with the problems that lie beyond its scope. If Hamilton had done as he would have had him, the calculus of quaternions could not have come into being, because division would not generally have had a determinate result. (...) When I found that my brother had purchased the [Jordan's] book in 1874, I told him it was the very book he needed to study, and that he would get a flood of illumination from it. But he only cut the leaves of the first sheet, and remained to his dying day a superstitious worshipper of two hostile gods, Hamilton and the scalar √(- 1) (PEIRCE *apud* ARCHIBALD, 1927, p. 525-527).

This family quarrel is quite simple, in mathematical terms. A complex number, whose set is denoted by **C**, is compound by $a + bi$ where a and b are natural numbers and i is the imaginary unit, since $i^2 = -1$, that is, i is the square root of minus 1. On the right side of the plus sign, that is a, we have the *real part*. On the left side, that is bi, we have the *imaginary part*. A *quaternion*, as we shall see, is an expansion of complex numbers. Its structure is defined by Hamilton in the first lines of his theory, one of the most direct definitions in the history of mathematics:

> Let an expression of the form $Q = w + ix + jy + kz$ be called a *quaternion*, when w, x, y, z, which we shall call the four *constituents* of the quaternion Q, denote any real quantities, positive or negative or null, but i, j, k are symbols of three imaginary quantities, which we shall call *imaginary units*, and shall suppose to be unconnected by any linear relation with each other (HAMILTON, 1844, p.10).

Also, this is the very moment that the Peirce family begins to disagree. In the footnote mentioned in C. S. Peirce's letter found in Camille Jordan's book, Benjamin Peirce criticizes Hamilton's method:

> Hamilton's total exclusion of the imaginary of ordinary algebra from the calculus as well as from the interpretation of quaternions will not probably be accepted in the future development of this algebra. It evinces the resources of his genius that he was able to accomplish his investigations under these trammels. But like the restrictions of the ancient geometry, they are inconsistent with the generalizations and

> broad philosophy of modern science. With the restoration of the ordinary imaginary, quaternions becomes Hamilton's biquaternions. From this point of view, all the algebras of this research would be called bi-algebras. But with the ordinary imaginary is involved a vast power of research, and the distinction of names should correspond; and the algebra which loses it should have its restricted nature indicated by such a name as that of a semi-algebra (PEIRCE, 1881, p. 105).

What we have here is the quaternion's characterization, by Benjamin Peirce, as an algebraic structure incomplete. For Benjamin – an opinion also shared by his other son and successor in the chair of mathematics at Harvard, James Mills Peirce, as C. S. Peirce mentions in the letter – Hamilton has a misdirected attitude when dealing with his creation, especially when defining that the constituents *w*, *x*, *y*, *z* cannot be complex numbers, as they are in biquaternions.

Only with such characterization, as Benjamin Peirce writes, is that quaternions (in case, biquaternions) could compose a complete algebraic set and constituting an quadruple algebra, not a triple one (Peirce, 1881, p. 105). However C. S. Peirce believed that quaternions were a good example of a double *algebra*, following the nomenclature developed by Augustus De Morgan (1849). The distinction is simple in symbolic algebra's field:

> This particular mode of giving significance to symbolic algebra [double algebra] is named from its meaning requiring us to consider space of two dimensions (or area), whereas all that ordinary algebra requires can be represented in space of one dimension (or length). If the name be adopted, ordinary algebra must be called single (...). All the symbols which in single algebra denote numbers or magnitudes, in double algebra denotes *lines*, and not merely the *lengths* of lines, but their *directions*. Thus two lines of the same length, but in different directions, or two lines in the same direction, but of different lenghts, *must* have different symbols. Accordingly, each symbol is meant to convey a *double* signification: it describes the lenght, and direction, of its line (...).Thus, *A* and *B* being points, *AB* and *BA* are not entitled to the same symbol: and if a *A, B, C. D* be the points of a parallelogram in order, *AB* and *DC* gave the same symbol, but not *AB* and *CD*. Thus *AB* = *DC* is true: *AB* = *CD* is not (DE MORGAN, 1849, p. 117).

Thus, C S. Peirce regarded quaternions in its vector potential while Benjamin and James Mills Peirce maintained its position for linear algebra. Of course, the advent of vector analysis – driven by the book *Vector Analysis*, by Edwin Bidwell Wilson based on Josiah Willard Gibbs' lectures – did not pass unnoticed to the Peirce family, who loved Hamilton and scalar √(- 1), as described in Kennedy (1979, p. 426). But we can notice that biquaternions' properties are not lost on a *double algebra* dynamics. An example is the hyperbolic quaternions, designed by Alexander MacFarlane (1900), whose application is similar to Minkowski diagrams.

While his father and his brother James position themselves within the linear algebra, C. S. Peirce seems to indicate his interest in the condition caused by De Morgan's vectorial double algebra. To understand how it affects the mathematical imagination that Peirce used in the categorical degeneration and in the classes of signs, we must first introduce the mathematical concept of quaternion.

The Quaternion

As mentioned earlier, a quaternion is defined by Hamilton as:

> Let an expression of the form $Q = w + ix + jy + kz$ be called a *quaternion*, when w, x, y, z, which we shall call the four *constituents* of the quaternion Q, denote any real quantities, positive or negative or null, but i, j, k are symbols of three imaginary quantities, which we shall call *imaginary units*, and shall suppose to be unconnected by any linear relation with each other (HAMILTON, 1844, p.10).

A quaternion, thus, is an expansion of a complex number – those compound by $a + bi$ where a and b are natural numbers and i is the imaginary unit, since $i^2 = -1$, that is, i is the square root of minus 1 –, so w is its *real part* or *scalar* and rest of the equation is its *imaginary part* or *vector*.

As we can see, to expand $a + bi$, in case $w + ix$, Hamilton introduces the imaginary units j e k, transforming the imaginary part into a vector with the following assumptions: $j^2 = -1$; $k^2 = -1$; $ijk = -1$. In traditional way, we can reduce it on an assumption: $i^2 = j^2 = k^2 = -1$. It makes Hamilton conceive, in the quaternion, the idea of a noncommutative multiplication: $ij = k; jk = i; ki = j$, but $ji = -k; kj = -i; ik = -j$. Addition and subtraction follow the complex number's rules and, like multiplication, those presupposes a complete separation between the real and imaginary parts.

> The separation of the real and imaginary parts of a quaternion is an operation of such frequent occurrence, and may be regarded as being so fundamental in this theory, that it is convenient to introduce symbols which shall denote concisely the two separate results of this operation. The algebraically *real* part may receive, according to the

> question in which it occurs, all values contained on the one *scale* of progression of number from negative to positive infinity; we shall call it therefore the *scalar part*, or simply the *scalar* of the quaternion, and shall form its symbol by prefixing, to the symbol of the quaternion, the characteristic Scal., or simply S., where no confusion seems likely to arise from using this last abbreviation. On the other hand, the algebraically *imaginary* part, being geometrically constructed by a straight line, or radius vector, which has, in general, for each determined quaternion, a determined length and determined direction in space, may be called the *vector part*, or simply the *vector* of the quaternion; and may be denoted by prefixing the characteristic Vect., or V. We may therefore say that *a quaternion is in general the sum of its own scalar and vector parts*, and may write Q = Scal.Q + Vect.Q = S.Q+V. Q or simply Q = SQ+VQ (HAMILTON, 1846, p. 26).

However, there is one controversy regarding the nomenclature in a quaternion with a real part equal to zero – that is a quaternion as $Q = ix + jy + kz$, also known as *pure quaternion* – can be thought of as a vector.

> Of course, there is a one-to-one correspondence between the set of vectors and that of pure quaternions, because vectors and pure quaternions are both triplets. However, not every triplet can be regarded as a vector, because a vector is a triplet with some *specific properties*. The notation for vectors and quaternions also helps to increase the confusion. In both cases one employs *i, j, k*, and this notation conjures up an identification between pure quaternions and vectors. However, for vectors, *i, j, k* are unit vectors in three perpendicular directions. In the case of quaternions, *i, j, k* are imaginary units. Altmann has shown, from a modern point of view, how dangerous it is to identify uncritically a pure quaternion with a vector. A pure quaternion and a vector do not have the same symmetry proprieties (SILVA & ANDRADE MARTINS, 2002, p.958-959).

This was noticed by Hamilton, who describes that quaternions can represent *rotations*. To this, we call it *versor* that is nothing more than a directed arc from a circle with radius 1, representing the path of a point that is rotated by an angle a in an axis r. With this, a *versor* is $\mathbf{U}q = \exp(a\mathbf{r}) = \cos a + \mathbf{r} \sin a$ where $\mathbf{r}^2 = -1$ e $a \in [0,\pi]$. The rotations performed by a quaternion can be placed such as those placed by Euler angles, however, with one difference: the angle must be halved.

In this comparison with the Euler angles – demarcating its proximity to Euler-Rodrigues formula – we can redesign the entire matrix of rotation operator. Taking into account that q_1, q_2 e q_3 are the vector components and q_4 is scalar component, as well $E_1 = 2q_4q_1$, $E2 = 2q_4q_2$, $E3 = 2q_4q_3$ e $E4 = q_4^2 - (q_1^2 + q_2^2 + q_3^2)$, we have this matrix:

$$\begin{bmatrix} 2(q_1^2+q_4^2)-1 & 2(q_1q_2-q_3q_4) & 2(q_1q_3+q_2q_4) \\ 2(q_1q_2+q_3q_4) & 2(q_2^2+q_4^2)-1 & 2(q_2q_3-q_1q_4) \\ 2(q_1q_3-q_2q_4) & 2(q_2q_3+q_1q_4) & 2(q_3^2+q_4^2)-1 \end{bmatrix}$$

Simplifying, we can notice that the rotation's definition of vector *v* post by quaternion is calculated as qvq^{-1} where q^{-1} is the conjugate of the quaternion *q*. This mathematical property of quaternions makes them ideal in the construction of algorithms for 3D computer graphics. Through qvq^{-1} and the actual condition to reduce the angle by half, those arcs gain equivalent negative arcs, so the rotation is calculated both in clockwise and counterclockwise manner. The result of these arcs is a hypersphere, a 3-sphere in 4-D space.

A special form of the *versor* is the *right versor* where $a = \pi/2$. The major consequence is that they produce a scalar null and all vectors of imaginary part of size one. Thus, they form a sphere of square roots of -1 in a three dimensional space.

With this brief explanation, we want to show, in general, the reasoning put by Hamilton in his time. It is not the scope of this paper to explain the current state of quaternions' research that, if on one side is being forgotten within mathematics, more specifically the Analytic Theory of Numbers, but on the other side, provides some interesting insights in quantum physics as well as being a simple way to construct algorithms for the digital world.

To link this reasoning to that engendered by the Semiotics of C. S. Peirce, we must first understand the scope of the cenopythagorean categories as well as the role that their degeneration have in the trichotomies' development in Peirce's thought. This is the moment when mathematical intuition will join semiotic intuition.

Categories and Degeneration

In the broad project of semiotics, the cenopythagorean categories' issue – namely Firstness, Secondness and Thirdness – is the beginning of theorizing about the sign, but also attempt made by Peirce to discuss the tradition posited by Hegel and Kant. In other words, we find ourselves in a search through the stages of thought, how the mind operates what is before it. In his own words, in a letter to William James, Peirce states that “by the *phenomenon* I mean whatever is before our minds in any sense. The three categories are supposed to be the three kinds of elements that attentive perception can make out in the phenomenon” (CP 8.265).

With this, we have a set of functions operating on a single point. From all the many definitions and descriptions of the cenopythagorean categories that Peirce gave, the most vivid are those given in the letter to Lady Welby. But it is only in his lectures on Pragmatism at Harvard that we found a systematic way to describe them.

The construction begins by Firstness that "is the Idea of that which is such as it is regardless of anything else. That is to say, it is a *Quality* of Feeling (…). Category the First owing to its Extremely Rudimentary character is not susceptible of any degenerate or weakened modification" (CP 5.66-68).

Secondness “is the Idea of that which is such as it is as being Second to some First, regardless of anything else, and in particular regardless of any *Law*, although it may conform to a law. That is to say, it is *Reaction* as an element of the Phenomenon” (CP 5.66).

At last, Thridness “is the Idea of that which is such as it is as being a Third, or Medium, between a Second and its First. That is to say, it is *Representation* as an element of the Phenomenon” (CP 5.66).

Quality, Reaction, Representation. It is this progression that the mind deals with the phenomena of the world. Thus, we could say, ultimately, that the three semiotic axes live in such conditions, basing semiosis’ very own way.

However, it is also these Harvard lectures that Peirce introduces the idea of degeneration. Degeneration would be supplemental trichotomy, as the cenopythagorean categories are a genuine trichotomy. Incidentally, the degeneration would be a consequence of the categorical interdependence. The Firstness exists in itself and also in the degenerate form in Secondness and in Thirdness. There is not Secondness in Firstness, but there is Secondness on Thirdness. Finally, the last phase of thinking, Thirdness, exists only in itself.

What we have here is the sign’s development kickoff.

> Peirce's theory involves the use of five principles: 1. There is a single triadic set of categories in terms of which all phenomena are to be classified.' A phenomenon is either a First, something in itself; a Second, an existent in dyadic relation to something else; or a Third, a mean inseparable from a law or pur- pose. 2. A triadically determined object exemplifies all three cate- gories (CP 2.238). Since a sign is defined as "something which stands to somebody for something in some respect or capacity" (CP 2.228), it is triadic in nature. By principles 1 and 2, one obtains three divisions of signs: the sign in itself, the sign as related to its object, and the sign as interpreted to represent an object. (...). 3. Each division is subject to all three categories (CP 2.243). Accordingly, each division is trichotomous. For example, the sign as related to its object (the second of the three divisions) may be similar to, may be existentially connected with, or may be referred by means of a law to, its object (...). 4. Thirds have two degenerate forms, Seconds one degenerate form (CP 1.365). The application of this principle to the three divisions yields ten divisions (...). By principle, the ten divisions yield the ten trichotomies (...). 5. Whatever is a First determines only a First; whatever is a Second determines a Second or (degenerately) a First; whatever is a Third determines a Third, or (degenerately) a Second or a First (CP 2.235). The application of this principle to the three trichotomies (...) yields ten classes of signs (...). The application of this principle to the ten trichotomies (...) yields the sixty-six classes of signs (WEISS & BURKE, 1945, p. 384).

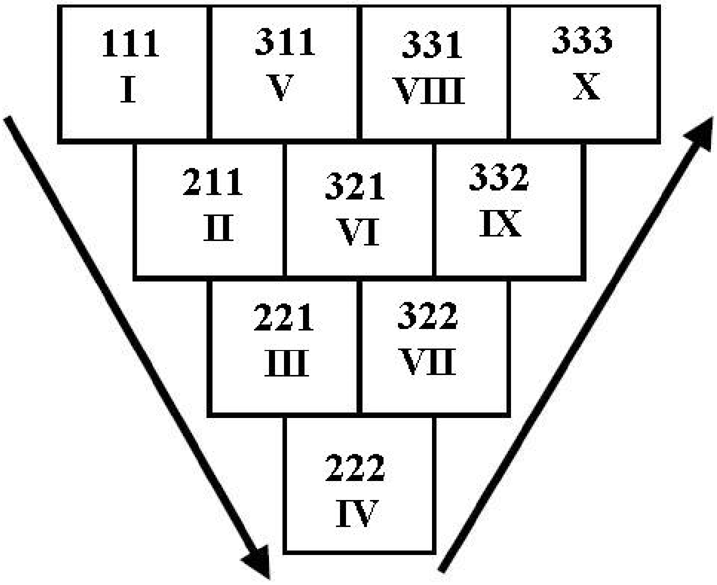
Thus, sign development becomes just a matter of combinatorial trichotomies governed by the equation where the number of classes is equal to $(n + 1)(n + 2)/2$, where n is the number of trichotomies. Thus, in accordance with Weiss and Burke’s consolidation, with 10 trichotomies, we have 66 signs. However, will this is the only way to understand the trichotomic reasoning designed by C. S. Peirce? Does it all lies in combinatorial analysis or there is more forms of mathematical intuition and imagination involved?

The complex approach to degeneration and classes of signs

We know that 10 trichotomies generate 66 signs from the reasoning set by Peirce (CP 2.254-65), which cenopythagorean categories obeys 3 factors: (1) the factor A, signs in themselves, (2) the factor B, signs in relation to objects, and (3) the factor C, signs interpreted to represent. Thus, it generates the well-known below, as well as its graphical triadic progression, both by Peirce:

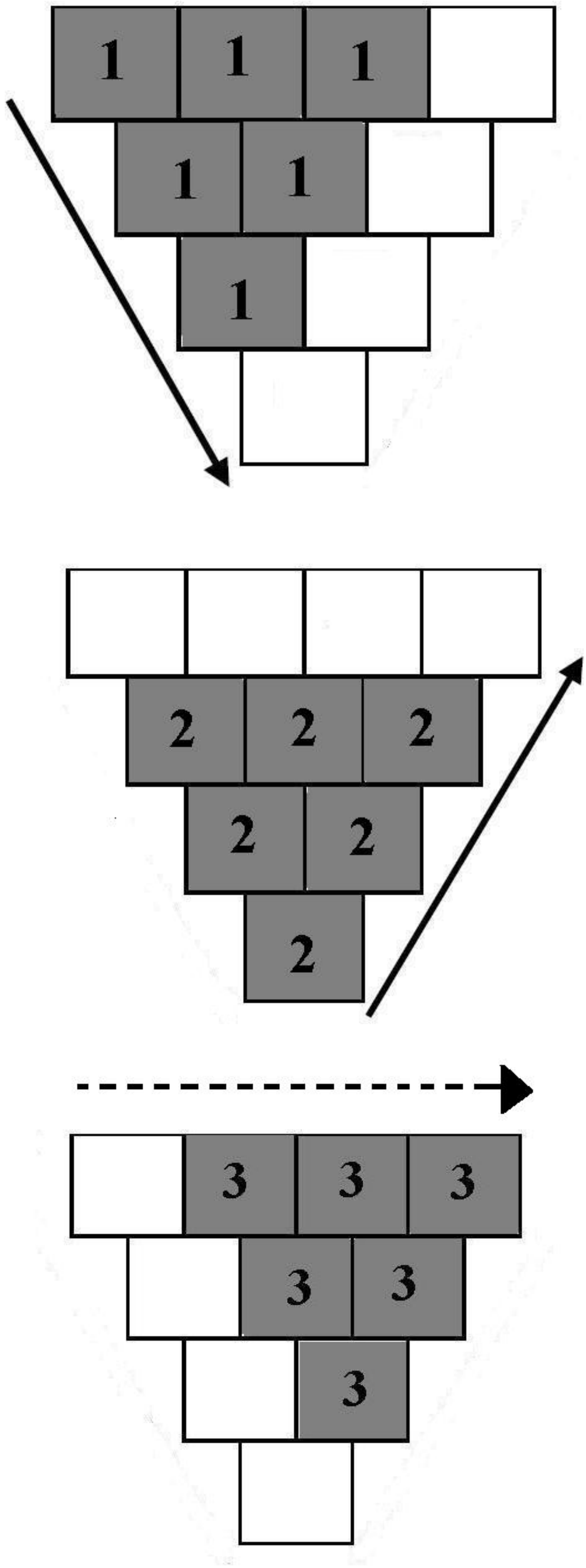
A	B	C	Name of Sign	Example

1	1	1	Qualisign (I)	A feeling of "red"
2	1	1	Iconic Sinsign (II)	An individual diagram
2	2	1	Rhematic Indexical Sinsign (III)	A spontaneous cry
2	2	2	Dicent Sinsign (IV)	A weathercock or photograph
3	1	1	Iconic Legisign (V)	A diagram, apart from its factual individuality
3	2	1	Rhematic Indexical Legisign (VI)	A demonstrative pronoun
3	2	2	Dicent Indexical Legisign(VII)	A street cry
3	3	1	Rhematic Symbol (Symbolic Rheme) (VIII)	A common noun
3	3	2	Dicent Symbol (Proposition) (IX)	Proposition
3	3	3	Argument (X)	Syllogism



Typically, this graphical progression is referred to be a lattice. But, if we think closer to the Order Theory, or even algebraic logic, we can see that its construction is not binary, ie, the lattice L does not follow the structure L = (L,R) – that means, L is partially ordered by R, a binary pair.

The logic developed by Peirce is triadic, but without being ternary. If we look deeply, we can see that triangle is form, in fact, by following of the movement of three triangles, namely: (1) the triangle of the classes that have at least a 1; (2) the triangle of the classes that have at least a 2; and (3) the triangle of classes that have at least a 3. Consider these three triangles separately:


1 1 1
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3 3 3
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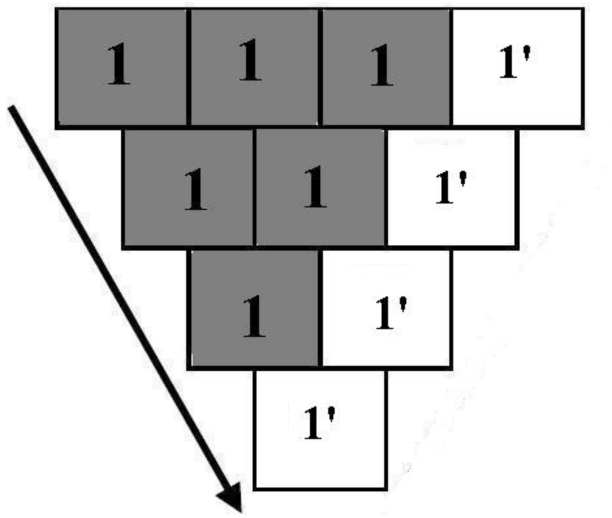
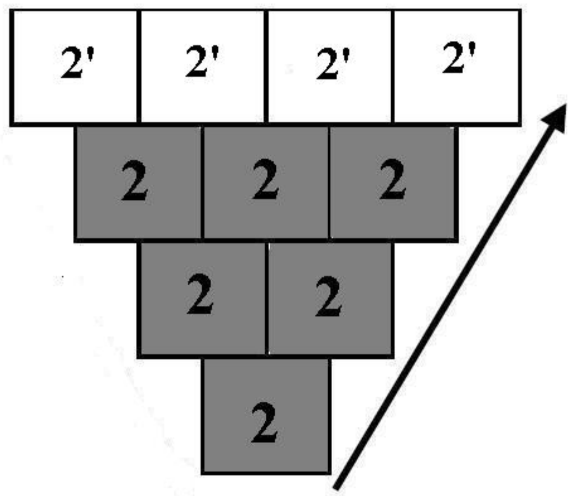
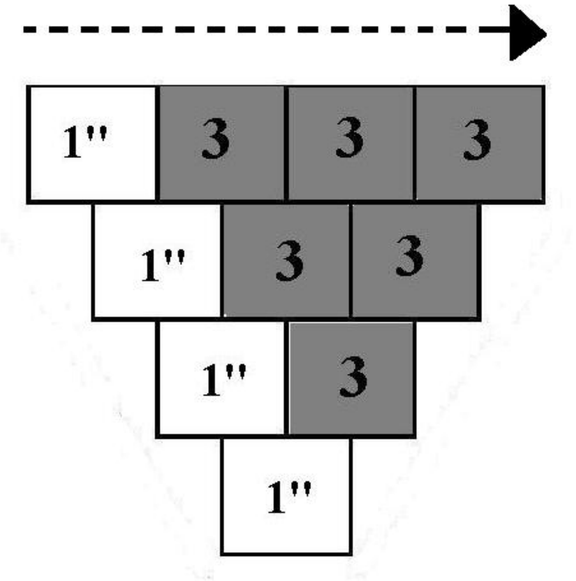
3

These triangles are flanked by a segment of the classes that have not the chosen number. It is interesting to note that these external classes to triangles seem to mark them against a possible exterior. With a much more close analysis and counting the movement between the triangles, we can, indeed, associate them with degenerate cenopythagorean categories, namely Firstness of Secondness (1'), Firstness of Thirdness (1'') and Secondness of Thirdness (2').

Therefore, our triangles would be like those:





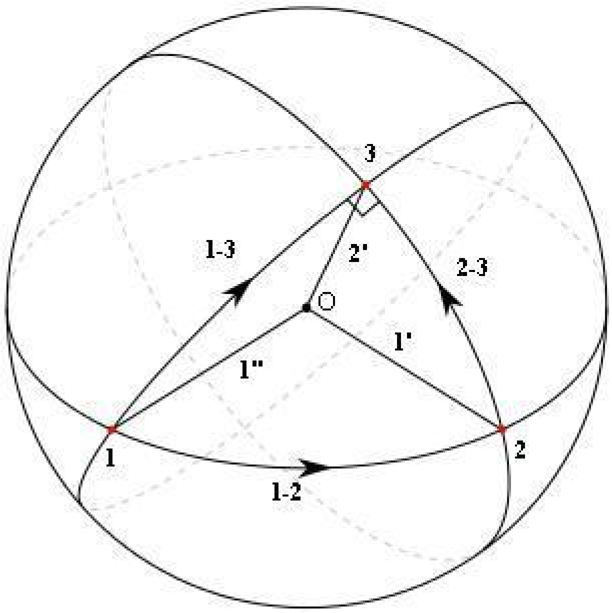


However, this does not seem enough, because we have not forgotten the classes involved. The difficulty of observing such consideration, coming from what we call in this article a *complex approach*, resides in the fact that we are considering these triangles in a two-dimensional condition. With the aid of complex numbers, the quaternions more specifically, it will allows us to see these 10 classes and their triangles in a three-dimensional condition.

We mentioned earlier about *right versor* which transforms the scalar in a null one and makes all the vectors in imaginary part being -1. We also said that they form a sphere of square roots of -1 in a three dimensional space. Thus, this sphere becomes an example of how to calculate the square root of -1 in the set **H** made of quaternions.

We conjecture that this sphere produced by the right versor is the best movement representation of semiotic degeneration, as well the best representation of classes of signs' internal logic. This sphere is built on a zero scalar, which is the center of the sphere, with its large arcs build by its vectors. This is justified in semiotics because classes of signs need the *phenomenon*, which is the guarantee of realism as defended by Peirce (CP 5.470, 8.16-17). The sphere's center (named here as O), the null scalar, is the anchoring of the *phenomenon* which, in turn, is the sphere as a whole.

Thus, this spherical representation of classes of signs would have the following graphical representation:



The classes of signs form a prism inserted in the sphere of *phenomenon*, three corresponding to the three classes of signs' triangles and one side corresponding to the excluded classes of signs in accordance with rules 4 and 5 by C. S. Peirce on the trichotomies. Let's recall them:

> 4. Thirds have two degenerate forms, Seconds one degenerate form (CP 1.365). The application of this principle to the three divisions yields ten divisions (...). By principle, the ten divisions yield the ten trichotomies (...). 5. Whatever is a First determines only a First; whatever is a Second determines a Second or (degenerately) a First; whatever is a Third determines a Third, or (degenerately) a Second or a First (CP 2.235). The application of this principle to the three trichotomies (...) yields ten classes of signs (...). The application of this principle to the ten trichotomies (...) yields the sixty-six classes of signs (WEISS & BURKE, 1945, p. 384).

Taking this spherical representation of classes of signs as a reference, we can think of the 66 classes of signs (and all others who fail in rules 4 and 5) while rotations in relation to the center of the sphere of *phenomenon* measured by such arcs. With this situation, we will can map out and build a more accurate model of the sign positioning, their interpenetration, as well as its relationship with the *phenomenon*.

Spherical representation of classes of signs as mathematical intuition and imagination

Our choice of a three-dimensional approach, going into the reasoning that are lead the two-dimensional triangles, only values the fabulous exercise within the philosophy of C. S. Peirce, composed by both mathematical intuition and imagination. In his mathematical works themselves or in his writings of logic and semiotics, Peirce has always shown the force of mathematical thinking in their elaborations.

Also, we may note that, in his writings, his thought is not only driven by the mathematical force, but also by the attempt to investigate deeply the logic of this force. As a member of his generation, Peirce put that search as an objective.

> He in fact developed a rich and systematic philosophical account of the nature of mathematical reasoning as centrally involving semeiotic experimentation. The effort to provide an account of the logic of inquiry, in general, had a central place among the many diverse intellectual interests that he cultivated through his life. And describing the logic of *mathematical* inquiry was especially important to him, as mathematics occupied a privileged place as the most general of sciences in his classification—a science that provides and exemplifies general patterns of reasoning for all other sciences. He in fact considered the search for general methods of discovery in mathematics — what he called a methodeutic — an endeavor worth a lifetime of research and study (CAMPOS, 2009, p. 136).

Campos claims that there are three guiding values in Peirce's mathematical reasoning: imagination, concentration and generalization. This group, which he calls "intellectual powers", he assigns three skills:

> (i) create a mathematical "icon"—a presentation of a hypothetical state of things that is of interest for its own intrinsic formal character to the inquirer *qua* mathematician; (ii) discriminate between mathematically essential and superfluous relations in the determination of the icon and focus the attention on the essential ones; and (iii) generalize on the basis of the characters and relations embodied in the icon (CAMPOS, 2009, 137).

Known to be systematic in the thinking-preparation of such forms, Peirce, along with other mathematicians such as Bernhard Riemann and Carl Friedrich Gauss, is remembered as someone with a powerful mind on the issue of mathematical imagination, especially in matter of analysis, analytical geometry and especially the field of non-Euclidean geometries.

Thus, mathematics, despite being the science of observation and experimentation on diagrams, resembling the physical sciences (CP 4.530), it is a science that requires the hypothesis-making. And for is subject, not only the imagination is needed, but also intuition in all forms that Polya (1954) describes such as induction and analogy.

There is a need in the field of semiotics, especially at the dawn of C. S. Peirce's centennial celebrations, to increasingly exercise these principles. The research field of complex numbers, which quaternions are an expansion, and their relationship with the sign construction needs further investigation. Especially because we know that they not only interested the other members in the Peirce Family. C S. Peirce (1879) himself used complex numbers to calculate its quincuncial projection of the sphere. Moreover, in the future of the complex approach designed in this article, the quincuncial projection method will be of enormous value when we start to calculate and categorize the 66 classes of signs through the spherical representation formulated here.

The task of the semiotics scholars, in this second new century after Peirce, is to go deeper and deeper into the semiotic imagination using mathematical intuition and imagination as reasoning companions. For this new centennial that begins in 2014, we can say the same famous David Hilbert's words: *Wir müssen wissen. Wir werden wissen.*

References

ARCHIBALD, R. C. "Benjamin Peirce's Linear Associative Algebra and C. S. Peirce". *The American Matematical Monthly*. Vol. 34, No. 10, Washington: Mathematical Association of America, 1927, pp. 525-527.

CAMPOS, D. G. "Imagination, Concentration, and Generalization: Peirce on the
Reasoning Abilities of the Mathematician". *Transactions of the Charles S. Peirce Society: A Quarterly Journal in American Philosophy*, Volume 45, Number 2, Indianapolis: Indiana University Press, 2009, pp. 135-156.

DE MORGAN, A. *Trigonometry and Double Algebra*. London: Taylor, Walton and Maberly, 1849.

HAMILTON, W. R. "On Quaternions, or on a new system of Imaginaries in Algebra (articles 1-5)". *The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science* (3rd Series), vol. XXV, London: Taylor and Francis, 1844, pp. 10-13.

HAMILTON, W. R. "On Quaternions, or on a new system of Imaginaries in Algebra (articles 18-21)". *The London, Edinburgh and Dublin Philosophical Magazine and Journal of Science* (3rd Series), vol. XXIX, London: Taylor and Francis, 1846, pp. 26-31.

KENNEDY, H. "James Mills Peirce and the cult of quaternions". *Historia Mathematica*. Vol. 6, issue 4. Amsterdan: Elsevier, 1979, pp. 423-429.

MACFARLANE, A. "Hyperbolic Quaternions". *Proceedings of the Royal Society at Edinburgh*, 1899–1900 session, Edinburgh: Royal Society, 1900, pp. 169–181.

PEIRCE, B. "Linear Associative Algebra" (with notes and addenda by C. S. Peirce). *American Journal of Mathematics.* Vol. 4, No. 1, Baltimore: The Johns Hopkins University Press, 1881, pp. 97-229.

PEIRCE, C. S. "A Quincuncial Projection of the Sphere". *American Journal of Mathematics.* Vol. 2, No. 4, Baltimore: The Johns Hopkins University Press, 1879, pp. 394-396.

PEIRCE, C. S. *Collected Papers of Charles Sanders Peirce* (8 volumes, vols. 1-6, eds. Charles Hartshorne and Paul Weiss, vols. 7-8, ed. Arthur W. Burks). Cambridge: Harvard University Press, 1931-1958.

POLYA, G. *Mathematics and Plausible Reasoning* (2 vol.). Princeton: Princeton University Press, 1954.

SILVA, C. C. & ANDRADE MARTINS, R. "Polar and axial vectors versus quaternions". *American Journal of Physics*. Vol. 70**,** No. 9, College Park: American Association of Physics Teachers, 2002, pp.958-963.

WEISS, P. & BURKE, A. "Peirce's Sixty-Six Signs". *The Journal of Philosophy,* Vol. 42, No. 14, New York : Journal of Philosophy 1945, pp. 383-388.