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'Now' and 'Then':

A Formal Study in the Logic of Tense Anaphora

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by

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The dissertation of Frank Vlach is approved, and it is acceptable in quality for publication on microfilm.

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In Memory of Richard Montague

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ABSTRACT OF THE DISSERTATION

'Now' and 'Then':

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Recent work by Hans Kamp ("Formal Properties of 'Now'", Theoria, 1971) and Arthur Prior ("'Now'", Noûs, 1968) has concentrated on the logical properties of 'now', taken as a tense operator that refers back always to the moment of utterance of the sentence in which it occurs. This paper extends the systems of Kamp and Prior to allow for temporal reference to previously established contexts other than the context of the sentence as a whole. An operator R, corresponding to 'then' is introduced into a tense logic with the familiar operators G and H. The function of R is to refer back to previously established contexts. An additional operator K is introduced in order to specify which previously established context an occurrence of R within its scope refers to.

A full formal semantics is provided for K and R within the framework of Montague ('Pragmatics' in R. Klibansky [editor], Contemporary Philosophy - La Philosophie Contemporaine, Florence, 1968). The primary distinguishing feature of the semantics is that formulas must be evaluated with respect to ordered pairs of moments rather than just single moments. This is because in evaluating the subformulas of a formula it is necessary to keep track not only of what might be called the primary moment, but also of a secondary moment with respect to which subformulas that occur within the scope of R are to be evaluated. This secondary moment is established by the syntactically closest occurrence of K within whose scope the subformula in question occurs. Thus the following truth conditions hold for a sentence ϕ and a pair of moments $\langle t,t'\rangle$:

$H\phi$ is true at $\langle t,t'\rangle$ if and only if ϕ is true at $\langle t'',t'\rangle$ for every moment t'' earlier than t.

$G\phi$ is true at $\langle t,t'\rangle$ if and only if ϕ is true at $\langle t'',t'\rangle$ for every moment t'' later than t.

$K\phi$ is true at $\langle t,t'\rangle$ if and only if ϕ is true at $\langle t,t\rangle$.

$R\phi$ is true at $\langle t,t'\rangle$ if and only if ϕ is true at $\langle t',t'\rangle$.

A set of axioms is presented for the system with K and R, and it is proved that the set of axioms is complete, in the sense that any formula is derivable from the axioms if and only if it is true at every point of reference $\langle t,t\rangle$ according to every interpretation. The set of axioms, although of course decidable, cannot be specified as the set of instances of any finite set of axiom schemata. In fact, the axioms are arrived at only as the end result of a long and rather complex construction. Some reasons are given for the suggestion that the system may not be finitely axiomatizable in the sense just specified, but the question remains open. The proof of completeness is a Henkin-type proof, and is based on the proof of Cocchiarella ('Tense and Modal Logic: A Study in the Topology of Temporal Reference', Ph.D. Thesis, UCLA, 1965), but certain features of the system with K and R seem to necessitate complicating Cocchiarella's methods.

CHAPTER I[1]

INTRODUCTION

A. The General Outline

1. The 'then' Operator

Hans Kamp [4], [5] and Arthur Prior [8], [9] have investigated a system of tense logic that contains a formal analogue of 'now'. This system has, in addition to the usual tense logical operators G (for 'it will always be the case that') and H (for 'it has always been the case that'), an operator N which is to be read 'it is now the case that'.

The semantics for N are fairly straightforward. In tense logic truth and satisfaction must always be defined with respect to a moment of time, and N is interpreted in such a way that $N\phi$ is true at any given moment just in case ϕ is true at the moment with respect to which the sentence in which $N\phi$ occurs is being evaluated. This latter moment is thought of as the moment of utterance of the sentence. This gives formulas like $\phi \leftrightarrow N\phi$ and $\phi \rightarrow GN\phi \wedge HN\phi$ as theorems, but not $G(\phi \leftrightarrow N\phi)$, which is false at a moment t if there is any moment later than t at which ϕ does not have the same truth value that it has at t.

In Prior [8] an axiomatization is set forward for a system with N. In Kamp [5] it is proved that this axiomatization is complete and also that similar axiomatizations can be given for a certain class of systems with N by extending in a certain way a complete set of axioms for the corresponding system without N.

The operator N always shifts the point of temporal reference back to the moment of utterance. This is a limitation, and removing that limitation (i.e., introducing an operator that can shift the point of reference to a moment other than the moment of utterance) produces a stronger system.

Consider for example the sentence

(1) Jones is going to cite everyone now driving too fast.[2]

interpreted as meaning that there is one future time at which the citing is done. This is symbolized in the 'now' system (under the obvious scheme of abbreviation, and defining $F\phi$ as $\neg G \neg \phi$, i.e. as 'it will sometime be the case that', and $P\phi$ as $\neg H \neg \phi$, i.e. as 'it has sometime been the case that') as

(2) $F \bigwedge x (N\ D(x) \rightarrow C(Jx))$

But consider instead the sentence

(3) Jones was once going to cite everyone then driving too fast.

The function of 'then' in 3 as of 'now' in 1 is to refer back to the moment corresponding to the main verb of the sentence. In 1 this is the same moment as the moment of utterance, so that the now operator is sufficient. But 3 cannot be symbolized in the 'now' system, because there is no way to shift the point of temporal reference to the right place. It is no use trying

(4) $PF\Lambda x(N\ D(x) \rightarrow C(Jx))$

because that says 'Jones was once going to cite everyone now driving too fast'.

In order to introduce an operator for the 'then' of 3, we must provide some means of indicating which previously established context the 'then' operator refers back to. There is more than one way to do this (see Appendix). We will simply introduce another operator, called the 'index operator', whose function is simply to indicate which moment the 'then' operator refers to. We call the 'then' operator 'R' and the index operator 'K'. R will always shift the temporal reference back to the point of the nearest (syntactically) occurrence of K within whose scope the occurrence of R lies. (If the occurrence of R does not lie within the scope of K, then R behaves exactly as N does.) In this way 3 can be symbolized quite simply as

(5) $PKF\Lambda x(R\ D(x) \rightarrow C(Jx))$

The preceding examples illustrate an interesting fact about the N system and the R system; the N system, in a certain sense, is not closed under future or past tenses. Suppose we define a past tense of a sentence ϕ as a sentence which is true at a moment t if and only if ϕ is true at some moment earlier than t.[3] In this sense, $P\phi$ is a past tense of any sentence ϕ that contains no tense operators other than G and H, and $PK\phi$ is a past tense of any sentence ϕ of the system with R. But some sentences of the N system have no past tense within the N system; 2 is one such sentence.[4] The R system contains a past tense and a future tense for any sentence of the R system, and hence for any sentence of the N system (since any sentence of the N system can be translated into the R system simply by substituting R for N). The R system can then be thought of as an extension of the N system that is closed under past and future tenses.

A surprising feature of the R system is that it is strengthened by the addition of a 'now' operator. This is discussed in the Appendix.

2. Semantics

Like the semantics for the N system, the semantics for the R system is quite straightforward. The primary change necessary is that we must evaluate formulas with respect to ordered pairs of moments rather than just individual moments.[5] The truth value of a given formula with respect to a given moment of utterance depends of course on the truth values of its subparts at moments other than the moment with respect to which the whole formula is being evaluated. In particular, we must arrange for the K operator to determine the moment at which subformulas beginning with R are to be evaluated. We do this by letting the second term of the ordered pair be the moment at which formulas beginning with R are to be evaluated, and by letting K change the second term or the ordered pair. More formally, we construct our semantics in such a way that the following truth conditions hold:

Kϕ is true at $\langle t,t' \rangle$ if and only if ϕ is true at $\langle t,t \rangle$

Rϕ is true at $\langle t,t' \rangle$ if and only if ϕ is true at $\langle t',t' \rangle$

G and H of course do not affect the second term, so that

Gϕ is true at $\langle t,t' \rangle$ if and only if ϕ is true at $\langle t'',t' \rangle$, for every moment t" later than t' (and analogously for Hϕ).

Changes in the second term of the ordered pairs are necessary only for the evaluation of subparts of a formula in the course of evaluating the whole formula. Pairs of the form $\langle t,t \rangle$ are thus taken as primary. This corresponds to the fact that R refers back to the moment of utterance when not within the scope of K, and to the definition of a logically valid formula as one which is true at every pair $\langle t,t \rangle$ in every interpretation.

We need take no special account of the K and R operators in the definition of an interpretation. An interpretation specifies a set of moments of time, an 'earlier than' relation among them, a set of possible objects, and a function that stipulates which relations hold among which objects at which times. We use essentially the definition in Cocchiarella [1], but there is nothing special about this choice. The K and R operators can be introduced into any ordinary system of modal or tense logic.

The definitions of truth and satisfaction are given in terms of intensions and extensions following Montague [1].

3. An Example from English

The final object of the study of formal tense logic is presumably its application to the study of (very broadly speaking) tensed discourse in natural language. We will not examine English tenses in this paper, but we ought at least to give some indication of the sort of English expression that can be symbolized in the system with K and R but not in the system with N. In order to do this we will simply quote one example from an actual text and make a few comments.

(6) When I looked that way I saw that, some way off, so far that they must mark some kind of settlement or farm well beyond the limits of the town, more lights showed.

I turned along the path at a trot, chewing at my chunk of barley bread as I went.

The lights turned out to belong to a fair-sized house whose buildings enclosed a courtyard.

The key phrase in this example is the phrase 'the lights' in the third paragraph. This can be symbolized only by relating it to 'more lights showed' in the first paragraph, using the index operator to indicate the point of temporal back-reference. In order to see this, let us simplify 6 by leaving out detail irrelevant to the present point:

(7) When Jones looked to the left, some lights showed. Later, the lights turned out to belong to a house.

Suppose we introduce the following scheme of abbreviation:

Q: Jones looks to the left

L(x): x is a light

S(x): x shows

T(xy): x turns out to belong to y

H(x): x is a house

Then 7 may be symbolized as

(8) $PK(Q \wedge \bigvee x(L(x) \wedge S(x)) \wedge$
$\bigvee yF(H(y) \wedge \bigwedge x(R(L(x) \wedge S(x)) \rightarrow T(xy))))$ [6]

It is easy to see why 7 cannot be symbolized without the index operator. If we begin

(9) $P(Q \wedge \bigvee x(L(x) \wedge S(x)) \wedge F \bigwedge x(. . .$

we are lost, because there is no way to specify the lights that showed at that one particular moment when Jones turned left--we can specify all the lights that ever showed on any occasion when Jones turned left, but that may include more lights.

We could try $P(Q \wedge \bigvee x(L(x) \wedge S(x) \wedge \bigvee yF(H(y) \wedge F\ T(xy))))$ but this fails to say that <u>all</u> of the lights that showed turned out to belong to a house.

We see no way to symbolize 7, and therefore 6, without the use of the index operator.

6 was chosen as an example because it is typical of the fragments of English that actually occur and that cannot be symbolized in a system that does not include something like the index operator. It is clearly not an odd piece of English; a few pages of reading in almost any continuing narrative will reveal a similar example. It shares the following features with most other examples:

(a) It occurs within a continuing narrative where the passage of time is indicated by the ordering of sentences. Subject to certain other indicators, it is as if we were to read each sentence as preceded by the phrase 'very shortly afterwards' or 'before the situation changed very much'. Our formal system is inadequate in that there is no way of indicating that there is a very limited amount of time between the moment when the lights were seen and the moment that they turned out to belong to the house, but even in a system that contained a formal analogue of 'very shortly afterwards' the index and 'then' operators would still be needed to express 6.

(b) The temporal back-reference extends over sentence and (in this case) even paragraph boundaries.

(c) The temporal back-reference is not actually made by any use of the word 'then'.

(d) The key phrase involved in the temporal back-reference is of the form 'the' followed by a plural noun phrase. It might be possible to read the phrase 'the lights' in the third paragraph of 6 as elliptical for 'the lights that I saw when I looked that way'; but the phrase 'when I looked that way' can only be taken as indicating a reference back to the context of the first paragraph. Since the fictional utterer of 6 may have looked the appropriate way many times, it is only this form of contextual back-reference that makes it possible to specify the intended lights.

4. Completeness

In Chapter III an axiom system is introduced and in Chapter IV it is proved that the axiomatization is complete, in the sense that the formulas that are derivable from the axioms are exactly those that are true at every moment of every interpretation. The axiomatization is rather unsatisfactory, since the axioms are a rather complex decidable set of formulas rather than a set that can be represented as the instances of some finite number of schemata. It is not known whether a more satisfying axiomatization can be found, but there are certain features of the system with K and R that suggest that a complete finite set of schemata may not be possible. This matter is discussed further in Chapter III.

The completeness proof is of the general sort of Henkin [3], as extended in Kripke [6] and most particularly in Cocchiarella [1], the methods of which were adapted as closely as possible. However, it was necessary to revise Cocchiarella's methods considerably, as is pointed out at the beginning of Chapter III.

B. Set Theoretical Background

This undertaking is carried out within Zermelo-Fraenkel set theory. We will set forward here only those definitions of set theoretical notions that may be of particular use in what follows, or the notation of which may not be fully standardized in the literature of set theory.

$\{A(x):F(x)\}$ is the set of objects y such that y is $A(x)$, for some x such that $F(x)$; in particular, $\{x:F(x)\}$ is the set of objects x such that $F(x)$. $S(A)$ is the power set of A, or the set of subsets of A. $A\sim B$ is the set of objects x such that $x \in A$ and $x \notin B$. $\bigcup_{F(x)} A(x)$ is $\bigcup\{A(x):F(x)\}$.

We assume that ω, or the set of natural numbers is defined in such a way that each natural number is identified with the set of smaller natural numbers. In particular, 0 is the empty set. $\max(m,n)$ is the larger of the two natural numbers m and n. $\bar{A}$ is the (finite) number of objects in A.

(x,y), or the improper pair of x and y, is $\{\{x\}\{xy\}\}$.

s is an I-sequence if and only if

(1) s is a set of improper pairs

(2) For each x,y and z, if $(x,y) \in s$ and $(x,z) \in s$, then y is z.

(3) I is the set of objects x such that for some y,

$(x,y) \in s$.

lh(s), or the _length_ of s, is that I such that s is an I-sequence.

s is a _finite sequence_ if and only if s is an n-sequence, for some $n \in \omega$. s is an _infinite sequence_ if and only if s is an ω-sequence.

If s is an I-sequence and $i \in I$, then s_i, or the $i^{\underline{th}}$ _term_ of s, is that object x such that $(i,x) \in s$.

A^I is the set of I-sequences s such that for each $i \in I$, $s_i \in A$.

If s and t are finite sequences, then $s^\frown t$, or the _concatenation_ of s and t, is $s \{(i+lh(s),t_i):i < lh(t)\}$.

If s is a sequence then s_x^i is $(s \sim \{(i,s_i)\}) \cup \{(i,x)\}$.

If s is a sequence, then $s \upharpoonright A$ is the set of objects (i,s_i) such that $i \in A$.

$\langle x \rangle$, or the 1-_tuple_ whose only term is x, is $\{(0,x)\}$; $\langle x,y \rangle$, or the 2-_tuple_ (or _ordered pair_) whose terms are, in order, x and y is $\{(0,x)(1,y)\}$; and so on for each natural number.

$A \times B$ is the set of ordered pairs $\langle x,y \rangle$ such that $x \in A$ and $y \in B$.

A _relation_ is a set of ordered pairs.

If R is a relation, then xRy if and only if $\langle x,y \rangle \in R$.

If R is a relation, then R _restricted_ to A is $R \cap (A \times A)$.

f is a _function_ if and only if f is a relation and for each x,y and z, if xfy and xfz, then y is z.

We define the notions of the _domain_ of R (Dom(R)), the _range_ of R (Rng(R)), the _field_ of R (Fld(R)), the _converse_ of R, ($\breve{R}$), and the _relative product_ of R and S, (R/S), in the usual way.

f is a one-to-one function if and only if f is a function and $\breve{f}$ is a function.

If f is a function and x $\in$ Dom(f), then f(x), or the _value_ of f for the argument x, is that object y such that xfy.

A set A is _denumerable_ if and only if there is a one-to-one function whose domain is A and whose range is ω.

A relation R is _reflexive_ if and only if for every x in the field of R, xRx.

A relation R is _symmetric_ if and only if for every x and y, if xRy then yRx.

A relation R is _antisymmetric_ if and only if for every x and y, if xRy and x $\neq$ y, then not yRx.

A relation is _transitive_ if and only if for every x,y and z, if xRy and yRz, then xRz.

A relation is _connected_ if and only if for every x and y in the field of R, either xRy or yRx or x is y.

R is an _equivalence relation_ if and only if R is a relation that is transitive, reflexive, and symmetric.

R is a _reflexive linear ordering_ if and only if R is a relation that is transitive, reflexive, antisymmetric and connected.

We note the following trivial theorem:

T1. If R is a reflexive linear ordering, then R restricted to A is a reflexive linear ordering.

C. Syntax

We now turn to the syntax of the object language. In this section we construct a language that consists of the ordinary predicate calculus together with the familiar one-place tense operators G and H and two new one-place operators K and R. The definitions of all the necessary syntactic notions are included here. Most of these notions are familiar and will be defined without comment.

1. The Basic Syntactic Notions

D1. The following are the symbols:

$\underline{n}$ (read 'the negation symbol')

$\underline{c}$ (read 'the conditional symbol')

$\underline{u}$ (read 'the universal quantifier symbol')

$\underline{v}_j$ (read 'the j^{th} variable symbol', for each $j \in \omega$)

$\underline{p}_{j,k}$ (read 'the j^{th} k-place predicate symbol', for each $j,k \in \omega$)

$\underline{o}_{j,k}$ (read 'the j^{th} k-place operation symbol', for each $j,k \in \omega$)

$\underline{e}$ (read 'the identity symbol')

$\underline{g}$ (read 'the future tense operator')

$\underline{h}$ (read 'the past tense operator')

$\underline{r}$ (read 'the 'then' operator')

$\underline{k}$ (read 'the index operator')

We must ensure, of course, that all of the different symbol-denoting expressions used above do, in fact, denote distinct symbols. We may do this by identifying each symbol with a distinct natural number, or by adding all of the necessary distinctness axioms to those of the set theory.

D2. (1) π is a <u>k-place predicate letter</u> if and only if π is $\langle \underline{p}_{j,k} \rangle$, for some $j \in \omega$.

(2) δ is a <u>k-place operation letter</u> if and only if δ is $\langle \underline{o}_{j,k} \rangle$, for some $j \in \omega$.

(3) π is a <u>non-logical constant</u> if and only if π is a predicate letter or an operation letter.

(4) π is a <u>propositional constant</u> if and only if π is a 0-place predicate letter.

(5) δ is an <u>individual constant</u> if and only if δ is a 0-place operation letter.

(6) v_k, or the $k^{\underline{th}}$ <u>individual variable</u>, or just <u>variable</u>, is $\langle \underline{v}_k \rangle$.

(7) α is an <u>individual variable</u> if and only if α is v_k, for some $k \in \omega$.

(8) Iv is the set of (individual) variables.

(9) An <u>expression</u> is a finite (possibly empty) sequence of symbols.

D3. $\neg\phi$ (read 'it is not the case that ϕ') is $\langle \underline{n} \rangle ^\frown \phi$

$(\phi \rightarrow \psi)$ (read 'if ϕ, then ψ') is $\langle \underline{c} \rangle ^\frown \phi ^\frown \psi$

$\eta = \zeta$ (read 'η is identical with ζ') is $\langle \underline{e} \rangle ^\frown \eta ^\frown \zeta$

$\Lambda \alpha \phi$ (read 'for all α, ϕ') is $\langle \underline{u} \rangle ^\frown \alpha ^\frown \phi$

$H\phi$ (read 'it has always been the case that ϕ') is $\langle \underline{h} \rangle ^\frown \phi$

$G\phi$ (read 'it will always be the case that ϕ') is $\langle \underline{g} \rangle ^\frown \phi$

$R\phi$ is $\langle \underline{r} \rangle ^\frown \phi$

$K\phi$ is $\langle \underline{k} \rangle ^\frown \phi$

From this point forward, '$^\frown$' will often be omitted.

D4. $V\alpha\phi$ (read 'for some α, ϕ') is $\neg \Lambda \alpha \neg \phi$

$(\phi \vee \psi)$ (read 'ϕ or ψ') is $(\neg \phi \rightarrow \psi)$

$(\phi \wedge \psi)$ (read 'ϕ and ψ') is $\neg(\phi \rightarrow \neg \psi)$

$(\phi \leftrightarrow \psi)$ (read 'ϕ if and only if ψ') is $((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi))$

$L\phi$ (read 'it is always the case that ϕ') is

$(H\phi \wedge (\phi \wedge G\phi))$

$M\phi$ (read 'it is sometimes the case that ϕ') is $\neg L \neg \phi$

$P\phi$ (read 'it has sometime been the case that ϕ') is

$\neg H \neg \phi$

$F\phi$ (read 'it will sometime be the case that ϕ') is

$\neg G \neg \phi$

D5. A <u>language</u> is a set of non-logical constants.

D6. Tm, or the set of <u>terms</u>, is the intersection of all the sets Γ such that

(1) Every variable is in Γ, and

(2) If δ is a k-place operation letter and

$\zeta_0, \ldots, \zeta_{k-1} \in \Gamma$, then $\delta^\frown\zeta_0{}^\frown\ldots{}^\frown\zeta_{k-1} \in \Gamma$.

The following trivial theorem justifies a form of induction on the set of terms.

T2. If (1) every variable is in Γ and

(2) for each $k \in \omega$, k-place operation letter δ and terms $\zeta_0, \ldots, \zeta_{k-1} \in \Gamma$, $\delta\zeta_0\ldots\zeta_{k-1} \in \Gamma$,

then every term is in Γ.

D7. AFm, or the set of <u>atomic formulas</u> is the set of expressions $\eta=\zeta$ such that η and ζ are terms, together with the set of expressions $\pi^\frown\zeta_0{}^\frown\ldots{}^\frown\zeta_{k-1}$ where π is a k-place predicate letter and $\zeta_0, \ldots, \zeta_{k-1}$ are terms.

D8. Fm, or the set of <u>formulas</u>, is the intersection of all the sets Γ such that

(1) Every atomic formula is in Γ

(2) If $\phi, \psi \in \Gamma$ and α is a variable, then $\neg\phi$, $(\phi \rightarrow \psi)$, $\Lambda\alpha\phi$, $H\phi$, $G\phi$, $K\phi$ and $R\phi \in \Gamma$.

The following theorem is a simple consequence of the preceding definitions. It will be used throughout without further acknowledgement both in the proofs of further theorems and as a justification for inductive definitions.

T3. If (1) every atomic formula is in Γ and

(2) for all formulas ϕ, ψ and variables α, if $\phi \in \Gamma$ and $\psi \in \Gamma$, then $\neg\phi$, $(\phi \rightarrow \psi)$, $\Lambda\alpha\phi$, $H\phi$, $G\phi$, $K\phi$ and

$R\phi \in \Gamma$;

then every formula is in Γ.

Since the set of expressions is denumerable, we will allow ourselves to speak of the k^{th} expression; and we will also speak of the k^{th} member of various subsets of the set of expressions.

D9. If θ, ξ are expressions, then θ occurs in ξ if and only if there are expressions ζ, η such that ξ is $\zeta^\frown\theta^\frown\eta$.

D10. If ϕ is a formula, then $fv(\phi)$, or the set of free variables of ϕ, is defined recursively as follows:

(1) If ϕ is an atomic formula, then $fv(\phi)$ is the set of variables that occur in ϕ

(2) $fv(\phi \rightarrow \psi)$ is $fv(\phi) \cup fv(\psi)$

(3) $fv(\Lambda\alpha\phi)$ is $fv(\phi) \sim \{\alpha\}$

(4) $fv(\neg\phi)$, $fv(H\phi)$, $fv(G\phi)$, $fv(K\phi)$ and $fv(R\phi)$ are all $fv(\phi)$

Convention: From here forward it is assumed that α, β, γ (also α', etc.) are variables; η, ζ are terms; ϕ, ψ and χ are formulas; and Γ is a set of formulas.

D11. St, or the set of sentences, is the set of formulas such that $fv(\phi)$ is 0.

D12. If L is a language, then Tm_L, or the set of terms of L, is the set of terms η such that every non-logical constant that occurs in η is in L. AFm_L, Fm_L, and St_L are defined in the analogous manner.

D13. $L(\Gamma)$, or the language of Γ is the set of non-logical constants that occur in some formula in Γ.

D14. We define ϕ contains R outside the scope of K by the following recursion:

(1) If ϕ is atomic, then ϕ does not contain R outside the scope of K.

(2) $\phi \rightarrow \psi$ contains R outside the scope of K if and only if ϕ contains R outside the scope of K or ψ contains R outside the scope of K.

(3) $K\phi$ does not contain R outside the scope of K.

(4) $R\phi$ contains R outside the scope of K.

(5) $\neg\phi$, $H\phi$, $G\phi$, and $\Lambda\alpha\phi$ each contain R outside the scope of K if and only if ϕ contains R outside the scope of K.

2. Substitution

D15. (Recursive) If ξ, ξ' are terms or atomic formulas, then ξ' is obtained from ξ by replacing 0 or more occurrences of (the term) η by η', if and only if one or more of the following holds:

(1) ξ is ξ'

(2) ξ is η and ξ' is η'

(3) There are k, π, $\zeta_0, \ldots, \zeta_{k-1}$, $\zeta'_0, \ldots, \zeta'_{k-1}$ such

that $k \in \omega$, π is a k-place predicate letter or π is a k-place operation letter, ξ is $\pi\zeta_0 \ldots \zeta_{k-1}$, ξ' is $\pi\zeta'_0 \ldots \zeta'_{k-1}$, and, for each $i < k$, ζ'_i is obtained from ζ_i by replacing 0 or more occurrences of η by η'.

(4) There are terms ζ, ζ', θ and θ' such that ζ' is obtained from ζ by replacing 0 or more occurrences of η by η', θ' is obtained from θ by replacing 0 or more occurrences of η by η', ξ is $\zeta=\theta$ and ξ' is $\zeta'=\theta'$.

D16. $R(\phi',\phi,\psi',\psi)$, or <u>ϕ' is obtained from ϕ by replacing 0 or more occurrences of (the formula) ψ by ψ'</u>, if and only if one or more of the following holds:

(1) ϕ' is ϕ

(2) ϕ' is ψ' and ϕ is ψ

(3) There are formulas χ, χ', θ, θ' such that ϕ' is $\chi' \rightarrow \theta'$, ϕ is $\chi \rightarrow \theta$, $R(\chi',\chi,\psi',\psi)$ and $R(\theta',\theta,\psi',\psi)$

(4) There are formulas χ,χ' such that $R(\chi',\chi,\psi',\psi)$ and either ϕ' is $\neg\chi'$ and ϕ is $\neg\chi$, ϕ' is $H\chi'$ and ϕ is $H\chi$, ϕ' is $G\chi'$ and ϕ is $G\chi$, ϕ' is $K\chi'$ and ϕ' is $R\chi'$ and ϕ is $R\chi$, or there is a variable α such that ϕ' is $\Lambda\alpha\chi'$ and ϕ is $\Lambda\alpha\chi$.

D17. We define $RP(\phi',\phi,\psi',\psi)$, or <u>$\phi'$ results from ϕ by replacing 0 or more positive occurrences of ψ by ψ'</u>, and $RN(\phi',\phi,\psi',\psi)$, or <u>$\phi'$ results from ϕ by replacing 0 or more</u>

negative occurrences of ψ by ψ', together recursively as follows:

(1) If ϕ' is ϕ, then $RP(\phi',\phi,\psi',\psi)$ and $RN(\phi',\phi,\psi',\psi)$

(2) If ϕ' is ψ' and ϕ is ψ, then $RP(\phi',\phi,\psi',\psi)$

(3) If ϕ' is not ϕ, and it is not the case that both ϕ' is ψ' and ϕ is ψ, then

(a) If there are formulas $\chi,\chi',\theta,\theta'$ such that ϕ is $\chi \rightarrow \theta$ and ϕ' is $\chi' \rightarrow \theta'$, then

(i) $RP(\phi',\phi,\psi',\psi)$ if and only if both $RP(\theta',\theta,\psi',\psi)$ and $RN(\chi',\chi,\psi',\psi)$

(ii) $RN(\phi',\phi,\psi',\psi)$ if and only if both $RN(\theta',\theta,\psi',\psi)$ and $RP(\chi',\chi,\psi',\psi)$

(b) If there are formulas χ,χ' such that ϕ' is $\neg\chi'$ and ϕ is $\neg\chi$, then

(i) $RP(\phi',\phi,\psi',\psi)$ if and only if $RN(\chi',\chi,\psi',\psi)$

(ii) $RN(\phi',\phi,\psi',\psi)$ if and only if $RP(\chi',\chi,\psi',\psi)$

(c) If there are formulas χ,χ' such that either ϕ' is $H\chi'$ and ϕ is $H\chi$, ϕ' is $G\chi'$ and ϕ is $G\chi$, ϕ' is $K\chi'$ and ϕ is $K\chi$, or ϕ' is $R\chi'$ and ϕ is $R\chi$, then

(i) $RP(\phi',\phi,\psi',\psi)$ if and only if $RP(\chi',\chi,\psi',\psi)$

(ii) $RN(\phi',\phi,\psi',\psi)$ if and only if $RN(\chi',\chi,\psi',\psi)$

(d) If there are formulas χ, χ' and a variable α such that ϕ' is $\Lambda\alpha\chi'$ and ϕ is $\Lambda\alpha\chi$, then

(i) $RP(\phi', \phi, \psi', \psi)$ if and only if $RP(\chi', \chi, \psi', \psi)$

(ii) $RN(\phi', \phi, \psi', \psi)$ if and only if $RN(\chi', \chi, \psi', \psi)$

D18. If f is a function, Dom(f) $\leq$ Iv, and Rng(f) $\leq$ Tm, then rep(ϕ,f), or the <u>replacement of variables in ϕ according to f</u>, is introduced recursively as follows:

(1) If α is a variable, then

(a) If $\alpha \in$ Dom(f), rep(α,f) is f(α)

(b) If $\alpha \notin$ Dom(f), rep(α,f) is α

(2) If π is a k-place operation letter or a k-place predicate letter and $\zeta_0, \ldots, \zeta_{k-1}$ are terms, then rep($\pi\zeta_0 \ldots \zeta_{k-1}$,f) is rep($\zeta_0$,f)...rep($\zeta_{k-1}$,f)

(3) If ζ, ζ' are terms, then rep($\zeta=\zeta'$,f) is rep(ζ,f)=rep(ζ',f)

(4) (a) rep($\neg\phi$,f) is $\neg$rep(ϕ,f)

(b) rep($\phi \rightarrow \psi$,f) is (rep(ϕ,f) $\rightarrow$ rep(ψ,f))

(c) rep($\Lambda\alpha\phi$,f) is Λrep(α,f)rep(ϕ,f)

(d) rep(Hϕ,f) is H rep(ϕ,f)

(e) rep(Gϕ,f) is G rep(ϕ,f)

(f) rep(Kϕ,f) is K rep(ϕ,f)

(g) rep(Rϕ,f) is R rep(ϕ,f)

T4. If ξ is a term or a formula, f and g are one-to-one functions, Fld(f) $\leq$ Iv, Fld(g) $\leq$ Iv, Rng(f) $\leq$ Dom(g), and each variable that occurs in ξ is in Dom(f), then rep(ξ,g/f) is rep(rep(ξ,g),f).

Proof: A trivial induction, using T2 and T3.

D19. If f is a function, Fld f $\leq$ Iv, and each variable that occurs in some formula in Γ is in Dom(f), then REP(Γ,f) is the set of formulas ϕ such that ϕ is rep(ψ,f), for some $\psi \in \Gamma$.

D20. If ξ is a term or a formula, then ra(α,η,ξ), or <u>the result of replacing all occurrences of α by η in ξ</u> is rep(ξ,f), where f is $\{\langle\alpha,\eta\rangle\}$.

T5. If ξ is a term or a formula, then

(a) ra(α,α,ξ) is ξ

(b) If α does not occur in ξ, then ra(α,η,ξ) is ξ

(c) If α does not occur in η, then α does not occur in ra(α,η,ξ)

Proof: (a), (b) and (c) are all immediate consequences of D20, T2 and T3.

T6. If β does not occur in ξ, and ξ is a term or an atomic formula, then ra(β,α,ra(α,β,ξ)) is ξ.

Proof: A trivial induction.

T7. If β does not occur in ϕ, then

(a) If $\alpha \notin$ fv(ϕ), then fv(ra(α,β,ϕ)) is fv(ϕ)

(b) If $\alpha \in fv(\phi)$, then $fv(ra(\alpha,\beta,\phi))$ is $(fv(\phi) \dot{\sim} \{\alpha\}) \dot{\cup} \{\beta\}$

Proof: Let Γ be the set of formulas ϕ such that the theorem holds for all α and β.

(1) Suppose that ϕ is an atomic formula and β does not occur in ϕ.

(a) If $\alpha \notin fv(\phi)$, then α does not occur in ϕ and, by T5b, $ra(\alpha,\beta,\phi)$ is ϕ

(b) If $\alpha \in fv(\phi)$, then α is not β. By T5c, $fv(ra(\alpha,\beta,\phi))$ is $(fv(\phi) \dot{\sim} \{\alpha\}) \dot{\cup} \{\beta\})$

(2) Suppose that $\phi \in \Gamma$ and β does not occur in $\neg\phi$. Then β does not occur in ϕ.

(a) If $\alpha \notin fv(\neg\phi)$, then $\alpha \notin fv(\phi)$ and $fv(ra(\alpha,\beta,\neg\phi))$ is $fv(\neg ra(\alpha,\beta,\phi))$ is $fv(ra(\alpha,\beta,\phi))$ is $fv(\phi)$ is $fv(\neg\phi)$.

(b) If $\alpha \in fv(\neg\phi)$, then $\alpha \in fv(\phi)$ and $fv(ra(\alpha,\beta,\neg\phi))$ is $fv(\neg ra(\alpha,\beta,\phi))$ is $fv(ra(\alpha,\beta,\phi))$ is $(fv(\phi) \dot{\sim} \{\alpha\}) \dot{\cup} \{\beta\}$ is $(fv(\neg\phi) \dot{\sim} \{\alpha\}) \dot{\cup} \{\beta\}$.

(3) Suppose that $\phi,\psi \in \Gamma$ and β does not occur in $(\phi \rightarrow \psi)$. Then β occurs neither in ϕ nor in ψ.

(a) If $\alpha \notin fv(\phi \rightarrow \psi)$ then $\alpha \notin fv(\phi)$, $\alpha \notin fv(\psi)$ and $fv(ra(\alpha,\beta,\phi \rightarrow \psi))$ is $fv(ra(\alpha,\beta,\phi) \rightarrow ra(\alpha,\beta,\psi))$ is $fv(ra(\alpha,\beta,\phi)) \dot{\cup} fv(ra(\alpha,\beta,\psi))$ is $fv(\phi) \dot{\cup} fv(\psi)$ is $fv(\phi \rightarrow \psi)$.

(b) Suppose $\alpha \in fv(\phi \rightarrow \psi)$; suppose further, without loss of generality, that $\alpha \in fv(\phi)$ and $\alpha \in fv(\psi)$.

Then $\mathrm{fv}(\mathrm{ra}(\alpha,\beta,\phi\rightarrow\psi))$ is $\mathrm{fv}(\mathrm{ra}(\alpha,\beta,\phi)\rightarrow\mathrm{ra}(\alpha,\beta,\psi))$ is $\mathrm{fv}(\mathrm{ra}(\alpha,\beta,\phi))\cup\mathrm{fv}(\mathrm{ra}(\alpha,\beta,\psi))$ is $(\mathrm{fv}(\phi)\sim\{\alpha\})\cup\{\beta\}\cup\mathrm{fv}(\psi)$ is $((\mathrm{fv}(\phi)\cup\mathrm{fv}(\psi))\sim\{\alpha\})\cup\{\beta\}$ is $(\mathrm{fv}(\phi\rightarrow\psi)\sim\{\alpha\})\cup\{\beta\}$.

(4) Suppose that $\phi \notin \Gamma$ and β does not occur in $\Lambda\gamma\phi$.

(a) Suppose that $\alpha \notin \mathrm{fv}(\Lambda\gamma\phi)$. We take three cases:

First, suppose that α is γ and $\alpha \in \mathrm{fv}(\phi)$. Then, since $\phi \in \Gamma$, $\mathrm{fv}(\mathrm{ra}(\alpha,\beta,\phi))$ is $\mathrm{fv}(\phi)$ and $\mathrm{fv}(\mathrm{ra}(\alpha,\beta,\Lambda\gamma\phi))$ is $\mathrm{fv}(\Lambda\beta\mathrm{ra}(\alpha,\beta,\phi))$ is $\mathrm{fv}(\mathrm{ra}(\alpha,\beta,\phi))\sim\{\beta\}$ is $\mathrm{fv}(\phi)\sim\{\beta\}$ is (since β does not occur in ϕ) $\mathrm{fv}(\phi)$.

Second, suppose that α is γ and $\alpha \in \mathrm{fv}(\phi)$. Then, since $\phi \in \Gamma$, $\mathrm{fv}(\mathrm{ra}(\alpha,\beta,\phi))$ is $(\mathrm{fv}(\phi)\sim\{\alpha\})\cup\{\beta\}$ and $\mathrm{fv}(\mathrm{ra}(\alpha,\beta,\Lambda\gamma\phi))$ is $\mathrm{fv}(\Lambda\beta\mathrm{ra}(\alpha,\beta,\phi))$ is $\mathrm{fv}(\mathrm{ra}(\alpha,\beta,\phi))\sim\{\beta\}$ is $((\mathrm{fv}(\phi)\sim\{\alpha\})\cup\{\beta\})\sim\{\beta\}$ is $(\mathrm{fv}(\Lambda\gamma\phi)\cup\{\beta\})\sim\{\beta\}$. Since β does not occur in $\Lambda\gamma\phi$, this is $\mathrm{fv}(\Lambda\gamma\phi)$.

Third, suppose that α is not γ. Then $\alpha \in \mathrm{fv}(\phi)$. Since $\phi \in \Gamma$, $\mathrm{fv}(\mathrm{ra}(\alpha,\beta,\phi))$ is $\mathrm{fv}(\phi)$ and $\mathrm{fv}(\mathrm{ra}(\alpha,\beta,\Lambda\gamma\phi))$ is $\mathrm{fv}(\Lambda\gamma\mathrm{ra}(\alpha,\beta,\phi))$ is $\mathrm{fv}(\mathrm{ra}(\alpha,\beta,\phi))\sim\{\gamma\}$ is $\mathrm{fv}(\phi)\sim\{\gamma\}$ is $\mathrm{fv}(\Lambda\gamma\phi)$.

(b) Suppose that $\alpha \in \mathrm{fv}(\Lambda\gamma\phi)$. Then α is not γ and $\alpha \in \mathrm{fv}(\phi)$. Since $\phi \in \Gamma$, $\mathrm{fv}(\mathrm{ra}(\alpha,\beta,\phi))$ is $(\mathrm{fv}(\phi)\sim\{\alpha\})\cup\{\beta\}$. Then $\mathrm{fv}(\mathrm{ra}(\alpha,\beta,\Lambda\gamma\phi))$ is $\mathrm{fv}(\Lambda\gamma\mathrm{ra}(\alpha,\beta,\phi))$ is $\mathrm{fv}(\mathrm{ra}(\alpha,\beta,\phi))\sim\{\gamma\}$ is $((\mathrm{fv}(\phi)\sim\{\alpha\})\cup\{\beta\})\sim\{\gamma\}$. Since α, β and γ are distinct, this is $((fv(\phi)\sim\{\gamma\})\cup\{\beta\})\sim\{\alpha\}$, which is $(fv(\Lambda\gamma\phi)\cup\{\beta\})\sim\{\alpha\}$.

(5) Suppose that $\phi \in \Gamma$; then (as in case (2)) $H\phi$, $G\phi$, $K\phi$ and $R\phi$ are all in Γ.

Proper substitution is defined in such a way that any term may be properly substituted for any variable in any formula. This is done by using the familiar device of replacing any bound variables that lead to a conflict of variables.

D21. $ps(\eta,\alpha,\phi)$, or <u>the result of proper substitution of η for α in ϕ</u>, is introduced recursively as follows:

(1) If ϕ is an atomic formula, then $ps(\eta,\alpha,\phi)$ is $ra(\alpha,\eta,\phi)$

(2) (a) $ps(\eta,\alpha,\neg\phi)$ is $\neg ps(\eta,\alpha,\phi)$

(b) $ps(\eta,\alpha,\phi\rightarrow\psi)$ is $(ps(\eta,\alpha,\phi)\rightarrow ps(\eta,\alpha,\psi))$

(c) (i) $ps(\eta,\alpha,\Lambda\beta\phi)$ is $\Lambda\beta\phi$ if $\alpha \notin fv(\Lambda\beta\phi)$

(ii) $ps(\eta,\alpha,\Lambda\beta\phi)$ is $\Lambda\beta ps(\eta,\alpha,\phi)$ if $\alpha \in fv(\Lambda\beta\phi)$ and β does not occur in η

(iii) $ps(\eta,\alpha,\Lambda\beta\phi)$ is $\Lambda\gamma ps(\eta,\alpha,ra(\beta,\gamma,\phi))$, where γ is the first variable such that γ occurs neither in η nor in ϕ, if $\alpha \in fv(\Lambda\beta\phi)$ and β occurs in η

(d) $ps(\eta,\alpha,H\phi)$ is $H\ ps(\eta,\alpha,\phi)$

(e) $ps(\eta,\alpha,G\phi)$ is $G\ ps(\eta,\alpha,\phi)$

(f) $ps(\eta,\alpha,K\phi)$ is $K\ ps(\eta,\alpha,\phi)$

(g) $ps(\eta,\alpha,R\phi)$ is R $ps(\eta,\alpha,\phi)$

The rank of ϕ is the number of sentential connectives and quantifiers in ϕ. It is sometimes convenient to prove that all formulas belong to a certain class by induction on the ranks of formulas.

D22. $rk(\phi)$, or the <u>rank of ϕ</u>, is introduced recursively as follows:

(1) If ϕ is an atomic formula, then $rk(\phi)$ is 0

(2) $rk(\neg\phi)$, $rk(\bigwedge\alpha\phi)$, $rk(H\phi)$, $rk(G\phi)$, $rk(K\phi)$, and $rk(R\phi)$ are all $rk(\phi)+1$

(3) $rk(\phi \rightarrow \psi)$ is $rk(\phi)+rk(\psi)+1$

T8. (a) $rk(ra(\alpha,\eta,\phi))$ is $rk(\phi)$

(b) $rk(ps(\eta,\alpha,\phi))$ is $rk(\phi)$

Proof: A trivial induction using T3

T9. (a) $ps(\alpha,\alpha,\phi)$ is ϕ

(b) If $\alpha \notin fv(\phi)$, then $ps(\eta,\alpha,\phi)$ is ϕ

(c) If α does not occur in η, then $\alpha \notin fv(ps(\gamma,\alpha,\phi))$

(d) If $\alpha \in fv(\phi)$ and γ is a variable, then $fv(ps(\gamma,\alpha,\phi))$ is $(fv(\phi)\sim\{\alpha\})\cup\{\gamma\}$

Proof: (We proceed by induction on the rank of ϕ.) Let Γ be the set of formulas ϕ such that (a), (b) and (c) all hold for ϕ.

(1) Suppose that ϕ is an atomic formula

(a) $ps(\alpha,\alpha,\phi)$ is ϕ, by T5a

(b) If $\alpha \notin fv(\phi)$, then α does not occur in ϕ, and $ps(\eta,\alpha,\phi)$ is ϕ, by T5b.

(c) Follows immediately from T5c.

(d) Follows immediately from T7b.

(2) Suppose that ϕ is $\neg\psi$, for some $\psi \in \Gamma$.

(a) $ps(\alpha,\alpha,\neg\psi)$ is $\neg ps(\alpha,\alpha,\psi)$ is $\neg\psi$.

(b) If $\alpha \notin fv(\neg\psi)$, then $\alpha \notin fv(\psi)$ and $ps(\eta,\alpha,\psi)$ is ψ. Then $ps(\eta,\alpha,\neg\psi)$ is $\neg ps(\eta,\alpha,\psi)$ is $\neg\psi$.

(c) If α does not occur in η, then $\alpha \notin fv(ps(\eta,\alpha,\psi))$ and $\alpha \notin fv(ps(\eta,\alpha,\neg\psi))$.

(d) If $\alpha \in fv(\phi)$, then $fv(ps(\gamma,\alpha,\phi))$ is $fv(\neg ps(\gamma,\alpha,\psi))$ is $fv(ps(\gamma,\alpha,\psi))$ is $(fv(\psi) \sim \{\alpha\}) \cup \{\gamma\}$ is $(fv(\phi) \sim \{\alpha\}) \cup \{\gamma\}$.

(3) Suppose that ϕ is $\psi \to \chi$, for some $\psi,\chi \in \Gamma$. This case is similar to case (2).

(4) Suppose that ϕ is $\Lambda\beta\psi$ for some $\psi \in \Gamma$. We take two cases:

First, suppose that α is β.

(a) By D21, $ps(\alpha,\alpha,\Lambda\beta\psi)$ is $\Lambda\beta\psi$.

(b) By D21 again, $ps(\eta,\alpha,\Lambda\beta\psi)$ is $\Lambda\beta\psi$.

(c) Suppose that α does not occur in η. Since $\alpha \notin fv(\Lambda\beta\psi)$, $ps(\eta,\alpha,\Lambda\beta\psi)$ is $\Lambda\beta\psi$ and $\alpha \notin fv(ps(\eta,\alpha,\Lambda\beta\psi))$.

(d) Since α is β, $\alpha \notin fv(\phi)$ and this case holds vacuously.

Secondly, suppose that α is not β.

(a) If $\alpha \notin fv(\Lambda\beta\psi)$, then $ps(\alpha,\alpha,\Lambda\beta\psi)$ is $\Lambda\beta\phi$, by D21. If $\alpha \in fv(\Lambda\beta\psi)$, then (since β does not occur in α) $ps(\alpha,\alpha,\Lambda\beta\psi)$ is $\Lambda\beta ps(\alpha,\alpha,\psi)$. Since $\psi \in \Gamma$, this is $\Lambda\beta\psi$.

(b) Suppose that $\alpha \notin fv(\Lambda\beta\psi)$. Then, by D21, $ps(\eta,\alpha,\Lambda\beta\psi)$ is $\Lambda\beta\psi$.

(c) Suppose that α does not occur in η. Since $\psi \in \Gamma$, $\alpha \notin fv(ps(\eta,\alpha,\psi))$. We take three subcases:

(i) If $\alpha \notin fv(\Lambda\beta\psi)$, then (D21) $\alpha \notin fv(ps(\eta,\alpha,\Lambda\beta\psi))$.

(ii) If $\alpha \in fv(\Lambda\beta\psi)$ and β does not occur in η, then $ps(\eta,\alpha,\Lambda\beta\psi)$ is $\Lambda\beta ps(\eta,\alpha,\psi)$, and $\alpha \notin fv(ps(\eta,\alpha,\Lambda\beta\psi))$

(iii) Suppose that $\alpha \in fv(\Lambda\beta\psi)$ and β occurs in η. Let γ be the first variable that occurs neither in η nor in ψ. Then $ps(\eta,\alpha,\Lambda\beta\psi)$ is $\Lambda\gamma ps(\eta,\alpha,ra(\beta,\gamma,\psi))$. By T8 and the inductive hypothesis, $\alpha \notin fv(ps(\eta,\alpha,ra(\beta,\gamma,\psi)))$, and hence $\alpha \notin fv(ps(\eta,\alpha,\Lambda\beta\psi))$.

(d) Suppose that $\alpha \in fv(\phi)$. We take two subcases:

(i) γ is β; then $ps(\gamma,\alpha,\phi)$ is $\Lambda\gamma' ps(\gamma,\alpha,ra(\beta,\gamma',\psi))$ (where γ' is the first variable that occurs neither in γ nor in ϕ), and $fv(ps(\gamma,\alpha,\phi))$ is $fv(ps(\gamma,\alpha,ra(\beta,\gamma',\psi))) \sim \{\gamma'\}$ is (by the inductive hypothesis and T8) $((fv(ra(\beta,\gamma',\psi)) \sim \{\alpha\}) \cup \{\gamma\}) \sim \{\gamma'\}$ is

$((fv(ra(\beta,\gamma',\psi))\sim\{\alpha\})\sim\{\gamma'\})\cup\{\gamma\}$ is

$((fv(ra(\beta,\gamma',\psi))\sim\{\gamma'\})\sim\{\alpha\})\cup\{\gamma\}$ is (by T7)

$((fv(\psi)\sim\{\beta\})\sim\{\alpha\})\cup\{\gamma\}$ is $(fv(\phi)\sim\{\alpha\})\cup\{\gamma\}$.

(ii) γ is not β; then $ps(\gamma,\alpha,\phi)$ is $\Lambda\beta(ps(\gamma,\alpha,\phi))$ and $fv(ps(\gamma,\alpha,\phi))$ is $fv(ps(\gamma,\alpha,\psi))\sim\{\beta\}$ is $((fv(\psi)\sim\{\alpha\})\cup\{\gamma\})\sim\{\beta\}$ is $((fv(\psi)\sim\{\beta\})\sim\{\alpha\})\cup\{\gamma\}$ is $(fv(\phi)\sim\{\alpha\})\cup\{\gamma\}$.

(5) If ϕ is $H\psi$, $G\psi$, $K\psi$ or $R\psi$ and $\psi \in \Gamma$, the proof is analogous to that for case (2).

3. Other Syntactic Notions

It will be convenient in the following chapters to think of (for instance) the formula $H\Lambda\alpha K\Lambda\beta G\phi$ as a generalization of ϕ, obtained by prefixing the generalizer $H\Lambda\alpha K\Lambda\beta G0$ to the formula ϕ. We also wish to distinguish the universal part and the tense part of a generalizer. The universal part of $H\Lambda\alpha K\Lambda\beta G0$ is $\Lambda\alpha\Lambda\beta 0$ and the tense part is $HKG0$. The following definitions introduce these notions formally.

D23. (1) We define the class of n-level generalizers recursively as follows:

(a) τ is a 0-level generalizer if and only if τ is 0.

(b) τ is an n+1-level generalizer if and only if τ is $H\xi$, $G\xi$, $K\xi$ or $\Lambda\alpha\xi$, where ξ is an n-level generalizer and α is a variable.

(2) ψ is an <u>n-level generalization</u> of ϕ if and only if ψ is $\tau^\frown\phi$, where τ is an n-level generalizer.

(3) ψ is a <u>generalization</u> of ϕ if and only if ψ is an n-level generalization of ϕ for some $n \in \omega$ (including 0).

D24. (1) τ is a <u>universal generalizer</u> if and only if τ is a generalizer and neither $H\xi$, $G\xi$, nor $K\xi$ occur in τ, for any ξ.

(2) ψ is a <u>universal generalization</u> of ϕ if and only if ψ is $\tau^\frown\phi$, where τ is a universal generalizer.

D25. If τ is a generalizer, then the <u>universal part</u> of and the <u>tense part</u> of τ are defined together recursively as follows:

(1) The universal part of 0 is 0.
The tense part of 0 is 0.

(2) If τ is a generalizer, ξ is the universal part of τ and τ' is the tense part of τ, then

(a) If α is a variable, then the universal part of $\Lambda\alpha\tau$ is $\Lambda\alpha\xi$, and the tense part of $\Lambda\alpha\tau$ is τ'.

(b) The universal part of $H\tau$ is ξ.
The tense part of $H\tau$ is $H\tau'$.

(c) The universal part of $G\tau$ is ξ.
The tense part of $G\tau$ is $G\tau'$.

(d) The universal part of $K\tau$ is ξ.

The tense part of $K\tau$ is $K\tau'$.

D26. If τ is a generalizer, then τ' is a <u>subgeneralizer</u> of τ if and only if τ and τ' are generalizers and there is a generalizer ξ such that τ is $\xi^\frown\tau'$.

D27. (1) ψ is a <u>closure</u> of ϕ if and only if ψ is $\Lambda\alpha_0 \ldots \Lambda\alpha_{n-1}\phi$, where $fv(\phi)$ is $\{\alpha_0, \ldots, \alpha_{n-1}\}$.

(2) The <u>standard closure</u> of ϕ is the formula $\Lambda\alpha_0 \ldots \Lambda\alpha_{n-1}\phi$, where $\alpha_0, \ldots, \alpha_{n-1}$ are (in order) the free variables of ϕ.

If Γ is a finite set of formulas, we will need the (standard) conjunction and disjunction of the formulas in Γ. If Γ is empty, we choose a tautology for the conjunction of the formulas in Γ and a contradiction for the disjunction of the formulas in Γ, in order to preserve the rule that a conjunction is true if and only if all of its conjuncts are true, and a disjunction is true if and only if at least one of its disjuncts is true.

D28. (1) If Γ is finite, then $CJ(\Gamma)$, or the <u>conjunction (in order) of the formulas in Γ</u>, is defined recursively as follows:

(a) If $\bar{\Gamma}$ is 0, the conjunction (in order) of the formulas of Γ is $\langle \underline{pr}_{0,0}\rangle \rightarrow \langle \underline{pr}_{0,0}\rangle$

(b) If $\bar{\Gamma}$ is 1, the conjunction (in order) of the formulas of Γ is ϕ, where ϕ is the formula in Γ.

(c) If $\bar{\Gamma} \geq 2$, then the conjunction (in order) of the formulas in Γ is $(\phi \wedge \psi)$, where ϕ is the first formula in Γ and ψ is the conjunction (in order) of the formulas in $\Gamma \sim \{\phi\}$.

(2) If Γ is finite, then $DJ(\Gamma)$, or the <u>disjunction (in order) of the formulas in Γ</u>, is defined recursively as follows:

(a) If $\bar{\Gamma}$ is 0, the disjunction (in order) of the formulas of Γ is $\langle \underline{pr}_{0,0} \rangle \rightarrow \neg \langle \underline{pr}_{0,0} \rangle$.

(b) If $\bar{\Gamma}$ is 1, the disjunction (in order) of the formulas of Γ is ϕ, where ϕ is the formula in Γ.

(c) If $\bar{\Gamma} \geq 2$, then the disjunction (in order) of the formulas in Γ is $(\phi \vee \psi)$ where ϕ is the first formula in Γ and ψ is the disjunction (in order) of the formulas in $\Gamma \sim \{\phi\}$.

CHAPTER II

SEMANTICS

A. The Formal System

1. Interpretations; Intension and Extension

We begin this chapter by setting out formally the system described in the Introduction. The following definitions (D29-D36) are constructed according to the general framework provided in Montague [7]. In particular, we follow Montague with regard to the notions of intension[7] and extension.

We begin by defining the notion of an interpretation. This definition is essentially unchanged from that presented in Cocchiarella [1], and we will not argue in this paper for the definition given. However, this is chosen only as one possible beginning; systems corresponding to ours could be constructed analogously building upon many systems of tense logic or modal logic. In the case of modal logic, it would be natural to read the analogue of the 'now' operator as 'actually'. The index operator seems to have no one natural reading either within tense logic or within modal logic.

D29. If L is a language, then $\mathcal{A}$ is an interpretation for L if and only if there are T, $\leq$, U, G such that:

(1) $\mathfrak{A}$ is $\langle T, \leq, U, G\rangle$

(2) G is a function with domain L

(3) $\leq$ is a reflexive linear ordering, the field of $\leq$ is T and $T \neq 0$

(4) $U \neq 0$

(5) For each $\pi \in L$, $G(\pi)$ is a function with domain T

(6) For each k-place predicate letter $\pi \in L$ and each $t \in T$, $G(\pi)(t) \subseteq U^k$

(7) For each k-place operation letter $\delta \in L$ and each $t \in T$, $G(\delta)(t)$ is a function from U^k into U

D30. If $\mathfrak{A}$ is an interpretation and $\mathfrak{A}$ is $\langle T, \leq, U, G\rangle$, then

(1) The set of <u>moments</u> of $\mathfrak{A}$ is T

(2) The set of <u>possible objects</u> of $\mathfrak{A}$, or $U_{\mathfrak{A}}$, is U

(3) The set of <u>points of reference</u> of $\mathfrak{A}$ is T×T

(4) The <u>language</u> of $\mathfrak{A}$, or $L_{\mathfrak{A}}$, is Dom(g)

D31. If $\mathfrak{A}$ is an interpretation for L, $\mathfrak{A}$ is $\langle T, \leq, U, G\rangle$ and $t \in T$, then $Ext_{t,\mathfrak{A}}(\zeta)$, or the <u>extension of ζ at t (according to $\mathfrak{A}$)</u> is introduced for an arbitrary term ζ of L by the following recursion:

(1) $Ext_{t,\mathfrak{A}}(v_n)$ is that function H with domain U^ω such that for each $x \in Dom(H)$, $H(x)$ is x_n

(2) If δ is a k-place operation letter in L and $\eta_0, \ldots, \eta_{k-1}$ are terms of L, then $Ext_{t,\mathfrak{A}}(\delta\eta_0 \ldots \eta_{k-1})$ is that function H with domain U^ω such that for each $x \in Dom(H)$, $H(x)$ is

$$G(\delta)(t)(\langle Ext_{t,\mathfrak{a}}(\eta_0)(x), \ldots, Ext_{t,\mathfrak{a}}(\eta_{k-1})(x)\rangle)$$

We think of $Ext_{t,\mathfrak{a}}(\eta)(x)$ as the object denoted by the term η at t, if the variables in η are assigned values according to x.

D32. If $\mathfrak{a}$ is an interpretation for L and η is a term of L, then $Int_{\mathfrak{a}}(\eta)$, or the <u>intension of η (according to $\mathfrak{a}$)</u> is that function H whose domain is the set of points of reference of $\mathfrak{a}$, and such that for each $\langle t,t'\rangle \in Dom(H)$, $H(\langle t,t'\rangle)$ is $Ext_{t,\mathfrak{a}}(\eta)$.

D33. If $\mathfrak{a}$ is an interpretation and $\mathfrak{a}$ is $\langle T, \leq, U, G\rangle$ then

(1) $t <_{\mathfrak{a}} t'$, or <u>t is earlier than t' (with respect to $\mathfrak{a}$)</u>, if and only if $t \leq t'$ and $t \neq t'$

(2) $t >_{\mathfrak{a}} t'$, or <u>t is later than t' (with respect to $\mathfrak{a}$)</u>, if and only if $t' <_{\mathfrak{a}} t$

D34. If $\mathfrak{a}$ is an interpretation for L and $\mathfrak{a}$ is $\langle T, \leq, U, G\rangle$, then $Int_{\mathfrak{a}}(\phi)$, or the <u>intension of ϕ (according to $\mathfrak{a}$)</u> is introduced for an arbitrary formula ϕ of L by the following recursive definition:

(1) If ζ, η are terms of L, then $Int_{\mathfrak{a}}(\zeta=\eta)$ is that function H with domain $T\times T$ such that for each $\langle t,t'\rangle \in T\times T$, $H(\langle t,t'\rangle)$ is the set of $x \in U^{\omega}$ such that $Ext_{t,\mathfrak{a}}(\zeta)(x)$ is $Ext_{t,\mathfrak{a}}(\eta)(x)$

(2) If π is a k-place predicate letter in L and $\eta_0, \ldots, \eta_{k-1}$ are terms of L, then $Int_{\mathfrak{a}}(\pi\eta_0 \ldots \eta_{k-1})$

is that function H with domain $T \times T$ such that for each $\langle t,t' \rangle \in T \times T$, $H(\langle t,t' \rangle)$ is the set of $x \in U^{\omega}$ such that $\langle Ext_{t,\mathcal{a}}(\eta_0)(x), \ldots, Ext_{t,\mathcal{a}}(\eta_{k-1})(x) \rangle \in G(\pi)(t)$

(3) If ϕ is a formula of L, then $Int_{\mathcal{a}}(\neg\phi)$ is that function H with domain $T \times T$ such that for each $i \in T \times T$, $H(i)$ is $U^{\omega} \sim Int_{\mathcal{a}}(\phi)(i)$

(4) If ϕ, ψ are formulas of L, then $Int_{\mathcal{a}}(\phi \rightarrow \psi)$ is that function H with domain $T \times T$ such that for each $i \in T \times T$, $H(i)$ is the set of $x \in U^{\omega}$ such that either $x \in Int_{\mathcal{a}}(\psi)(i)$ or $x \notin Int_{\mathcal{a}}(\phi)(i)$

(5) If ϕ is a formula of L, then $Int_{\mathcal{a}}(\Lambda v_n \phi)$ is that function H with domain $T \times T$ such that for each $i \in T \times T$, $H(i)$ is the set of $x \in U^{\omega}$ such that for each $y \in U$, $x^n_y \in Int_{\mathcal{a}}(\phi)(i)$

(6) If ϕ is a formula of L, then $Int_{\mathcal{a}}(H\phi)$ is that function H with domain $T \times T$ such that for each $\langle t,t' \rangle \in T \times T$, $H(\langle t,t' \rangle)$ is the set of $x \in U^{\omega}$ such that for each $t'' <_{\mathcal{a}} t$, $x \in Int_{\mathcal{a}}(\phi)(\langle t'',t' \rangle)$

(7) If ϕ is a formula of L, then $Int_{\mathcal{a}}(G\phi)$ is that function H with domain $T \times T$ such that for each $\langle t,t' \rangle \in T \times T$, $H(\langle t,t' \rangle)$ is the set of $x \in U^{\omega}$ such that for each $t'' >_{\mathcal{a}} t$, $x \in Int_{\mathcal{a}}(\phi)(\langle t'',t' \rangle)$

(8) If ϕ is a formula of L, then $Int_{\mathcal{a}}(K\phi)$ is that function H with domain $T \times T$ such that for each $\langle t,t' \rangle \in T \times T$, $H(\langle t,t' \rangle)$ is $Int_{\mathcal{a}}(\phi)(\langle t,t \rangle)$

(9) If ϕ is a formula of L, then $Int_{\mathcal{A}}(R\phi)$ is that function H with domain T×T such that for each $\langle t,t'\rangle \in T\times T$, $H(\langle t,t'\rangle)$ is $Int_{\mathcal{A}}(\phi)(\langle t',t'\rangle)$

2. Satisfaction and Truth

We list the following definition primarily in order to clarify the relation between intensions and the more familiar notion of satisfaction.

D35. If $\mathcal{A}$ is an interpretation for L, ϕ is a formula of L and $\langle t,t'\rangle$ is a point of reference of $\mathcal{A}$, then x satisfies ϕ at $\langle t,t'\rangle$ (according to $\mathcal{A}$) if and only if $x \in Int_{\mathcal{A}}(\phi)(\langle t,t'\rangle)$.

D36. If $\mathcal{A}$ is an interpretation for L, $\mathcal{A}$ is $\langle T,\leq,U,G\rangle$, ϕ is a formula of L and $i \in T\times T$, then ϕ is true at i (according to $\mathcal{A}$) if and only if $Int_{\mathcal{A}}(\phi)(i)$ is U^{ω}.

We list some immediate consequences of D34.

T10. If $\mathcal{A}$ is an interpretation, t and t' are moments of $\mathcal{A}$, $x \in U^{\omega}$, ϕ and ψ are formulas, and $n \in \omega$, then

(a) $x \in Int_{\mathcal{A}}((\phi \vee \psi))(\langle t,t'\rangle)$ if and only if either $x \in Int_{\mathcal{A}}(\phi)(\langle t,t'\rangle)$ or $x \in Int_{\mathcal{A}}(\psi)(\langle t,t'\rangle)$

(b) $x \in Int_{\mathcal{A}}((\phi \wedge \psi))(\langle t,t'\rangle)$ if and only if both $x \in Int_{\mathcal{A}}(\phi)(\langle t,t'\rangle)$ and $x \in Int_{\mathcal{A}}(\psi)(\langle t,t'\rangle)$

(c) $x \in Int_{\mathcal{A}}((\phi \leftrightarrow \psi))(\langle t,t'\rangle)$ if and only if either $x \in Int_{\mathcal{A}}(\phi)(\langle t,t'\rangle)$ and $x \in Int_{\mathcal{A}}(\psi)(\langle t,t'\rangle)$ or $x \notin Int_{\mathcal{A}}(\phi)(\langle t,t'\rangle)$ and $x \notin Int_{\mathcal{A}}(\psi)(\langle t,t'\rangle)$

(d) $x \in Int_{\mathfrak{a}}(\forall v_n \phi)(\langle t,t'\rangle)$ if and only if there is an $a \in U_{\mathfrak{a}}$ such that $x^n_a \in Int_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$

(e) $x \in Int_{\mathfrak{a}}(P\phi)(\langle t,t'\rangle)$ if and only if there is a $t'' <_{\mathfrak{a}} t$ such that $x \in Int_{\mathfrak{a}}(\phi)(\langle t'',t'\rangle)$

(f) $x \in Int_{\mathfrak{a}}(F\phi)(\langle t,t'\rangle)$ if and only if there is a $t'' >_{\mathfrak{a}} t$ such that $x \in Int_{\mathfrak{a}}(\phi)(\langle t'',t'\rangle)$

(g) $x \in Int_{\mathfrak{a}}(L\phi)(\langle t,t'\rangle)$ if and only if for each moment t'' of $\mathfrak{a}$, $x \in Int_{\mathfrak{a}}(\phi)(\langle t'',t'\rangle)$

(h) $x \in Int_{\mathfrak{a}}(M\phi)(\langle t,t'\rangle)$ if and only if, for some moment t'' of $\mathfrak{a}$, $x \in Int_{\mathfrak{a}}(\phi)(\langle t'',t'\rangle)$

Proof: Each part is an immediate consequence of the appropriate clauses of D34. Parts (g) and (h) also use the fact that $\leq_{\mathfrak{a}}$ is a connected relation.

The following remarks ((a)-(g)) are simple consequences of D34 and D36; we list them here in order to facilitate the discussion of the intuitive motivation for the semantics presented above. Suppose that $\mathfrak{a}$ is an interpretation for L, $\mathfrak{a}$ is $\langle T,\leq,U,G\rangle$, t and t' are moments of $\mathfrak{a}$, u is a variable, ϕ is a sentence of L, c and d are individual constants in L and S is a one-place predicate in L. Then (omitting mention of the relativization to $\mathfrak{a}$):

(a) c=d is true at $\langle t,t'\rangle$ if and only if G(c)(t)(0) is G(d)(t)(0).

(b) Sc is true at $\langle t,t'\rangle$ if and only if $\langle G(c)(t)(0)\rangle \in$ G(S)(t).

(c) $\bigwedge u S u$ is true at $\langle t,t'\rangle$ if and only if for every $x \in U$, $\langle x\rangle \in G(S)(t)$.

(d) $H\phi$ is true at $\langle t,t'\rangle$ if and only if ϕ is true at $\langle t'',t'\rangle$, for every moment t" earlier than t.

(e) $G\phi$ is true at $\langle t,t'\rangle$ if and only if ϕ is true at $\langle t'',t'\rangle$, for every moment t" later than t.

(f) $K\phi$ is true at $\langle t,t'\rangle$ if and only if ϕ is true at $\langle t,t\rangle$.

(g) $R\phi$ is true at $\langle t,t'\rangle$ if and only if ϕ is true at $\langle t',t'\rangle$.

3. What the 'then' Operator Does

If we omit clauses (8) and (9) from D34, and K and R from the language, we have the system of Cocchiarella [1]. If we omit only clause (8) from D34 and the index operator from the language, then we have the result of adding the 'now' operator (here written 'R') with the semantics given in Kamp [4], to the system of Cocchiarella [1].

In the full system with K and R, the interpretation of R remains the same, but it is important to notice that it is no longer possible to read it as 'now', except in certain contexts.

In the introduction, it was stated that 5 ($PKF\bigwedge x(R\,D(x) \rightarrow C(Jx))$) is a correct symbolization of 3 ('Jones was once going to cite everyone then driving too fast.'). As a further intuitive justification and clarification of our semantics, let us now, step by step, show how to find the truth value of 5 at a point of reference ⟨t,t⟩ according to an interpretation $\mathcal{A}$.

The idea is simple; in order to evaluate a formula, we must keep track of two moments--the moment (let us call it the 'primary moment') at which formulas that do not begin with R are evaluated, and the one (let us call it the 'secondary moment') at which formulas preceded by R are evaluated. We thus must take as points of reference ordered pairs of moments, the first members of which are taken as the primary moments, and the second members of which are taken as the secondary moments. The function of the operator K is to fix the secondary moment so that R will refer back to the context of K. The function of R is to make the secondary moment become the primary moment.

To continue with the evaluation, 5 is true at ⟨t,t⟩ according to $\mathcal{A}$

if and only if

There is a moment $t' <_{\mathcal{A}} t$ such that $KF\Lambda x(R\ D(x) \rightarrow C(Jx))$ is true at $\langle t',t\rangle$ (according to $\mathcal{A}$)

if and only if

There is a moment $t' <_{\mathcal{A}} t$ such that $F\Lambda x(R\ D(x) \rightarrow C(Jx))$ is true at $\langle t',t'\rangle$

if and only if

There are moments $t' <_{\mathcal{A}} t$ and $t'' >_{\mathcal{A}} t'$ such that $\Lambda x(R\ D(x) \rightarrow C(Jx))$ is true at $\langle t'',t'\rangle$

if and only if

There are moments $t' <_a t$ and $t'' >_a t'$ such that for all $y \in U^{\omega}$, if y satisfies R D(x) at $\langle t'', t' \rangle$, then y satisfies C(Jx) at $\langle t'', t' \rangle$

if and only if

There are moments $t' <_a t$ and $t'' >_a t'$ such that for all $y \in U^{\omega}$, if y satisfies D(x) at $\langle t', t' \rangle$, then y satisfies C(Jx) at $\langle t'', t' \rangle$

if and only if

There are moments $t' <_a t$ and $t'' >_a t$ such that for all $y \in U$, if $\langle y \rangle \in G(D)(t')$, then $\langle G(J)(t''), y \rangle \in G(C)(t'')$

if and only if (intuitively speaking)

There are moments t' before t and t" after t' such that everyone driving too fast at t' is cited by Jones at t".

The latter would seem to state the conditions under which 5 is true, and hence our semantics seems to be intuitively correct, at least as far as sentences like 5 are concerned. We see, in fact, that for any formula ϕ of the language with K and R, $PK\phi$ is the past tense of ϕ and $FK\phi$ is the future tense of ϕ.

B. Some Theorems about the Formal System

1. Validity

In the following two definitions the notions of satisfiability and validity, as opposed to weak satisfiability and strong validity, are primary. This is because sentences are evaluated with respect to their own moments of utterance, and reflects the fact that R refers back to the moment of utterance of the sentence when not within the scope of K.

The notions of weak satisfiability and strong validity are introduced for purely technical reasons, namely to deal with (proper) subformulas of sentences which are being evaluated with respect to their moments of utterance. Operators which occur in the original utterance might not occur in the subformula under consideration, but will affect the point of reference at which the subformula is evaluated.

D37. If Γ is a set of formulas, then

(1) Γ is <u>satisfiable</u> if and only there is an interpretation $\mathcal{A}$ for the language of Γ, a moment t of $\mathcal{A}$, and an $x \in U^{\omega}$ such that for each formula $\phi \in \Gamma$, $x \in \mathrm{Int}_{\mathcal{A}}(\phi)(\langle t,t\rangle)$

(2) Γ is <u>weakly satisfiable</u> if and only if there is an interpretation $\mathcal{A}$ for the language of Γ, moments t and t' of $\mathcal{A}$, and an $x \in U^{\omega}_{\mathcal{A}}$ such that for each formula $\phi \in \Gamma$, $x \in \mathrm{Int}_{\mathcal{A}}(\phi)(\langle t,t'\rangle)$

D38. If ϕ is a formula, then

(1) ϕ is <u>logically valid</u> if and only if for each interpretation $\mathcal{A}$ and moment t of $\mathcal{A}$, ϕ is true at

$\langle t,t\rangle$ (according to $\mathfrak{A}$)

(2) ϕ is <u>strongly logically valid</u> if and only if for all interpretations $\mathfrak{A}$ and moments t,t' of $\mathfrak{A}$, ϕ is true at $\langle t,t'\rangle$ (according to $\mathfrak{A}$)

T11. If ϕ is a formula, then

(a) ϕ is logically valid if and only if $\{\neg\phi\}$ is not satisfiable

(b) ϕ is strongly logically valid if and only if $\{\neg\phi\}$ is not weakly satisfiable.

Proof: A trivial consequence of D37 and D38

T12. If Γ is finite, $\langle T,\leq,U,G\rangle$ is an interpretation, $x \in U^{\omega}$ and $t,t' \in T$, then

(a) $x \in Int_{\mathfrak{A}}(CJ(\Gamma))(\langle t,t'\rangle)$ if and only if for each $\phi \in \Gamma$, $x \in Int_{\mathfrak{A}}(\phi)(\langle t,t'\rangle)$

(b) $x \in Int_{\mathfrak{A}}(DJ(\Gamma))(\langle t,t'\rangle)$ if and only if for some $\phi \in \Gamma$, $x \in Int_{\mathfrak{A}}(\phi)(\langle t,t'\rangle)$

Proof: A simple induction (on the size of Γ) using D28, D34 and T10.

T13. (a) If ϕ is logically valid, then $K\phi$ is strongly logically valid.

(b) If ϕ is strongly logically valid and ψ is a generalization of ϕ, then ψ is strongly logically valid.

(c) If ϕ is logically valid and ψ is a universal generalization of ϕ, then ψ is logically valid.

(d) If ϕ is logically valid and $\phi \to \psi$ is logically valid, then ψ is logically valid.

Proof:

(a) Suppose ϕ is logically valid and $K\phi$ is not strongly logically valid. Then $\{\neg K\phi\}$ is weakly satisfiable, and there are an interpretation $\mathfrak{A}$, moments t,t' of $\mathfrak{A}$ and $x \in U_{\mathfrak{A}}^{\omega}$ such that $x \in Int_{\mathfrak{A}}(\neg K\phi)(\langle t,t'\rangle)$; then $x \notin Int_{\mathfrak{A}}(K\phi)(\langle t,t'\rangle)$ and, by clause (8) of D34, $x \notin Int_{\mathfrak{A}}(\phi)(\langle t,t\rangle)$. But this contradicts the hypothesis.

(b) Suppose that ϕ is strongly logically valid. We will show by induction that, for any generalizer τ, $\tau\phi$ is strongly logically valid.

By hypothesis $0^\frown\phi$ is strongly logically valid.

Suppose that, for any n-level generalizer τ', $\tau'^\frown\phi$ is strongly logically valid. Suppose also that τ is an n+1-level generalizer. Then (D23) there is an n-level generalizer τ' such that τ is $H\tau'$, $G\tau'$, $K\tau'$ or $\Lambda\alpha\tau'$, for some variable α. By the inductive hypothesis, $\tau'^\frown\phi$ is strongly logically valid.

Suppose that $\tau\phi$ is not strongly logically valid. Then $\{\neg\tau\phi\}$ is weakly satisfiable, and there are an interpretation $\mathfrak{A}$, moments t and t' of $\mathfrak{A}$ and $x \in U_{\mathfrak{A}}^{\omega}$ such that $x \in Int_{\mathfrak{A}}(\neg\tau\phi)(\langle t,t'\rangle)$ and hence $x \notin Int_{\mathfrak{A}}(\tau\phi)(\langle t,t'\rangle)$. We now take four cases, according to D23.

(i) $\tau\phi$ is $H\tau'\phi$; by D34 clause (6), there is a $t'' <_{\mathcal{A}} t$ such that $x \notin Int_{\mathcal{A}}(\tau'\phi)(\langle t'', t'\rangle)$. But then $x \in Int_{\mathcal{A}}(\neg\tau'\phi)(\langle t'', t'\rangle)$, $\neg\tau'\phi$ is weakly satisfiable, and $\tau'\phi$ is not strongly logically valid, contradicting the hypothesis.

(ii) $\tau\phi$ is $G\tau'\phi$; this case is analogous to (i) using D34, clause (7).

(iii) $\tau\phi$ is $K\tau'\phi$; by D34 clauses (8) and (3) $x \in Int_{\mathcal{A}}(\neg\tau'\phi)(\langle t, t\rangle)$, again contradicting the hypothesis.

(iv) $\tau\phi$ is $\Lambda v_n\phi$, for some $n \in \omega$; then by D34 clauses (5) and (3), there is an $a \in U_{\mathcal{A}}$ such that $x_a^n \in Int_{\mathcal{A}}(\neg\tau'\phi)(\langle t, t\rangle)$, contradicting the hypothesis.

(c) Suppose ϕ is logically valid and there is a universal generalizer τ such that $\tau\phi$ is not logically valid. Let n be the smallest $n \in \omega$ such that there is an n-level universal generalizer τ such that $\tau\phi$ is not logically valid. Then there is a $k \in \omega$ and a formula ψ such that $\tau\phi$ is $\Lambda v_k\psi$ and ψ is logically valid. There are an interpretation $\mathcal{A}$, a moment t of $\mathcal{A}$ and $x \in U_{\mathcal{A}}^{\omega}$ such that $x \notin Int_{\mathcal{A}}(\Lambda v_k\psi)(\langle t, t\rangle)$. By clauses (5) and (3) of D34, there is an $a \in U_{\mathcal{A}}$ such that $x_a^k \notin Int_{\mathcal{A}}(\psi)(\langle t, t\rangle)$. But this is impossible, since ψ is logically valid.

(d) Suppose ϕ is logically valid and $\phi \rightarrow \psi$ is logically valid, but ψ is not logically valid. Then there are an interpretation $\mathcal{A}$, a moment t of $\mathcal{A}$ and an $x \in U^{\omega}_{\mathcal{A}}$ such that $x \notin Int_{\mathcal{A}}(\psi)(\langle t,t\rangle)$. By the hypothesis, $x \in Int_{\mathcal{A}}(\phi)(\langle t,t\rangle)$ and $x \in Int_{\mathcal{A}}((\phi \rightarrow \psi))(\langle t,t\rangle)$. By clause (4) of D34, $x \in Int_{\mathcal{A}}(\psi)(\langle t,t\rangle)$, which is a contradiction.

2. Basic Theorems about Intension and Extension

The next five theorems correspond to familiar theorems about the predicate calculus.

T14. If $\mathcal{A}$ is an interpretation, $\mathcal{A}$ is $\langle T,\leq,U,G\rangle$, $x,x' \in U^{\omega}$, $\eta \in Tm_{L_{\mathcal{A}}}$ and for each n such that v_n occurs in η, x_n is x'_n, then $Ext_{t,\mathcal{A}}(\eta)(x)$ is $Ext_{t,\mathcal{A}}(\eta)(x')$.

Proof: Assume that $\mathcal{A}$ is an interpretation, $\mathcal{A}$ is $\langle T,\leq,U,G\rangle$, $x,x' \in U^{\omega}$ and $t \in T$. Let Γ be the set of $Tm_{L_{\mathcal{A}}}$ such that, if for each n such that v_n occurs in η, x_n is x'_n, then $Ext_{t,\mathcal{A}}(\eta)(x)$ is $Ext_{t,\mathcal{A}}(\eta)(x')$. It is sufficient to show that $Tm_{L_{\mathcal{A}}} \subseteq \Gamma$. We proceed by induction, using T2.

(1) Suppose that $m \in \omega$ and for each n such that v_n occurs in v_m, x_n is x'_n. Then $Ext_{t,\mathcal{A}}(v_m)(x)$ is x_m is x'_m is $Ext_{t,\mathcal{A}}(v_m)(x')$.

(2) Suppose that δ is a k-place operation letter, $\delta \in L_{\mathcal{A}}$ and $\zeta_0,\ldots,\zeta_{k-1} \in \Gamma$. Suppose also that for each n such that v_n occurs in $\delta\zeta_0\ldots\zeta_{k-1}$, x_n is x'_n. Since

$\zeta_0,\ldots,\zeta_{k-1} \in \Gamma$, for each $i < k$, $Ext_{t,\mathcal{A}}(\zeta_i)(x)$ is $Ext_{t,\mathcal{A}}(\zeta_i)(x')$. Then $Ext_{t,\mathcal{A}}(\delta\zeta_0\ldots\zeta_{k-1})(x)$ is $G(\delta)(t)(\langle Ext_{t,\mathcal{A}}(\zeta_0)(x),\ldots,Ext_{t,\mathcal{A}}(\zeta_{k-1})(x)\rangle)$ is $G(\delta)(t)(\langle Ext_{t,\mathcal{A}}(\zeta_0)(x'),\ldots,Ext_{t,\mathcal{A}}(\zeta_{k-1})(x')\rangle)$ is $Ext_{t,\mathcal{A}}(\delta\zeta_0\ldots\zeta_{k-1})(x')$.

T15. If $\mathcal{A}$ is an interpretation, $\mathcal{A}$ is $\langle T,\leq,U,G\rangle$, $x,x' \in U^{\omega}$, $t,t' \in T$, $\phi \in Fm_{L_{\mathcal{A}}}$ and for each n such that $v_n \in fv(\phi)$, x_n is x'_n, then $x \in Int_{\mathcal{A}}(\phi)(\langle t,t'\rangle)$ if and only if $x' \in Int_{\mathcal{A}}(\phi)(\langle t,t'\rangle)$.

Proof: Suppose that $\mathcal{A}$ is an interpretation and $\mathcal{A}$ is $\langle T,\leq,U,G\rangle$. Let Γ be the set of $\phi \in Fm_{L_{\mathcal{A}}}$ such that for all $t,t' \in T$ and $x,x' \in U^{\omega}$, if for each n such that $v_n \in fv(\phi)$, x_n is x'_n, then $x \in Int_{\mathcal{A}}(\phi)(\langle t,t'\rangle)$ if and only if $x' \in Int_{\mathcal{A}}(\phi)(\langle t,t'\rangle)$. We proceed by induction, using T3 to show that $Fm_{L_{\mathcal{A}}} \subseteq \Gamma$.

(1) Suppose $\phi \in AFm_{L_{\mathcal{A}}}$, $t,t' \in T$, $x,x' \in U^{\omega}$ and for each n such that $v_n \in fv(\phi)$, x_n is x'_n.

(a) Suppose there is a k-place predicate letter π and terms $\zeta_0,\ldots,\zeta_{k-1}$ such that ϕ is $\pi\zeta_0,\ldots,\zeta_{k-1}$. Then $x \in Int_{\mathcal{A}}(\phi)(\langle t,t'\rangle)$ if and only if $\langle Ext_{t,\mathcal{A}}(\zeta_0)(x),\ldots,Ext_{t,\mathcal{A}}(\zeta_{k-1})(x)\rangle \in G(\pi)(t)$ if and only if (T14) $\langle Ext_{t,\mathcal{A}}(\zeta_0)(x'),\ldots,$ $Ext_{t,\mathcal{A}}(\zeta_{k-1})(x')\rangle \in G(\pi)(t)$ if and only if $x' \in Int_{\mathcal{A}}(\phi)(\langle t,t'\rangle)$

(b) The case where ϕ is $\zeta=\eta$ for some $\zeta,\eta \in Tm_{L_{\mathcal{A}}}$ is similar to case (a).

(2) Suppose $\phi, \psi \in \Gamma$ and suppose, for each of the following seven cases, that t, t', x, x' satisfy the appropriate conditions.

(a) $x \in \mathrm{Int}(\neg\phi)(\langle t,t'\rangle)$ if and only if $x \notin \mathrm{Int}_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$ if and only if (since $\phi \in \Gamma$) $x' \notin \mathrm{Int}_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$ if and only if $x' \in \mathrm{Int}_{\mathfrak{a}}(\neg\phi)(\langle t,t'\rangle)$

(b) $x \in \mathrm{Int}_{\mathfrak{a}}(\phi \rightarrow \psi)(\langle t,t'\rangle)$ if and only if either $x \notin \mathrm{Int}_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$ or $x \in \mathrm{Int}_{\mathfrak{a}}(\psi)(\langle t,t'\rangle)$ if and only if either $x' \notin \mathrm{Int}_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$ or $x' \in \mathrm{Int}_{\mathfrak{a}}(\psi)(\langle t,t'\rangle)$ if and only if $x' \in \mathrm{Int}_{\mathfrak{a}}(\phi \rightarrow \psi)(\langle t,t'\rangle)$

(c) Suppose that $x \in \mathrm{Int}_{\mathfrak{a}}(\Lambda v_n\phi)(\langle t,t'\rangle)$ (the other implication is similar). Then, for each $a \in U$, $x^m_a \in \mathrm{Int}_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$. Since $fv(\phi) \subseteq fv(\Lambda v_m\phi) \cup \{v_m\}$, for each n such that $v_n \in fv(\phi)$ and each $a \in U$, $(x^m_a)_n$ is $(x'^m_a)_n$. Hence (since $\phi \in \Gamma$) for each $a \in U$, $x'^m_a \in \mathrm{Int}_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$, and $x' \in \mathrm{Int}_{\mathfrak{a}}(\Lambda v_m\phi)(\langle t,t'\rangle)$

(d) $x \in \mathrm{Int}_{\mathfrak{a}}(H\phi)(\langle t,t'\rangle)$ if and only if for each $t'' <_{\mathfrak{a}} t$, $x \in \mathrm{Int}_{\mathfrak{a}}(\phi)(\langle t'',t'\rangle)$ if and only if (since $\phi \in \Gamma$) for each $t'' <_{\mathfrak{a}} t$, $x' \in \mathrm{Int}_{\mathfrak{a}}(\phi)(\langle t'',t'\rangle)$ if and only if $x' \in \mathrm{Int}_{\mathfrak{a}}(H\phi)(\langle t,t'\rangle)$.

(e) The case for $G\phi$ is similar to the case for $H\phi$.

(f) $x \in Int_{\mathfrak{a}}(K\phi)(\langle t,t'\rangle)$ if and only if $x \in Int_{\mathfrak{a}}(\phi)(\langle t,t\rangle)$ if and only if (since $\phi \in \Gamma$) $x' \in Int_{\mathfrak{a}}(\phi)(\langle t,t\rangle)$ if and only if $x' \in Int_{\mathfrak{a}}(K\phi)(\langle t,t\rangle)$

(g) The case for $R\phi$ is similar to the case for $K\phi$.

By T3, this completes the proof.

T16. If $\mathfrak{a}$ is an interpretation, $\mathfrak{a}$ is $\langle T,\leq,U,G\rangle$, ζ and η are terms of the language of $\mathfrak{a}$, $t \in T$, and $x \in U^{\omega}$ and $p \in \omega$, then $Ext_{t,\mathfrak{a}}(ra(v_p,\eta,\zeta))(x)$ is $Ext_{t,\mathfrak{a}}(\zeta)(x^{p}_{Ext_{t,\mathfrak{a}}(\eta)(x)})$

Proof: Suppose $\mathfrak{a}$ is an interpretation, $\mathfrak{a}$ is $\langle T,\leq,U,G\rangle$, η is a term of the language of $\mathfrak{a}$, $t \in T$, $x \in U^{\omega}$ and $p \in \omega$. Let a be $Ext_{t,\mathfrak{a}}(\eta)(x)$, and let Γ be the set of terms ζ of the language of $\mathfrak{a}$ such that $Ext_{t,\mathfrak{a}}(ra(v_p,\eta,\zeta))(x)$ is $Ext_{t,\mathfrak{a}}(\zeta)(x^{p}_{a})$. It is sufficient to show, using T2, that the set of terms of the language of $\mathfrak{a}$ is included in Γ.

(1) Suppose ζ is a variable. We take two cases:

(a) ζ is v_p; then (D20) $Ext_{t,\mathfrak{a}}(ra(v_p,\eta,\zeta))(x)$ is $Ext_{t,\mathfrak{a}}(\eta)(x)$ is a is $Ext_{t,\mathfrak{a}}(v_p)(x^{p}_{a})$

(b) ζ is not v_p; then $Ext_{t,\mathfrak{a}}(ra(v_p,\eta,\zeta))(x)$ is (T5b) $Ext_{t,\mathfrak{a}}(\zeta)(x)$ is (T14) $Ext_{t,\mathfrak{a}}(\zeta)(x^{p}_{a})$

(2) Suppose δ is a k-place operation letter of L and $\zeta_0,\ldots,\zeta_{k-1} \in \Gamma$; then $Ext_{t,\mathfrak{a}}(ra(v_p,\eta,\delta\zeta_0\ldots\zeta_{k-1})(x)$ is (D20) $Ext_{t,\mathfrak{a}}(\delta ra(v_p,\eta,\zeta_0\ldots ra(v_p,\eta,\zeta_{k-1}))(x)$ is $G(\delta)(t)(\langle Ext_{t,\mathfrak{a}}(ra(v_p,\eta,\zeta_0))(x),\ldots,ra(v_p,\eta,\zeta_{k-1})(x)\rangle)$ is (by the hypothesis) $G(\delta)(t)(\langle Ext_{t,\mathfrak{a}}(\zeta_0)(x^{p}_{a}),\ldots,$

$Ext_{t,\mathcal{a}}(\zeta_{k-1})(x^p_a)\rangle)$ is $Ext_{t,\mathcal{a}}(\delta\zeta_0\ldots\zeta_{k-1})(x^p_a)$.

T17. If $\mathcal{a}$ is an interpretation, $\mathcal{a}$ is $\langle T,\leq,U,G\rangle$, $a \in U$, $x \in U^\omega$, $t,t' \in T$ and v_m does not occur in ϕ, then $x^n_a \in Int_{\mathcal{a}}(\phi)(\langle t,t'\rangle)$ if and only if $x^m_a \in Int_{\mathcal{a}}(ra(v_n,v_m,\phi))(\langle t,t'\rangle)$.

Proof: Assume that $\mathcal{a}$ is an interpretation, and $\mathcal{a}$ is $\langle T,\leq,U,G\rangle$. By T14 and T16, we have this lemma: If ζ is a term of $L_{\mathcal{a}}$, v_m does not occur in ζ, $a \in U$, $x \in U^\omega$ and $t \in T$, then $Ext_{t,\mathcal{a}}(\zeta)(x^n_a)$ is $Ext_{t,\mathcal{a}}(ra(v_n,v_m,\zeta))(x^m_a)$.

Let Γ be the set of formulas ϕ of $L_{\mathcal{a}}$ such that, if v_m does not occur in ϕ, then for all $a \in U$, $x \in U^\omega$ and $t,t' \in T$, $x^n_a \in Int_{\mathcal{a}}(\phi)(\langle t,t'\rangle)$ if and only if $x^m_a \in Int_{\mathcal{a}}(ra(v_n,v_m,\phi))(\langle t,t'\rangle)$. It will be sufficient to show, using T3, that $Fm_{L_{\mathcal{a}}} \subseteq \Gamma$. For each of the following cases we will assume that $x \in U^\omega$ and $t,t' \in T$.

(1a) If π is a k-place predicate letter, $\zeta_0,\ldots,\zeta_{k-1}$ are terms of L, and v_m does not occur in $\pi\zeta_0\ldots\zeta_{k-1}$, then $x^n_a \in Int_{\mathcal{a}}(\pi\zeta_0\ldots\zeta_{k-1})(\langle t,t'\rangle)$ if and only if $\langle Ext_{t,\mathcal{a}}(\zeta_0)(x^n_a),\ldots,Ext_{t,\mathcal{a}}(\zeta_{k-1})(x^n_a)\rangle \in G(\pi)(t)$ if and only if (by the lemma) $\langle Ext_{t,\mathcal{a}}(ra(v_n,v_m,\zeta_0))(x^m_a),\ldots,Ext_{t,\mathcal{a}}(ra(v_n,v_m,\zeta_0)(x^m_a)\rangle \in G(\pi)(t)$ if and only if $x^m_a \in Int_{\mathcal{a}}(\pi ra(v_n,v_m,\zeta_0)\ldots ra(v_n,v_m,\zeta_{k-1}))(\langle t,t'\rangle)$ if and only if (D20) $x^m_a \in Int_{\mathcal{a}}(ra(v_n,v_m,\pi\zeta_0,\ldots,\zeta_{k-1}))(\langle t,t'\rangle)$.

(1b) The case for $\zeta=\eta$, where ζ and η are terms of $L_{\mathcal{a}}$ is similar.

(2) Suppose that $\phi, \psi \in \Gamma$.

(a) Suppose that v_m does not occur in $\neg\phi$. Then $x^n_a \in \mathrm{Int}_{\mathfrak{a}}(\neg\phi)(\langle t,t'\rangle)$ if and only if $x^n_a \notin \mathrm{Int}_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$ if and only if (since $\phi \in \Gamma$) $x^m_a \notin \mathrm{Int}_{\mathfrak{a}}(\mathrm{ra}(v_n,v_m,\phi))(\langle t,t'\rangle)$ if and only if $x^m_a \in \mathrm{Int}_{\mathfrak{a}}(\neg\mathrm{ra}(v_n,v_m,\phi))(\langle t,t'\rangle)$ if and only if (D20) $x^m_a \in \mathrm{Int}_{\mathfrak{a}}(\mathrm{ra}(v_n,v_m,\neg\phi))(\langle t,t'\rangle)$.

(b) The case for $\phi \rightarrow \psi$ is similar.

(c) Suppose that v_m does not occur in $\Lambda\alpha\phi$. We take two cases:

(i) α is v_n; then $x^n_a \in \mathrm{Int}_{\mathfrak{a}}(\Lambda\alpha\phi)(\langle t,t'\rangle)$ if and only if for all $b \in U$, $x^{n\ n}_{a\ b} \in \mathrm{Int}_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$ if and only if for all $b \in U$, $x^n_b \in \mathrm{Int}_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$ if and only if (since $\phi \in \Gamma$) for all $b \in U$, $x^m_b \in \mathrm{Int}_{\mathfrak{a}}(\mathrm{ra}(v_n,v_m,\phi))(\langle t,t'\rangle)$ if and only if $x \in \mathrm{Int}_{\mathfrak{a}}(\Lambda v_m \mathrm{ra}(v_n,v_m,\phi))(\langle t,t'\rangle)$ if and only if (D20) $x \in \mathrm{Int}_{\mathfrak{a}}(\mathrm{ra}(v_n,v_m,\Lambda\alpha\phi))(\langle t,t'\rangle)$

(ii) α is not v_n; then (letting p be the $p \in \omega$ such that α is v_p) $x^n_a \in \mathrm{Int}_{\mathfrak{a}}(\Lambda\alpha\phi)(\langle t,t'\rangle)$ if and only if for all $b \in U$, $x^{n\ p}_{a\ b} \in \mathrm{Int}_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$ if and only if for all $b \in U$, $x^{p\ n}_{b\ a} \in \mathrm{Int}_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$ if and only if (since $\phi \in \Gamma$) for all $b \in U$, $x^{p\ m}_{b\ a} \in \mathrm{Int}_{\mathfrak{a}}(\mathrm{ra}(v_n,v_m,\phi))(\langle t,t'\rangle)$ if and only if for all $b \in U$, $x^{m\ p}_{a\ b} \in \mathrm{Int}_{\mathfrak{a}}(\mathrm{ra}(v_n,v_m,\phi))(\langle t,t'\rangle)$

if and only if $x^m_a \in$

$Int_{\mathfrak{a}}(\Lambda \alpha ra(v_n, v_m, \phi))(\langle t, t' \rangle)$ if and only if

$x^m_a \in Int\ (ra(v_n, v_m, \Lambda\alpha\phi))(\langle t, t' \rangle)$

(d) Suppose that v_m does not occur in $H\phi$. Then $x^n_a \in Int_{\mathfrak{a}}(H\phi)(\langle t, t' \rangle)$ if and only if for all $t'' <_{\mathfrak{a}} t$, $x^n_a \in Int_{\mathfrak{a}}(\phi)(\langle t, t' \rangle)$ if and only if (since $\phi \in \Gamma$) for all $t'' <_{\mathfrak{a}} t$, $x^m_a \in Int_{\mathfrak{a}}(ra(v_n, v_m, \phi))(\langle t, t' \rangle)$ if and only if $x^m_a \in Int_{\mathfrak{a}}(H\ ra(v_n, v_m, \phi))(\langle t, t' \rangle)$ if and only if (D20) $x^m_a \in Int_{\mathfrak{a}}(ra(v_n, v_m, H\phi))(\langle t, t' \rangle)$.

(e) The cases for $G\phi$, $K\phi$ and $R\phi$ are all similar to the $H\phi$ case.

This completes the proof.

T18. If $\mathfrak{a}$ is an interpretation for L, $\mathfrak{a}$ is $\langle T, \leq, U, G \rangle$, $t, t' \in T$, $x \in U^\omega$, $n, m \in \omega$, and $\phi \in Fm_L$, then $x^n_{x_m} \in Int_{\mathfrak{a}}(\phi)(\langle t, t' \rangle)$ if and only if $x \in Int_{\mathfrak{a}}(ps(v_m, v_n, \phi))(\langle t, t' \rangle)$.

Proof: Suppose $\mathfrak{a}$ is an interpretation for L, $\mathfrak{a}$ is $\langle T, \leq, U, G \rangle$ and $n, m \in \omega$. Let Γ be the set of $\phi \in Fm_L$ such that for each $t, t' \in T$ and $x \in U^\omega$, $x^n_{x_m} \in Int_{\mathfrak{a}}(\phi)(\langle t, t' \rangle)$ if and only if $x \in Int_{\mathfrak{a}}(ps(v_m, v_n, \phi))(\langle t, t' \rangle)$. It is sufficient to show, using induction on the rank of ϕ, that $Fm_L \subseteq \Gamma$. In each of the following cases, we suppose that $t, t' \in T$ and $x \in U^\omega$.

(1a) Suppose that π is a k-place predicate letter and $\zeta_0, \ldots, \zeta_{k-1} \in Tm_L$; then $x^n_{x_m} \in$

$\mathrm{Int}_{\mathfrak{a}}(\pi\zeta_0 \ldots \zeta_{k-1})(\langle t,t'\rangle)$ if and only if $\langle \mathrm{Ext}_{t,\mathfrak{a}}(\zeta_0)(x^n_{x_m}),\ldots,\mathrm{Ext}_{t,\mathfrak{a}}(\zeta_{k-1})(x^n_{x_m})\rangle \in G(\pi)(t)$ if and only if (T16) $\langle \mathrm{Ext}_{t,\mathfrak{a}}(\mathrm{ra}(v_n,v_m,\zeta_0))(x),\ldots,$ $\mathrm{Ext}_{t,\mathfrak{a}}(\mathrm{ra}(v_n,v_m,\zeta_{k-1}))(x)\rangle \in G(\pi)(t)$ if and only if $x \in \mathrm{Int}_{\mathfrak{a}}(\pi\mathrm{ra}(v_n,v_m,\zeta_0)\ldots\mathrm{ra}(v_n,v_m,\zeta_{k-1}))(\langle t,t'\rangle)$ if and only if (D20) $x \in$ $\mathrm{Int}_{\mathfrak{a}}(\mathrm{ra}(v_n,v_m,\pi\zeta_0\ldots\zeta_{k-1}))(\langle t,t'\rangle)$ if and only if (D21) $x \in \mathrm{Int}_{\mathfrak{a}}(\mathrm{ps}(v_m,v_n,\pi\zeta_0\ldots\zeta_{k-1}))(\langle t,t'\rangle)$.

(1b) The case for $\zeta=\eta$, where $\zeta,\eta \in \mathrm{Tm}_L$, is similar.

(2a) $x^n_{x_m} \in \mathrm{Int}_{\mathfrak{a}}(\neg\phi)(\langle t,t'\rangle)$ if and only if $x^n_{x_m} \notin$ $\mathrm{Int}_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$ if and only if (by the inductive hypothesis) $x \notin \mathrm{Int}_{\mathfrak{a}}(\mathrm{ps}(v_m,v_n,\phi))(\langle t,t'\rangle)$ if and only if $x \in \mathrm{Int}\ (\neg\mathrm{ps}(v_m,v_n,\phi))(\langle t,t'\rangle)$ if and only if (D21) $x \in \mathrm{Int}_{\mathfrak{a}}(\mathrm{ps}(v_m,v_n,\neg\phi))(\langle t,t'\rangle)$.

(2b) The case for $\phi \to \psi$ is similar.

(2c) Suppose that β is a variable. Let p be that $p \in \omega$ such that β is v_p. We take two subcases:

(i) $v_n \notin \mathrm{fv}(\Lambda\beta\phi)$; we take three subcases:

First, suppose that $v_n \notin \mathrm{fv}(\phi)$. Then $x^n_{x_m} \in \mathrm{Int}_{\mathfrak{a}}(\Lambda\beta\phi)(\langle t,t'\rangle)$ if and only if (T15) $x \in \mathrm{Int}_{\mathfrak{a}}(\Lambda\beta\phi)(\langle t,t'\rangle)$ if and only if (D21) $x \in \mathrm{Int}_{\mathfrak{a}}(\mathrm{ps}(v_m,v_n,\Lambda\beta\phi))(\langle t,t'\rangle)$.

Second, suppose that v_n is β. Then n is p and $x^n_{x_m} \in \mathrm{Int}_{\mathfrak{a}}(\Lambda\beta\phi)(\langle t,t'\rangle)$ if and only if for all $a \in U$, $x^{n\ p}_{x_m a} \in \mathrm{Int}_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$ if and only if for all $a \in U$, $x^p_a \in \mathrm{Int}_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$ if and only if $x \in Int_{\mathcal{a}}(\Lambda\beta\phi)(\langle t,t'\rangle)$ if and only if (D21) $x \in Int_{\mathcal{a}}(ps(v_m,v_n,\Lambda\beta\phi))(\langle t,t'\rangle)$.

(ii) $v_n \in fv(\Lambda\beta\phi)$ and β does not occur in v_m; then β is not v_m and p is not m. Also, since $\beta \notin fv(\Lambda\beta\phi)$, v_n is not β, n is not p and $x^n_{x_m} \in Int_{\mathcal{a}}(\Lambda\beta\phi)(\langle t,t'\rangle)$ if and only if for all $a \in U$, $x^{n\ p}_{x_m a} \in Int_{\mathcal{a}}(\phi)(\langle t,t'\rangle)$ if and only if for all $a \in U$, $x^{p\ n}_{a\ x_m} \in Int_{\mathcal{a}}(\phi)(\langle t,t'\rangle)$ if and only if for all $a \in U$, $x^{p\ \ n}_{a\ (x^p_a)_m} \in Int_{\mathcal{a}}(\phi)(\langle t,t'\rangle)$ if and only if (by the inductive hypothesis) for all $a \in U$, $x^p_a \in Int_{\mathcal{a}}(ps(v_m,v_n,\phi))(\langle t,t'\rangle)$ if and only if (D21) $x \in Int_{\mathcal{a}}(ps(v_m,v_n,\phi))(\langle t,t'\rangle)$.

(iii) $v_n \in fv(\Lambda\beta\phi)$ and β occurs in v_m; then p is m and n is not p. Let γ be the first variable such that γ occurs neither in v_m nor in ϕ and let j be that $j \in \omega$ such that γ is v_j. Then j is not m, j is not n, and $x^n_{x_m} \in Int_{\mathcal{a}}(\Lambda\beta\phi)(\langle t,t'\rangle)$ if and only if for all $a \in U$, $x^{n\ p}_{x_m a} \in Int_{\mathcal{a}}(\phi)(\langle t,t'\rangle)$ if and only if (T17) for all $a \in U$, $x^{n\ j}_{x_m a} \in Int_{\mathcal{a}}(ra(\beta,\gamma,\phi))(\langle t,t'\rangle)$ if and only if for all $a \in U$, $x^{j\ n}_{a\ x_m} \in Int_{\mathcal{a}}(ra(\beta,\gamma,\phi))(\langle t,t'\rangle)$ if and only if for all $a \in U$, $x^{j\ \ n}_{a\ (x^j_a)_m} \in Int_{\mathcal{a}}(ra(\beta,\gamma,\phi))(\langle t,t'\rangle)$ if and only if (by the inductive hypothesis and T8) for all $a \in U$, $x^j_a \in Int_{\mathcal{a}}(ps(v_m,v_n,ra(\beta,\gamma,\phi)))(\langle t,t'\rangle)$ if and only if

$x \in Int_{\mathcal{a}}(\wedge\gamma ps(v_m, v_n, ra(\beta,\gamma,\phi)))(\langle t,t'\rangle)$ if and only if (D21) $x \in Int\ (ps(v_m, v_n, \wedge\beta\phi))(\langle t,t'\rangle)$.

(2d) $x^n_{x_m} \in Int_{\mathcal{a}}(H\phi)(\langle t,t'\rangle)$ if and only if for each $t'' <_{\mathcal{a}} t$, $x^n_{x_m} \in Int_{\mathcal{a}}(\phi)(\langle t'',t'\rangle)$ if and only if (by the inductive hypothesis) for each $t'' <_{\mathcal{a}} t$, $x \in Int_{\mathcal{a}}(ps(v_m, v_n, \phi))(\langle t'',t'\rangle)$ if and only if $x \in Int_{\mathcal{a}}(H\ ps(v_m, v_n, \phi))(\langle t,t'\rangle)$ if and only if (D21) $x \in Int_{\mathcal{a}}(ps(v_m, v_n, H\phi))(\langle t,t'\rangle)$.

(2e) The cases for $G\phi$, $K\phi$ and $R\phi$ are similar to the case for $H\phi$.

This completes the proof.

T19. If $\mathcal{a}$ is an interpretation, t is a moment of $\mathcal{a}$, $x \in U^\omega$, η, η', ζ and ζ' are terms, ζ' is obtained from ζ by replacing 0 or more occurrences of η by η' and $Ext_{t,\mathcal{a}}(\eta)(x)$ is $Ext_{t,\mathcal{a}}(\eta')(x)$, then $Ext_{t,\mathcal{a}}(\zeta)(x)$ is $Ext_{t,\mathcal{a}}(\zeta')(x)$.

Proof: Suppose that $\mathcal{a}$ is an interpretation, t is a moment of $\mathcal{a}$, $x \in U^\omega$, η and η' are terms, and $Ext_{t,\mathcal{a}}(\eta)(x)$ is $Ext_{t,\mathcal{a}}(\eta')(x)$. Let Γ be the set of terms ζ of L such that for any term ζ', if ζ' is obtained from ζ by replacing 0 or more occurrences of η by η', then $Ext_{t,\mathcal{a}}(\zeta)(x)$ is $Ext_{t,\mathcal{a}}(\zeta')(x)$. We will show by induction (according to the clauses of D15) that every term of $L_{\mathcal{a}}$ is a member of Γ.

(1) ζ is ζ'; then $Ext_{t,\mathcal{a}}(\zeta)(x)$ is $Ext_{t,\mathcal{a}}(\zeta')(x)$

(2) ζ is η and ζ' is η'; then, by hypothesis, $Ext_{t,\mathcal{a}}(\zeta)(x)$ is $Ext_{t,\mathcal{a}}(\zeta')(x)$

(3) There are $k, \delta, \xi_0, \ldots, \xi_{k-1}, \xi'_0, \ldots, \xi'_{k-1}$ such that $k \in \omega$, δ is a k-place operation letter, ζ is $\delta\xi_0, \ldots, \xi_{k-1}$, ζ' is $\delta\xi'_0 \ldots \xi'_{k-1}$, and for each $i < k$, ξ'_i is obtained from ξ_i by replacing 0 or more occurrences of η by η'. By the inductive hypothesis, for each $i < k$, $Ext_{t,\mathcal{A}}(\xi'_i)(x)$ is $Ext_{t,\mathcal{A}}(\xi_i)(x)$. Then $Ext_{t,\mathcal{A}}(\zeta)(x)$ is $G(\delta)(t)(\langle Ext_{t,\mathcal{A}}(\xi_0)(x), \ldots, Ext_{t,\mathcal{A}}(\xi_{k-1})(x)\rangle)$ is $G(\delta)(t)(\langle Ext_{t,\mathcal{A}}(\xi'_0)(x), \ldots, Ext_{t,\mathcal{A}}(\xi'_{k-1})(x)\rangle)$ is $Ext_{t,\mathcal{A}}(\zeta')(x)$.

This completes the proof.

T20. If ϕ is an atomic formula, $\mathcal{A}$ is an interpretation and t and t' are moments of $\mathcal{A}$, then $Int_{\mathcal{A}}(\phi)(\langle t,t'\rangle)$ is $Int_{\mathcal{A}}(\phi)(\langle t,t\rangle)$.

Proof: This is an immediate consequence of clauses (1) and (2) of D34.

CHAPTER III

AN AXIOMATIZATION

A. Arrangements

In this chapter we will construct an axiomatization for the system of Chapter II. In Chapter IV it will be proved that this axiomatization is complete, in the sense that any formula is logically valid if and only if it is derivable from the axioms. Unfortunately, the set of axioms cannot be represented as the set of instances of a finite set of axiom schemata. We do not know whether there is a complete set of axioms for the present system that can be so represented. The set of axioms will be presented as a number of axiom schemata, together with an additional decidable set of axioms which is arrived at by a complicated construction that begins here.

We begin with a description of the method used to construct the set of axioms, and an attempt to explain why this method seems to be necessary. The completeness proofs in Henkin [3], Kripke [6], and Cocchiarella [1] all follow what can be seen as special cases of a general method. The main part of the completeness proof in each case is to show that each consistent set of formulas has an interpretation.

In each of the proofs mentioned above, the interpretation is constructed step by step by gradually adding to some structure made up of sets of formulas. In Henkin [3] the structure is simply a single set of formulas; in Kripke [6] it is a tree with sets of formulas assigned to the nodes; in Cocchiarella [1] it is a set of 'moments' with a linear ordering and with one moment selected as the present one.

The structure in each of the above cases is gradually expanded; that is, an infinite sequence of finite structures is assembled such that each one of these structures is more nearly complete than its predecessor. Then an infinite structure is defined in terms of this infinite sequence of finite structures in such a way that the infinite structure includes each of the finite structures and is complete. An interpretation is then constructed from the infinite structure in a natural way, and it is shown that the original consistent set of formulas is satisfiable in that interpretation. The present completeness proof shares all of these features with the three mentioned above.

We turn now to a comparison between our completeness proof and Cocchiarella's, since his system is closest to ours. In Cocchiarella's proof, the final infinite structure must be complete in three ways.[8]

(i) For each formula ϕ of the language of the final interpretation, and each of the sets of formulas Γ that correspond to the moments, either ϕ or $\neg\phi$ must belong to Γ.

(ii) For each formula $\neg\Lambda\alpha\phi$ that belongs to one of the sets of formulas Γ, there must be a variable β such that $\neg ps(\beta,\alpha,\phi) \in \Gamma$.

(iii) For each sentence $\neg H\phi$ that belongs to one of the sets Γ, $\neg\phi$ must belong to one of the sets that are assigned to the moments that precede the moment to which Γ is assigned (and similarly for formulas of the form $\neg G\phi$).

In the infinite sequence of structures, then, each structure is constructed from its predecessor by adding formulas to sets of formulas in order to partially satisfy one of the three requirements above. In order to satisfy requirement (iii), it is also sometimes necessary to add moments. Our completeness proof will also follow Cocchiarella's in this respect.

We come now to the main point in which Cocchiarella's proof differs from our own. In Cocchiarella's system it is possible to characterize any one of his finite structures by a formula. For instance, suppose we consider the structure whose moments are t_1,t_2,t_3,t_4 in that order, with t_3 as the present moment, and such that $\{\phi_1\},\{\phi_2\},\{\phi_3\},\{\phi_4\}$ are assigned to t_1,t_2,t_3,t_4, respectively. The formula that characterizes that structure is $\phi_3 \wedge P(\phi_2 \wedge P\phi_1) \wedge F\phi_4$ (call it χ_0): i.e., the formula that specifies that those formulas hold in the order specified by the structure. A key fact about this correspondence is that any structure in which χ_0 'holds' includes the structure defined above. That is, it has ϕ_3 in the set of formulas for the present moment, ϕ_2 in a preceding set, ϕ_1 in a still earlier set, and ϕ_4 in a set that comes after the present moment.

This correspondence between formulas and structures is of central importance in Cocchiarella's proof, as well as in those of Henkin and Kripke. In order to ensure that the initial consistent set of formulas is satisfiable in the final interpretation, it is necessary that the characteristic formula of each term of the infinite sequence of finite structures is consistent with the initial consistent set of formulas.

In order to construct the sequence of structures Σ, then, it is only necessary to show that for any structure B whose characteristic formula is consistent with Γ, there is another structure B' which is more complete than B (in a certain way) and whose characteristic formula is also consistent with Γ. Cocchiarella is able to find a nice set of axioms that ensure that this latter proof can be carried through. As a particular case consider the structure specified above and suppose we want to partially satisfy requirement (i) by adding either ψ or $\neg\psi$ to the set of formulas assigned to t_1. The characteristic formula of the result would be either $\phi_3 \wedge (P\phi_2 \wedge P(\phi_1 \wedge \psi)) \wedge F\phi_4$ (call it χ_1) or $\phi_3 \wedge P(\phi_2 \wedge P(\phi_1 \wedge \neg\psi)) \wedge F\phi_4$ (call it χ_2). To ensure that one of the formulas ψ or $\neg\psi$ can be added, it is only necessary that $\chi_0 \rightarrow \chi_1 \vee \chi_2$ can be derived from the axioms (since if the latter is derivable, and χ_0 is consistent with Γ, then either χ_1 is consistent with Γ or χ_2 is consistent with Γ). This is relatively simple once it is shown that all instances of $P(\phi \vee \psi) \rightarrow P\phi \vee P\psi$ are derivable. Similar remarks can be made for requirements (ii) and (iii), although the situation is more complex, expecially for (iii).

Now consider the system with K and R. Here, since the points of reference are ordered pairs of moments, our structures must be essentially square matrices of sets of formulas, rather than just linear orderings of sets of formulas. And the fundamental difficulty with the system with K and R that differentiates it from Cocchiarella's system is that there seem to be no formulas that characterize our structures (which we call 'arrangements', since each of them is just a way of arranging a set of formulas among a square matirx of points of reference).

In order to facilitate this discussion, we now introduce the definitions of an arrangement and of some related notions. We also state now two basic theorems about arrangements.

D39. A is an <u>arrangement</u> if and only if there are j,R,F such that

(1) A is $\langle j, R, F \rangle$,

(2) $j \in \omega \cup \{\omega\}$ and j is not 0,

(3) R is a reflexive linear ordering on j,

(4) F is a function with domain $j \times j$,

and (5) For each $i \in \mathrm{Dom}(F)$, $F(i)$ is a set of formulas.

D40. (a) $i \leq_A j$ if and only if $\langle i,j \rangle \in A_1$.

(b) $i <_A j$ if and only if $i \leq_A j$ and i is not j.

D41. A is a <u>finite arrangement</u> if and only if

(1) A is an arrangement,

(2) $A_0 \in \omega$,

and (3) For each $i \in A_0 \times A_0$, $A_2(i)$ is finite.

D42. A is <u>part of</u> B if and only if

(1) A and B are arrangements,

(2) $A_0 \leq B_0$,

(3) $A_1 \subseteq B_1$,

and (4) For each $i \in A_0 \times A_0$, $A_2(i) \subseteq B_2(i)$.

D43. If A is an arrangement, then (a) the set of <u>formulas of</u> A is the union of the range of A_2, and (b) the <u>language of</u> (the arrangement) A is the language of the set of formulas of A.

T21. The set of finite arrangements is denumerable.

Proof: Since the union of a denumerable set of denumerable sets is denumerable, it is sufficient to show that, for each $n \in \omega$, the set of arrangements A such that A_0 is n is denumerable. Hence, suppose that $n \in \omega$, and let

$\mathcal{A}$ be the set of arrangements A such that A_0 is n. Let $\mathcal{F}$ be the set of functions from n×n into S(Fm). Since n×n is finite and Fm is denumerable, $\mathcal{F}$ is denumerable, and so is $\{n\} \cup S(n \times n) \cup \mathcal{F}$, and so is $(\{n\} \cup S(n \times n) \cup \mathcal{F})^3$. But it is easy to see that $\mathcal{A} \subseteq (\{n\} \cup S(n \times n) \cup \mathcal{F})^3$.

T22. If Γ is finite and $k \in \omega$, then the set of arrangements A such that $A_0 \leq k$ and the set of formulas of A is included in Γ is finite.

Proof: Suppose Γ is finite. Since the union of a finite number of finite sets is finite, it is sufficient to show that, for each $n \leq k$, the set of arrangements A such that A_0 is n and the union of the range of $A_2 \subseteq \Gamma$ is finite. Suppose $n \leq k$ and let Δ be the set of arrangements A such that A_0 is n and the set of formulas of A is included in Γ. It is sufficient to show that Δ is finite. Let $\mathcal{F}$ be the set of functions from n×n into S(Γ). {n} is finite, and so is S(n×n), and so is S(Γ), and so is $\mathcal{F}$, and so is $(\{n\} \cup S(n \times n) \cup \mathcal{F})^3$. Hence it is sufficient to show that $\Delta \subseteq (\{n\} \cup S(n \times n) \cup \mathcal{F})^3$. Suppose $A \in \Delta$; then $A_0 \in \{n\}$, and $A_1 \in S(n \times n)$, and $A_2 \in \mathcal{F}$. Hence $A \in (\{n\} \cup S(n \times n) \cup \mathcal{F})^3$.

Consider now the arrangement $\langle j,R,F \rangle$ (call it A), where j is 2, 0 is ordered before 1, and the sets of formulas $\{\phi_1\},\{\phi_2\},\{\phi_3\},\{\phi_4\}$ are assigned to $\langle 0,0 \rangle, \langle 0,1 \rangle, \langle 1,0 \rangle$, $\langle 1,1 \rangle$ respectively. Graphically, we have

$\langle 0,0 \rangle$ -- $\{\phi_1\}$	$\langle 0,1 \rangle$ -- $\{\phi_2\}$
$\langle 1,0 \rangle$ -- $\{\phi_3\}$	$\langle 1,1 \rangle$ -- $\{\phi_4\}$

We will take 0 as the present moment of every arrangement, so one formula that holds in A is $\phi_1 \wedge F(\phi_3 \wedge K(\phi_4 \wedge P\phi_2))$. But this formula does not characterize A, because it also holds in the arrangement

$\langle 2,2 \rangle$ -- 0	$\langle 2,0 \rangle$ -- $\{\phi_1\}$	$\langle 2,1 \rangle$ -- $\{\phi_2\}$
$\langle 0,2 \rangle$ -- 0	$\langle 0,0 \rangle$ -- $\{\phi_1\}$	$\langle 0,1 \rangle$ -- 0
$\langle 1,2 \rangle$ -- 0	$\langle 1,0 \rangle$ -- $\{\phi_3\}$	$\langle 1,1 \rangle$ -- $\{\phi_4\}$

which does not include A. In order for a formula to represent A, it must (for one thing) hold only in those arrangements where ϕ_2 is in one of the sets of formulas on the same row as $\langle 0,0 \rangle$, and there seems to be no formula for which this is the case. Adding more detail to the formula above does not help; for instance, the formula $\phi_1 \wedge F(\phi_3 \wedge K(\phi_4 \wedge P(\phi_2 \wedge K\phi_1)))$ also holds in the 3×3 arrangement shown above. In fact, it seems to be the case that every formula that holds in A also holds in some arrangement that does not include A. It even seems that every infinite set of formulas that all hold in A also all hold in some arrangement that does not include A.

What all this suggests is that it is not possible to proceed as Cocchiarella does, by associating a characteristic formula with each term of the finite sequence Σ of arrangements, because there are no such characteristic formulas. This complicates the process of constructing a set of axioms. In Cocchiarella's system, in order to show that the characteristic formula of each Σ_n is consistent with the original consistent set of formulas one can proceed inductively, showing that the characteristic formula of Σ_n entails the disjunction of the characteristic formulas of the possible choices for Σ_{n+1}. In our system, no such procedure is available. Our axioms, so to speak, can't just say that the characteristic formula of Σ_n entails the disjunction of the possible characteristic formulas of Σ_{n+1}.

Since we do not have the absolute notion of the characteristic formula of an arrangement we must speak of the n-level characteristic formula of an arrangement (D51). The idea is that as n increases, the n-level characteristic formula of A more nearly characterizes A. What we want our axioms to say is (roughly) that for any consistent set of formulas Γ and any $n \in \omega$, there is an n-place sequence Σ of arrangements such that each Σ_i is more complete (in a certain way) than its predecessor, and such that the n-level characteristic formula of Σ_{n-1} is consistent with Γ. It will turn out that this is sufficient to prove completeness.

Calling these sequences <u>extension sequences</u>, we would need only to take, for each n, the disjunction of the n-level characteristic formulas of all the n-place extension sequences as an axiom, but there is an infinite number of such n-place sequences. So we must define instead the notion of an <u>n-place minimal extension sequence with respect</u>

to Γ and Δ (D50). These latter sequences will do the job of the wider class of n-place extension sequences, but there are only a finite number of them (for a given Γ, Δ, and n) so that we can state the disjunctions of their n-level characteristic formulas (actually, of certain existential generalizations of their n-level characteristic formulas) as an axiom.

We begin this procedure with the notion of an extension of an arrangement.

1. The Extensions of an Arrangement

We think of an extension of an arrangement as being more nearly complete, in that an extension of A (so to speak) specifies which individuals make the existential formulas of A hold and which moments make the formulas of A of the form $P\phi$ and $F\phi$ hold.

D44. B is an extension of A with respect to Γ if and only if

(1) A is part of B

(2) Γ is a set of formulas

(3) For each $i \in B_0 \times B_0$ and each $\phi \in \Gamma$, either $\phi \in B_2(i)$ or $\neg\phi \in B_2(i)$

(4) For each $i \in B_0 \times B_0$, variable α and formula ϕ, if $\Lambda\alpha\phi \in \Gamma$ then either $\Lambda\alpha\phi \in B_2(i)$ or there is a variable β such that $\neg ps(\beta,\alpha,\phi) \in B_2(i)$

(5) For each $m,n \in A_0$ and each $\phi \in \Gamma$

(a) Either $H\phi \in B_2(\langle m,n \rangle)$ or there is a p such that $p <_B m$ and $\neg\phi \in B_2(\langle p,n\rangle)$

(b) Either $G\phi \in B_2(\langle m,n\rangle)$ or there is a p such that $m <_B p$ and $\neg\phi \in B_2(\langle p,n\rangle)$

A complete arrangement is one that already specifies everything about itself.

D45. A is a complete arrangement if and only if A is an extension of A with respect to the set of formulas of the language of A.

In building up our set of axioms, we cannot take account of all the extensions of a given finite arrangement, since these will not form a finite set. We must instead consider the set of minimal extensions. The set of variables Δ serves to indicate which variables may be introduced into the arrangement. By limiting the new variables and new moments of the extension to finite sets determined by the arrangement being extended, we ensure that the set of minimal extensions of that arrangement is finite. In clause (3d) of the following definition, we allow $\overline{B_0}^2\overline{\Gamma}$ new variables because this would be the number necessary if all the formulas of Γ were of the form $\Lambda\alpha\phi$ and each one had to be instantiated at each of the $\overline{B_0}^2$ 'points of reference' of B. Similarly in clause (4) we must allow for $2\overline{A_0}^2\overline{\Gamma}$ new moments, because this would be the number needed to provide a new moment for $H\phi$ and $G\phi$ for every formula ϕ in Γ and every 'point of reference' of A. What we are really doing here is ensuring that the set of axioms is recursive by putting an upper bound on n in the 'the least n such that...' operator.

D46. If Δ is a set of variables, then <u>B is a minimal extension of A with respect to Γ and Δ</u> if and only if

(1) A is a finite arrangement

(2) B is an extension of A with respect to Γ

(3) For each $i,j \in B_0$ and each $\phi \in B_2(\langle i,j \rangle)$ either

(a) $\phi \in A_2(\langle i,j \rangle)$

(b) $\phi \in \Gamma$

(c) For some $\psi \in \Gamma$, ϕ is $\neg\psi$ or ϕ is $H\psi$ or ϕ is $G\psi$

or (d) There are a formula ψ and variables α and β such that $\Lambda\alpha\psi \in \Gamma$, β is one of the first ${B_0}^2\overline{\Gamma}$ variables in Δ occurring free neither in any formula in Γ nor in any formula of A, and ϕ is $\neg ps(\beta,\alpha,\psi)$

(4) $B_0 - A_0 \leq 2{A_0}^2\overline{\Gamma}$

T23. If B is a minimal extension of A with respect to Γ and Δ, and Γ is finite, then B is a finite arrangement.

Proof: Suppose that B is a minimal extension of A with respect to Γ and Δ. Then A is a finite arrangement. Let j, R, F be A_0, A_1, A_2 respectively. Let n be the number of formulas in Γ. Then $B_0 \leq 2j^2n+j$, and $B_0 \in \omega$.

Also, by D46 clause (3), if $i \in Dom(F)$, then

$\overline{F(i)} \leq \overline{A_2(i)}+n+3n+nB_0{}^2n$, and F(i) is finite.

T24. If A is a finite arrangement and Γ is finite, then the set of minimal extensions of A with respect to Γ and Δ is finite.

Proof: Assume the hypothesis, and let n,j,R,F be $\overline{\Gamma}, A_0, A_1, A_2$ respectively. Let k be $2j^2+n+j+1$. Then for every minimal extension of B of A with respect to Γ and Δ, $B_0 < k$.

Now let Γ' be the set of formulas φ such that either

(a) φ is a formula of A

(b) φ ∈ Γ

(c) For some ψ ∈ Γ, φ is ¬ψ or φ is Hψ or φ is Gψ

(d) There are a formula ψ and variables α,β such that Λαψ ∈ Γ, β is one of the first j^2n variables in Δ occuring free neither in Γ nor in any formula of A, and φ is ¬ps(β,α,ψ).

Then Γ' is finite and for every minimal extension B of A with respect to Γ and Δ, the set of formulas of B is included in Γ'. By T22, then, the set of minimal extensions of A with respect to Γ and Δ is finite.

2. The n-level Formulas of an Arrangement

Given an arrangement and a point i of the arrangement, we would like to specify the formulas that we think of as 'holding' at that point. This of course depends on what formulas are assigned to all of the other points of the arrangement. The set of such formulas would in general be infinite, but we can restrict ourselves to a finite set by speaking of the formulas of a certain level. The level of a formula may be thought of as the number of steps involved in constructing that formula from the formulas of the arrangement.

D47. If A is an arrangement, $i \in A_0 \times A_0$ and $k \in \omega$, then $lev(\phi,k,A,i)$, or ϕ is a k-level formula of A at i, if and only if there are $m,n \in A_0$ and a finite set of formulas Γ such that i is $\langle m,n \rangle$, ϕ is the conjunction (in order) of the formulas in Γ, and for each $\psi \in \Gamma$, either

(a) $\psi \in A_2(i)$,

(b) k is not 0, and there are a formula χ and $p \in A_0$ such that $p <_A m$, $lev(\chi,k-1,A,\langle p,n \rangle)$ and ψ is $P\chi$,

(c) k is not 0, and there are a formula χ and $p \in A_0$ such that $m <_A p$, $lev(\chi,k-1,A,\langle p,n \rangle)$ and ψ is $F\chi$,

(d) k is not 0, and there is a formula χ such that $lev(\chi,k-1,A,\langle m,m \rangle)$ and ψ is $K\chi$,

or (e) k is not 0, and $lev(\psi,k-1,A,\langle m,n \rangle)$.

It is convenient to single out one formula as the strongest k-level formula of A at i. It will turn out that all the k-level formulas of A at i are derivable from the k-level formula of A at i.

D48. If A is a finite arrangement, $i \in A_0 \times A_0$ and $k \in \omega$, then $LEV(k,A,i)$, or the k-level formula of A at i is that formula ϕ such that there are $m,n \in A_0$ such that i is $\langle m,n\rangle$ and ϕ is the conjunction (in order) of the formulas ψ such that either

(a) $\psi \in A_2(i)$

(b) k is not 0, and there is a p such that $p <_A m$ and ψ is P LEV(k-1,A,$\langle p,n\rangle$)

(c) k is not 0, and there is a p such that $m <_A p$ and ψ is F LEV(k-1,A,$\langle p,n\rangle$)

(d) k is not 0, and ψ is K LEV(k-1,A,$\langle m,m\rangle$)

(e) k is not 0, and ψ is LEV(k-1,A,$\langle m,n\rangle$)

T25. (a) If A is a finite arrangement, $i \in A_0 \times A_0$ and $k \in \omega$, then lev(LEV(k,A,i),k,A,i)

(b) If A is part of B and lev(ϕ,k,A,i), then lev(ϕ,k,B,i)

(c) If lev(ϕ,k,A,i) and $k < n$, then lev(ϕ,n,A,i)

Proof: These are all trivial consequences of D47 and D48. (c) requires a simple induction using (e) of D47.

The following theorem will be important in Chapter IV.

T26. If A is an arrangement and lev(ϕ,k,A,$\langle m,n\rangle$), then there is a finite arrangement B such that B is part of A and lev(ϕ,k,B,$\langle m,n\rangle$).

Proof: Assume that A is an arrangement, and let N be the set of $k \in \omega$ such that, for any formula ϕ and m,n

$\in A_0$, if lev(ϕ,k,A,$\langle$m,n$\rangle$), then there is a finite arrangement B such that B is part of A and lev(ϕ,k,B,$\langle$m,n$\rangle$). We will show by induction that N is ω.

Suppose first that lev(ϕ,0,A,$\langle$m,n$\rangle$). Then there is a finite set of formulas $\Gamma \subseteq A_2(\langle m,n\rangle)$ such that ϕ is CJ(Γ).

Let j be 1+max(m,n). Let R be the set of ordered pairs $\langle p,p'\rangle$ such that p,p' < j and $p \leq_A p'$. Let F be that function with domain j×j such that, for each $\langle p,p'\rangle \in$ j×j;
(i) if $\langle p,p'\rangle$ is not $\langle m,n\rangle$, F($\langle p,p'\rangle$) is 0, and
(ii) F($\langle m,n\rangle$) is Γ.

Then B is a finite arrangement, B is part of A and lev(ϕ,0,B,$\langle$m,n$\rangle$). Hence $0 \in$ N.

For the inductive step, suppose that $k \in$ N; suppose also that ϕ is a formula, m,n $\in A_0$, and lev(ϕ,k+1,A,$\langle$m,n$\rangle$).

We will assume (T21) that all of the finite arrangements have been enumerated in some standard way.

Let f be that function whose domain is Γ, and such that (i) If $\psi \in \Gamma$ and lev(ψ,k,A,$\langle$m,n$\rangle$), f(ψ) is the first finite arrangement B such that B is part of A and lev(ψ,k,B,$\langle$m,n$\rangle$)

Otherwise,

(ii) If ψ is Pχ, f(ψ) is the first finite arrangement B such that B is part of A and lev(ψ,k,B,$\langle$p,n$\rangle$), where p is the first $p \in \omega$ such that $p <_A m$ and lev(χ,k,A,$\langle$p,n$\rangle$)

(iii) If ψ is Fχ, f(ψ) is the first finite arrangement

B such that B is part of A and $lev(\chi,k,B,\langle p,n\rangle)$, where p is the first $p \in \omega$ such that $m <_A p$ and $lev(\chi,k,A,\langle p,n\rangle)$

(iv) If ψ is $K\chi$, $f(\psi)$ is the first finite arrangement B such that B is part of A and $lev(\chi,k,B,\langle m,m\rangle)$

The existence of the finite arrangement B is guaranteed in each case by the inductive hypothesis.

Let j be the union of the sets $(f(\psi))_0$, for $\psi \in \Gamma$.

Let R be A_1 restricted to j.

Let F be that function whose domain is $j\times j$, and such that, for each $\langle p,p'\rangle \in j\times j$, $F(\langle p,p'\rangle)$ is the union of the sets $B_2(\langle p,p'\rangle)$, where $B \in Rng(f)$ and $p,p' \in B_0$.

Let B be $\langle j,R,F\rangle$. Then B is a finite arrangement and B is part of A. It is sufficient to show that for each formula $\psi \in \Gamma$, if ψ satisfies one of the conditions (a)-(e) of D47 with respect to A, it also satisfies one of those five conditions with respect to B. We have the following five cases:

(a) $\psi \in A_2(\langle m,n\rangle)$; then $lev(\psi,0,A,\langle m,n\rangle)$. By T25c, $lev(\psi,k,A,\langle m,n\rangle)$. Then $lev(\psi,k,f(\psi),\langle m,n\rangle)$. By T25b, $lev(\psi,k,B,\langle m,n\rangle)$.

(b) There are a formula χ and $p \in A_0$ such that $p <_A m$, $lev(\chi,k,A,\langle p,n\rangle)$ and ψ is $P\chi$. Let p' be the first such p; then $lev(\chi,k,f(\psi),\langle p',n\rangle)$ and, by T25b, $lev(\chi,k,B,\langle p',n\rangle)$. ψ satisfies condition (b).

(c) There are a formula χ and $p \in A_0$ such that $m <_A p$,

$lev(\chi,k,A,\langle p,n\rangle)$ and ψ is $F\chi$. This case is exactly analogous to (b).

(d) There is a formula χ such that $lev(\chi,k,A,\langle m,m\rangle)$ and ψ is $K\chi$. Then $lev(\chi,k,f(\psi),\langle m,m\rangle)$, and (T25b) $lev(\chi,k,B,\langle m,m\rangle)$. ψ satisfies condition (d).

(e) $lev(\psi,k,A,\langle m,n\rangle)$. Then $lev(\psi,k,f(\psi),\langle m,n\rangle)$ and (T25b) $lev(\psi,k,B,\langle m,n\rangle)$.

3. Minimal Extension Sequences

D49. If Γ is a set of formulas and $n \in \omega$, $\Gamma * n$ is the set of the first n formulas in Γ, if there are more than n formulas in Γ; otherwise $\Gamma * n$ is Γ.

The main part of the completeness proof will be the construction of a certain ω-place minimal extension sequence. In the following definition $\langle\{0\},\{\langle 0,0\rangle\},\{\langle\langle 0,0\rangle,0\rangle\}\rangle$ is the null arrangement, which is part of every arrangement.

D50. <u>Σ is a p-place minimal extension sequence with respect to Γ and Δ</u> if and only if

(1) Σ is a p-sequence; $p \in \omega\cup\{\omega\}$ and p is not 0,

(2) Σ_0 is $\langle\{0\},\{\langle 0,0\rangle\},\{\langle\langle 0,0\rangle,0\rangle\}\rangle$,

and (3) For each k such that $k+1 \in p$, Σ_{k+1} is a minimal extension of Σ_k with respect to $\Gamma*(k+1)$ and Δ.

T27. If Σ is a p-place minimal extension sequence, $m,m' \in p$, and $m \leq m'$, then Σ_m is part of $\Sigma_{m'}$.

Proof: A trivial induction.

T28. If Σ is an n-place minimal extension sequence with respect to Γ and Δ, then the set of n+1-place minimal extension sequences Σ' with respect to Γ and Δ such that $\Sigma \subseteq \Sigma'$ is finite and non-empty.

Proof: Assume Σ is an n-place minimal extension sequence with respect to Γ and Δ. By D50, n is not 0. Also, either Σ_{n-1} is $\langle\{0\},\{\langle 0,0\rangle\},\{\langle\langle 0,0\rangle,0\rangle\}\rangle$ or Σ_{n-1} is a minimal extension of Σ_{n-2} with respect to $\Gamma^*(n-1)$ and Δ; in either case

(1) For each $i,j \in \Sigma_{n-1,0}$ and each $\phi \in \Sigma_{n-1,2}(\langle i,j\rangle)$ either

(a) $\phi \in \Sigma_{n-2,2}(\langle i,j\rangle)$

(b) $\phi \in \Gamma^*(n-1)$

(c) For some $\psi \in \Gamma^*(n-1)$, ϕ is $\neg\psi$ or ϕ is $H\psi$ or ϕ is $G\psi$

(d) There are a formula ψ and variables α and β such that $\Lambda\alpha\psi \in \Gamma^*(n-1)$, $\Lambda\alpha\psi$ is a formula of Σ_{n-2}, β is one of the first $\Sigma_{n-1,0}{}^{2\overline{\Gamma^*(n-1)}}$ variables in Δ occurring free neither in any formula in $\Gamma^*(n-1)$ nor in any formula of Σ_{n-1}, and ϕ is $\neg ps(\beta,\alpha,\psi)$.

We construct the new arrangement B as follows: Let j be $\Sigma_{n-1,0}$. Let R be $\Sigma_{n-1,1}$. Let F be that function whose domain is $j\times j$ and such that, for each $m,k \in j$,

$F(\langle m,k\rangle)$ is $\Sigma_{n-1,2}(\langle m,k\rangle) \cup (\Gamma^{*}n) \cup \{H\phi : \phi \in \Gamma^{*}n\} \cup \{G\phi : \phi \in \Gamma^{*}n\}$.

Note that B is an extension of Σ_{n-1} with respect to $\Gamma^{*}n$. In addition, by (1) and the above, B is a minimal extension of Σ_{n-1}. Hence $\Sigma^\frown\langle B\rangle$ is an n+1-place minimal extension sequence with respect to Γ.

It remains only to show that there are only finitely many n+1-place minimal extension sequences with respect to Γ and Δ such that $\Sigma \leq \Sigma'$, but this follows immediately from T24.

T29. If Δ is a set of variables, then there is an ω-place minimal extension sequence with respect to Γ and Δ.

Proof: By T21, there is an enumeration f of the set of finite arrangements. Let Σ be that sequence such that

(a) Σ_0 is $\langle\{0\},\{\langle 0,0\rangle\},\{\langle\langle 0,0\rangle,0\rangle\}\rangle$

(b) For each $n \in \omega$, Σ_{n+1} is the first (according to f) arrangement A such that $\langle\Sigma_0,\ldots,\Sigma_n\rangle^\frown\langle A\rangle$ is a minimal extension sequence with respect to Γ and Δ.

By T28, Σ is a minimal extension sequence with respect to Γ and Δ.

T30. If Δ is a set of variables, then the set of n+1-place minimal extension sequences with respect to Γ and Δ is finite and non-empty.

Proof: (By Induction) The set of 1-place minimal extension sequences with respect to Γ and Δ is $\{\langle\langle\{0\},\{\langle 0,0\rangle\},\{\langle\langle 0,0\rangle,0\rangle\}\rangle\rangle\}$.

Suppose $n \in \omega$ and the set of n+1-place minimal extension sequences with respect to Γ and Δ is finite and non-empty. The set of n+2-place minimal extension sequences with respect to Γ and Δ is the union of the sets S such that for some n+1-place minimal extension sequence Σ with respect to Γ and Δ, S is the set of n+2-place minimal extension sequences Σ' with respect to Γ and Δ such that $\Sigma \leq \Sigma'$. But then (by the inductive hypothesis and T28) the set of n+2-place minimal extension sequences with respect to Γ and Δ is the union of a finite and non-empty set of finite and non-empty sets, and hence is itself finite and non-empty.

4. Characteristic Formulas

The following definition reflects the decision to regard 0 as the 'present moment' of any arrangement.

D51. If A is a finite arrangement and $n \in \omega$, then CH(A,n) or <u>the n-level characteristic formula of A</u> is LEV(n,A,⟨0,0⟩).

The set of axioms that result from the construction in this section will be specified according to the following definition.

D52. If Δ is a denumerable set of variables and $m,n \in \omega$, then CH*(Γ,Δ,n,m) or <u>the m-level characteristic formula of the n^{th} extension of Γ and Δ</u>, is the disjunction (in order) of the formulas $\vee\alpha_0 \ldots \vee\alpha_{k-1}$CH($\Sigma_n$,m) where Σ is an n+1-place minimal extension sequence with respect to Γ and Δ, and

$\alpha_0, \ldots, \alpha_{k-1}$ are (in order) the variables in Δ that occur free in $CH(\Sigma_n, m)$.

T31. If Δ is a denumerable set of variables, then $CH^*(\Gamma, \Delta, n, m)$ is logically valid.

Proof: Suppose that Δ is a denumerable set of variables, and let L be the language of Γ. Suppose also that $\mathfrak{a}$ is an interpretation for L, $\mathfrak{a}$ is $\langle T, \leq, U, G\rangle$, $t \in T$ and $x \in U^{\omega}$. Then, by D38, it is sufficient to show that for each $m, n \in \omega$, $x \in Int_{\mathfrak{a}}(CH^*(\Gamma, \Delta, n, m))(\langle t, t\rangle)$.

We will show this by constructing an ω-place minimal extension sequence Σ such that for each $m, n \in \omega$, $x \in Int_{\mathfrak{a}}(\bigvee\alpha_0 \ldots \bigvee\alpha_{k-1} CH(\Sigma_n, m))(\langle t, t\rangle)$ (where $\alpha_0, \ldots, \alpha_{k-1}$ are as in D52). Thus, for each $m, n \in \omega$, x will satisfy some disjunct of $CH^*(\Gamma, \Delta, n, m)$ at $\langle t, t\rangle$.

For this construction we will need to progressively select members of T and U. Let C be a choice function on $ST \sim \{0\}$, and let C' be a choice function on $SU \sim \{0\}$.

In the following definition, $\mathfrak{F}(n, \Xi)$ is a relation that is to hold if Ξ is an entity that specifies the way in which Σ_{n+1} is constructed from Σ_n. Ξ is always a 15-tuple, since there are many aspects of this construction that must be kept track of. In the definition A is the arrangement corresponding to Σ_n in the construction and B is the one corresponding to Σ_{n+1}.

The construction of Σ must of course depend on reflecting at each stage a greater part of the structure of the interpretation $\mathfrak{A}$. In order to do this we must keep track of functions that relate the moments of the arrangement at each step to the corresponding moments of $\mathfrak{A}$; in the definition below, r and s are such functions. Each variable that is introduced into one of the Σ_n as an instantiation (i.e., according to clause (3d) of D46) must be related to a member of the universe of $\mathfrak{A}$; a and b are the functions that do this. Here b is an extension of a, just as s is an extension of r and B is an extension of A. We need not have included y and z in Ξ, since y is determined by x and a, and z by x and b, but it is done as a matter of convenience. y is a sequence that satisfies CH(A,m) and similarly for z and CH(B,m).

The terms g,h,c,d and e are instrumental in building up Θ' from Θ. The function g specifies what new moments of $\mathfrak{A}$ must be reflected into B_0 in order to 'account for' formulas of the form $G\phi$ not being satisfied, and similarly for h and formulas of the form $H\phi$. The function C associates new moments of B with new moments of $\mathfrak{A}$. The function d specifies what new individuals of $\mathfrak{A}$ must be assigned to variables that occur in formulas of A to 'account for' formulas of the form $\Lambda\alpha\phi$ not being satisfied, and e assigns one of the individuals specified by d to each of the new variables introduced into B.

Clause (12) of the following definition indicates how B is to be specified in terms of A and Ξ. Put briefly, B is the interpretation formed by adding the moments Dom(c) to A_0, using the order induced by $\mathfrak{a}$ and s, and assigning to each $\langle p,p'\rangle \in B_0$ a certain set of formulas specified by A, $\mathfrak{a}$, z and the already defined functions s,g,h,d and e.

Definition A. We say $\mathfrak{J}(n,\Xi)$ if and only if there are Θ,A,r,y,a,g,h,c,s,d,e,b,z,B and Θ' such that all of (1)-(12) hold:

(1) Ξ is the 15-tuple $\langle \Theta,A,r,y,a,g,h,c,s,d,e,b,z,B,\Theta'\rangle$

(2) Θ is $\langle A,r,y,a\rangle$ and Θ' is $\langle B,s,z,b\rangle$

(3) A is a finite arrangement, r is a one-to-one function, Dom(r) is A_0, Rng(r) $\subseteq$ T, y $\in U^\omega$, a is a function, Dom(a) $\subseteq$ Iv and Rng(a) $\subseteq$ U

(4) h is that function such that

(a) Dom(h) is the set of triples $\langle p,p',\phi\rangle$ such that p,p' $\in A_0$, $\phi \in \Gamma^*(n+1)$, and y $\notin$ $Int_{\mathfrak{a}}(H\phi)(\langle r(p),r(p')\rangle)$

(b) For each $\langle p,p',\phi\rangle \in$ Dom(h), h($\langle p,p',\phi\rangle$) is C(T'), where T' is the set of t $\in$ T such that r(p) $>_{\mathfrak{a}}$ t and y $\in$ $Int_{\mathfrak{a}}(\neg\phi)(\langle t,r(p')\rangle)$

(5) g is that function such that

(a) Dom(g) is the set of triples $\langle p,p',\phi\rangle$ such that p,p' $\in A_0$, $\phi \in \Gamma^*(n+1)$, and y $\notin$ $Int_{\mathfrak{a}}(G\phi)(\langle r(p),r(p')\rangle)$

(b) For each $\langle p,p',\phi\rangle \in$ Dom(g), g($\langle p,p',\phi\rangle$) is C(T'), where T' is the set of t $\in$ T such that t $>_{\mathfrak{a}}$ r(p) and y $\in$ $Int_{\mathfrak{a}}(\neg\phi)(\langle t,r(p')\rangle)$

(6) c is a one-to-one function from the first $\overline{\overline{(\mathrm{Rng}(g) \cup \mathrm{Rng}(h)) \sim \mathrm{Rng}(r)}}$ natural numbers not in A_0 onto $(\mathrm{Rng}(g) \cup \mathrm{Rng}(h)) \sim \mathrm{Rng}(r)$

(7) s is $r \cup c$

(8) d is that function such that

(a) Dom(d) is the set of quadruples $\langle p, p', \phi, \alpha \rangle$ such that $p, p' \in \mathrm{Dom}(s)$, $\alpha \in \mathrm{fv}(\phi)$, $y \notin \mathrm{Int}_{\mathfrak{A}}(\Lambda \alpha \phi)(\langle s(p), s(p') \rangle)$ and $\Lambda \alpha \phi \in \Gamma^*(n+1)$

(b) For each $\langle p, p', \phi, v_j \rangle \in \mathrm{Dom}(d)$, $d(\langle p, p', \phi, v_j \rangle)$ is $C'(B)$, where B is the set of $w \in J$ such that $y^j_w \in \mathrm{Int}_{\mathfrak{A}}(\neg \phi)(\langle s(p), s(p') \rangle)$

(9) e is a one-to-one function such that

(a) Rng(e) is Rng(d)

(b) Dom(e) is the set that contains the first $\overline{\overline{\mathrm{Rng}(d)}}$ variables $\beta \in \Delta$ such that β occurs free neither in any formula in $\Gamma^*(n+1)$ nor in any formula of A

(10) b is $a \cup e$

(11) z is that infinite sequence such that

(a) z_j is y_j, if $v_j \in Iv \sim \mathrm{Dom}(b)$

(b) z_j is $b(v_j)$, if $v_j \in \mathrm{Dom}(b)$

(12) $\mathfrak{B}$ is that arrangement such that

(a) B_0 is $A_0 \cup \mathrm{Dom}(c)$

(b) B_1 is the set of pairs $\langle p, p' \rangle$ such that $p, p' \in B_0$ and $s(p) \leq_{\mathfrak{A}} s(p')$

(c) B_2 is that function F with domain $B_0 \times B_0$

such that for each $p, p' \in B_0$, $F(\langle p, p' \rangle)$ is the union of the following seven sets:

(i) $A_2(\langle p, p' \rangle)$, if $p, p' \in A_0$

(ii) the set of formulas $\phi \in \Gamma^*(n+1)$ such that $z \in Int_{\mathfrak{a}}(\phi)(\langle s(p), s(p') \rangle)$

(iii) the set of formulas $\neg\phi$ such that $\phi \in \Gamma^*(n+1)$ and $z \notin Int_{\mathfrak{a}}(\phi)(\langle s(p), s(p') \rangle)$

(iv) the set of formulas $H\phi$ such that $\phi \in \Gamma^*(n+1)$ and $z \in Int_{\mathfrak{a}}(H\phi)(\langle s(p), s(p') \rangle)$

(v) the set of formulas $G\phi$ such that $\phi \in \Gamma^*(n+1)$ and $z \in Int_{\mathfrak{a}}(G\phi)(\langle s(p), s(p') \rangle)$

(vi) the set of formulas $\neg ps(\beta, \alpha, \phi)$, where $\langle p, p', \phi, \alpha \rangle \in Dom(d)$ and β is $\breve{e}(d(\langle p, p', \phi, \alpha \rangle))$

(vii) the set of formulas $\neg\phi$ such that for some α, $\Lambda\alpha\phi \in \Gamma^*(n+1)$, $y \notin Int_{\mathfrak{a}}(\Lambda\alpha\phi)(\langle s(p), s(p') \rangle)$, and $\alpha \notin fv(\phi)$.

Let C" be a choice function on $SB \sim \{0\}$, where B is the set of 4-tuples $\langle A, r, y, a \rangle$ such that A is an arrangement, r is a function from a natural number into T, $y \in U^{(\iota)}$, and a is a function such that $Dom(a) \subseteq Iv$ and $Rng(a) \subseteq U$.

We use C" and Definition A to define a sequence of 4-tuples Λ. For each n, Σ_n will be the first term of the 4-tuple Λ_n.

Let Λ be that infinite sequence defined as follows:

(1) Λ_0 is $\langle\langle\{0\}, \{\langle 0,0 \rangle\}, \{\langle\langle 0,0 \rangle, 0 \rangle\}\rangle, \{\langle 0, t \rangle\}, x, 0 \rangle$

(2) For each $n \in \omega$, Λ_{n+1} is C"(B), where B is the set of 4-tuples $\langle A,r,y,a \rangle$ such that there is a Ξ such that $\mathfrak{F}(n+1,\Xi)$, Ξ_0 is Λ_n and Ξ_{14} is $\langle A,r,y,a \rangle$.

Let Σ be that infinite sequence such that for each $n \in \omega$, Σ_n is $\Lambda_{n,0}$.

The following Lemma states that the sequences Λ and Σ behave in the manner claimed before Definition A.

Lemma B. The following seven clauses hold for each $n \in \omega$. In order to make the statement of the lemma more easily readable, let A,r,y,a be those objects such that Λ_n is $\langle A,r,y,a \rangle$, and let B,s,z,b be those objects such that Λ_{n+1} is $\langle B,s,z,b \rangle$. Then of course Σ_n is A and Σ_{n+1} is B.

(1) There is a Ξ such that $\mathfrak{F}(n+1,\Xi)$, Λ_n is Ξ_0 and Λ_{n+1} is Ξ_{14}

(2) B is a minimal extension of A with respect to $\Gamma^*(n+1)$ and Δ

(3) s is a one-to-one function from B_0 into T, $r \subseteq s$, and for each $p,p' \in B_0$, $\langle p,p' \rangle \in B_1$ if and only if $s(p) \leq_{\mathfrak{a}} s(p')$

(4) (a) $a \subseteq b$

(b) b is a function

(c) For each $\beta \in Dom(b)$, $\beta \in \Delta$

(d) For each $\beta \in Dom(b)$, β occurs free in some formula of B

(e) For each $\beta \in Dom(b \sim a)$, β does not occur free in any formula of A nor any formula in $\Gamma^*(n+1)$

(5) For each j such that $v_j \notin \Delta$, z_j is x_j

(6) For each $\phi \in \Gamma^*(n+1)$ and j such that $v_j \in fv(\phi)$, z_j is y_j

(7) For each $p,p' \in B_0$ and $\phi \in B_2(\langle p,p' \rangle)$, $z \in Int_{\mathcal{a}}(\phi)(\langle s(p),s(p') \rangle)$

Proof: We will prove Lemma B by induction on n. We omit the case where n is 0, since the construction involved is entirely similar to the one involved in the inductive step. Therefore, we suppose that (1)-(7) all hold for n, and we will show that they also hold for n+1.

Let Ξ be a 15-tuple such that $\mathcal{F}(n+1,\Xi)$, Λ_n is Ξ_0 and Λ_{n+1} is Ξ_{14}. Let Θ,A,r,y,a,g,h,c,s,d,e,b,z,B and Θ' be those objects such that Ξ is $\langle \Theta,A,r,y,a,g,h,c,s,d,e,b,z,B,\Theta' \rangle$; then Λ_n is θ and Λ_{n+1} is Θ'.

Since A is a finite arrangement, $A_0 \in \omega$. By Definition A parts (6) and (12a), $B_0 \in \omega$. By Definition A parts (3)-(7), s is a one-to-one function from B_0 into T. By Definition A part (12b), B_1 is a reflexive linear ordering on B. By Definition A part (12c), B is a finite arrangement. By Definition A parts (9), (10), and (11), $z \in U^{\omega}$. By Definition A parts (10), (9), and (8), b is a function, $Dom(b) \subseteq Iv$ and $Rng(b) \subseteq U$. Following through the clauses of Definition A, we see that there are g',h',c',s',d',e', b',z',B',Θ'', and Ξ' such that Ξ' is $\langle \Theta',B,s,z,b,g',h',c',s',d',e',b',z',B',\Theta'' \rangle$ and $\mathcal{F}(n+2,\Xi')$. Then the set $\mathcal{a}$ is non-empty, where $\mathcal{a}$ is the set of 4-tuples $\langle B',s',z',b' \rangle$ such that there is a Ξ' such that $\mathcal{J}(n+2,\Xi')$, Ξ_0 is $\langle B,s,z,b\rangle$ and Ξ_{14} is $\langle B',s',z',b'\rangle$. Let Θ'' be $C''(\mathfrak{a})$. Let Ξ' be a 15-tuple such that $\mathcal{J}(n+2,\Xi')$, Ξ'_0 is $\langle B,s,z,b\rangle$ and Ξ'_{14} is Θ''. Let g',h',c',s',d',e',b',z' and B' be those objects such that Ξ' is $\langle\Theta',B,s,z,b,g',h',c',s',d',e',b',z',B',\Theta''\rangle$. Then Λ_{n+2} is $\langle B',s',z',b'\rangle$ and Σ_{n+2} is B'. Thus (1) holds for n+1.

We must now prove (2); that is, that B' is a minimal extension of B with respect to $\Gamma^*(n+2)$ and Δ.

By Definition A part (12), and the fact that B is a minimal extension of A with respect to $\Gamma^*(n+1)$ and Δ, it follows easily that clauses (1)-(4) of Definition 44 hold. In order to show that clause (5a) of D44 holds ((5b) is of course analogous), suppose that $p,p' \in B_0$, $\phi \in \Gamma^*(n+2)$, and $H\phi \notin B'_2(\langle p,p'\rangle)$. By Definition A part (12c iv), $z' \in Int_{\mathfrak{a}}(H\phi)(\langle s'(p),s'(p')\rangle)$. By the inductive hypothesis clause (6) and T15, $z \notin Int_{\mathfrak{a}}(H\phi)(\langle s'(p),s'(p')\rangle)$. By the inductive hypothesis clause (3), $z \notin Int_{\mathfrak{a}}(H\phi)(\langle s(p),s(p')\rangle)$. By Definition A part (4a), $\langle p,p',\phi\rangle \in Dom(h')$. Since $z \notin Int_{\mathfrak{a}}(H\phi)(\langle s(p),s(p')\rangle)$, there is a $t \in T$ such that $s(p) >_{\mathfrak{a}} t$ and $z \in Int_{\mathfrak{a}}(\neg\phi)(\langle t,s(p')\rangle)$, and hence $h(\langle p,p',\phi\rangle)$ is one such. Let t' be $h(\langle p,p',\phi\rangle)$. By Definition A parts (6) and (7), $t' \in Rng(s')$. Let q be $s'(t')$. Then $z \in Int_{\mathfrak{a}}(\neg\phi)(\langle s(q),s(p')\rangle)$. Again, by the inductive hypothesis clauses (3) and (6) and T15, $z' \in Int_{\mathfrak{a}}(\neg\phi)(\langle s'(q),s'(p')\rangle)$. By Definition A part (12c iii), $\neg\phi \in B'_2(\langle q,p'\rangle)$. By Definition A part (12b), $p >_{B'} q$, completing the proof that B' is an extension of B.

Now in order to show that B' is a minimal extension of B, we need only show that clauses (3) and (4) of D46 hold.

For clause (3) of D46, suppose that $p, p' \in B'_0$ and $\phi \in B'_2(\langle p, p' \rangle)$. Then ϕ must belong to one of the seven sets listed under Definition A part (12c). If ϕ belongs to some set other than (vi) or (vii) it follows immediately that ϕ satisfies clause (3) of D46.

Suppose first, then, that $\phi \in$ (vi); then there is a formula ψ and variables α and β such that ϕ is $\neg ps(\beta, \alpha, \psi)$, $\langle p, p', \psi, \alpha \rangle \in Dom(d')$ and β is $\breve{e}'(d'(\langle p, p', \psi, \alpha \rangle))$. By Definition A parts (9) and (8), $\overline{\overline{Dom(e')}} = \overline{\overline{Rng(e')}} = \overline{\overline{Rng(d')}} \leq \overline{\overline{Dom(d')}} \leq B'_0{}^2 \cdot \overline{\overline{\Gamma^*(n+2)}}$; that is, $\overline{\overline{Rng(d')}} \leq B'_0{}^2 \cdot \overline{\overline{\Gamma^*(n+2)}}$. Hence, (Definition A part (9)) β is one of the first $B'_0{}^2 \cdot \overline{\overline{\Gamma^*(n+2)}}$ variables in Δ that occur neither in any formula of B nor in any formula in $\Gamma^*(n+2)$.

Second, suppose that $\phi \in$ (vii); then there is a formula ψ and a variable α such that ϕ is $\neg\psi$, $\Lambda\alpha\psi \in \Gamma^*(n+2)$, $z \notin Int_{\alpha}(\Lambda\alpha\psi)(\langle s'(p), s'(p') \rangle)$, and $\alpha \notin fv(\psi)$. Let β be the first variable in Δ such that β occurs free neither in any formula in $\Gamma^*(n+2)$ nor in any formula of B. Then, by T9b, ϕ is $\neg ps(\beta, \alpha, \psi)$, and ϕ satisfies clause (3d) of D46.

For clause (4) of D46, we proceed as follows: By Definition A part (4), $\overline{\overline{Dom(h')}} \leq B_0{}^2 \cdot \overline{\overline{\Gamma^*(n+2)}}$. By Definition part (5), $\overline{\text{Dom}(g')} \leq B_0{}^2\ \overline{\Gamma^*(n+2)}$. Hence we have $\overline{\text{Dom}(c')} \leq$ (by Definition A part (6)) $\overline{\text{Rng}(g') \cup \text{Rng}(h')} \leq \overline{\text{Rng}(g')}+\overline{\text{Rng}(h')}$ $\leq \overline{\text{Dom}(g')}+\overline{\text{Dom}(h')} \leq 2B_0{}^2 \cdot \overline{\Gamma^*(n+2)}$; that is, $\overline{\text{Dom}(c')} \leq$ $2B_0{}^2 \cdot \overline{\Gamma^*(n+2)}$. By Definition A part (12a), B'_0 is $B_0+\overline{\text{Dom}(c')}$. Hence $B'_0 \leq B_0+2B_0{}^2 \cdot \overline{\Gamma^*(n+2)}$, and $B'_0-B_0 \leq 2B_0{}^2 \cdot \overline{\Gamma^*(n+2)}$. This completes the proof of (2).

By the inductive hypothesis s is a one-to-one function from B_0 into T. By Definition A parts (12) and (4)-(7), s' is a one-to-one function from B'_0 into T. By Definition A part (7), $s \leq s'$; also, by Definition A part (12b), for each $p,p' \in B'_0$, $\langle p,p' \rangle \in B'_1$ if and only if $s'(p) <_a s'(p')$. Hence Lemma B part (3) holds.

By Definition A part (10), $b \leq b'$ and Lemma B part (4a) holds. By the inductive hypothesis (clause (d)), for each $\beta \in \text{Dom}(b)$, β occurs free in some formula of B. By Definition A part (9b), for each $\beta \in \text{Dom}(e')$, β does not occur free in any formula of B. Hence Dom(b) and Dom(e') are disjoint, b' is a function, and Lemma B part (4b) holds. Lemma B part (4c) holds also, since if $\beta \in \text{Dom}(b')$, then either $\beta \in \text{Dom}(b)$, in which case (by the inductive hypothesis) $\beta \in \Delta$; or $\beta \in \text{Dom}(e')$, in which case (by Definition A part 9(b)) $\beta \in \Delta$.

For Lemma B part (4d), suppose that $\beta \in \text{Dom}(b')$. Then either $\beta \in \text{Dom}(b)$ or $\beta \in \text{Dom}(e')$. If $\beta \in \text{Dom}(b)$, then (by the inductive hypothesis) β occurs free in some formula of B and (since B' is an extension of B) β occurs free in some formula of B'. Suppose, on the other hand, that $\beta \in$ Dom(e'). Then (Definition A part (9)) $e'(\beta) \in$ Rng(d'), and there is a 4-tuple $\langle p,p',\phi,\alpha\rangle$ such that $p,p' \in$ Dom(s'), $\alpha \in$ fv(ϕ), $z \notin \mathrm{Int}_{\mathfrak{A}}(\Lambda\alpha\phi)(\langle s'(p),s'(p')\rangle)$, $\Lambda\alpha\phi \in \Gamma^*(n+2)$, and $e'(\beta)$ is $d'(\langle p,p',\phi,\alpha\rangle)$. Then β is $\breve{e}'(d'(\langle p,p',\phi,\alpha\rangle))$. By Definition A part (12c vi) $\neg ps(\beta\cdot\alpha,\phi) \in B'_2(\langle p,p'\rangle)$, and hence $\neg ps(\beta,\alpha,\phi)$ is a formula of B'. By T9d, $\beta \in$ fv($\neg ps(\beta,\alpha,\phi)$), and β occurs free in some formula of B'.

Lemma B part (4e) follows immediately from Definition A parts (9) and (10). This completes the proof of Lemma B part (4).

To show Lemma B part (5), suppose that $v_j \notin \Delta$. Then $v_j \notin$ Dom(b'), and z'_j is z_j. But by the inductive hypothesis, z_j is x_j.

For Lemma B part (6), suppose that $\phi \in \Gamma^*(n+2)$ and $v_j \in$ fv(ϕ). If $v_j \notin$ Dom(b'), then (by Definition A part (11)) z'_j is z_j. Suppose, then, that $v_j \in$ Dom(b'). By Definition A part (9), $v_j \notin$ Dom(e'), and hence $v_j \in$ Dom(b). Then z_j is b(j), by the inductive hypothesis; and z'_j is b'(j) is b(j), since $v_j \notin$ Dom(e'). Therefore z'_j is z_j.

It remains only to show Lemma B part (7). Suppose that $p,p' \in B'_0$ and $\phi \in B'_2(\langle p,p'\rangle)$. We must show that $z' \in \mathrm{Int}_{\mathfrak{A}}(\phi)(\langle s'(p),s'(p')\rangle)$. We take seven cases, according to which of the seven sets listed under Definition A part (12c) ϕ belongs to:

(i) $p,p' \in B_0$ and $\phi \in B_2(\langle p,p'\rangle)$; by the inductive hypothesis, $z \in Int_{\mathfrak{a}}(\phi)(\langle s(p),s'(p')\rangle)$; by Definition A part (7), $z \in Int_{\mathfrak{a}}(\phi)(\langle s'(p),s'(p')\rangle)$. By Definition A parts (9), (10), and (11), for each j such that v_j occurs free in ϕ, z_j is z'_j. Hence, by T15, $z' \in Int_{\mathfrak{a}}(\phi)(\langle s'(p),s'(p')\rangle)$.

(ii) Follows immediately

(iii) There is a formula ψ such that ϕ is $\neg\psi$, $\psi \in \Gamma^*(n+2)$, and $z \notin Int_{\mathfrak{a}}(\psi)(\langle s'(p),s'(p')\rangle)$. Then $z' \in Int_{\mathfrak{a}}(\phi)(\langle s'(p),s'(p')\rangle)$.

(iv) Follows immediately

(v) Follows immediately

(vi) There are a formula ψ and variables α and β such that ϕ is $\neg ps(\beta,\alpha,\psi)$, $\langle p,p',\psi,\alpha\rangle \in Dom(d')$ and β is $\breve{e}'(d'(\langle p,p',\psi,\alpha\rangle))$. Let j be that number such that α is v_j and let u be $d'(\langle p,p',\psi,\alpha\rangle)$. By Definition A part (8), $z \notin Int_{\mathfrak{a}}(\Lambda v_j\psi)(\langle s'(p),s'(p')\rangle)$; hence there is a $w \in U$ such that $z^j_w \in Int_{\mathfrak{a}}(\neg\psi)(\langle s'(p),s'(p')\rangle)$, and $z^j_u \in Int_{\mathfrak{a}}(\neg\psi)(\langle s'(p),s'(p')\rangle)$. Let $v_{j'}$ be $\breve{e}'(u)$. By Definition A part (9b) (since $\Lambda\alpha\psi$ is a formula of B) $v_{j'}$ does not occur in ψ; then (by T15) $z^{j}_{u}{}^{j'}_{u} \in Int_{\mathfrak{a}}(\neg\psi)(\langle s'(p),s'(p')\rangle)$ and $z^{j'}_{u}{}^{j}_{u} \in Int_{\mathfrak{a}}(\neg\psi)(\langle s'(p),s'(p')\rangle)$. Since u is $(z^{j'}_u)_{j'}$, $z^{j'}_{u}{}^{j}_{(z^{j'}_u)_{j'}} \in Int_{\mathfrak{a}}(\neg\psi)(\langle s'(p),s'(p')\rangle)$. By T18, $z^{j'}_u \in Int_{\mathfrak{a}}(ps(v_{j'},v_j,\neg\psi))(\langle s'(p),s'(p')\rangle)$; that is, $z^{j'}_u \in Int_{\mathfrak{a}}(\phi)(\langle s'(p),s'(p')\rangle)$. By T9d and Definition A parts (9)-(11), and since $z_{j'}$ is u, for each j" such that $v_{j''}$ occurs free in ϕ, $(z^{j'}_u)_{j''}$ is $z'_{j''}$; hence, by T15, $z' \in Int_{\mathcal{A}}(\phi)(\langle s'(p), s'(p')\rangle)$.

(vii) Suppose there is a formula ψ and a variable α such that ϕ is $\neg\psi$, $\Lambda\alpha\psi \in \Gamma^*(n+2)$, $z \notin Int_{\mathcal{A}}(\Lambda\alpha\psi)(\langle s'(p), s'(p')\rangle)$ and $\alpha \notin fv(\phi)$. Let j be that number such that α is v_j. Then there is a $w \in U$ such that $z^j_w \notin Int_{\mathcal{A}}(\Lambda\alpha\psi)(\langle s'(p), s'(p')\rangle)$. But by T15, since $v_j \in fv(\phi)$, $z \notin Int_{\mathcal{A}}(\psi)(\langle s'(p), s'(p')\rangle)$. Then $z \in Int_{\mathcal{A}}(\neg\psi)(\langle s'(p), s'(p')\rangle)$ and, by Lemma B part (6) and T15, $z' \in Int_{\mathcal{A}}(\phi)(\langle s'(p), s'(p')\rangle)$.

This completes the proof of Lemma B part (7) and also the proof of Lemma B.

We were to show that for each $m, n \in \omega$, $x \in Int_{\mathcal{A}}(CH^*(\Gamma, \Delta, n, m))(\langle t, t\rangle)$. Suppose then that $n \in \omega$, and let A, r, y, a be those objects such that Λ_n is $\langle A, r, y, a\rangle$; then Σ_n is A.

If n is 0 and $m \in \omega$, then $x \in Int_{\mathcal{A}}(CH^*(\Gamma, \Delta, m, n))(\langle t, t\rangle)$ by T12a, since $CH^*(\Gamma, \Delta, 0, m)$ is CJ(0). Suppose then that n is not 0. By Lemma B part (7), for each $p, p' \in A_0$ and each $\phi \in A_2(\langle p, p'\rangle)$, $y \in Int_{\mathcal{A}}(\phi)(\langle r(p), r(p')\rangle)$. Also, by Lemma B part (3), for each $p, p' \in A_0$, $p <_A p'$ if and only if $r(p) <_{\mathcal{A}} r(p')$.

Lemma C. For each $m \in \omega$, each $p, p' \in A_0$ and each formula ϕ such that $lev(\phi, m, A, \langle p, p'\rangle)$, $y \in Int_{\mathcal{A}}(\phi)(\langle r(p), r(p')\rangle)$.

We will prove Lemma C by induction on m. Suppose first that $p, p' \in A_0$ and $\text{lev}(\phi, 0, A, \langle p, p' \rangle)$. Then by Lemma B part (7) and T12a, $y \in \text{Int}_{\mathcal{A}}(\phi)(\langle r(p), r(p') \rangle)$.

For the inductive step, suppose that for each $p, p' \in A_0$ and each formula ϕ such that $\text{lev}(\phi, m, A, \langle p, p' \rangle)$, $y \in \text{Int}_{\mathcal{A}}(\phi)(\langle r(p), r(p') \rangle)$. Suppose also that $p, p' \in A_0$ and $\text{lev}(\phi, m+1, A, \langle p, p' \rangle)$. Then there is a finite set of formulas Γ' such that ϕ is $CJ(\Gamma')$, and for each $\psi \in \Gamma'$, ψ satisfies one of the conditions (a)-(e) of D47. By T12, it is sufficient to show that for each $\psi \in \Gamma'$, $y \in \text{Int}_{\mathcal{A}}(\psi)(\langle r(p), r(p') \rangle)$. Suppose then that $\psi \in \Gamma'$; we take five cases, according to the clause of D51 that ψ satisfies:

(a) $\psi \in A_2(\langle r(p), r(p') \rangle)$; then by Lemma B part (7), $y \in \text{Int}_{\mathcal{A}}(\psi)(\langle r(p), r(p') \rangle)$

(b) There are a formula χ and $p'' \in A_0$ such that $p <_A p''$, $\text{lev}(\chi, m, A, \langle p'', p' \rangle)$ and ψ is $P\chi$. By the inductive hypothesis $y \in \text{Int}_{\mathcal{A}}(\chi)(\langle r(p''), r(p') \rangle)$. By Lemma B part (3), $r(p) <_{\mathcal{A}} r(p'')$. By T10, $y \in \text{Int}_{\mathcal{A}}(\psi)(\langle r(p), r(p') \rangle)$

(c) This case is similar to case (b).

(d) There is a formula χ such that $\text{lev}(\chi, m, A, \langle p, p \rangle)$ and ψ is $K\chi$. By the inductive hypothesis, $y \in \text{Int}_{\mathcal{A}}(\chi)(\langle r(p), r(p) \rangle)$. By D34, $y \in \text{Int}_{\mathcal{A}}(\psi)(\langle r(p), r(p') \rangle)$

(e) $\text{lev}(\psi, m, A, \langle p, p' \rangle)$; then, by the inductive hypothesis, $y \in \text{Int}_{\mathcal{A}}(\psi)(\langle r(p), r(p') \rangle)$

This completes the proof of Lemma C.

Suppose now that $m \in \omega$; it is sufficient to show that $x \in Int_{\mathfrak{a}}(CH^*(\Gamma,\Delta,n,m))(\langle t,t\rangle)$.

By Lemma C, T25a, and D51, $y \in Int_{\mathfrak{a}}(CH(\Sigma_n,m))(\langle r(0),r(0)\rangle)$. By Lemma B part (3) and the definition of Λ, $r(0)$ is t and $y \in Int_{\mathfrak{a}}(CH(\Sigma_n,m))(\langle t,t\rangle)$. Let $\alpha_0,\ldots,\alpha_{k-1}$ be (in order) the variables in Δ that occur free in $CH(\Sigma_n,m)$.

By k applications of T10, $y \in Int_{\mathfrak{a}}(\forall\alpha_0\ldots\forall\alpha_{k-1}CH(\Sigma_n,m))(\langle t,t\rangle)$. By Lemma B part (5), for each j such that $v_j \notin \Delta$, y_j is x_j. Therefore, by T15, and since no free variable of $\forall\alpha_0\ldots\forall\alpha_{k-1}CH(\Sigma_n,m)$ is in Δ, $x \in Int_{\mathfrak{a}}(\forall\alpha_0\ldots\forall\alpha_{k-1}CH(\Sigma_n,m))(\langle t,t\rangle)$. But now, by T12 and D53, $x \in Int_{\mathfrak{a}}(CH^*(\Gamma,\Delta,m,n))(\langle t,t\rangle)$. This completes the proof of T31.

The following theorem will be proved in an informal way, since a strict proof would require the introduction of a good deal of formal apparatus.

T32. The set of formulas $CH^*(\Gamma,\Delta,n,m)$, where Δ is a denumerable set of variables and $n,m \in \omega$, is decidable.

Proof: First we give a procedure for deciding whether a formula ϕ is $CH^*(0,\Delta,n,m)$, for some Δ,n,m such that Δ is a denumerable set of variables and $n,m \in \omega$. By D46 and D50, for each n-place minimal extension sequence Σ

with respect to Γ and Δ and each $k \le n$, Σ_k is $\langle\{0\},\{\langle 0,0\rangle\},\{\langle\langle 0,0\rangle,0\rangle\}\rangle$. Hence, for any denumerable set of variables Δ and $m,n \in \omega$, ϕ is CH*$(0,\Delta,n,m)$ if and only if ϕ is CH$(\langle\{0\},\{\langle 0,0\rangle\},\{\langle\langle 0,0\rangle,0\rangle\}\rangle,m)$; but the latter formula is easy to construct using D48.

Now we give a procudere for deciding whether a formula ϕ is CH*(Γ,Δ,n,m) for some Γ,Δ,n,m such that Γ is non-empty, Δ is a denumerable set of variables and $n,m \in \omega$. We note that for any such Γ,Δ,n,m, CH*$(\Gamma,\Delta,n+1,m)$ and CH*$(\Gamma,\Delta,n,m+1)$ are both longer than CH*(Γ,Δ,n,m). This puts an upper bound k (the length of ϕ, say) on the values of both n and m that need be considered; that is, ϕ cannot be CH*(Γ,Δ,n,m) where $k < n$ or $k < m$.

As for Γ, it is a simple consequence of D46 that for any n,m, CH*(Γ,Δ,n,m) is CH*$(\Gamma{*}n,\Delta,n,m)$. Also, for any Γ,Δ,n,m, if ϕ is CH*$(\Gamma{*}n,\Delta,n,m)$, then every member of $\Gamma{*}n$ occurs in ϕ. Thus, if ϕ is CH*(Γ,Δ,n,m) for any Γ,Δ,n,m, it is CH*(Γ',Δ,n,m) for some subset Γ' of the set of formulas that occur in ϕ.

For any possible denumerable set of variables Δ, only those variables in Δ that occur in ϕ can be relevant to the construction. Similarly to the preceding case, if ϕ is CH*(Γ,Δ,n,m) for any Γ,Δ,n,m, it is CH*(Γ,Δ,n,m) for some subset Δ' of the set of variables that occur in ϕ.

Putting all of this together, in order to see whether ϕ is CH*(Γ,Δ,n,m) for any appropriate Γ,Δ,n,m, it is only necessary to see whether ϕ is CH*(Γ,Δ,n,m) for some subset Γ of the set of formulas that occur in ϕ, some subset Δ of the set of variables that occur in ϕ, and some $n,m < k$. But this leaves only a finite number of tests, each of which is accomplished by mechanically constructing one of the formulas CH*(Γ,Δ,n,m), where Γ,Δ,n,m are as just specified.

B. Derivations

In this section we present an axiom system and define the notion of a derivation. Then we look into the matter of which formulas follow from the axioms.

1. The Axioms

D53. (a) θ is a <u>restricted axiom</u> if and only if there is a formula ϕ such that θ is a universal generalization of

(RA1) $\phi \rightarrow K\phi$

(b) θ is a <u>general axiom</u> if and only if there are formulas ϕ,ψ and χ, variables α and β, and terms ζ and η such that θ is a generalization of one of the following:

(GA1) $\phi \rightarrow (\psi \rightarrow \phi)$

(GA2) $(\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$

(GA3) $(\neg\phi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \phi)$

(GA4) $\Lambda\alpha(\phi \rightarrow \psi) \rightarrow (\Lambda\alpha\phi \rightarrow \Lambda\alpha\psi)$

(GA5) $\phi \rightarrow \Lambda\alpha\phi$, where $\alpha \notin fv(\phi)$

(GA6) $\eta=\zeta \rightarrow (\phi \rightarrow \psi)$, where ϕ,ψ are atomic formulas and ψ is obtained from ϕ by replacing 0 or more occurrences of η by ζ

(GA7) $\vee\alpha\alpha=\eta$, where α does not occur in η

(GA8) $\alpha=\beta \rightarrow L\alpha=\beta$

(GA9) $H(\phi \rightarrow \psi) \rightarrow (H\phi \rightarrow H\psi)$

(GA10) $G(\phi \rightarrow \psi) \rightarrow (G\phi \rightarrow G\psi)$

(GA11) $H\phi \rightarrow HH\phi$

(GA12) $G\phi \rightarrow GG\phi$

(GA13) $\phi \rightarrow GP\phi$

(GA14) $\phi \rightarrow HF\phi$

(GA15) $\phi \rightarrow K\phi$, where ϕ does not contain R outside the scope of K

(GA16) $\neg K\phi \leftrightarrow K\neg\phi$

(GA17) $K(\phi \rightarrow \psi) \rightarrow (K\phi \rightarrow K\psi)$

(GA18) $\bigwedge\alpha K\phi \rightarrow K\bigwedge\alpha\phi$

(GA19) $K(\phi \leftrightarrow R\phi)$

(GA20) $R(\phi \rightarrow \psi) \rightarrow (R\phi \rightarrow R\psi)$

(GA21) $L\phi \rightarrow R\phi$

(GA22) $R\phi \rightarrow LR\phi$

(GA23) $K\ CH^*(\Gamma,\Delta,n,m)$, where Δ is a denumerable set of variables

This set of axioms is by no means independent. In fact, about half of the axioms GA1-GA22 can be proved from the remaining axioms by a procedure similar to that used in the proof of T88. It is possible to weaken GA23 in such a way as to eliminate most of this redundancy, but this would complicate things considerably without offering any compensating advantages.

The redundant axioms are left in for two reasons. First, to leave them out and then re-introduce them as theorems would necessitate a number of rather tedious proofs without any purpose except to make an axiom list, which is rather unsatisfying in any case, somewhat neater.

Second, it is not known that there is no finite set of axiom schemata that is a complete set of axioms for the present system, and it is hoped that this set of axioms might provide a clue to such a finite set of axiom schemata, if there is one. The present set of axioms is constructed on the set presented in Cocchiarella [1]. Our axiom GA23 is meant to correspond to Cocchiarella's axiom $P\phi \wedge P\psi \rightarrow P(\phi \wedge \psi) \vee P(P\phi \wedge \psi) \vee P(\phi \wedge P\psi)$ and the corresponding one with F, in that it corresponds to the fact that the 'earlier than' relation among moments is a connected one.

But GA23 also provides for the more complex relations among points of reference that are involved in the matrix structure of our interpretations. There may be a finite set of axiom schemata that accounts for these relations. If so, of course, that finite set of axiom schemata together with GA1-GA22 would be a complete set of axioms.

We take modus ponens as the only inference rule. The axioms have been constructed in such a way as to make other inference rules superfluous.

2. Derivations and Theorems

D54. Δ is a derivation of ϕ from Γ if and only if Δ is a finite sequence, $\Delta_{lh(\Delta)-1}$ is ϕ and for each $n < lh(\Delta)$ either

(1) $\Delta_n \in \Gamma$,

(2) Δ_n is an axiom,

or (3) There are $j,k < n$ such that Δ_j is $\Delta_k \rightarrow \Delta_n$

In the following definition, parts (a) and (b) are taken to define the natural notions, corresponding to the fact that the operator R refers back to the present moment when not within the scope of K. The notions defined in (c) and (d) are introduced for purely technical reasons. They turn out to be useful in the completeness proof.

D55. (a) If Γ is a set of formulas and ϕ is a formula, then $\Gamma \vdash \phi$, or <u>Γ yields ϕ</u>, if and only if there is a derivation of ϕ from Γ.

(b) If ϕ is a formula, then $\vdash \phi$, or ϕ is a <u>theorem</u> if and only if $0 \vdash \phi$.

(c) If ϕ is a formula and Γ is a set of formulas, then $\Gamma \vdash_s \phi$, or <u>Γ strongly yields ϕ</u>, if and only if $\Gamma \vdash \psi$ for every generalization ψ of ϕ.

(d) If ϕ is a formula, then $\vdash_s \phi$, or ϕ is a <u>strong theorem</u>, if and only if $0 \vdash_s \phi$.

T33. If $\vdash \phi$, then ϕ is logically valid.

Proof: By a trivial induction using T13d, it is sufficient to show that every axiom is logically valid. This is trivial for several of the axioms, so we will omit the proof for some. In this proof all references to clauses are to the clauses of D34.

RA1: By T13c, it is sufficient to show that $\phi \to K\phi$ is logically valid, for any formula ϕ. Suppose that $\phi \to K\phi$ is not logically valid, for some formula ϕ. Then there is an interpretation $\mathcal{A}$, a moment t of $\mathcal{A}$, and $x \in U^{\omega}_{\mathcal{A}}$ such that $x \notin Int_{\mathcal{A}}(\phi \to K\phi)(\langle t,t\rangle)$. Then by clauses (3) and (4), $x \in Int_{\mathcal{A}}(\phi)(\langle t,t\rangle)$ and $x \notin Int_{\mathcal{A}}(K\phi)(\langle t,t\rangle)$. By clause (3), $x \notin Int_{\mathcal{A}}(\phi)(\langle t,t\rangle)$, which is a contradiction.

GA3: By T13b, it is sufficient to show that, for any formulas ϕ,ψ, $((\neg\phi \to \neg\psi) \to (\psi \to \phi))$ is strongly logically valid. Suppose that (for some formulas ϕ,ψ) $((\neg\phi \to \neg\psi) \to (\psi \to \phi))$ is not strongly logically valid. Then there are an interpretation $\mathcal{A}$, moments t and t' of $\mathcal{A}$, and $x \in U^{\omega}_{\mathcal{A}}$, such that

that (i) $x \in Int_{\mathcal{A}}(\neg((\neg\phi \to \neg\psi) \to (\psi \to \phi)))(\langle t,t'\rangle)$

By clauses (3) and (4)

(ii) $x \in Int_{\mathcal{A}}(\neg\phi \to \neg\psi)(\langle t,t'\rangle)$

and (iii) $x \notin Int_{\mathcal{A}}(\psi \to \phi)(\langle t,t'\rangle)$

By clause (4)

(iv) $x \in Int_{\mathcal{A}}(\psi)(\langle t,t'\rangle)$

and (v) $x \notin Int_{\mathcal{A}}(\phi)(\langle t,t'\rangle)$

By clause (3)

(vi) $x \in Int_{\mathcal{A}}(\neg\phi)(\langle t,t'\rangle)$

By (ii) and clause (4)

(vii) $x \in Int_{\mathfrak{a}}(\neg\psi)(\langle t,t'\rangle)$

By clause (3)

(viii) $x \notin Int_{\mathfrak{a}}(\psi)(\langle t,t'\rangle)$, which contradicts (iv).

GA4: By T13b, it is sufficient to show that, for any variable α and formulas ϕ and ψ, $\Lambda\alpha(\phi\rightarrow\psi)\rightarrow(\Lambda\alpha\phi\rightarrow\Lambda\alpha\psi)$ is strongly logically valid. Suppose that (for some $n \in \omega$ and formulas ϕ and ψ) $(\Lambda v_n(\phi\rightarrow\psi)\rightarrow(\Lambda v_n\phi\rightarrow\Lambda v_n\psi))$ is not strongly logically valid. Then there are an interpretation $\mathfrak{a}$, moments t and t' of $\mathfrak{a}$, and $x \in U^{\omega}_{\mathfrak{a}}$, such that $x \notin Int_{\mathfrak{a}}(\Lambda v_n(\phi\rightarrow\psi)\rightarrow(\Lambda v_n\phi\rightarrow\Lambda v_n\psi))(\langle t,t'\rangle)$

By clauses (3) and (4)

(i) $x \in Int_{\mathfrak{a}}(\Lambda v_n(\phi\rightarrow\psi))(\langle t,t'\rangle)$

(ii) $x \in Int_{\mathfrak{a}}(\Lambda v_n\phi)(\langle t,t'\rangle)$

and (iii) $x \notin Int_{\mathfrak{a}}(\Lambda v_n\psi)(\langle t,t'\rangle)$

By clause (5), there is an $a \in U^{\omega}_{\mathfrak{a}}$ such that

(iv) $x^n_a \notin Int_{\mathfrak{a}}(\psi)(\langle t,t'\rangle)$

By clause (5) again

(v) $x^n_a \in Int_{\mathfrak{a}}(\phi\rightarrow\psi)(\langle t,t'\rangle)$

and (vi) $x^n_a \in Int_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$

By clause (4)

(vii) $x^n_a \in Int_{\mathfrak{a}}(\psi)(\langle t,t'\rangle)$, which contradicts (iv).

GA5: By T13b, it is sufficient to show that, for any variable α and formula ϕ, if α is not free in ϕ, then $(\phi\rightarrow\Lambda\alpha\phi)$ is strongly logically valid. Suppose that (for some $n \in \omega$ and formula ϕ) $v_n \notin fv(\phi)$ and $(\phi\rightarrow\Lambda v_n\phi)$ is not strongly logically valid. Then there are an interpretation

$\mathcal{A}$, moments t and t' of $\mathcal{A}$, and $x \in U_{\mathcal{A}}^{\omega}$ such that

(i) $x \in Int_{\mathcal{A}}(\neg(\phi \rightarrow \Lambda v_n \phi))(\langle t,t' \rangle)$

By clauses (3) and (4)

(ii) $x \in Int_{\mathcal{A}}(\phi)(\langle t,t' \rangle)$

and (iii) $x \notin Int_{\mathcal{A}}(\Lambda v_n \phi)(\langle t,t' \rangle)$

By clause (5), there is an $a \in U_{\mathcal{A}}$ such that

(iv) $x_a^n \notin Int_{\mathcal{A}}(\phi)(\langle t,t' \rangle)$

But, by (ii) and T15

(v) $x_a^n \in Int_{\mathcal{A}}(\phi)(\langle t,t' \rangle)$

GA6: By T13b, it is sufficient to show that, for any terms η and ζ and atomic formulas ϕ and ψ, if ψ is obtained from ϕ by replacing 0 or more occurrences of η by ζ, then $(\eta=\zeta \rightarrow (\phi \rightarrow \psi))$ is strongly logically valid. Suppose that (for some terms η and ζ and atomic formulas ϕ and ψ) ψ is obtained from ϕ by replacing 0 or more occurrences of η by ζ, and $(\eta=\zeta \rightarrow (\phi \rightarrow \psi))$ is not strongly logically valid. Then there are an interpretation $\mathcal{A}$, moments t and t' of $\mathcal{A}$, and $x \in U_{\mathcal{A}}^{\omega}$ such that

(i) $x \in Int_{\mathcal{A}}(\neg(\eta=\zeta \rightarrow (\phi \rightarrow \psi)))(\langle t,t' \rangle)$

By clauses (3) and (4)

(ii) $x \in Int_{\mathcal{A}}(\eta=\zeta)(\langle t,t' \rangle)$

(iii) $x \in Int_{\mathcal{A}}(\phi)(\langle t,t' \rangle)$

and (iv) $x \notin Int_{\mathcal{A}}(\psi)(\langle t,t' \rangle)$

By clause (1)

(v) $Ext_{t,\mathcal{A}}(\eta)(x)$ is $Ext_{t,\mathcal{A}}(\zeta)(x)$

We take two cases:

(a) There are terms ξ,θ such that ϕ is $\xi=\theta$; then there are terms ξ',θ' such that ψ is $\xi'=\theta'$, ξ' is obtained from ξ by replacing 0 or more occurrences of η by ζ, and θ' is obtained from θ by replacing 0 or more occurrences of η by ζ. By (iii) and clause (1)

(vi) $Ext_{t,\mathfrak{a}}(\xi)(x)$ is $Ext_{t,\mathfrak{a}}(\theta)(x)$

By (v), (vi) and T19

(vii) $Ext_{t,\mathfrak{a}}(\xi)(x)$ is $Ext_{t,\mathfrak{a}}(\xi')(x)$

and (viii) $Ext_{t,\mathfrak{a}}(\theta)(x)$ is $Ext_{t,\mathfrak{a}}(\theta')(x)$

By (vi), (vii) and (viii)

(ix) $Ext_{t,\mathfrak{a}}(\xi')(x)$ is $Ext_{t,\mathfrak{a}}(\theta')(x)$

By clause (1)

(x) $x \in Int_{\mathfrak{a}}(\xi'=\theta')(\langle t,t'\rangle)$, which contradicts (iv)

(b) There are $k,\pi,\xi_0,\ldots,\xi_{k-1},\xi'_0,\ldots,\xi'_{k-1}$ such that $k \in \omega$, π is a k-place predicate letter, ϕ is $\pi\xi_0\ldots\xi_{k-1}$, ψ is $\pi\xi'_0\ldots\xi'_{k-1}$, and, for each $i < k$, ξ'_i is obtained from ξ_i by replacing 0 or more occurrences of η by ζ.

By (iii) and clause (2)

(xi) $\langle Ext_{t,\mathfrak{a}}(\xi_0)(x),\ldots,Ext_{t,\mathfrak{a}}(\xi_{k-1})(x)\rangle \in G(\pi)(t)$

By (v) and T19

(xii) For each $i < k$, $Ext_{t,\mathfrak{a}}(\xi_i)(x)$ is $Ext_{t,\mathfrak{a}}(\xi'_i)(x)$

By (xi) and (xii)

(xiii) $\langle Ext_{t,\mathfrak{a}}(\xi'_0)(x),\ldots,Ext_{t,\mathfrak{a}}(\xi'_{k-1})(x)\rangle \in G(\pi)(t)$

By clause (2)

(xiv) $x \in Int_{\mathfrak{a}}(\psi)(\langle t,t'\rangle)$, contradicting (iv)

GA7: By T13b, it is sufficient to show that for any variable α and term η such that α does not occur in η, $\Lambda\alpha\alpha=\eta$ is strongly logically valid. Suppose that $n \in \omega$, η is a term and v_n does not occur in η. Then there are an interpretation $\mathcal{a}$, moments t and t' of $\mathcal{a}$, and $x \in U_{\mathcal{a}}^{\omega}$ such that

$x \in Int_{\mathcal{a}}(\neg V v_n v_n=\eta)(\langle t,t'\rangle)$

By clause (3)

$x \in Int_{\mathcal{a}}(\Lambda v_n \neg v_n=\eta)(\langle t,t'\rangle)$

By clause (4)

$x^n_{Ext_{t,\mathcal{a}}(\eta)(x)} \in Int_{\mathcal{a}}(\neg v_n=\eta)(\langle t,t'\rangle)$

By clauses (3) and (1)

$Ext_{t,\mathcal{a}}(v_n)(x^n_{Ext_{t,\mathcal{a}}(\eta)(x)})$ is not $Ext_{t,\mathcal{a}}(\eta)(x^n_{Ext_{t,\mathcal{a}}(\eta)(x)})$

By D31

$Ext_{t,\mathcal{a}}(\eta)(x)$ is not $Ext_{t,\mathcal{a}}(\eta)(x^n_{Ext_{t,\mathcal{a}}(\eta)(x)})$

This contradicts T14.

GA8: Suppose $n,m \in \omega$, and

(i) $x \notin Int_{\mathcal{a}}(v_n=v_m \rightarrow Lv_n=v_m)(\langle t,t'\rangle)$

Then

(ii) $x \in Int_{\mathcal{a}}(v_n=v_m)(\langle t,t'\rangle)$

(iii) $x \notin Int_{\mathcal{a}}(Lv_n=v_m)(\langle t,t'\rangle)$

By (ii), clause (1) and D31

(iv) x_n is x_m

By (iii) and T10g, there is a moment t" of $\mathcal{a}$ such that

(v) $x \notin Int_{\mathcal{a}}(v_n=v_m)(\langle t'',t'\rangle)$

By clause (1) and D31,

(vi) x_n is not x_m, contradicting (iv)

GA9: Suppose ϕ and ψ are formulas, $\mathcal{A}$ is an interpretation, t and t' are moments of $\mathcal{A}$, and $x \in U^{\omega}_{\mathcal{A}}$. By T13b it is sufficient to show that $x \in$ $Int_{\mathcal{A}}((H(\phi \rightarrow \psi) \rightarrow (H\phi \rightarrow H\psi)))(\langle t, t' \rangle)$. Suppose not; then

(i) $x \in Int_{\mathcal{A}}(H(\phi \rightarrow \psi))(\langle t, t' \rangle)$

(ii) $x \in Int_{\mathcal{A}}(H\phi)(\langle t, t' \rangle)$

(iii) $x \notin Int_{\mathcal{A}}(H\psi)(\langle t, t' \rangle)$

By clause (7), there is a t" $<_{\mathcal{A}}$ t such that

(iv) $x \notin Int_{\mathcal{A}}(\psi)(\langle t'', t \rangle)$

By (i) and clause (6)

(v) $x \in Int_{\mathcal{A}}(\phi \rightarrow \psi)(\langle t'', t \rangle)$

and (vi) $x \in Int_{\mathcal{A}}(\phi)(\langle t'', t \rangle)$

But then

(vii) $x \in Int_{\mathcal{A}}(\psi)(\langle t'', t \rangle)$, contradicting (iv)

GA11: Suppose ϕ is a formula, $\mathcal{A}$ is an interpretation, t and t' are moments of $\mathcal{A}$, and $x \in U^{\omega}_{\mathcal{A}}$. Suppose also that $x \notin Int_{\mathcal{A}}((H\phi \rightarrow HH\phi))(\langle t, t' \rangle)$

then

(i) $x \in Int_{\mathcal{A}}(H\phi)(\langle t, t' \rangle)$

and (ii) $x \notin Int_{\mathcal{A}}(HH\phi)(\langle t, t' \rangle)$

By clause (6), there is a moment t" $<_{\mathcal{A}}$ t such that

(iii) $x \notin Int_{\mathcal{A}}(H\phi)(\langle t'', t' \rangle)$

By clause (6) again, there is a moment s $<_{\mathcal{A}}$ t" such that

(iv) $x \notin Int_{\mathfrak{A}}(\phi)(\langle s,t'\rangle)$

But since $<_{\mathfrak{A}}$ is transitive, $s >_{\mathfrak{A}} t$ and hence, by clause (6) and (i)

(v) $x \in Int_{\mathfrak{A}}(\phi)(\langle s,t'\rangle)$

GA13: Suppose

(i) $x \notin Int_{\mathfrak{A}}(\phi \rightarrow GP\phi)(\langle t,t'\rangle)$

Then

(ii) $x \in Int_{\mathfrak{A}}(\phi)(\langle t,t'\rangle)$

but (iii) $x \notin Int_{\mathfrak{A}}(GP\phi)(\langle t,t'\rangle)$

By clause (7), there is a $t'' >_{\mathfrak{A}} t$ such that

(iv) $x \notin Int_{\mathfrak{A}}(P\phi)(\langle t'',t'\rangle)$

Hence,

(v) $x \in Int_{\mathfrak{A}}(H\neg\phi)(\langle t'',t'\rangle)$

Since $t <_{\mathfrak{A}} t''$, by clause (6)

(vi) $x \in Int_{\mathfrak{A}}(\neg\phi)(\langle t,t'\rangle)$

Hence,

(vii) $x \notin Int_{\mathfrak{A}}(\phi)(\langle t,t'\rangle)$, contradicting (ii)

GA15: Let Γ be the set of formulas ϕ such that if ϕ does not contain R outside the scope of K, then for any interpretation $\mathfrak{A}$, moments t,t' and t'' of $\mathfrak{A}$, $Int_{\mathfrak{A}}(\phi)(\langle t,t'\rangle)$ is $Int_{\mathfrak{A}}(\phi)(\langle t,t''\rangle)$. We will show by induction that every formula is in Γ.

(a) If ϕ is an atomic formula, then ϕ is in Γ by T20.

(b) Suppose ϕ is $\psi \rightarrow \chi$ and ϕ does not contain R outside the scope of K. Then ψ and χ do not contain R outside the scope of K. Suppose also that $\mathfrak{A}$ is an interpretation, t and t' are moments of $\mathfrak{A}$ and $x \in U^{\omega}_{\mathfrak{A}}$. Then $x \in$ $Int_{\mathfrak{A}}(\psi \rightarrow \chi)(\langle t,t' \rangle)$

if and only if

$$x \notin Int_{\mathfrak{A}}(\psi)(\langle t,t' \rangle) \text{ or } x \in Int_{\mathfrak{A}}(\chi)(\langle t,t' \rangle)$$

if and only if (by the inductive hypothesis)

$$x \notin Int_{\mathfrak{A}}(\psi)(\langle t,t'' \rangle) \text{ or } x \in Int_{\mathfrak{A}}(\chi)(\langle t,t'' \rangle)$$

if and only if

$$x \in Int_{\mathfrak{A}}(\psi \rightarrow \chi)(\langle t,t'' \rangle)$$

Hence, $Int_{\mathfrak{A}}(\phi)(\langle t,t' \rangle)$ is $Int_{\mathfrak{A}}(\phi)(\langle t,t'' \rangle)$

(c) Suppose ϕ is $K\psi$, $\mathfrak{A}$ is an interpretation, and t, t' and t" are moments of $\mathfrak{A}$. By clause (8), $Int_{\mathfrak{A}}(\phi)(\langle t,t' \rangle)$ is $Int_{\mathfrak{A}}(\psi)(\langle t,t \rangle)$ is $Int_{\mathfrak{A}}(\phi)(\langle t,t'' \rangle)$

(d) Suppose ϕ does not contain R outside the scope of K and ϕ is $\neg\psi$, $H\psi$, $G\psi$ or $\Lambda v_n \psi$. Then ψ does not contain R outside the scope of K. Suppose that $\mathfrak{A}$ is an interpretation and t,t' and t" are moments of $\mathfrak{A}$. We take four cases:

(i) ϕ is $\neg\psi$; then $Int_{\mathfrak{A}}(\phi)(\langle t,t' \rangle)$ is $U^{\omega} \sim Int_{\mathfrak{A}}(\psi)(\langle t,t' \rangle)$ is (by the inductive hypothesis) $U^{\omega} \sim Int_{\mathfrak{A}}(\psi)(\langle t,t'' \rangle)$ is $Int_{\mathfrak{A}}(\phi)(\langle t,t'' \rangle)$

(ii) ϕ is $H\psi$; then $Int_{\mathfrak{A}}(\phi)(\langle t,t' \rangle)$ is the set of $x \in U^{\omega}_{\mathfrak{A}}$ such that for each s, if $s <_{\mathfrak{A}} t$, then $x \in Int_{\mathfrak{A}}(\psi)(\langle s,t' \rangle)$ which is (by the inductive hypothesis) the set of $x \in U^{\omega}_{\mathfrak{A}}$ such that for each s, if $s <_{\mathfrak{A}} t$, then $x \in Int_{\mathfrak{A}}(\psi)(\langle s,t'' \rangle)$ which is $Int_{\mathfrak{A}}(\phi)(\langle t,t'' \rangle)$

(iii) ϕ is $G\psi$; this is analogous to (ii)

(iv) ϕ is $\Lambda v_n \psi$; then $Int_{\mathcal{A}}(\phi)(\langle t,t'\rangle)$ is the set of $x \in U_{\mathcal{A}}^{\omega}$ such that, for each $y \in U_{\mathcal{A}}$, $x_y^n \in Int_{\mathcal{A}}(\psi)(\langle t,t'\rangle)$ is (by the inductive hypothesis) the set of $x \in U_{\mathcal{A}}^{\omega}$ such that, for each $y \in U_{\mathcal{A}}$, $x_y^n \in Int_{\mathcal{A}}(\psi)(\langle t,t''\rangle)$ is $Int_{\mathcal{A}}(\phi)(\langle t,t''\rangle)$

This completes the induction. We show now that, for any formula ϕ that does not contain R outside the scope of K, $\phi \rightarrow K\phi$ is strongly logically valid. Suppose $\mathcal{A}$ is an interpretation, t,t' are moments of $\mathcal{A}$, $x \in U_{\mathcal{A}}^{\omega}$, and $x \in Int_{\mathcal{A}}(\neg(\phi \rightarrow K\phi))(\langle t,t'\rangle)$.

By clauses (3) and (4)

(i) $x \in Int_{\mathcal{A}}(\phi)(\langle t,t'\rangle)$

and (ii) $x \notin Int_{\mathcal{A}}(K\phi)(\langle t,t'\rangle)$

By clause (8)

(iii) $x \notin Int_{\mathcal{A}}(\phi)(\langle t,t\rangle)$

But, by the above induction and (i)

(iv) $x \in Int_{\mathcal{A}}(\phi)(\langle t,t\rangle)$

By T13b, every instance of GA15 is logically valid.

GA18: Assume $n \in \omega$, ϕ is a formula, $\mathcal{A}$ is an interpretation, t and t' are moments of $\mathcal{A}$, and $x \in U_{\mathcal{A}}^{\omega}$. By T13b it is sufficient to show that $x \in Int_{\mathcal{A}}(\Lambda v_n K\phi \rightarrow K\Lambda v_n \phi)(\langle t,t'\rangle)$. Suppose that $x \notin Int_{\mathcal{A}}(\Lambda v_n K\phi \rightarrow K\Lambda v_n \phi)(\langle t,t'\rangle)$. Then

(i) $x \in Int_{\mathcal{A}}(\Lambda v_n K\phi)(\langle t,t'\rangle)$

and (ii) $x \notin Int_{\mathcal{A}}(K\Lambda v_n \phi)(\langle t,t'\rangle)$

By clause (8)

(iii) $x \notin Int_{\mathfrak{A}}(\wedge v_n \phi)(\langle t,t \rangle)$

By clause (4), there is an $a \in U^{\omega}_{\mathfrak{A}}$ such that

(iv) $x^n_a \notin Int_{\mathfrak{A}}(\phi)(\langle t,t \rangle)$

By (i) and clause (4)

(v) $x^n_a \in Int_{\mathfrak{A}}(K\phi)(\langle t,t' \rangle)$

By clause (8)

(vi) $x^n_a \in Int_{\mathfrak{A}}(\phi)(\langle t,t \rangle)$, contradicting (iv)

GA19: By T13a and T13b, it is sufficient to show that, for any formula ϕ, $\phi \rightarrow R\phi$ is logically valid. Suppose $\mathfrak{A}$ is an interpretation, t is a moment of $\mathfrak{A}$, and $x \in U^{\omega}_{\mathfrak{A}}$. By T10c, it is sufficient to show that $x \in Int_{\mathfrak{A}}(\phi)(\langle t,t \rangle)$ if and only if $x \in Int_{\mathfrak{A}}(R\phi)(\langle t,t \rangle)$. But this follows immediately from clause (9).

GA21: Suppose

(i) $x \notin Int_{\mathfrak{A}}(L\phi \rightarrow R\phi)(\langle t,t' \rangle)$

Then

(ii) $x \in Int_{\mathfrak{A}}(L\phi)(\langle t,t' \rangle)$

and (iii) $x \notin Int_{\mathfrak{A}}(R\phi)(\langle t,t' \rangle)$

By (iii) and clause (9)

(iv) $x \notin Int_{\mathfrak{A}}(\phi)(\langle t',t' \rangle)$

But by (ii) and T10g

(v) $x \in Int_{\mathfrak{A}}(\phi)(\langle t',t' \rangle)$

GA23: This follows from T31, T13a, and T13b.

3. Basic Theorems about Derivability

The theorems on the following list all correspond to basic theorems of the predicate calculus. By taking all universal generalizations of axioms as axioms we ensure the truth of the deduction theorem in the unrestricted form (f).

T34. (a) If ϕ is a restricted axiom or a general axiom, then $\vdash \phi$

(b) If $\Gamma \vdash \phi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \phi$

(c) If $\phi \in \Gamma$, then $\Gamma \vdash \phi$

(d) If $\Gamma \vdash \phi$ and α is not free in any formula of ϕ, then $\Gamma \vdash \Lambda\alpha\phi$

(e) If $\Gamma \vdash \phi$ and $\Gamma \vdash \phi \rightarrow \psi$, then $\Gamma \vdash \psi$

(f) If $\Gamma \cup \{\phi\} \vdash \psi$, then $\Gamma \vdash \phi \rightarrow \psi$

(g) If $\Gamma \vdash \phi$, then there is a finite set of formulas Γ' such that $\Gamma' \subseteq \Gamma$ and $\Gamma' \vdash \phi$

(h) If $\Gamma \vdash \phi \rightarrow \psi$, then $\Gamma \cup \{\phi\} \vdash \psi$

(i) If $\Gamma \vdash \phi$, then $\Gamma' \vdash \phi$, where Γ' is the set of formulas in Γ that are not axioms.

Proof: (a), (b), and (c) are trivial consequences of D55.

(d) Proof by induction on the length of derivations. Assume the hypothesis, and assume that Δ is a derivation of ϕ from Γ. Let N be the set of $n < lh(\Delta)$ such that $\Gamma \vdash \Lambda\alpha\Delta_n$. We will show by strong induction that N is $lh(\Delta)$. There are three cases.

(i) $\Delta_n \in \Gamma$. Then $\langle \Delta_n, \Delta_n \rightarrow \Lambda\alpha\Delta_n, \Lambda\alpha\Delta_n \rangle$ is a derivation of $\Lambda\alpha\Delta_n$ from Γ, since $\Delta_n \rightarrow \Lambda\alpha\Delta_n$ is an instance of GA5.

(ii) Δ_n is an axiom; then $\Lambda\alpha\Delta_n$ is an axiom and, by (a), $\vdash \Lambda\alpha\Delta_n$.

(iii) There are $j,k < n$ such that Δ_j is $\Delta_k \to \Delta_n$. By the inductive hypothesis, there are derivations Δ' of $\Lambda\alpha(\Delta_k \to \Delta_n)$ from Γ and Δ'' of $\Lambda\alpha\Delta_k$ from Γ. Then $\Delta'{}^\frown\Delta''{}^\frown\langle\Lambda\alpha(\Delta_k \to \Delta_n) \to (\Lambda\alpha\Delta_k \to \Lambda\alpha\Delta_n), \Lambda\alpha\Delta_k \to \Lambda\alpha\Delta_n, \Lambda\alpha\Delta_n\rangle$ is a derivation of $\Lambda\alpha\Delta_n$ from Γ, since $\Lambda\alpha(\Delta_k \to \Delta_n) \to (\Lambda\alpha\Delta_k \to \Lambda\alpha\Delta_n)$ is an instance of GA4.

This completes the induction; therefore, since ϕ is $\Delta_{\mathrm{lh}(\Delta)-1}$, there is a derivation of $\Lambda\alpha\phi$ from Γ.

(e) A trivial consequence of D55.

(f) Suppose Δ is a derivation of ψ from $\Gamma\cup\{\phi\}$. We will show by strong induction that, for each $n < \mathrm{lh}(\Delta)$, $\Gamma \vdash \phi \to \Delta_n$. There are three cases.

(i) $\Delta_n \in \Gamma\cup\{\phi\}$. We take two subcases.

(a) $\Delta_n \in \Gamma$; then $\langle\Delta_n, (\Delta_n \to (\phi \to \Delta_n)), (\phi \to \Delta_n)\rangle$ is a derivation of $(\phi \to \Delta_n)$ from Γ, since $(\Delta_n \to (\phi \to \Delta_n))$ is an instance of GA1.

(b) Δ_n is ϕ; then $\langle((\phi \to ((\phi \to \phi) \to \phi)) \to ((\phi \to (\phi \to \phi)) \to (\phi \to \phi))), (\phi \to ((\phi \to \phi) \to \phi)), ((\phi \to (\phi \to \phi)) \to (\phi \to \phi)), (\phi \to (\phi \to \phi)), (\phi \to \phi)\rangle$ is a derivation of $(\phi \to \Delta_n)$ from Γ, since the first formula is an instance of GA2, and the second and fourth formulas are instances of GA1.

(ii) Δ_n is an axiom; then $\langle\Delta_n, (\Delta_n \to (\phi \to \Delta_n)), (\phi \to \Delta_n)\rangle$

is a derivation of $(\phi \rightarrow \Delta_n)$ from Γ.

(iii) There are $j,k < n$ such that Δ_j is $\Delta_k \rightarrow \Delta_n$. By the inductive hypothesis, there are derivations Δ' of $(\phi \rightarrow \Delta_k)$ from Γ and Δ'' of $(\phi \rightarrow (\Delta_k \rightarrow \Delta_n))$ from Γ. Then $\Delta'^\frown\Delta''^\frown\langle((\phi \rightarrow (\Delta_k \rightarrow \Delta_n)) \rightarrow ((\phi \rightarrow \Delta_k) \rightarrow (\phi \rightarrow \Delta_n))), ((\phi \rightarrow \Delta_k) \rightarrow (\phi \rightarrow \Delta_n)), (\phi \rightarrow \Delta_n)\rangle$ is a derivation of $(\phi \rightarrow \Delta_n)$ from Γ.

This completes the induction. Since ψ is $\Delta_{lh(\Lambda)-1}$, there is a derivation of ψ from Γ.

(g) Suppose that Δ is a derivation of ϕ from Γ. Let Γ' be the set of formulas Δ_n, for $n < lh(\Delta)$. Then $\Gamma \cap \Gamma'$ is finite, and Δ is a derivation of ϕ from $\Gamma \cap \Gamma'$.

(h) Suppose that Δ is a derivation of $\phi \rightarrow \psi$ from Γ. Then $\Delta^\frown\langle\phi,\psi\rangle$ is a derivation of ψ from $\Gamma \cup \{\phi\}$.

(i) Suppose that Δ is a derivation of ϕ from Γ. Then Δ is also a derivation of ϕ from the set of formulas in Γ that are not axioms.

T35. (a) If $\Gamma \vdash_S \phi$, then $\Gamma \vdash \phi$

(b) If ϕ is a general axiom, then $\vdash_S \phi$

(c) If $\Gamma \vdash_S \phi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash_S \phi$

(d) If $\Gamma \vdash_S \phi$, then $\Gamma \vdash_S H\phi$, $\Gamma \vdash_S G\phi$, $\Gamma \vdash_S K\phi$, and for every variable α, $\Gamma \vdash_S \alpha\phi$

Proof:

(a) ϕ is a generalization of ϕ

(b) If ϕ is a general axiom and ψ is a generalization of

ϕ, then ψ is also a general axiom and, by T34a, $\vdash \psi$

(c) Assume the hypothesis and that ψ is a generalization of ϕ. Then $\Gamma \vdash \psi$ and, by T34b, $\Gamma' \vdash \psi$

(d) Assume that $\Gamma \vdash_S \phi$, and that ψ is a generalization of $H\phi$, $G\phi$, $K\phi$, or $\bigwedge\alpha\phi$ (for some variable α). Then ψ is a generalization of ϕ and $\Gamma \vdash \psi$

T36. If τ is a generalizer and ϕ,ψ are formulas, then $\vdash \tau(\phi \to \psi) \to (\tau\phi \to \tau\psi)$

Proof: By induction on the level of τ.

(A) By T34c and T34f, $\vdash (\phi \to \psi) \to (\phi \to \psi)$. Hence, if τ is 0, $\vdash \tau(\phi \to \psi) \to (\tau\phi \to \tau\psi)$

(B) Suppose that $n \in \omega$, and for any n-level generalizer τ and any formulas ϕ,ψ, $\vdash \tau(\phi \to \psi) \to (\tau\phi \to \tau\psi)$. Also suppose τ' is an n+1-level generalizer and ϕ,ψ are formulas. Let τ and σ be those expressions such that τ' is $\tau^\frown\sigma$ and τ is an n-level generalizer. Then σ is a 1-level generalizer.

By the inductive hypothesis

(1) $\vdash \tau(\sigma\phi \to \sigma\psi) \to (\tau\sigma\phi \to \tau\sigma\psi)$

By GA4, GA9, GA10 or GA17 (depending on σ)

(2) $\vdash_S \sigma(\phi \to \psi) \to (\sigma\phi \to \sigma\psi)$

Hence,

(3) $\vdash \tau(\sigma(\phi \to \psi) \to (\sigma\phi \to \sigma\psi))$

By the inductive hypothesis again,

(4) $\vdash \tau(\sigma(\phi \to \psi) \to (\sigma\phi \to \sigma\psi)) \to (\tau\sigma(\phi \to \psi) \to \tau(\sigma\phi \to \sigma\psi))$

From (3) and (4), there follows

(5) $\vdash \tau\sigma(\phi \to \psi) \to \tau(\sigma\phi \to \sigma\psi)$

From (1) and (5), there follows

(6) $\vdash \tau\sigma(\phi \to \psi) \to (\tau\sigma\phi \to \tau\sigma\psi)$

But this is just

(7) $\vdash \tau'(\phi \to \psi) \to (\tau'\phi \to \tau'\psi)$

T37. If $\Gamma \vdash_S \phi \to \psi$ and $\Gamma \vdash_S \phi$, then $\Gamma \vdash_S \psi$.

Proof: Suppose $\Gamma \vdash_S \phi \to \psi$ and $\Gamma \vdash_S \phi$. Suppose also that τ is a generalizer; it is sufficient to show that $\Gamma \vdash \tau\psi$.

By the hypothesis

(1) $\Gamma \vdash \tau(\phi \to \psi)$

and (2) $\Gamma \vdash \tau\phi$

From (1) and T36

(3) $\Gamma \vdash \tau\phi \to \tau\psi$

From (2) and (3)

(4) $\Gamma \vdash \tau\psi$

4. Theorems about Tautologies

T38. If ϕ is a tautology, then $\vdash_S \phi$.

Proof: We will not formally introduce the notion of a propositional tautology, and hence we cannot provide a full formal proof of this theorem. We note, however, that GA1, GA2, and GA3 correspond to a complete set of axioms for the propositional calculus.

Suppose that ϕ is a tautology; then, by the completeness proof for the propositional calculus, there is a derivation of ϕ from GA1, GA2, and GA3. Suppose that τ is a generalizer; by a simple induction using T37, there is a derivation of $\tau\phi$ from those formulas $\tau\psi$, where ψ is one of the instances of GA1, GA2, and GA3 used in the proof of ϕ. Since all of those formulas $\tau\psi$ are axioms, then (by T34) $\vdash \tau\phi$.

T39. If $1 \leq n$ and $\phi_0 \wedge \ldots \wedge \phi_{n-1} \rightarrow \psi$ is a tautology, then

(a) If for each $i < n$, $\Gamma \vdash \phi_i$, then $\Gamma \vdash \psi$

(b) If for each $i < n$, $\Gamma \vdash_s \phi_i$, then $\Gamma \vdash_s \psi$

Proof: Assume the hypothesis. We define the sequence of formulas χ recursively as follows:

χ_0 is $(\phi_0 \wedge \ldots \wedge \phi_{n-1})$

χ_{k+1} is $\phi_k \rightarrow \chi_k$

(e.g., if χ_0 is $P \wedge Q \wedge R$, then χ_3 is

$R \rightarrow (Q \rightarrow (P \rightarrow (P \wedge Q \wedge R)))$)

(a) Suppose, for each $i < n$, $\Gamma \vdash \phi_i$. We will show, by induction on j, that for each $j \leq n$, $\Gamma \vdash \chi_{n-j}$. By T38, $\Gamma \vdash \chi_{n-0}$.

Suppose $\Gamma \vdash \chi_{n-j}$, that is, $\Gamma \vdash \phi_{n-(j+1)} \rightarrow \chi_{n-(j+1)}$. By hypothesis, $\Gamma \vdash \phi_{n-(j+1)}$, and hence (T34e) $\Gamma \vdash \chi_{n-(j+1)}$. This completes the induction.

Then $\Gamma \vdash \chi_{n-n}$, that is, $\Gamma \vdash \chi_0$. By T38, $\Gamma \vdash \chi_0 \rightarrow \psi$. By T34e, $\Gamma \vdash \psi$.

(b) This proof is analogous, except that we use T37 in place of T34e.

T40. If Γ is a finite set of formulas and $\Gamma' \subseteq \Gamma$, then

(a) $\vdash_s CJ(\Gamma) \rightarrow CJ(\Gamma')$

(b) $\vdash_s DJ(\Gamma') \rightarrow DJ(\Gamma)$

Proof: A trivial induction, using T38 and T39b

T41. If Γ is a finite set of formulas, then $\Gamma \vdash \phi$ if and only if $\vdash CJ(\Gamma) \rightarrow \phi$.

Proof: We will prove this theorem by induction on the number of formulas in Γ.

(a) $\overline{\Gamma}$ is 0; then the result follows by T39a.

(b) $\overline{\Gamma}$ is 1; then the result follows by T34f and T34h.

(c) Suppose $1 \leq n$ and, for each set of formulas Γ such that $\overline{\Gamma}$ is n, and each formula ϕ, $\Gamma \vdash \phi$ if and only if $\vdash CJ(\Gamma) \rightarrow \phi$. Suppose also that $\overline{\Gamma}$ is n+1. Let ψ be the first formula in Γ, and let Γ' be $\Gamma \sim \{\psi\}$. Then $\Gamma \vdash \phi$ if and only if (by T34f and T34h) $\Gamma \vdash \psi \rightarrow \phi$ if and only if (by the inductive hypothesis) $\vdash CJ(\Gamma') \rightarrow (\psi \rightarrow \phi)$ if and only if (by T39a) $\vdash (\psi \wedge CJ(\Gamma')) \rightarrow \phi$ if and only if (since $(\psi \wedge CJ(\Gamma'))$ is $CJ(\Gamma)$) $\vdash CJ(\Gamma) \rightarrow \phi$.

5. Theorems about Replacement of Formulas

T42. If $\Gamma \vdash_s \phi$, then

(a) $\Gamma \vdash_s L\phi$

(b) $\Gamma \vdash_s R\phi$

Proof: Suppose $\Gamma \vdash_S \phi$

(a) By T35d, $\Gamma \vdash_S G\phi$ and $\Gamma \vdash_S H\phi$. By T39b, $\Gamma \vdash_S L\phi$.

(b) By (a), $\Gamma \vdash_S L\phi$; by GA21 and T39b, $\Gamma \vdash_S R\phi$.

T43. If ϕ' results from ϕ by replacing 0 or more occurrences of ψ by ψ' and $\vdash_S \psi \leftrightarrow \psi'$, then $\vdash_S \phi \leftrightarrow \phi'$.

Proof: Assume $\vdash_S \psi \leftrightarrow \psi'$. The proof is by induction, following the four cases of D16.

(1) ϕ' is ϕ; then, by T38, $\vdash_S \phi \leftrightarrow \phi'$

(2) ϕ' is ψ' and ϕ is ψ; by hypothesis, $\vdash_S \phi \leftrightarrow \phi'$

(3) There are formulas $\chi, \chi', \theta, \theta'$ such that ϕ' is $\chi' \rightarrow \theta'$, ϕ is $\chi \rightarrow \theta$, $R(\chi', \chi, \psi', \psi)$, $R(\theta', \theta, \psi', \psi)$, $\vdash_S \chi \leftrightarrow \chi'$, and $\vdash_S \theta \leftrightarrow \theta'$. Then, by T39b, $\vdash_S \phi \leftrightarrow \phi'$.

(4) There are formulas χ, χ' such that $R(\chi', \chi, \psi', \psi)$ and $\vdash_S \chi \leftrightarrow \chi'$. We take five subcases.

(a) ϕ' is $\neg\chi'$ and ϕ is $\neg\chi$; then $\vdash_S \phi \leftrightarrow \phi'$, by T39b.

(b) ϕ' is $H\chi'$ and ϕ is $H\chi$; by T39b and T35d, $\vdash_S H(\chi \rightarrow \chi')$ and $\vdash_S H(\chi' \rightarrow \chi)$. By GA9 and T37, $\vdash_S H\chi \rightarrow H\chi'$ and $\vdash_S H\chi' \rightarrow H\chi$. By T39b, $\vdash_S \phi \leftrightarrow \phi'$.

(c) ϕ' is $G\chi'$ and ϕ is $G\chi$; this case is analogous to (b) except that GA10 is used in place of GA9.

(d) ϕ' is $K\chi'$ and ϕ is $K\chi$; this case is analogous to (b) except that GA17 is used in place of GA9.

(e) ϕ' is $R\chi'$ and ϕ is $R\chi$; this case is also analogous to (b) except that GA20 is used in place of GA9.

T44. (a) If ϕ' results from ϕ by replacing 0 or more positive occurrences of ψ by ψ' and $\vdash_S \psi \to \psi'$, then $\vdash_S \phi \to \phi'$.

(b) If ϕ' results from ϕ by replacing 0 or more negative occurrences of ψ by ψ' and $\vdash_S \psi \to \psi'$, then $\vdash_S \phi' \to \phi$.

Proof: Suppose that $\vdash_S \psi \to \psi'$. We prove (a) and (b) together by induction, following the five clauses of D17.

(1) ϕ' is ϕ; then $\vdash_S \phi \to \phi'$ and $\vdash_S \phi' \to \phi$

(2) ϕ' is ψ' and ϕ is ψ; by hypothesis, $\vdash_S \phi \to \phi'$

(3) There are formulas $\chi, \chi', \theta, \theta'$ such that ϕ is $\chi \to \theta$ and ϕ' is $\chi' \to \theta'$. We take two subcases:

(a) $RP(\phi', \phi, \psi', \psi)$; then $RP(\theta', \theta, \psi', \psi)$, $RN(\chi', \chi, \psi', \psi)$, $\vdash_S \chi' \to \chi$ and $\vdash_S \theta \to \theta'$. By T39b, $\vdash_S \phi \to \phi'$.

(b) $RN(\phi', \phi, \psi', \psi)$; then $RN(\theta', \theta, \psi', \psi)$, $RP(\chi', \chi, \psi', \psi)$, $\vdash_S \theta' \to \theta$ and $\vdash_S \chi \to \chi'$. By T39b, $\vdash_S \phi' \to \phi$.

(4) There are formulas χ, χ' such that ϕ' is $\neg\chi'$ and ϕ is $\neg\chi$. We take two subcases:

(a) $RP(\phi', \phi, \psi', \psi)$; then $RN(\chi', \chi, \psi', \psi)$ and $\vdash_S \chi' \to \chi$. By T39b, $\vdash_S \phi \to \phi'$.

(b) $RN(\phi', \phi, \psi', \psi)$; then $RP(\chi', \chi, \psi', \psi)$ and $\vdash_S \chi \to \chi'$. By T39b, $\vdash_S \phi' \to \phi$.

(5) There are formulas χ, χ' such that either ϕ' is $H\chi'$ and ϕ is $H\chi$, ϕ' is $G\chi'$ and ϕ is $G\chi$, ϕ' is $K\chi'$ and ϕ is $K\chi$, ϕ' is $R\chi'$ and ϕ is $R\chi$, or there is a variable α such that ϕ' is $\Lambda\alpha\chi'$ and ϕ is $\Lambda\alpha\chi$. We take two subcases:

(a) $RP(\phi',\phi,\psi',\psi)$; then $RP(\chi',\chi,\psi',\psi)$ and $\vdash_S \chi \to \chi'$. By T35d, $\vdash_S H(\chi\to\chi')$, $\vdash_S G(\chi\to\chi')$, $\vdash_S K(\chi\to\chi')$ and $\vdash_S \Lambda\alpha(\chi\to\chi')$. By T42b, $\vdash_S R(\chi\to\chi')$. By GA9, GA10, GA17, GA4, or GA20 respectively, and T37, $\vdash_S \phi\to\phi'$

(b) $RN(\phi',\phi,\psi',\psi)$; then $RN(\chi',\chi,\psi',\psi)$ and $\vdash_S \chi'\to\chi$. By T35d, $\vdash_S H(\chi'\to\chi)$, $\vdash_S G(\chi'\to\chi)$, $\vdash_S K(\chi'\to\chi)$ and $\vdash_S \Lambda\alpha(\chi'\to\chi)$. By T42b, $\vdash_S R(\chi'\to\chi)$. By GA9, GA10, GA17, GA4 or GA20, respectively, and T37, $\vdash_S \phi'\to\phi$

From this point forward, reference to T39 will often be omitted.

6. Theorems about Identity

T45. (a) $\vdash_S \eta=\eta$

(b) $\vdash_S \zeta=\eta \to \eta=\zeta$

(c) $\vdash_S \zeta=\eta \wedge \eta=\xi \to \zeta=\xi$

Proof:

(a) Let β be a variable not occurring in η. Then

	(i)	$\vdash_S \beta=\eta \to (\beta=\eta \to \eta=\eta)$	GA6
	(ii)	$\vdash_S \neg\eta=\eta \to \neg\beta=\eta$	T39b
	(iii)	$\vdash_S \Lambda\beta\neg\eta=\eta \to \Lambda\beta\neg\beta=\eta$	T35d, GA4, T37
	(iv)	$\vdash_S \neg\eta=\eta \to \Lambda\beta\neg\beta=\eta$	GA5, T39b
	(v)	$\vdash_S \eta=\eta$	GA7, T39b
(b)	(i)	$\vdash_S \zeta=\eta \to (\zeta=\zeta \to \eta=\zeta)$	GA6
	(ii)	$\vdash_S \zeta=\eta \to \eta=\zeta$	T45a
(c)	(i)	$\vdash_S \eta=\xi \to (\zeta=\eta \to \zeta=\xi)$	GA6

(ii) $\vdash_S \zeta=\eta \wedge \eta=\xi \rightarrow \zeta=\xi$ T39b

T46. If $\eta, \eta', \zeta, \zeta'$ are terms, and η' is obtained from η by replacing 0 or more occurrences of ζ by ζ', then $\vdash_S \zeta=\zeta' \rightarrow \eta=\eta'$.

Proof: Assume the hypothesis. By GA6, $\vdash_S \zeta=\zeta' \rightarrow (\eta=\eta \rightarrow \eta=\eta')$. By T45a, $\vdash_S \zeta=\zeta' \rightarrow \eta=\eta'$.

T47. If $1 \leq k$ and $\eta_0, \ldots, \eta_{k-1}, \zeta_0, \ldots, \zeta_{k-1}$ are terms, then

(a) If δ is a k-place operation letter, then

$$\vdash_S \eta_0=\zeta_0 \wedge \ldots \wedge \eta_{k-1}=\zeta_{k-1} \rightarrow \delta\eta_0\ldots\eta_{k-1}=\delta\zeta_0\ldots\zeta_{k-1}$$

(b) If π is a k-place predicate letter, then

$$\vdash_S \eta_0=\zeta_0 \wedge \ldots \wedge \eta_{k-1}=\zeta_{k-1} \rightarrow (\pi\eta_0\ldots\eta_{k-1} \leftrightarrow \pi\zeta_0\ldots\zeta_{k-1})$$

Proof: The proof is trivial. We will demonstrate the case for k is 3. Assume that $\eta_0, \eta_1, \eta_2, \zeta_0, \zeta_1, \zeta_2$ are terms.

(a) Suppose that δ is a k-place predicate letter.

(i) $\vdash_S \eta_0=\zeta_0 \rightarrow \delta\eta_0\eta_1\eta_2=\delta\zeta_0\zeta_1\zeta_2$ T46

(ii) $\vdash_S \eta_1=\zeta_1 \rightarrow \delta\zeta_0\eta_1\eta_2=\delta\zeta_0\zeta_1\eta_2$ T46

(iii) $\vdash_S \eta_2=\zeta_2 \rightarrow \delta\zeta_0\zeta_1\eta_2=\delta\zeta_0\zeta_1\zeta_2$ T46

(iv) $\vdash_S \eta_0=\zeta_0 \wedge \eta_1=\zeta_1 \wedge \eta_2=\zeta_2 \rightarrow$
$\delta\eta_0\eta_1\eta_2=\delta\zeta_0\zeta_1\zeta_2$ (i),(ii),(iii),T45c

(b) Suppose that π is a k-place predicate letter.

(i) $\vdash_S \eta_0=\zeta_0 \rightarrow (\pi\eta_0\eta_1\eta_2 \rightarrow \pi\zeta_0\eta_1\eta_2)$ GA6

(ii) $\vdash_S \eta_1=\zeta_1 \rightarrow (\pi\zeta_0\eta_1\eta_2 \rightarrow \pi\zeta_0\zeta_1\eta_2)$ GA6

(iii) $\vdash_S \eta_2=\zeta_2 \rightarrow (\pi\zeta_0\zeta_1\eta_2 \rightarrow \pi\zeta_0\zeta_1\zeta_2)$ GA6

(iv) $\vdash_S \eta_0=\zeta_0 \wedge \eta_1=\zeta_1 \wedge \eta_2=\zeta_2 \rightarrow$

$(\pi\eta_0\eta_1\eta_2 \rightarrow \pi\zeta_0\zeta_1\zeta_2)$ (i), (ii), (iii), T39b

(v) $\vdash_S \zeta_0=\eta_0 \rightarrow (\pi\zeta_0\zeta_1\zeta_2 \rightarrow \pi\eta_0\zeta_1\zeta_2)$ GA6

(vi) $\vdash_S \zeta_1=\eta_1 \rightarrow (\pi\eta_0\zeta_1\zeta_2 \rightarrow \pi\eta_0\eta_1\zeta_2)$ GA6

(vii) $\vdash_S \zeta_2=\eta_2 \rightarrow (\pi\eta_0\eta_1\zeta_2 \rightarrow \pi\eta_0\eta_1\eta_2)$ GA6

(viii) $\vdash_S \eta_0=\zeta_0 \wedge \eta_1=\zeta_1 \wedge \eta_2=\zeta_2 \rightarrow$

$(\pi\zeta_0\zeta_1\zeta_2 \rightarrow \pi\eta_0\eta_1\eta_2)$ (v), (vi), (vii), T39b

(ix) $\vdash_S \eta_0=\zeta_0 \wedge \eta_1=\zeta_1 \wedge \eta_2=\zeta_2 \rightarrow$

$(\pi\eta_0\eta_1\eta_2 \leftrightarrow \pi\zeta_0\zeta_1\zeta_2)$ (iv), (viii)

7. Theorems about G, H, and L

T48. (a) $\vdash_S L(\phi \rightarrow \psi) \rightarrow (L\phi \rightarrow L\psi)$

(b) $\vdash_S L(\phi \rightarrow \psi) \rightarrow (M\phi \rightarrow M\psi)$

(c) $\vdash_S H(\phi \rightarrow \psi) \rightarrow (P\phi \rightarrow P\psi)$

(d) $\vdash_S G(\phi \rightarrow \psi) \rightarrow (F\phi \rightarrow F\psi)$

(e) $\vdash_S M\phi \leftrightarrow (P\phi \vee \phi \vee F\phi)$

(f) $\vdash_S (H\phi \wedge P\psi) \rightarrow P(\phi \wedge \psi)$

(g) $\vdash_S (G\phi \wedge F\psi) \rightarrow F(\phi \wedge \psi)$

(h) $\vdash_S (L\phi \wedge M\psi) \rightarrow M(\phi \wedge \psi)$

(i) $\vdash_S L\phi \leftrightarrow \neg M \neg \phi$

Proof: All of the parts are trivial consequences of GA9 and GA10.

T49. (a) $\vdash_S P(\phi \vee \psi) \leftrightarrow (P\phi \vee P\psi)$

(b) $\vdash_S F(\phi \vee \psi) \leftrightarrow (F\phi \vee F\psi)$

(c) $\vdash_S M(\phi \vee \psi) \leftrightarrow (M\phi \vee M\psi)$

(d) $\vdash_S H(\phi \wedge \psi) \leftrightarrow (H\phi \wedge H\psi)$

(e) $\vdash_S G(\phi \wedge \psi) \leftrightarrow (G\phi \wedge G\psi)$

(f) $\vdash_S L(\phi \wedge \psi) \leftrightarrow (L\phi \wedge L\psi)$

Proof:

(a) (i) $\vdash_S H((\phi \vee \psi) \wedge \neg\phi \rightarrow \psi)$ T38,T35d

(ii) $\vdash_S P((\phi \vee \psi) \wedge \neg\phi) \rightarrow P\psi$ T48c,T37

(iii) $\vdash_S P(\phi \vee \psi) \wedge H\neg\phi \rightarrow P\psi$ T48c,T48f

(iv) $\vdash_S P(\phi \vee \psi) \rightarrow (P\phi \vee P\psi)$

(v) $\vdash_S P\phi \rightarrow P(\phi \vee \psi)$ T38,T35d,T48c,T37

(vi) $\vdash_S P\psi \rightarrow P(\phi \vee \psi)$ T38,T35d,T48c,T37

(vii) $\vdash_S P(\phi \vee \psi) \leftrightarrow (P\phi \vee P\psi)$

(b) Similar to (a)

(c) Follows from (a), (b), and T48e

(d) (i) $\vdash_S \neg H\neg(\neg\phi \vee \neg\psi) \leftrightarrow (\neg H\neg\neg\phi \vee \neg H\neg\neg\psi)$ from (a)

(ii) $\vdash_S H\neg(\neg\phi \vee \neg\psi) \leftrightarrow (H\neg\neg\phi \wedge H\neg\neg\psi)$

(iii) $\vdash_S H(\phi \wedge \psi) \leftrightarrow (H\phi \wedge H\psi)$ T43

(e) Similar to (d) using (b)

(f) Similar to (d) using (c)

8. Theorems about Quantification

T50. (a) $\vdash_S \Lambda\alpha(\phi \wedge \psi) \leftrightarrow (\Lambda\alpha\phi \wedge \Lambda\alpha\psi)$

(b) $\vdash_S \Lambda\alpha\phi \wedge V\alpha\psi \rightarrow V\alpha(\phi \wedge \psi)$

(c) $\vdash_S \Lambda\alpha(\phi \rightarrow \psi) \rightarrow (V\alpha\phi \rightarrow V\alpha\psi)$

(d) $\vdash_S \Lambda\alpha_0 \ldots \Lambda\alpha_{n-1}\neg\phi \leftrightarrow \neg V\alpha_0 \ldots V\alpha_{n-1}\phi$

Proof:

(a) (i) $\vdash_S \phi \wedge \psi \rightarrow \phi$ T38

(ii) $\vdash_S \Lambda\alpha(\phi \wedge \psi) \rightarrow \Lambda\alpha\phi$ T35d,GA4,T37

(iii) $\vdash_S \phi \wedge \psi \rightarrow \psi$ T38

(iv) $\vdash_S \Lambda\alpha(\phi \wedge \psi) \rightarrow \Lambda\alpha\psi$ T35d,GA4,T37

(v) $\vdash_S \Lambda\alpha(\phi \wedge \psi) \rightarrow \Lambda\alpha\phi \wedge \Lambda\alpha\psi$ (ii),(iv)

(vi) $\vdash_S \phi \rightarrow (\psi \rightarrow (\phi \wedge \psi))$ T38

(vii) $\vdash_S \Lambda\alpha\phi \rightarrow (\Lambda\alpha\psi \rightarrow \Lambda\alpha(\phi \wedge \psi))$ (vi),GA4

(viii) $\vdash_S \Lambda\alpha\phi \wedge \Lambda\alpha\psi \rightarrow \Lambda\alpha(\phi \wedge \psi)$

(ix) $\vdash_S \Lambda\alpha(\phi \wedge \psi) \leftrightarrow (\Lambda\alpha\phi \wedge \Lambda\alpha\psi)$ (v),(viii)

(b) (i) $\vdash_S \Lambda\alpha(\phi \rightarrow \neg\psi) \rightarrow (\Lambda\alpha\phi \rightarrow \Lambda\alpha\neg\psi)$ GA4

(ii) $\vdash_S \Lambda\alpha\phi \wedge V\alpha\psi \rightarrow \neg\Lambda\alpha(\phi \rightarrow \neg\psi)$

(iii) $\vdash_S \Lambda\alpha\phi \wedge V\alpha\psi \rightarrow V\alpha(\phi \wedge \psi)$ T43

(c) (i) $\vdash_S \Lambda\alpha(\neg\psi \rightarrow \neg\phi) \rightarrow (\Lambda\alpha\neg\psi \rightarrow \Lambda\alpha\neg\phi)$ GA4

(ii) $\vdash_S \Lambda\alpha(\phi \rightarrow \psi) \rightarrow (\Lambda\alpha\neg\psi \rightarrow \Lambda\alpha\neg\phi)$ T43

(iii) $\vdash_S \Lambda\alpha(\phi \rightarrow \psi) \rightarrow (V\alpha\phi \rightarrow V\alpha\psi)$

(d) Proof by induction; let N be the set of $n \in \omega$ such that, for any variables $\alpha_0, \ldots, \alpha_{n-1}$ and any formula ϕ $\vdash_S \Lambda\alpha_0 \ldots \Lambda\alpha_{n-1}\neg\phi \leftrightarrow \neg V\alpha_0 \ldots V\alpha_{n-1}\phi$. It is sufficient to show by induction that N is ω.

(i) $0 \in N$, since, by T38, $\vdash_S \neg\phi \rightarrow \neg\phi$, which is

$\vdash_S \Lambda\alpha_0 \ldots \Lambda\alpha_{0-1}\neg\phi \leftrightarrow \neg V\alpha_0 \ldots V\alpha_{0-1}\phi$

(ii) Suppose $n \in N$. By the inductive hypothesis $\vdash_S \Lambda\alpha_0 \ldots \Lambda\alpha_{n-1}\neg\neg\Lambda\alpha_n\neg\phi \leftrightarrow \neg V\alpha_0 \ldots V\alpha_{n-1}\neg\Lambda\alpha_n\neg\phi$. By T43, $\vdash_S \Lambda\alpha_0 \ldots \Lambda\alpha_n\neg\phi \leftrightarrow \neg V\alpha_0 \ldots V\alpha_n\phi$.

This completes the proof.

T51. If $\alpha \notin fv(\phi)$, then

(a) $\vdash_S \Lambda\alpha(\phi \rightarrow \psi) \leftrightarrow (\phi \rightarrow \Lambda\alpha\psi)$

(b) $\vdash_S \Lambda\alpha(\psi \to \phi) \leftrightarrow (\vee\alpha\psi \to \phi)$

Proof: Suppose $\alpha \notin fv(\phi)$

(a)	(i)	$\vdash_S \Lambda\alpha(\phi \to \psi) \to (\phi \to \Lambda\alpha\psi)$	GA4,GA5,T39b
	(ii)	$\vdash_S \Lambda\alpha(\psi \to (\phi \to \psi))$	GA1,T35b,T35d
	(iii)	$\vdash_S \Lambda\alpha\psi \to \Lambda\alpha(\phi \to \psi)$	(ii),GA4
	(iv)	$\vdash_S \neg\phi \to (\phi \to \psi)$	T38
	(v)	$\vdash_S \Lambda\alpha\neg\phi \to \Lambda\alpha(\phi \to \psi)$	(iv),T35d,GA4
	(vi)	$\vdash_S \neg\phi \to \Lambda\alpha(\phi \to \psi)$	(v),GA5
	(vii)	$\vdash_S (\phi \to \Lambda\alpha\psi) \to \Lambda\alpha(\phi \to \psi)$	(iii),(vi)
	(viii)	$\vdash_S \Lambda\alpha(\phi \to \psi) \leftrightarrow (\phi \to \Lambda\alpha\psi)$	(i),(vii)

((b) By (a), $\vdash_S \Lambda\alpha(\neg\phi \to \neg\psi) \leftrightarrow (\neg\phi \to \Lambda\alpha\neg\psi)$.

By T38 and T43, $\vdash_S \Lambda\alpha(\psi \to \phi) \leftrightarrow (\vee\alpha\psi \to \phi)$

T52. If β does not occur in ϕ, then $\vdash_S \alpha=\beta \to (\phi \leftrightarrow ra(\alpha,\beta,\phi))$.

Proof: (By induction on the rank of ϕ.) Suppose that β does not occur in ϕ. If β is α, then the theorem follows at once by T38 and T5a. Suppose then that β is not α.

(a) If ϕ is an atomic formula, then the conclusion follows by repeated applications of GA6 and T39b.

(b) If ϕ is $\neg\psi$ or ϕ is $\psi \to \chi$, the conclusion follows by the inductive hypothesis and T39b.

(c) Suppose ϕ is $\Lambda\gamma\psi$. Then β is not γ. By the inductive hypothesis, $\vdash_S \alpha=\beta \to (\psi \leftrightarrow ra(\alpha,\beta,\psi))$. We take two cases.

(i) Suppose α is not γ. Then $\vdash_S \alpha=\beta \to (\psi \to ra(\alpha,\beta,\psi))$ and $\vdash_S \alpha=\beta \to (ra(\alpha,\beta,\psi) \to \psi)$. By T51 and T35d,

$\vdash_S \alpha=\beta \rightarrow \Lambda\gamma(\psi \rightarrow ra(\alpha,\beta,\psi))$ and $\vdash_S \alpha=\beta \rightarrow \Lambda\gamma(ra(\alpha,\beta,\psi) \rightarrow \psi)$. By GA4 and T39, $\vdash_S \alpha=\beta \rightarrow (\Lambda\gamma\psi \rightarrow \Lambda\gamma ra(\alpha,\beta,\psi))$ and $\vdash_S \alpha=\beta \rightarrow (\Lambda\gamma ra(\alpha,\beta,\psi) \rightarrow \Lambda\gamma\psi)$. By T39 again, $\vdash_S \alpha=\beta \rightarrow (\Lambda\gamma\psi \leftrightarrow \Lambda\gamma ra(\alpha,\beta,\psi))$; but this is $\vdash_S \alpha=\beta \rightarrow (\phi \leftrightarrow ra(\alpha,\beta,\phi))$.

(ii) Suppose α is γ. By GA4, T39, and T45b, $\vdash_S \Lambda\alpha\psi \rightarrow (\Lambda\alpha\neg ra(\alpha,\beta,\psi) \rightarrow \Lambda\alpha\neg\alpha=\beta)$ and $\vdash_S \Lambda\beta ra(\alpha,\beta,\psi) \rightarrow (\Lambda\beta\neg\psi \rightarrow \Lambda\beta\neg\beta=\alpha)$. By GA7 and T39, $\vdash_S \Lambda\alpha\psi \rightarrow \neg\Lambda\alpha\neg ra(\alpha,\beta,\psi)$ and $\vdash_S \Lambda\beta ra(\alpha,\beta,\psi) \rightarrow \neg\Lambda\beta\neg\psi$. By GA5 and T5c, $\vdash_S \neg ra(\alpha,\beta,\psi) \rightarrow \Lambda\alpha\neg ra(\alpha,\beta,\psi)$ and $\vdash_S \neg\psi \rightarrow \Lambda\beta\neg\psi$. By T39, $\vdash_S \Lambda\alpha\psi \rightarrow ra(\alpha,\beta,\psi)$ and $\vdash_S \Lambda\beta ra(\alpha,\beta,\psi) \rightarrow \psi$. By T35d and T51a, $\vdash_S \Lambda\alpha\psi \rightarrow \Lambda\beta ra(\alpha,\beta,\psi)$ and $\vdash_S \Lambda\beta ra(\alpha,\beta,\psi) \rightarrow \Lambda\alpha\psi$. But then by T39b, $\vdash_S \phi \leftrightarrow ra(\alpha,\beta,\phi)$ and $\vdash_S \alpha=\beta \rightarrow (\phi \leftrightarrow ra(\alpha,\beta,\phi))$.

(d) Suppose ϕ is $H\psi$. By the inductive hypothesis, $\vdash_S \alpha=\beta \rightarrow (\psi \leftrightarrow ra(\alpha,\beta,\psi))$. By T35d and GA9, $\vdash_S H\alpha=\beta \rightarrow H(\psi \rightarrow ra(\alpha,\beta,\psi))$ and $\vdash_S H\alpha=\beta \rightarrow H(ra(\alpha,\beta,\psi) \rightarrow \psi)$. By GA8, $\vdash_S \alpha=\beta \rightarrow H\alpha=\beta$. By GA9, $\vdash_S \alpha=\beta \rightarrow (H\psi \leftrightarrow H ra(\alpha,\beta,\psi))$ which is $\vdash_S \alpha=\beta \rightarrow (\phi \leftrightarrow ra(\alpha,\beta,\phi))$.

(e) Suppose ϕ is $G\psi$. This is similar to (d).

(f) Suppose ϕ is $K\psi$. By the inductive hypothesis, $\vdash_S \alpha=\beta \rightarrow (\psi \leftrightarrow ra(\alpha,\beta,\psi))$. By T35d, GA17, and T39b, $\vdash_S K\alpha=\beta \rightarrow (K\psi \leftrightarrow K ra(\alpha,\beta,\psi))$. By GA15, $\vdash_S \alpha=\beta \rightarrow K\alpha=\beta$. By T39b, $\vdash_S \alpha=\beta \rightarrow (\phi \leftrightarrow ra(\alpha,\beta,\phi))$.

(g) Suppose ϕ is $R\psi$. By the inductive hypothesis, $\vdash_S \alpha=\beta \rightarrow (\psi \leftrightarrow ra(\alpha,\beta,\psi))$. By T35d, GA20, and T39b,

$\vdash_S R\alpha=\beta \rightarrow (R\psi \leftrightarrow Rra(\alpha,\beta,\psi))$. By GA8, GA21, and T39b, $\vdash_S \alpha=\beta \rightarrow R\alpha=\beta$. By T39b again, $\vdash_S \alpha=\beta \rightarrow (R\psi \leftrightarrow Rra(\alpha,\beta,\psi))$, which is $\vdash_S \alpha=\beta \rightarrow (\phi \leftrightarrow ra(\alpha,\beta,\phi))$.

T53. If β does not occur in ϕ and β is not α, then $\vdash_S \Lambda\alpha\phi \rightarrow ra(\alpha,\beta,\phi)$.

Proof: Suppose that β does not occur in ϕ and β is not α. By T52, $\vdash_S \alpha=\beta \rightarrow (\phi \rightarrow ra(\alpha,\beta,\phi))$. Hence, $\vdash_S \phi \rightarrow (\neg ra(\alpha,\beta,\phi) \rightarrow \neg\alpha=\beta)$. By GA4, $\vdash_S \Lambda\alpha\phi \rightarrow (\Lambda\alpha\neg ra(\alpha,\beta,\phi) \rightarrow \Lambda\alpha\neg\alpha=\beta)$. By GA7, $\vdash_S \Lambda\alpha\phi \rightarrow \neg\Lambda\alpha\neg ra(\alpha,\beta,\phi)$. By T5c, $\alpha \notin fv(ra(\alpha,\beta,\phi))$, and by GA5, $\vdash_S \neg ra(\alpha,\beta,\phi) \rightarrow \Lambda\alpha\neg ra(\alpha,\beta,\phi)$. By T39b, $\vdash_S \Lambda\alpha\phi \rightarrow ra(\alpha,\beta,\phi)$.

T54. $\vdash_S \Lambda\alpha\phi \rightarrow \phi$

Proof: Let β be some variable that does not occur in $\Lambda\alpha\phi$. Then, by T53, $\vdash_S \Lambda\alpha\phi \rightarrow ra(\alpha,\beta,\phi)$. By T35d and T51a, $\vdash_S \Lambda\alpha\phi \rightarrow \Lambda\beta ra(\alpha,\beta,\phi)$. By GA7, $\vdash_S V\beta\beta=\alpha$ and by T50b, $\vdash_S \Lambda\alpha\phi \rightarrow V\beta(\beta=\alpha \wedge ra(\alpha,\beta,\phi))$. By T52 and T35d, $\vdash_S \Lambda\beta(\beta=\alpha \wedge ra(\alpha,\beta,\phi) \rightarrow \phi)$. By T50c, $\vdash_S \Lambda\alpha\phi \rightarrow \neg\Lambda\beta\neg\phi$. By GA5 and T39b, $\vdash_S \neg\Lambda\beta\neg\phi \rightarrow \phi$. By T39b, $\vdash_S \Lambda\alpha\phi \rightarrow \phi$.

T55. (a) $\vdash_S \Lambda\alpha H\phi \leftrightarrow H\Lambda\alpha\phi$

(b) $\vdash_S \Lambda\alpha G\phi \leftrightarrow G\Lambda\alpha\phi$

(c) $\vdash_S \Lambda\alpha L\phi \leftrightarrow L\Lambda\alpha\phi$

Proof:

(a)	(i)	$\vdash_S H(\Lambda\alpha\phi \rightarrow \phi)$	T54,T35d
	(ii)	$\vdash_S H\Lambda\alpha\phi \rightarrow H\phi$	GA9,T37

(iii) $\vdash_S \Lambda\alpha(H\Lambda\alpha\phi \rightarrow H\phi)$ T35d

(iv) $\vdash_S \Lambda\alpha H\Lambda\alpha\phi \rightarrow \Lambda\alpha H\phi$ GA4, T37

(v) $\vdash_S H\Lambda\alpha\phi \rightarrow \Lambda\alpha H\Lambda\alpha\phi$ GA5

(vi) $\vdash_S H\Lambda\alpha\phi \rightarrow \Lambda\alpha H\phi$ (iv),(v)

(vii) $\vdash_S \Lambda\alpha H\phi \rightarrow H\phi$ T54

(viii) $\vdash_S G(\Lambda\alpha H\phi \rightarrow H\phi)$ T35d

(ix) $\vdash_S F\Lambda\alpha H\phi \rightarrow FH\phi$ T48d, T37

(x) $\vdash_S \Lambda\alpha F\Lambda\alpha H\phi \rightarrow \Lambda\alpha FH\phi$ T35d,GA4,T37

(xi) $\vdash_S F\Lambda\alpha H\phi \rightarrow \Lambda\alpha F\Lambda\alpha H\phi$ GA5

(xii) $\vdash_S F\Lambda\alpha H\phi \rightarrow \Lambda\alpha FH\phi$ (x),(xi)

(xiii) $\vdash_S HF\Lambda\alpha H\phi \rightarrow H\Lambda\alpha FH\phi$ T35d,GA9

(xiv) $\vdash_S \Lambda\alpha H\phi \rightarrow H\Lambda\alpha FH\phi$ (xiii),GA14

(xv) $\vdash_S \neg\phi \rightarrow G\neg H\neg\neg\phi$ GA13

(xvi) $\vdash_S FH\phi \rightarrow \phi$ T43

(xvii) $\vdash_S H\Lambda\alpha FH\phi \rightarrow H\Lambda\alpha\phi$ T35d,GA4,T37,T35d,GA9

(xviii) $\vdash_S \Lambda\alpha H\phi \rightarrow H\Lambda\alpha\phi$ (xiv),(xvii),T44a

(xix) $\vdash_S \Lambda\alpha H\phi \leftrightarrow H\Lambda\alpha\phi$ (vi),(xviii)

(b) Analogous to (a)

(c) (i) $\vdash_S \Lambda\alpha L\phi \rightarrow \Lambda\alpha H\phi$ T38,T44

(ii) $\vdash_S \Lambda\alpha L\phi \rightarrow \Lambda\alpha\phi$ T38,T44

(iii) $\vdash_S \Lambda\alpha L\phi \rightarrow \Lambda\alpha G\phi$ T38,T44

(iv) $\vdash_S \Lambda\alpha L\phi \rightarrow L\Lambda\alpha\phi$ (i),(ii),(iii),(a),(b)

(v) $\vdash_S L\Lambda\alpha\phi \rightarrow \Lambda\alpha H\phi \wedge \Lambda\alpha\phi \wedge \Lambda\alpha G\phi$ (a),(b)

(vi) $\vdash_S \Lambda\alpha H\phi \wedge \Lambda\alpha\phi \wedge \Lambda\alpha G\phi \rightarrow \Lambda\alpha L\phi$ T50a

(vii) $\vdash_S \Lambda\alpha L\phi \leftrightarrow L\Lambda\alpha\phi$ (iv),(v),(vi)

T56. (a) $\vdash_S \alpha=\beta \rightarrow (\phi \leftrightarrow ps(\beta,\alpha,\phi))$

(b) $\vdash_S \Lambda\alpha\phi \rightarrow ps(\beta,\alpha,\phi)$

(c) $\vdash_S \alpha=\beta \rightarrow (ps(\alpha,\gamma,\phi) \leftrightarrow ps(\beta,\gamma,\phi))$

(d) $\vdash_S \phi \leftrightarrow ps(\alpha,\beta,(ps(\beta,\alpha,\phi)))$

(e) $\vdash_S$ If $\beta \notin fv(\phi)$, then $\vdash_S \Lambda\alpha\phi \leftrightarrow \Lambda\beta ps(\beta,\alpha,\phi)$

Proof:

(a) By induction on the rank of ϕ

(1) If ϕ is an atomic formula, the conclusion follows by GA6.

(2) If (a) holds for ϕ and ψ, then it holds for $\neg\phi$ and $\phi \rightarrow \psi$ by T39b.

(3) Suppose, for each formula ψ such that $rk(\psi) < rk(\Lambda\xi\phi)$, $\vdash_S \alpha=\beta \rightarrow (\psi \leftrightarrow ps(\beta,\alpha,\psi))$. There are three cases.

(i) $\alpha \notin fv(\Lambda\xi\phi)$; then $\vdash_S \alpha=\beta \rightarrow (\Lambda\xi\phi \leftrightarrow ps(\beta,\alpha,\Lambda\xi\phi))$ by T9b and T38.

(ii) $\alpha \in fv(\Lambda\xi\phi)$ and ξ does not occur in β. Then ξ is not β, ξ is not α (since $\xi \notin fv(\Lambda\xi\phi)$), and ξ does not occur in $\alpha=\beta$. By the inductive hypothesis, $\vdash_S \alpha=\beta \rightarrow (\phi \rightarrow ps(\beta,\alpha,\phi))$ and $\vdash_S \alpha=\beta \rightarrow (ps(\beta,\alpha,\phi) \rightarrow \phi)$. By T35d and T51a, $\vdash_S \alpha=\beta \rightarrow \Lambda\xi(\phi \rightarrow ps(\beta,\alpha,\phi))$ and $\vdash_S \alpha=\beta \rightarrow \Lambda\xi(ps(\beta,\alpha,\phi) \rightarrow \phi)$. By GA4, $\vdash_S \alpha=\beta \rightarrow (\Lambda\xi\phi \leftrightarrow \Lambda\xi ps(\beta,\alpha,\phi))$. But by definition this is $\vdash_S \alpha=\beta \rightarrow (\Lambda\xi\phi \leftrightarrow ps(\beta,\alpha,\Lambda\xi\phi))$.

(iii) $\alpha \in fv(\Lambda\xi\phi)$ and ξ occurs in β (i.e. ξ is β). Let γ be the first variable that occurs neither in ϕ nor in β. Then $ps(\beta,\alpha,\Lambda\xi\phi)$ is $\Lambda\gamma ps(\beta,\alpha,ra(\xi,\gamma,\phi))$. By the inductive hypothesis and T8 $\vdash_S \alpha=\beta \rightarrow (ra(\xi,\gamma,\phi) \leftrightarrow ps(\beta,\alpha,ra(\xi,\gamma,\phi)))$. By T39b, $\vdash_S \alpha=\beta \rightarrow (ra(\xi,\gamma,\phi) \rightarrow ps(\beta,\alpha,ra(\xi,\gamma,\phi)))$ and $\vdash_S \alpha=\beta \rightarrow (ps(\beta,\alpha,ra(\xi,\gamma,\phi)) \rightarrow ra(\xi,\gamma,\phi))$. By T53, $\vdash_S \Lambda\xi\phi \rightarrow ra(\xi,\gamma,\phi)$. By T39b, $\vdash_S \alpha=\beta \rightarrow (\Lambda\xi\phi \rightarrow ps(\beta,\alpha,ra(\xi,\gamma,\phi)))$. By T35d, T51a (twice) and T49b, $\vdash_S \alpha=\beta \rightarrow (\Lambda\xi\phi \rightarrow \Lambda\gamma ps(\beta,\alpha,ra(\xi,\gamma,\phi)))$. By T53 and T5c, $\vdash_S \Lambda\gamma ra(\xi,\gamma,\phi) \rightarrow ra(\gamma,\xi,ra(\xi,\gamma,\phi))$. By T6, $\vdash_S \Lambda\gamma ra(\xi,\gamma,\phi) \rightarrow \phi$. By T51a and T5c, $\vdash_S \Lambda\gamma ra(\xi,\gamma,\phi) \rightarrow \Lambda\xi\phi$. By T51a and GA4, $\vdash_S \alpha=\beta \rightarrow (\Lambda\gamma ps(\beta,\alpha,ra(\xi,\gamma,\phi)) \rightarrow \Lambda\gamma ra(\xi,\gamma,\phi))$ and $\vdash_S \alpha=\beta \rightarrow (\Lambda\gamma ps(\beta,\alpha,ra(\xi,\gamma,\phi)) \rightarrow \Lambda\xi\phi)$. By T39b, $\vdash_S \alpha=\beta \rightarrow (\Lambda\xi\phi \leftrightarrow ps(\beta,\alpha,\Lambda\xi\phi))$.

(4) The cases for $H\phi$, $G\phi$, $K\phi$ and $R\phi$ are all analogous to the corresponding cases of T52.

(b) $\vdash_S \Lambda\alpha\phi \rightarrow ps(\beta,\alpha,\phi)$

(1) β is α; follows immediately from T9a and T54.

(2) β is not α. By (a) and T39b, $\vdash_S \phi \rightarrow (\neg ps(\beta,\alpha,\phi) \rightarrow \neg\alpha=\beta)$. By GA4 and T44, $\vdash_S \Lambda\alpha\phi \rightarrow (\Lambda\alpha\neg ps(\beta,\alpha,\phi) \rightarrow \Lambda\alpha\neg\alpha=\beta)$. By GA7, $\vdash_S \Lambda\alpha\phi \rightarrow \neg\Lambda\alpha\neg ps(\beta,\alpha,\phi)$. By GA5 and T9c, $\vdash_S \neg ps(\beta,\alpha,\phi) \rightarrow \Lambda\alpha\neg ps(\beta,\alpha,\phi)$ and hence, $\vdash_S \Lambda\alpha\phi \rightarrow ps(\beta,\alpha,\phi)$.

(c) $\vdash_S \alpha=\beta \rightarrow (ps(\alpha,\gamma,\phi) \rightarrow ps(\beta,\gamma,\phi))$

(1) α is not γ, β is not γ. By (a), $\vdash_S \gamma=\alpha \rightarrow (\phi \leftrightarrow$

$ps(\alpha,\gamma,\phi))$ and $\vdash_S \gamma=\beta \rightarrow (\phi \leftrightarrow ps(\beta,\gamma,\phi))$. By T39b and T35, $\vdash_S \Lambda\gamma(\gamma=\alpha \wedge \gamma=\beta \rightarrow (ps(\alpha,\gamma,\phi) \leftrightarrow ps(\beta,\gamma,\phi)))$. By T51b and T9c, $\vdash_S \vee\gamma(\gamma=\alpha \wedge \gamma=\beta) \rightarrow (ps(\alpha,\gamma,\phi) \leftrightarrow ps(\beta,\gamma,\phi))$. By T45a, $\vdash_S \alpha=\beta \rightarrow \alpha=\alpha \wedge \alpha=\beta$. By T53, $\vdash_S \Lambda\gamma\neg(\gamma=\alpha \wedge \gamma=\beta) \rightarrow \neg(\alpha=\alpha \wedge \alpha=\beta)$ and hence, $\vdash_S \alpha=\alpha \wedge \alpha=\beta \rightarrow \vee\gamma(\gamma=\alpha \wedge \gamma=\beta)$ and $\vdash_S \alpha=\beta \rightarrow \vee\gamma(\gamma=\alpha \wedge \gamma=\beta)$. By T39b, $\vdash_S \alpha=\beta \rightarrow (ps(\alpha,\gamma,\phi) \leftrightarrow ps(\beta,\gamma,\phi))$.

(2) α is γ. By (a), $\vdash_S \gamma=\beta \rightarrow (\phi \leftrightarrow ps(\beta,\gamma,\phi))$. By T9a, $\vdash_S \alpha=\beta \rightarrow (ps(\alpha,\gamma,\phi) \leftrightarrow ps(\beta,\gamma,\phi))$.

(3) β is γ. By (a), $\vdash_S \gamma=\alpha \rightarrow (\phi \leftrightarrow ps(\alpha,\gamma,\phi))$. By T45b, T39b, and T9a, $\vdash_S \alpha=\beta \rightarrow (ps(\alpha,\gamma,\phi) \leftrightarrow ps(\beta,\gamma,\phi))$.

(d) $\vdash_S \phi \leftrightarrow ps(\alpha,\beta,ps(\beta,\alpha,\phi))$

(1) α is β; then the conclusion follows immediately by T9a.

(2) α is not β. By (a) and T45b, $\vdash_S \beta=\alpha \rightarrow (\phi \leftrightarrow ps(\beta,\alpha,\phi))$. By (a), $\vdash_S \beta=\alpha \rightarrow (ps(\beta,\alpha,\phi) \leftrightarrow ps(\alpha,\beta,ps(\beta,\alpha,\phi)))$. Then $\vdash_S \beta=\alpha \rightarrow (\phi \rightarrow ps(\alpha,\beta,ps(\beta,\alpha,\phi)))$ and $\vdash_S \beta=\alpha \rightarrow (\neg\phi \rightarrow \neg ps(\alpha,\beta,ps(\beta,\alpha,\phi)))$. By T50c, $\vdash_S \vee\beta\beta=\alpha \rightarrow (\vee\beta\phi \rightarrow \vee\beta ps(\alpha,\beta,ps(\beta,\alpha,\phi)))$ and $\vdash_S \vee\beta\beta=\alpha \rightarrow (\vee\beta\neg\phi \rightarrow \vee\beta\neg ps(\alpha,\beta,ps(\beta,\alpha,\phi)))$. By GA7, $\vdash_S \vee\beta\phi \rightarrow \vee\beta ps(\alpha,\beta,ps(\beta,\alpha,\phi))$ and $\vdash_S \vee\beta\neg\phi \rightarrow \vee\beta\neg ps(\alpha,\beta,ps(\beta,\alpha,\phi))$. By T54, $\vdash_S \phi \rightarrow \vee\beta\phi$ and $\vdash_S \neg\phi \rightarrow \vee\beta\neg\phi$. Hence, $\vdash_S \phi \rightarrow \vee\beta ps(\alpha,\beta,ps(\beta,\alpha,\phi))$ and $\vdash_S \neg\phi \rightarrow \vee\beta\neg ps(\alpha,\beta,ps(\beta,\alpha,\phi))$. By GA5 and T9c, $\vdash_S \vee\beta ps(\alpha,\beta,ps(\beta,\alpha,\phi)) \rightarrow ps(\alpha,\beta,ps(\beta,\alpha,\phi))$ and

$\vdash_S \forall\beta\neg ps(\alpha,\beta,ps(\beta,\alpha,\phi)) \rightarrow \neg ps(\alpha,\beta,ps(\beta,\alpha,\phi))$. Hence, $\vdash_S \phi \rightarrow ps(\alpha,\beta,ps(\beta,\alpha,\phi))$ and $\vdash_S \neg\phi \rightarrow \neg ps(\alpha,\beta,ps(\beta,\alpha,\phi))$. By T39b, $\vdash_S \phi \leftrightarrow ps(\alpha,\beta,ps(\beta,\alpha,\phi))$.

(e) If $\beta \notin fv(\phi)$, then $\vdash_S \Lambda\alpha\phi \leftrightarrow \Lambda\beta ps(\beta,\alpha,\phi)$. Assume $\beta \notin fv(\phi)$. By (b), $\vdash_S \Lambda\alpha\phi \rightarrow ps(\beta,\alpha,\phi)$. By GA4 and T37, $\vdash_S \Lambda\beta\Lambda\alpha\phi \rightarrow \Lambda\beta ps(\beta,\alpha,\phi)$. By GA5, $\vdash_S \Lambda\alpha\phi \rightarrow \Lambda\beta\Lambda\alpha\phi$ and hence, $\vdash_S \Lambda\alpha\phi \rightarrow \Lambda\beta ps(\beta,\alpha,\phi)$. By an argument similar to the above, $\vdash_S \Lambda\beta ps(\beta,\alpha,\phi) \rightarrow \Lambda\alpha ps(\alpha,\beta,ps(\beta,\alpha,\phi))$. By (c) and T43, $\vdash_S \Lambda\beta ps(\beta,\alpha,\phi) \rightarrow \Lambda\alpha\phi$ and hence $\vdash_S \Lambda\alpha\phi \leftrightarrow \Lambda\beta ps(\beta,\alpha,\phi)$.

9. Theorems about Replacement of Variables

T57. If f is a one-to-one function, Fld(f) $\subseteq$ Iv, Rng(f) and Dom(f) are disjoint, all the variables that occur in ϕ are in Dom(f) and $\alpha_0,\ldots,\alpha_{n-1}$ are all the free variables of ϕ, then (a) If n is not 0, then

$$\vdash_S \alpha_0=f(\alpha_0) \wedge \ldots \wedge \alpha_{n-1}=f(\alpha_{n-1}) \rightarrow (\phi \leftrightarrow rep(\phi,f))$$

(b) If n is 0, then $\vdash_S \phi \leftrightarrow rep(\phi,f)$

Proof: (By induction on the rank of ϕ)

(1) If ϕ is an atomic formula, the theorem holds by repeated applications of GA6 and T39b.

(2) If ϕ is $\neg\psi$ or ϕ is $\psi \rightarrow \chi$, the result follows by the inductive hypothesis and T39b.

(3) Suppose ϕ is $\Lambda\alpha\psi$. We take three cases:

(a) $\alpha \notin fv(\psi)$. Then $f(\alpha) \notin fv(rep(\psi,f))$. By GA5 and T54, $\vdash_S \Lambda\alpha\psi \leftrightarrow \psi$ and $\vdash_S \Lambda f(\alpha) rep(\psi,f) \leftrightarrow rep(\psi,f)$. And by T39b and the inductive hypothesis, $\vdash_S \Lambda\alpha\psi \leftrightarrow \Lambda f(\alpha) rep(\psi,f)$.

(b) Suppose $fv(\psi)$ is $\{\alpha\}$. Then $fv(rep(\psi,f))$ is $\{f(\alpha)\}$. By the inductive hypothesis, $\vdash_S \alpha=f(\alpha)\rightarrow(\psi\leftrightarrow rep(\psi,f))$. By T39b, $\vdash_S \psi\rightarrow(\neg rep(\psi,f)\rightarrow\neg\alpha=f(\alpha))$ and $\vdash_S rep(\psi,f)\rightarrow(\neg\psi\rightarrow\neg f(\alpha)=\alpha)$. By GA4, $\vdash_S \Lambda\alpha\psi\rightarrow(\Lambda\alpha\neg rep(\psi,f)\rightarrow\Lambda\alpha\neg\alpha=f(\alpha))$ and $\vdash_S \Lambda f(\alpha) rep(\psi,f)\rightarrow(\Lambda f(\alpha)\neg\psi\rightarrow\Lambda f(\alpha)\neg f(\alpha)=\alpha)$. By GA7 and T39b, $\vdash_S \Lambda\alpha\psi\rightarrow\neg\Lambda\alpha\neg rep(\psi,f)$ and $\vdash_S \Lambda f(\alpha) rep(\psi,f)\rightarrow\neg\Lambda f(\alpha)\neg\psi$. By GA5 (since $f(\alpha)\notin fv(\psi)$ and $\alpha\notin fv(rep(\phi,f))$) $\vdash_S \neg rep(\psi,f)\rightarrow\Lambda\alpha\neg rep(\psi,f)$ and $\vdash_S \neg\psi\rightarrow\Lambda f(\alpha)\neg\psi$. By T39b, $\vdash_S \Lambda\alpha\psi\rightarrow rep(\psi,f)$ and $\vdash_S \Lambda f(\alpha) rep(\psi,f)\rightarrow\psi$. By T35d and T51a, $\vdash_S \Lambda\alpha\psi\rightarrow\Lambda f(\alpha) rep(\psi,f)$ and $\vdash_S \Lambda f(\alpha) rep(\psi,f)\rightarrow\Lambda\alpha\psi$; hence $\vdash_S \Lambda\alpha\psi\rightarrow rep(\Lambda\alpha\psi,f)$.

(c) Suppose $\alpha\in fv(\phi)$ and $fv(\phi)$ is not $\{\alpha\}$. Let $\beta_0,\ldots,\beta_{n-1}$ be (in order) the free variables (other than α) that occur in ψ. By the inductive hypothesis and T39b, $\vdash_S \beta_0=f(\beta_0)\wedge\ldots\wedge\beta_{n-1}=f(\beta_{n-1})\rightarrow(\psi\rightarrow(\neg rep(\psi,f)\rightarrow\neg\alpha=f(\alpha)))$ and $\vdash_S \beta_0=f(\beta_0)\wedge\ldots\wedge\beta_{n-1}=f(\beta_{n-1})\rightarrow(rep(\psi,f)\rightarrow(\neg\psi\rightarrow\neg f(\alpha)=\alpha))$. Let χ be the formula $\beta_0=f(\beta_0)\wedge\ldots\wedge\beta_{n-1}=f(\beta_{n-1})$. By GA4 and T39b, $\vdash_S \Lambda\alpha\chi\rightarrow(\Lambda\alpha\psi\rightarrow(\Lambda\alpha\neg rep(\psi,f)\rightarrow\Lambda\alpha\neg\alpha=f(\alpha)))$ and $\vdash_S \Lambda f(\alpha)\chi\rightarrow(\Lambda f(\alpha) rep(\psi,f)\rightarrow(\Lambda f(\alpha)\neg\psi\rightarrow\Lambda f(\alpha)\neg f(\alpha)=\alpha))$. By GA7 and T39b, $\vdash_S \Lambda\alpha\chi\rightarrow(\Lambda\alpha\psi\rightarrow\neg\Lambda\alpha\neg rep(\psi,f))$ and

$\vdash_S \bigwedge f(\alpha)\chi \rightarrow (\bigwedge f(\alpha)rep(\psi,f) \rightarrow \neg\bigwedge f(\alpha)\neg\psi)$. Since $Dom(f) \cap Rng(f)$ is 0; $\alpha \notin fv(\chi)$, $f(\alpha) \notin fv(\chi)$, $\alpha \notin fv(rep(\psi,f))$ and $f(\alpha) \notin fv(\psi)$. Hence, by GA5 and T39b, $\vdash_S \chi \rightarrow (\bigwedge\alpha\psi \rightarrow rep(\psi,f))$ and $\vdash_S \chi \rightarrow (\bigwedge f(\alpha)rep(\psi,f) \rightarrow \psi)$. By T51a (twice) and T39b, $\vdash_S \chi \rightarrow (\bigwedge\alpha\psi \rightarrow \bigwedge f(\alpha)rep(\psi,f))$ and $\vdash_S \chi \rightarrow (\bigwedge f(\alpha)rep(\psi,f) \rightarrow \bigwedge\alpha\psi)$. By T39b, $\vdash_S \chi \rightarrow (\bigwedge\alpha\psi \leftrightarrow rep(\bigwedge\alpha\psi,f))$.

(4) The cases where ϕ is $H\psi$, $G\psi$, $K\psi$ or $R\psi$ are analogous to the corresponding cases of T52.

T58. If f is a one-to-one function, $Fld(f) \subseteq Iv$, all the variables that occur in ϕ are in Dom(f) and $\alpha_0,\ldots,\alpha_{n-1}$ are all the free variables of ϕ, then

(a) If n is not 0, then

$$\vdash_S \alpha_0=f(\alpha_0) \wedge \ldots \wedge \alpha_{n-1}=f(\alpha_{n-1}) \rightarrow (\phi \leftrightarrow rep(\phi,f))$$

(b) If n is 0, then $\vdash_S \phi \leftrightarrow rep(\phi,f)$

Proof: Assume the hypothesis. Let Θ be the set of variables in ϕ, and let f' be $f \cap (\Theta \times Rng(f))$. Then the hypothesis holds for f' also. Since $Fld(f')$ is finite, there is a set of variables Θ' such that Θ is equinumerous with Θ' and $Fld(f) \cap \Theta'$ is 0. Let g be a one-to-one function from Θ onto Θ'. Let h be that one-to-one function from Θ' onto $Rng(f')$ such that, for each $\beta \in \Theta'$, $h(\beta)$ is $f'(\breve{g}(\beta))$. By T4, rep(rep(ϕ,g),h) is rep(ϕ,f). Let $\alpha_0,\ldots,\alpha_{n-1}$ be the free variables of ϕ. We will prove the case where n is not 0; the other case is similar.

By T57, $\vdash_s \alpha_0 = g(\alpha_0) \wedge \ldots \wedge \alpha_{n-1} = g(\alpha_{n-1}) \rightarrow (\phi \leftrightarrow rep(\phi, g))$. $g(\alpha_0), \ldots, g(\alpha_{n-1})$, of course, are the free variables of $rep(\phi, g)$.

By T57 again (since $h(g(\alpha_i))$ is $f(\alpha_i)$), $\vdash_s g(\alpha_0) = f(\alpha_0) \wedge \ldots \wedge g(\alpha_{n-1}) = f(\alpha_{n-1}) \rightarrow (rep(\phi, g) \leftrightarrow rep(rep(\phi, g), h))$.

By T39b, we have $\vdash_s \alpha_0 = g(\alpha_0) \wedge \ldots \wedge \alpha_{n-1} = g(\alpha_{n-1}) \wedge g(\alpha_0) = f(\alpha_0) \wedge \ldots \wedge g(\alpha_{n-1}) = f(\alpha_{n-1}) \rightarrow (\phi \leftrightarrow rep(\phi, f))$.

Since none of the variables $g(\alpha_0), \ldots, g(\alpha_{n-1})$ occur in ϕ or in $rep(\phi, f)$, we have, by repeated use of T35d and T51b, $\vdash_s \vee g(\alpha_0) \ldots \vee g(\alpha_{n-1}) (\alpha_0 = g(\alpha_0) \wedge \ldots \wedge \alpha_{n-1} = g(\alpha_{n-1}) \wedge g(\alpha_0) = f(\alpha_0) \wedge \ldots \wedge g(\alpha_{n-1}) = f(\alpha_{n-1})) \rightarrow (\phi \leftrightarrow rep(\phi, f))$.

By T53, $\vdash_s \wedge g(\alpha_0) \ldots \wedge g(\alpha_{n-1}) (\alpha_0 = g(\alpha_0) \wedge \ldots \wedge \alpha_{n-1} = g(\alpha_{n-1}) \wedge g(\alpha_0) = f(\alpha_0) \wedge \ldots \wedge g(\alpha_{n-1}) = f(\alpha_{n-1}) \rightarrow (\alpha_0 = \alpha_0 \wedge \ldots \wedge \alpha_{n-1} = \alpha_{n-1} \wedge \alpha_0 = f(\alpha_0) \wedge \ldots \wedge \alpha_{n-1} = f(\alpha_{n-1}))$.

By T45a, T50d, and T39b, $\vdash_s \alpha_0 = f(\alpha_0) \wedge \ldots \wedge \alpha_{n-1} = f(\alpha_{n-1}) \rightarrow \vee g(\alpha_0) \ldots \vee g(\alpha_{n-1}) (\alpha_0 = g(\alpha_0) \wedge \ldots \wedge \alpha_{n-1} = g(\alpha_{n-1}) \wedge g(\alpha_0) = f(\alpha_0) \wedge \ldots \wedge g(\alpha_{n-1}) = f(\alpha_{n-1}))$, and by T39b, $\vdash_s \alpha_0 = f(\alpha_0) \wedge \ldots \wedge \alpha_{n-1} = f(\alpha_{n-1}) \rightarrow (\phi \leftrightarrow rep(\phi, f))$.

T59. If f is a one-to-one function, $Fld(f) \subseteq Iv$, and every variable that occurs in ϕ or in some formula of Γ is in $Dom(f)$, then

(a) $\Gamma \vdash \phi$ if and only if $REP(\Gamma, f) \vdash rep(\phi, f)$

(b) $\Gamma \vdash_s \phi$ if and only if $REP(\Gamma, f) \vdash_s rep(\phi, f)$

Proof:

(a) Assume the hypothesis. First, suppose that $\Gamma \vdash \phi$.

By T34a, there is a finite set of formulas Γ' such that $\Gamma' \subseteq \Gamma$ and $\Gamma' \vdash \phi$. There is a derivation Δ of ϕ from Γ'. Let g be f $(\Theta \times \mathrm{Rng}(f))$, where Θ is the (finite) set of variables that occur in Γ' or in ϕ. There is a function h such that $g \subseteq h$, h is a one-to-one function, $\mathrm{Fld}(h) \subseteq \mathrm{Iv}$, and any variable that occurs in some formula Δ_n, for $n < \mathrm{lh}(\Delta)$, is in Dom(h). We will prove by strong induction that, for any $n < \mathrm{lh}(\Delta)$, $\mathrm{REP}(\Gamma',h) \vdash \mathrm{rep}(\Delta_n,h)$. Suppose that $n < \mathrm{lh}(\Delta)$: there are three cases:

(i) Δ_n is an axiom; then $\vdash_s \Delta_n$ and, by T39b and T58, $\vdash_s \mathrm{rep}(\Delta_n,h)$. Hence, $\mathrm{REP}(\Gamma',h) \vdash \mathrm{rep}(\Delta_n,h)$.

(ii) $\Delta_n \in \Gamma'$; then $\mathrm{rep}(\Delta_n,h) \in \mathrm{REP}(\Gamma',h)$ and, by T34c, $\mathrm{REP}(\Gamma',h) \vdash \mathrm{rep}(\Delta_n,h)$.

(iii) There are $j,k < n$ such that Δ_j is $\Delta_k \rightarrow \Delta_n$. By the inductive hypothesis, $\mathrm{REP}(\Gamma',h) \vdash \mathrm{rep}(\Delta_k,h)$ and $\mathrm{REP}(\Gamma',h) \vdash \mathrm{rep}(\Delta_k \rightarrow \Delta_n,h)$.

But then $\mathrm{REP}(\Gamma',h) \vdash \mathrm{rep}(\Delta_k,h) \rightarrow \mathrm{rep}(\Delta_n,h)$ and, by T34e, $\mathrm{REP}(\Gamma',h) \vdash \mathrm{rep}(\Delta_n,h)$.

This completes the induction; for any $n < \mathrm{lh}(\Delta)$, $\mathrm{REP}(\Gamma',h) \vdash \mathrm{rep}(\Delta_n,h)$. In particular, $\mathrm{REP}(\Gamma',h) \vdash \mathrm{rep}(\phi,h)$. Since $g \subseteq h$, $\mathrm{REP}(\Gamma',g) \vdash \mathrm{rep}(\phi,g)$. Since g agrees with f for the variables in Θ, $\mathrm{REP}(\Gamma',f) \vdash \mathrm{rep}(\phi,f)$. By T34b, $\mathrm{REP}(\Gamma,f) \vdash \mathrm{rep}(\phi,f)$.

Conversely, suppose that $\mathrm{REP}(\Gamma,f) \vdash \mathrm{rep}(\phi,f)$.

By an argument similar to the one above, $REP(REP(\Gamma,f),\breve{f}) \vdash rep(rep(\phi,f),\breve{f})$, i.e. $\Gamma \vdash \phi$.

(b) First, suppose $\Gamma \vdash_{S} \phi$ and that τ is a generalizer. Let τ' be that generalizer which is the result of deleting from τ all occurrences of $\langle \underline{u} \rangle^{\frown}\alpha$, where α does not occur in $rep(\phi,f)$. By a simple induction using GA5, T54 and T43, $\vdash_{S} \tau rep(\phi,f) \leftrightarrow \tau' rep(\phi,f)$. By hypothesis $\Gamma \vdash rep(\tau' rep(\phi,f),\breve{f})$. By (a), $REP(\Gamma,f) \vdash rep(rep(\tau' rep(\phi,f),\breve{f}),f)$. That is, $REP(\Gamma,f) \vdash \tau' rep(\phi,f)$. By T39a, $REP(\Gamma,f) \vdash \tau rep(\phi,f)$.

For the other conditional, suppose that $REP(\Gamma,f) \vdash_{S} rep(\phi,f)$ and that τ is a generalizer. Let τ' be obtained as before, this time by omitting the vacuous quantifiers over ϕ. As before, $\vdash_{S} \tau\phi \leftrightarrow \tau'\phi$. By hypothesis, $REP(\Gamma,f) \vdash rep(\tau'\phi,f)$. By (a), $\Gamma \vdash \tau'\phi$. By T39a, $\Gamma \vdash \tau\phi$.

10. Consistency

D56. (a) Γ is <u>consistent</u> if and only if there is a formula ϕ such that not $\Gamma \vdash \phi$.

(b) Γ is <u>inconsistent</u> if and only if Γ is not consistent.

T60. All of the following conditions are equivalent:

(a) Γ is inconsistent.

(b) There is a finite subset Γ' of Γ such that Γ' is inconsistent.

(c) There is a formula $\phi \in \Gamma$ such that $\Gamma \vdash \neg\phi$.

(d) For some formula ϕ, $\Gamma \vdash \phi \wedge \neg\phi$.

Proof: The proof is trivial, and analogous to the case for the ordinary predicate calculus.

T61. If Γ is a finite set of formulas, then Γ is inconsistent if and only if $\vdash \neg CJ(\Gamma)$.

Proof: Suppose Γ is a finite set of formulas. Suppose first that Γ is inconsistent; then (T60c) there is a formula $\phi \in \Gamma$ such that $\Gamma \vdash \neg\phi$. By T41, $\vdash CJ(\Gamma) \rightarrow \neg\phi$. By T38, $\vdash CJ(\Gamma) \rightarrow \phi$. Hence, $\vdash \neg CJ(\Gamma)$.

Conversely, suppose that $\vdash \neg CJ(\Gamma)$. Γ cannot be 0, since then we have (by T38) $\vdash CJ(\Gamma)$; this is impossible (by T33) since $CJ(\Gamma)$ and $\neg CJ(\Gamma)$ cannot both be logically valid. Hence, there is a formula $\phi \in \Gamma$. By T39a and the hypothesis, $\vdash CJ(\Gamma) \rightarrow \neg\phi$. By T41, $\Gamma \vdash \neg\phi$, and Γ is inconsistent.

T62. If $\Gamma \cup \{\bigvee\alpha\phi\}$ is consistent, where α does not occur in any formula of Γ, then $\Gamma \cup \{\phi\}$ is consistent.

Proof: Assume the hypothesis. Then (T60c) not $\Gamma \cup \{\bigvee\alpha\phi\} \vdash \neg\bigvee\alpha\phi$. By T34h, not $\Gamma \vdash \bigvee\alpha\phi \rightarrow \neg\bigvee\alpha\phi$, and hence not $\Gamma \vdash \neg\bigvee\alpha\phi$. Then not $\Gamma \vdash \bigwedge\alpha\neg\phi$. By T34d, not $\Gamma \vdash \neg\phi$.

Suppose, for reductio, that $\Gamma \cup \{\phi\}$ is not consistent. Then (T60c) there is a formula $\psi \in \Gamma \cup \{\phi\}$ such that $\Gamma \cup \{\phi\} \vdash \neg\psi$. There are two cases.

(i) ψ is ϕ; then by T34f, $\Gamma \vdash \phi \rightarrow \neg\phi$ and $\Gamma \vdash \neg\phi$, contradicting the above paragraph.

(ii) ψ is not ϕ; then, by T34f, $\Gamma \vdash \phi \rightarrow \neg\psi$. Hence $\Gamma \vdash \psi \rightarrow \neg\phi$. By T34h, $\Gamma \cup \{\psi\} \vdash \neg\phi$; since $\psi \in \Gamma$, $\Gamma \vdash \neg\phi$, contradicting the above.

T63. If $\Gamma \cup \{\phi\}$ is inconsistent and $\vdash \psi \rightarrow \phi$, then $\Gamma \cup \{\psi\}$ is inconsistent.

Proof: Assume the hypothesis. Then (T60c) there is a formula $\chi \in \Gamma \cup \{\phi\}$ such that $\Gamma \cup \{\phi\} \vdash \neg\chi$. There are two cases.

(i) χ is ϕ; then by T34f, $\Gamma \vdash \phi \rightarrow \neg\phi$, and $\Gamma \vdash \neg\phi$. By T34b, $\Gamma \vdash \neg\psi$. By T34b, $\Gamma \cup \{\psi\} \vdash \neg\psi$, and $\Gamma \cup \{\psi\}$ is inconsistent (by T60c).

(ii) χ is not ϕ; then, by T34f, $\Gamma \vdash \chi \rightarrow \neg\phi$. By T34b, $\Gamma \vdash \chi \rightarrow \neg\psi$, and $\Gamma \vdash \psi \rightarrow \neg\chi$. By T34h, $\Gamma \cup \{\psi\} \vdash \neg\chi$. Since $\chi \in \Gamma$, $\Gamma \cup \{\psi\}$ is inconsistent.

T64. If Γ is a set of formulas, f is a one-to-one function, Fld(f) $\subseteq$ Iv, and every variable that occurs in some formula of Γ is in Dom(f), then Γ is consistent if and only if REP(Γ,f) is consistent.

Proof: Assume the hypothesis.

First, suppose that REP(Γ,f) is not consistent. Then (T60c) there is a formula $\psi \in$ REP(Γ,f) such that REP(Γ,f) $\vdash \neg\psi$. There is a formula $\phi \in \Gamma$ such that ψ is rep(ϕ,f); then $\neg\psi$ is rep($\neg\phi$,f). Hence, by T59, $\Gamma \vdash \neg\phi$, and Γ is inconsistent.

On the other hand, suppose Γ is inconsistent. Then there is a $\phi \in \Gamma$ such that $\Gamma \vdash \neg\phi$. By T59, REP(Γ,f) $\vdash$ $\neg$rep(ϕ,f), and therefore REP(Γ,f) is inconsistent.

11. Theorems about Arrangements

We now begin a series of theorems that will be used in the final completeness proof in Chapter IV.

D57. <u>A is an arrangement consistent with Γ</u> if and only if

(1) A is an arrangement

(2) For each formula ϕ and each $k \in \omega$ such that lev(ϕ,k,A,$\langle 0,0\rangle$), $\Gamma \cup \{\phi\}$ is consistent.

D58. <u>A is a consistent arrangement</u> if and only if A is an arrangement consistent with 0.

T65. If A is a complete arrangement and A is an arrangement consistent with Γ and the language of Γ is included in the language of A, then $\Gamma \subseteq A_2(\langle 0,0\rangle)$.

Proof: Assume the hypothesis, and assume that $\phi \in \Gamma$. Since the language of Γ is included in the language of A, ϕ is a formula of the language of A. Since A is an extension of A with respect to the set of formulas of the language of A, either $\phi \in A_2(\langle 0,0\rangle)$ or $\neg\phi \in A_2(\langle 0,0\rangle)$. Suppose, for reductio, that $\neg\phi \in A_2(\langle 0,0\rangle)$. Since A is consistent with Γ, and lev($\neg\phi$,0,A,$\langle 0,0\rangle$), $\Gamma \cup \{\phi\}$ is consistent. But this (by T34c) is impossible.

T66. If A is an arrangement, m,n $\in A_0$, Γ is finite and for each $\phi \in \Gamma$ there is a $k \leq p$ such that lev(ϕ,k,A,$\langle$m,n$\rangle$), then there are a generalizer τ and a formula ψ such that lev(ψ,p+4,A,$\langle$0,0$\rangle$) and $\vdash_S \psi \rightarrow \neg\tau\neg$CJ($\Gamma$).

Assume the hypothesis. Let h be that function whose domain is Γ, and such that, for each $\phi \in \Gamma$, h(ϕ) is the least k such that lev(ϕ,k,A,$\langle$m,n$\rangle$). Let p be the largest number in Rng(h). Then, by T25c, for each $\phi \in \Gamma$, lev(ϕ,p,A,$\langle$m,n$\rangle$). By D47, lev(CJ(Γ),p+1,A,$\langle$m,n$\rangle$).

We now take nine cases, according to whether 0 $<_A$ n, 0 is n, or n $<_A$ 0; and also according to whether m $<_A$ n, m is n, or n $<_A$ m.

(1) 0 $<_A$ n and m $<_A$ n. Then lev (PCJ(Γ),p+2,A,$\langle$n,n$\rangle$); lev(KPCJ(Γ),p+3,A,$\langle$n,0$\rangle$); lev(FKPCJ(Γ),p+4,A,$\langle$0,0$\rangle$). Since (by GA16, T38, and T43) $\vdash_S$ FKPCJ(Γ)$\rightarrow \neg$GKH$\neg$CJ(Γ), this completes case (1).

(2) 0 $<_A$ n and m is n. Then lev(KCJ(Γ),p+2,A,$\langle$n,0$\rangle$); lev(FKCJ(Γ),p+3,A,$\langle$n,0$\rangle$). Since (by GA16 and T43) $\vdash_S$ FKCJ(Γ)$\rightarrow \neg$GK$\neg$CJ(Γ), this completes (2).

(3) 0 $<_A$ n and n $<_A$ m. Then lev(FCJ(Γ),p+2,A,$\langle$n,m$\rangle$); lev(KFCJ(Γ),p+3,A,$\langle$n,0$\rangle$); lev(FKFCJ(Γ),p+4,A,$\langle$0,0$\rangle$). By GA16, T38, and T43, $\vdash_S$ FKFCJ(Γ)$\rightarrow \neg$GKG$\neg$CJ(Γ).

(4) 0 is n and m $<_A$ n. Then lev(PCJ(Γ),p+2,A,$\langle$n,n$\rangle$). By T38, $\vdash_S$ PCJ(Γ)$\rightarrow \neg$H$\neg$CJ(Γ).

(5) 0 is n and m is n; then lev(CJ(Γ), p+1,A,$\langle$0,0$\rangle$). By T38, $\vdash_S$ CJ(Γ)$\rightarrow \neg\neg$CJ(Γ).

(6) 0 is n and n $<_A$ m; then lev(FCJ(Γ),p+2,A,$\langle$n,m$\rangle$). By

T38, $\vdash_S$ FCJ(Γ) $\rightarrow$ ¬G¬CJ(Γ).

(7) n $<_A$ 0 and m $<_A$ n. Then lev(PCJ(Γ),p+2,A,⟨n,n⟩); lev(KPCJ(Γ),p+3,A,⟨n,0⟩); lev(PKPCJ(Γ),p+3,A,⟨0,0⟩). By GA16, T38, and T43, $\vdash_S$ PKPCJ(Γ) $\rightarrow$ ¬HKH¬CJ(Γ).

(8) n $<_A$ 0 and m is n. Then lev(KCJ(Γ),p+2,A,⟨n,0⟩); lev(PKCJ(Γ),p+3,A,⟨0,0⟩). By GA16 and T43, $\vdash_S$ PKCJ(Γ) $\rightarrow$ ¬HK¬CJ(Γ).

(9) n $<_A$ 0 and n $<_A$ m. Then lev(FCJ(Γ),p+2,A,⟨n,n⟩); lev(KFCJ(Γ),p+3,A,⟨n,0⟩); lev(PKFCJ(Γ),p+4,A,⟨0,0⟩). By GA9, T38, and T43, $\vdash_S$ PKCJ(Γ) $\rightarrow$ ¬HKG¬CJ(Γ).

This completes the proof. Note that there are just nine possibilities for the generalizer τ -- GKH0, GK0, GKG0, H0, 0, G0, HKH0, HK0, and HKG0.

T67. If A is a consistent, complete arrangement, m,n $\in A_0$ and lev(ϕ,k,A,⟨m,n⟩), then $\phi \in A_2$(⟨m,n⟩).

Proof: Assume the hypothesis, and assume that $\phi \notin A_2$(⟨m,n⟩). Since A is complete, $\neg\phi \in A_2$(⟨m,n⟩). By T25c, lev(¬ϕ,k,A,⟨m,n⟩). By T66, there are a generalizer τ, a formula ψ and a j $\in \omega$ such that lev(ψ,j,A,⟨0,0⟩) and $\vdash_S \psi \rightarrow$ ¬τ¬CJ({ϕ,¬ϕ}). By T38, $\vdash_S$ τ¬CJ({ϕ,¬ϕ}). Hence, $\vdash_S \neg\psi$, and {ϕ} is inconsistent, contradicting the hypothesis.

T68. If A is a consistent, complete arrangement, m,n $\in A_0$ and ϕ is a formula of the language of A, then

(a) If $\vdash_S \phi$, $\phi \in A_2$(m,n)

(b) $\phi \in A_2$(⟨m,n⟩) if and only if $\neg\phi \notin A_2$(⟨m,n⟩).

Proof: Assume the hypothesis.

(a) Suppose $\vdash_S \phi$ and $\phi \notin A_2(\langle m,n\rangle)$. Since A is complete, $\neg\phi \in A_2(\langle m,n\rangle)$. By T66, there is a generalizer τ, a formula ψ and a $j \in \omega$ such that $lev(\psi,j,A,\langle 0,0\rangle)$ and $\vdash_S \psi \rightarrow \neg\tau\neg\neg\phi$. By T39b, $\vdash_S \tau\neg\neg\phi$. By T39b again, $\vdash_S \neg\psi$. Hence $\{\psi\}$ is inconsistent, contradicting the hypothesis.

(b) Since A is complete, either $\phi \in A_2(\langle m,n\rangle)$ or $\neg\phi \in A_2(\langle m,n\rangle)$. We need only show that not both $\phi \in A_2(\langle m,n\rangle)$ and $\neg\phi \in A_2(\langle m,n\rangle)$. Suppose, then, that $\phi \in A_2(\langle m,n\rangle)$ and $\neg\phi \in A_2(\langle m,n\rangle)$. By T66, there is a generalizer τ, a formula ψ and a $j \in \omega$ such that $lev(\psi,j,A,\langle 0,0\rangle)$ and $\vdash_S \psi \rightarrow \neg\tau\neg CJ(\{\phi,\neg\phi\})$. By T38, $\vdash_S \tau\neg CJ(\{\phi,\neg\phi\})$. By T39b, $\vdash_S \neg\psi$. Hence $\{\psi\}$ is inconsistent, contradicting the hypothesis.

T69. If A is a consistent, complete arrangement and $m,n,p \in A_0$, then

(a) If $\phi \in A_2(\langle m,n\rangle)$ and $\phi \rightarrow \psi \in A_2(\langle m,n\rangle)$, then $\psi \in A_2(\ m,n\ )$

(b) If ϕ does not contain R outside the scope of K and $\phi \in A_2(\langle m,n\rangle)$, then $\phi \in A_2(\langle m,p\rangle)$

(c) If $0 < k$ and $\phi_0,\ldots,\phi_{k-1} \in A_2(\langle m,n\rangle)$, ψ is a formula of the language of A and $\vdash_S \phi_0 \wedge \ldots \wedge \phi_{k-1} \rightarrow \psi$, then $\psi \in A_2(\langle m,n\rangle)$.

Proof: Suppose that A is a consistent, complete arrangement and $m,n,p \in A_0$.

(a) Suppose $\phi \in A_2(\langle m,n\rangle)$ and $\phi \rightarrow \psi \in A_2(\langle m,n\rangle)$ but $\psi \notin A_2(\langle m,n\rangle)$. By T68b, $\neg\psi \in A_2(\langle m,n\rangle)$. Let Γ be $\{\phi, \phi \rightarrow \psi, \neg\psi\}$. By T66 there is a generalizer τ, a formula ψ and a $j \in \omega$ such that $lev(\psi, j, A, \langle 0,0\rangle)$ and $\vdash_S \psi \rightarrow \neg\tau\neg CJ(\Gamma)$. By T38, $\vdash_S \tau\neg CJ(\Gamma)$. By T39b, $\vdash_S \neg\psi$. Hence $\{\psi\}$ is inconsistent and A is not a consistent arrangement, contradicting the hypothesis.

(b) Suppose ϕ does not contain R outside the scope of K, $\phi \in A_2(\langle m,n\rangle)$ and $\phi \notin A_2(\langle m,p\rangle)$. By T68b, $\neg\phi \in A_2(\langle m,p\rangle)$. By GA16, GA17, T68a, and T69a, $K\phi \in A_2(\langle m,n\rangle)$ and $\neg K\phi \in A_2(\langle m,p\rangle)$.

We will show that $\phi \notin A_2(\langle m,m\rangle)$. Suppose, for reductio, that $\phi \in A_2(\langle m,m\rangle)$. By D47, $lev(K\phi, 1, A, \langle m,p\rangle)$. By T67, $K\phi \in A_2(\langle m,p\rangle)$. But this is a contradiction, by T68b.

Since A is complete, $\neg\phi \in A_2(\langle m,m\rangle)$. By D47, $lev(K\neg\phi, 1, A, \langle m,n\rangle)$. By T67, $K\neg\phi \in A_2(\langle m,n\rangle)$. By GA16, T68a, and T69a, $\neg K\phi \in A_2(\langle m,n\rangle)$. But this is again a contradiction, by T68b.

(c) Suppose $0 < k$ and $\phi_0, \ldots, \phi_{k-1} \in A_2(\langle m,n\rangle)$, ψ is a formula of the language of A and $\vdash_S \phi \wedge \ldots \wedge \phi_{k-1} \rightarrow \psi$. By D47, $lev(CJ(\{\phi_0, \ldots, \phi_{k-1}\}), 0, A, \langle m,n\rangle)$. By T67, $CJ(\{\phi_0, \ldots, \phi_{k-1}\}) \in A_2(\langle m,n\rangle)$. By T38 and T68a, $CJ(\{\phi_0, \ldots, \phi_{k-1}\}) \rightarrow (\phi_0 \wedge \ldots \wedge \phi_{k-1}) \in A_2(\langle m,n\rangle)$. By T69a, $\phi_0 \wedge \ldots \wedge \phi_{k-1} \in A_2(\langle m,n\rangle)$. By T68a and T69a,

$\psi \in A_2(\langle m,n\rangle)$.

T70. If A is a consistent, complete arrangement and $m \in A_0$, then (a) If ϕ is a formula of the language of A and $\vdash_S K\phi$, then $\phi \in A_2(\langle m,m\rangle)$

(b) $\phi \in A_2(\langle m,m\rangle)$ if and only if $K\phi \in A_2(\langle m,m\rangle)$

(c) $\phi \in A_2(\langle m,m\rangle)$ if and only if $R\phi \in A_2(\langle m,m\rangle)$

Proof: Suppose that A is a consistent, complete arrangement and $m \in A_0$.

(a) Suppose that ϕ is a formula of the language of A and $\vdash_S K\phi$. By T68a, $K\phi \in A_2(\langle m,m\rangle)$. Suppose $\phi \notin A_2(\langle m,m\rangle)$; then $\neg\phi \in A_2(\langle m,m\rangle)$. By D47, $lev(K\neg\phi,1,A,\langle m,m\rangle)$. By T67, $K\neg\phi \in A_2(\langle m,m\rangle)$. By GA16, T68a, and T69a, $\neg K\phi \in A_2(\langle m,m\rangle)$. By T68b, $\neg K\phi \notin A_2(\langle m,m\rangle)$.

(b) First, we will show $\vdash_S K(\phi \leftrightarrow K\phi)$.

(i)	$\vdash_S K(\phi \rightarrow K\phi) \rightarrow (K(K\phi \rightarrow \phi) \rightarrow K(\phi \leftrightarrow K\phi))$	T38,GA17
(ii)	$\vdash_S K\phi \rightarrow K(K\phi \rightarrow \phi)$	T38,GA17
(iii)	$\vdash_S \neg K\phi \rightarrow K\neg\phi$	GA16
(iv)	$\vdash_S K\neg\phi \rightarrow K(\phi \rightarrow K\phi)$	T38,GA17
(v)	$\vdash_S K\phi \rightarrow KK\phi$	GA15
(vi)	$\vdash_S K\neg K\phi \rightarrow K(K\phi \rightarrow \phi)$	T38,GA17
(vii)	$\vdash_S KK\phi \rightarrow K(\phi \rightarrow K\phi)$	
(viii)	$\vdash_S \neg K\phi \rightarrow K\neg K\phi$	GA15
(ix)	$\vdash_S K(\phi \rightarrow K\phi)$	(iii),(iv),(v),(vii)
(x)	$\vdash_S K(K\phi \rightarrow \phi)$	(vi),(viii),(iii),(ii)

(xi) $\vdash_S K(\phi \leftrightarrow K\phi)$ (ix), (x), (i)

By part (a), $\phi \leftrightarrow K\phi \in A_2(\langle m,m\rangle)$. By T69c, $\phi \rightarrow K\phi \in A_2(\langle m,m\rangle)$ and $K\phi \rightarrow \phi \in A_2(\langle m,m\rangle)$. By T69a, $\phi \in A_2(\langle m,m\rangle)$ if and only if $K\phi \in A_2(\langle m,m\rangle)$.

(c) This is similar to (b) except that GA19 is used in place of the proof that $\vdash_S K(\phi \leftrightarrow K\phi)$.

T71. If A is a consistent, complete arrangement and $m,n,p \in A_0$, then

(a) If $L\phi \in A_2(\langle m,p\rangle)$, then $\phi \in A_2(\langle n,p\rangle)$

(b) If $\alpha=\beta \in A_2(\langle m,p\rangle)$, then $\alpha=\beta \in A_2(\langle n,p\rangle)$

(c) $R\phi \in A_2(\langle m,p\rangle)$ if and only if $\phi \in A_2(\langle p,p\rangle)$.

Proof: Assume the hypothesis.

(a) Suppose that $L\phi \in A_2(\langle m,p\rangle)$. There are three cases:

(1) $n <_A m$; then by T69c, $H\phi \in A_2(\langle m,p\rangle)$. Suppose that $\phi \notin A_2(\langle n,p\rangle)$. Since A is complete, $\neg\phi \in A_2(\langle n,p\rangle)$. By D47, $lev(P\neg\phi, 1, A, \langle m,p\rangle)$. By T67, $P\neg\phi \in A_2(\langle m,p\rangle)$. By T38, T43, T68a, and T69a, $\neg H\phi \in A_2(\langle m,p\rangle)$. But this is impossible, by T68b.

(2) m is n; then $\phi \in A_2(\langle m,p\rangle)$, by T69c.

(3) $m <_A n$; this case is similar to case (1).

(b) Suppose that $\alpha=\beta \in A_2(\langle m,p\rangle)$. Then by GA8 and T69c, $L\alpha=\beta \in A_2(\langle m,p\rangle)$. But then by (a), $\alpha=\beta \in A_2(\langle n,p\rangle)$.

(c) First, suppose that $R\phi \in A_2(\langle m,p\rangle)$. By GA22 and T69c, $LR\phi \in A_2(\langle m,p\rangle)$. By T71a, $R\phi \in A_2(\langle p,p\rangle)$. By T70c,

$\phi \in A_2(\langle p,p\rangle)$.

Second, suppose that $\phi \in A_2(\langle p,p\rangle)$. By T70c, $R\phi \in A_2(\langle p,p\rangle)$. By GA22 and T71c, $LR\phi \in A_2(\langle p,p\rangle)$. By T71a, $k\phi \in A_2(\langle m,p\rangle)$.

T72. If $k < n$, A is a finite arrangement, and $i \in A_0 \times A_0$, then $\vdash_S$ LEV(n,A,i) $\rightarrow$ LEV(k,A,i).

Proof: Assume that A is a finite arrangement. Let K be the set of $k \in \omega$ such that, for each $m,n \in A_0 \times A_0$, $\vdash_S$ LEV(k+1,A,$\langle m,n\rangle$) LEV(k,A,$\langle m,n\rangle$). By T39b, it is sufficient to show that K is ω.

First, suppose that $\langle m,n\rangle \in A_0 \times A_0$. There are Γ,Γ' such that LEV(0,A,$\langle m,n\rangle$) is CJ(Γ) and LEV(1,A,$\langle m,n\rangle$) is CJ(Γ'). Since $\Gamma \subseteq \Gamma'$ (by T40a), $\vdash_S$ LEV(1,A,$\langle m,n\rangle$) $\rightarrow$ LEV(0,A,$\langle m,n\rangle$).

Now, suppose (for the inductive step) that for each $\langle m,n\rangle \in A_0 \times A_0$, $\vdash_S$ LEV(k+1,A,$\langle m,n\rangle$) $\rightarrow$ LEV(k,A,$\langle m,n\rangle$). Also suppose that $m,n \in A_0$. There are Γ,Γ' such that LEV(k+2,A,$\langle m,n\rangle$) is CJ(Γ') and LEV(k+1,A,$\langle m,n\rangle$) is CJ(Γ). By T40a and T39b, it is sufficient to show that, for each $\psi \in \Gamma$, there is a $\psi' \in \Gamma'$ such that $\vdash_S \psi' \rightarrow \psi$. Suppose that $\psi \in \Gamma$; there are five cases:

(1) $\psi \in A_2(\langle m,n\rangle)$; then $\psi \in \Gamma'$.

(2) There is a p such that $p <_A m$ and ψ is PLEV(k,A,$\langle p,n\rangle$). By the inductive hypothesis, $\vdash_S$ LEV(k+1,A,$\langle p,n\rangle$) $\rightarrow$ LEV(k,A,$\langle p,n\rangle$). By T35d, T48c, and T37,

$\vdash_S$ PLEV(k+1,A,⟨p,n⟩) → ψ. Since PLEV(k+1,A,⟨p,n⟩) ∈ Γ', there is a ψ' ∈ Γ' such that $\vdash_S$ ψ' → ψ.

(3) There is a p such that m $<_A$ p and ψ is FLEV(k,A,⟨p,n⟩). This case is similar to (2) except that T49d is used in place of T48c.

(4) ψ is KLEV(k,A,⟨m,m⟩). By the inductive hypothesis, $\vdash_S$ LEV(k+1,A,⟨m,m⟩) → LEV(k,A,⟨m,m⟩). By T35d, GA17, and T37, $\vdash_S$ KLEV(k+1,A,⟨m,m⟩) → ψ. Since KLEV(k+1,A,⟨m,m⟩) ∈ Γ', there is a ψ' ∈ Γ' such that $\vdash_S$ ψ' → ψ.

(5) ψ is LEV(k,A,⟨m,n⟩). By the inductive hypothesis, $\vdash_S$ LEV(k+1,A,⟨m,n⟩) → LEV(k,A,⟨m,n⟩). Since LEV(k+1,A,⟨m,n⟩) ∈ Γ', there is a ψ' ∈ Γ' such that $\vdash_S$ ψ' → ψ. This completes the proof.

T73. If k < n and A is a finite arrangement, then $\vdash_S$ CH(A,n) → CH(A,k).

Proof: A special case of T72.

T74. If B is a finite arrangement and A is part of B and i ∈ $A_0 \times A_0$, then $\vdash_S$ LEV(k,B,i) → LEV(k,A,i).

Proof: Suppose that A is part of B. Let N be the set of k ∈ ω such that, for each ⟨m,n⟩ ∈ $A_0 \times A_0$, $\vdash_S$ LEV(k,B,⟨m,n⟩) → LEV(k,A,⟨m,n⟩).

First, suppose that ⟨m,n⟩ ∈ $A_0 \times A_0$; then $\vdash_S$ LEV(0,B,⟨m,n⟩) → LEV(0,A,⟨m,n⟩), by T40a.

For the inductive step, suppose that for all ⟨m,n⟩ ∈

$A_0 \times A_0$, $\vdash_S$ LEV(k,B,⟨m,n⟩) → LEV(k,A,⟨m,n⟩).

Suppose, in addition, that m,n ∈ A_0. There are finite sets of formulas Γ,Γ' such that LEV(k+1,A,⟨m,n⟩) is CJ(Γ) and LEV(k+1,B,⟨m,n⟩) is CJ(Γ'). By T40a and T39b, it is sufficient to show that, for each ψ ∈ Γ, there is a ψ' ∈ Γ' such that $\vdash_S$ ψ' → ψ. Suppose that ψ ∈ Γ. There are five cases:

(1) ψ ∈ A_2(i); then ψ ∈ B_2(i) and ψ ∈ Γ'.

(2) There is a p such that p $<_A$ m and ψ is PLEV(k,A,⟨p,n⟩). By the inductive hypothesis, $\vdash_S$ LEV(k,B,⟨p,n⟩) → LEV(k,A,⟨p,n⟩). By T35d, T48c, and T37, $\vdash_S$ PLEV(k,B,⟨p,n⟩) → PLEV(k,A,⟨p,n⟩). Since PLEV(k,B,⟨p,n⟩) ∈ Γ', there is a ψ' ∈ Γ such that $\vdash_S$ ψ' → ψ.

(3) There is a p such that m $<_A$ p and ψ is FLEV(k,A,⟨p,n⟩). This case is analogous to (2) except that T48d is used instead of T48c.

(4) ψ is KLEV(k,A,⟨m,m⟩). By the inductive hypothesis, $\vdash_S$ LEV(k,B,⟨m,m⟩) → LEV(k,A,⟨m,m⟩). By T35d, GA17, and T37, $\vdash_S$ KLEV(k,B,⟨m,m⟩) → ψ. Since KLEV(k,B,⟨m,m⟩) ∈ Γ', there is a ψ' ∈ Γ' such that $\vdash_S$ ψ' → ψ.

(5) ψ is LEV(k,A,⟨m,n⟩). By the inductive hypothesis, $\vdash_S$ LEV(k,B,⟨m,n⟩) → LEV(k,A,⟨m,n⟩). Since LEV(k,B,⟨m,n⟩) ∈ Γ', there is a ψ' ∈ Γ' such that $\vdash_S$ ψ' → ψ.

T75. If B is a finite arrangement and A is part of B, then

$\vdash_S$ CH(B,n) → CH(A,n).

Proof: An immediate consequence of T74.

T76. If A is a finite arrangement, $i \in A_0 \times A_0$, and lev(ϕ,n,A,i), then $\vdash_S$ LEV(n,A,i) → ϕ.

Proof: Suppose that A is a finite arrangement. Let N be the set of $n \in \omega$ such that, for each $i \in A_0 \times A_0$ and each formula ϕ, if lev(ϕ,n,A,i) then $\vdash_S$ LEV(n,A,i) → ϕ. It is sufficient to prove by induction that N is ω. First suppose $i \in A_0 \times A_0$, and lev(ϕ,0,A,i): then $\vdash_S$ LEV(0,A,i) → ϕ, by T40a.

Next suppose that, for each $i \in A_0 \times A_0$ and each formula ϕ, if lev(ϕ,n,A,i), then $\vdash_S$ LEV(n,A,i) → ϕ. Suppose also that $i \in A_0 \times A_0$ and lev(ϕ,n+1,A,i). There are $j,k \in A_0$ such that i is ⟨j,k⟩. There are sets Γ and Γ' such that LEV(n,A,i) is CJ(Γ) and ϕ is CJ(Γ'). By T39b, it is sufficient to show that, for each formula ψ' ∈ Γ', there is a formula ψ ∈ Γ such that $\vdash_S$ ψ → ψ'. Suppose, therefore, that ψ' ∈ Γ'. Then (D47) there are five cases:

(1) $\psi' \in A_2(i)$; then ψ' ∈ Γ.

(2) There are a formula χ and $p \in A_0$ such that $p <_A j$, lev(χ,n,A,⟨p,k⟩) and ψ' is Pχ. By the inductive hypothesis, $\vdash_S$ LEV(n,A,⟨p,k⟩) → χ. By T35d, T48c, and T37, $\vdash_S$ PLEV(n,A,⟨p,k⟩) → Pχ. Since PLEV(n,A,⟨p,k⟩) ∈ Γ, there is a ψ ∈ Γ such that $\vdash_S$ ψ → ψ'.

(3) There are a formula χ and $p \in A_0$ such that $j <_A p$, lev(χ,n,A,⟨p,k⟩) and ψ' is Fχ. This case is analogous

to (2).

(4) There is a formula χ such that lev(χ,n,A,$\langle$j,j$\rangle$) and is Kχ. By the inductive hypothesis, $\vdash_S$ LEV(n,A,$\langle$j,j$\rangle$)$\rightarrow$ χ. By T35d, GA17, and T37, $\vdash_S$ KLEV(n,A,$\langle$j,j$\rangle$)$\rightarrow$ Kχ. Since KLEV(n,A,$\langle$j,j$\rangle$) $\in$ Γ, there is a $\psi \in \Gamma$ such that $\vdash_S$ $\psi \in \psi'$.

(5) lev(ψ,n,A,$\langle$j,k$\rangle$). By the inductive hypothesis, $\vdash_S$ LEV(n,A,$\langle$j,k$\rangle$) $\rightarrow \psi$. Since LEV(n,A,$\langle$j,k$\rangle$) $\in$ Γ, there is a $\psi \in \Gamma$ such that $\vdash_S$ $\psi \rightarrow \psi'$.

T77. If A is a finite arrangement and lev(ϕ,n,A,$\langle$0,0$\rangle$), then $\vdash_S$ CH(A,n) $\rightarrow \phi$.

Proof: Follows immediately from T76.

T78. If A is an arrangement, {CH(A,k+4)} is consistent, i $\in A_0 \times A_0$, and $\vdash_S \neg\phi$, then not lev(ϕ,k,A,i).

Proof: Suppose A is an arrangement, {CH(A,k+4)} is consistent, i $\in A_0 \times A_0$, $\vdash_S \neg\phi$, and lev(ϕ,k,A,i).

By D47 and T25c, there is a finite set of formulas Γ such that, for each formula $\psi \in \Gamma$, lev(ψ,k,A,i); and such that ϕ is CJ(Γ).

By T66, there is a generalizer τ and a formula χ such that lev(χ,k+4,A,$\langle$0,0$\rangle$) and $\vdash_S$ $\chi \rightarrow \neg\tau\neg\phi$. By T35d, $\vdash_S$ $\tau\neg\phi$. By T39b, $\vdash_S \neg\chi$. By T77, $\vdash_S$ CH(A,k+4) $\rightarrow \chi$. By T39b, $\vdash_S$ $\neg$CH(A,k+4), contradicting the hypothesis.

12. More Theorems about G, H, and L

T79. If τ is a generalizer, ξ is the universal part of τ and τ' is the tense part of τ, then $\vdash_S \xi\tau'\phi \rightarrow \tau\phi$.

Proof: Follows by a trivial induction, using T44, T55a, T55b, and GA18.

T80. If $\vdash \phi$ and ψ is a universal generalization of ϕ, then $\vdash \psi$.

Proof: An immediate consequence of T34d

The following theorem establishes that every axiom of Cocchiarella's system is a strong theorem. It is a simple consequence of this that every theorem of Cocchiarella's system is a strong theorem of our system.

T81. (a) $\vdash_S P\phi \wedge P\psi \rightarrow P(\phi \wedge \psi) \vee P(P\phi \wedge \psi) \vee P(\phi \wedge P\psi)$

(b) $\vdash_S F\phi \wedge F\psi \rightarrow F(\phi \wedge \psi) \vee F(F\phi \wedge \psi) \vee F(\phi \wedge F\psi)$

Proof: We will prove part (a) only, since part (b) is analogous. Let χ be $P\phi \wedge P\psi \rightarrow P(\phi \wedge \psi) \vee P(P\phi \wedge \psi) \vee P(\phi \wedge P\psi)$. Assume $\vdash_S \chi$. Then there is a generalizer τ such that not $\vdash \tau\chi$. Let ξ be the universal part of τ and let τ' be the tense part of τ. By T79, not $\vdash \xi\tau'\chi$. By T80, not $\vdash \tau'\chi$.

Let n be that $n \in \omega$ such that τ' is an n-level generalizer. Let Γ be the set of formulas that occur in $\tau'\chi$, and let Δ be the set of variables that do not occur in $\tau'\chi$. Let m be $\overline{\Gamma}$. Let θ be $CH^*(\Gamma,\Delta,m+n+1,5)$. Then $\vdash_S K\theta$, since

$K\theta$ is a general axiom.

θ is the disjunction (in order) of the formulas $\vee\alpha_0 \ldots \vee\alpha_{k-1}CH(\Sigma_{m+n+1},5)$, where Σ is an m+n+2-place minimal extension sequence with respect to Γ and Δ and $\alpha_0, \ldots \alpha_{k-1}$ are (in order) the variables in Δ that occur free in $CH(\Sigma_{m+n+1},5)$.

By T34e and T35a, since not $\vdash \tau'\chi$, not $\vdash \theta \rightarrow \tau'\chi$. By T39a, there is an m+n+2-place minimal extension sequence Σ with respect to Γ and Δ and variables $\alpha_0, \ldots, \alpha_{k-1}$ such that $\alpha_0, \ldots, \alpha_{k-1}$ are (in order) the variables in Δ that occur free in $CH(\Sigma_{m+n+1},5)$, and not $\vdash \vee\alpha_0 \ldots \vee\alpha_{k-1}CH(\Sigma_{m+n+1},5) \rightarrow \tau'\chi$. By k applications of T51b, not $\vdash \wedge\alpha_0 \ldots \wedge\alpha_{k-1}(CH(\Sigma_{m+n+1},5) \rightarrow \tau'\chi)$. By T80, not $\vdash CH(\Sigma_{m+n+1},5) \rightarrow \tau'\chi$. By T39a, not $\vdash \neg CJ(\{CH(\Sigma_{m+n+1},5), \neg\tau'\chi\})$. By T61, $\{CH(\Sigma_{m+n+1},5), \neg\tau'\chi\}$ is consistent.

Lemma A. For each $i \leq n+1$, each $p,p' \in \Sigma_{m+i,0}$, and all formulas ξ,ξ'

(1) If $\models_S \neg\xi$, then $\neg lev(\xi,1,\Sigma_{m+i},\langle p,p'\rangle)$

(2) If $\xi \in \Gamma$, then $\xi \in \Sigma_{m+i,2}(\langle p,p'\rangle)$ if and only if $\neg\xi \notin \Sigma_{m+i,2}(\langle p,p'\rangle)$

(3) If $\xi \in \Gamma$, $\models_S \xi' \rightarrow \xi$, and $\xi' \in \Sigma_{m+i,2}(\langle p,p'\rangle)$, then $\xi \in \Sigma_{m+i,2}(\langle p,p'\rangle)$.

Proof: Suppose that $i \leq n+1$ and $p,p' \in \Sigma_{m+i,0}$.

(1) Suppose that $\models_S \neg\xi$. Since $\{CH(\Sigma_{m+n+1},5), \neg\tau'\chi\}$ is con-

sistent, $\{CH(\Sigma_{m+n+1},5)\}$ is consistent. By T75, T63, and T27, $\{CH(\Sigma_{m+i},5)\}$ is consistent. Hence, by T78, not $lev(\xi,1,\Sigma_{m+i},\langle p,p'\rangle)$.

(2) Suppose that $\xi \in \Gamma$. Then, since Σ_{m+i} is a minimal extension of Σ_{m+i-1} with respect to Γ and Δ, either $\xi \in \Sigma_{m+i,2}(\langle p,p'\rangle)$ or $\neg\xi \in \Sigma_{m+i,2}$. And it cannot be the case that both $\xi \in \Sigma_{m+i,2}(\langle p,p'\rangle)$ and $\neg\xi \in \Sigma_{m+i,2}(\langle p,p'\rangle)$, because then $lev(CJ(\{\xi,\neg\xi\}),1,\Sigma_{m+i},\langle p,p'\rangle)$, which contradicts part (1) (since, by T38, $\vdash_S \neg CJ(\{\xi,\neg\xi\})$.

(3) Suppose $\xi \in \Gamma$, $\vdash_S \xi' \to \xi$, $\xi' \in \Sigma_{m+i,2}(\langle p,p'\rangle)$ and $\xi \notin \Sigma_{m+i,2}(\langle p,p'\rangle)$. By (2), $\neg\xi \in \Sigma_{m+i,2}(\langle p,p'\rangle)$. Hence, $lev(CJ(\{\xi',\neg\xi\},0,\Sigma_{m+i},\langle p,p'\rangle)$. By T25c, $lev(CJ(\{\xi',\neg\xi\},1,\Sigma_{m+i},\langle p,p'\rangle)$. But this is impossible by (1), since $\vdash_S \neg CJ(\{\xi',\neg\xi\})$.

Lemma B. For each $i \leq n$, there are $p,p' \in \Sigma_{m+i,0}$ such that $\neg\tau''\chi \in \Sigma_{m+i,2}(\langle p,p'\rangle)$, where τ'' is the (n-1)-level subgeneralizer of τ'.

Proof: We proceed by induction. Suppose first that i is 0. Σ_m is a minimal extension of Σ_{m-1} with respect to Γ^*m and Δ; since Γ^*m is Γ and $\tau'\chi \in \Gamma$, either $\tau'\chi \in \Sigma_{m,2}(\langle 0,0\rangle)$ or $\neg\tau'\chi \in \Sigma_{m,2}(\langle 0,0\rangle)$. It is sufficient for this case to show that $\neg\tau'\chi \in \Sigma_{m,2}(\langle 0,0\rangle)$, so suppose for reductio that $\tau'\chi \in \Sigma_{m,2}(\langle 0,0\rangle)$. By D47, $lev(\tau'\chi,0,\Sigma_m,\langle 0,0\rangle)$. By T77, $\vdash_S CH(\Sigma_m,0) \to \tau'\chi$. By T75,

$\vdash_S CH(\Sigma_{m+n+1},0) \rightarrow \tau'\chi$. By T73, $\vdash_S CH(\Sigma_{m+n+1},5) \rightarrow \tau'\chi$, which is a contradiction.

Suppose now that Lemma B holds for i and i < n; that is, that there are $p,p' \in \Sigma_{m+i,0}$ such that $\neg\tau''\chi \in \Sigma_{m+i,2}(\langle p,p'\rangle)$ where τ'' is the (n-i)-level subgeneralizer of τ'. Let σ be the (n-(i+1))-level subgeneralizer of τ'. Then there are three cases:

(1) τ'' is $H\sigma$. By Lemma A2, $H\sigma\chi \notin \Sigma_{m+i+1,2}(\langle p,p'\rangle)$. Hence, since Σ_{m+i+1} is a minimal extension of Σ_{m+i} with respect to Γ and $\{\neg\tau'\chi\}$, there is a q such that $q <_{\Sigma_{m+i+1}} p$ and $\chi \in \Sigma_{m+i+1,2}(\langle q,p'\rangle)$.

(2) τ'' is $G\sigma$. This case is exactly analogous to the preceding one.

(3) τ'' is $K\sigma$. We will show that $\neg\sigma\chi \in \Sigma_{m+i+1,2}(\langle p,p\rangle)$. Suppose that $\neg\sigma\chi \notin \Sigma_{m+i+1,2}(\langle p,p\rangle)$. Then, by Lemma A2, $\sigma\chi \in \Sigma_{m+i+1,2}(\langle p,p\rangle)$, and $lev(CJ(\{\tau''\chi,\neg\tau''\chi\}),1,\Sigma_{m+i+1},\langle p,p'\rangle)$. But this is impossible, by Lemma A1.

This completes the induction.

Hence, by Lemma B, there are $p,p' \in \Sigma_{m+n,0}$ such that $\neg\chi \in \Sigma_{m+n,2}(\langle p,p'\rangle)$. That is,
$\neg(P\phi \wedge P\psi \rightarrow P(\phi \wedge \psi) \vee P(P\phi \wedge \psi) \vee P(\phi \wedge P\psi)) \in \Sigma_{m+n,2}(\langle p,p'\rangle)$.

By Lemma A3, $P\phi \in \Sigma_{m+n,2}(\langle p,p'\rangle)$ and $P\psi \in \Sigma_{m+n,2}(\langle p,p'\rangle)$. Since Σ_{m+n} is part of Σ_{m+n+1}, $P\phi \in \Sigma_{m+n+1,2}(\langle p,p'\rangle)$ and $P\psi \in \Sigma_{m+n+1,2}(\langle p,p'\rangle)$. By Lemma A2,

$H\neg\phi \notin \Sigma_{m+n+1,2}(\langle p,p'\rangle)$ and $H\neg\psi \notin \Sigma_{m+n+1,2}(\langle p,p'\rangle)$. Since Σ_{m+n+1} is a minimal extension of Σ_{m+n} with respect to Γ and Δ, there are $q,q' \in \Sigma_{m+n+1,0}$ such that $\neg\neg\phi \in \Sigma_{m+n+1,2}(\langle q,p'\rangle)$ and $\neg\neg\psi \in \Sigma_{m+n+1,2}(\langle q',p'\rangle)$. By Lemma A3, $\phi \in \Sigma_{m+n+1,2}(\langle q,p'\rangle)$ and $\psi \in \Sigma_{m+n+1,2}(\langle q',p'\rangle)$.

Tnere are three cases, each leading to a contradiction:

(1) $q <_{\Sigma_{m+n+1}} q'$. By Lemma A3, (since $\neg\chi \in \Sigma_{m+n+1,2}(\langle p,p'\rangle)$) $H\neg(P\phi \wedge \psi) \in \Sigma_{m+n+1,2}(\langle p,p'\rangle)$.

We will show that $P\phi \wedge \psi \in \Sigma_{m+n+1,2}(\langle q',p'\rangle)$. Suppose that $P\phi \wedge \psi \notin \Sigma_{m+n+1,2}(\langle q',p'\rangle)$. By Lemma A2, $\neg(P\phi \wedge \psi) \in \Sigma_{m+n+1,2}(\langle q',p'\rangle)$. By D47, $lev(CJ(\{P\phi,\psi,\neg(P\phi \wedge \psi)\}),1,\Sigma_{m+n+1},\langle q',p'\rangle)$. But this is impossible by Lemma A1, since $\vdash_S \neg CH(\{P\phi,\psi,\neg(P\phi \wedge \psi)\})$.

Since $P\phi \wedge \psi \in \Sigma_{m+n+1,2}(\langle q',p'\rangle)$, $lev(CJ(\{H\neg(P\phi \wedge \psi),P(P\phi \wedge \psi)\}),1,\Sigma_{m+n+1},\langle p,p'\rangle)$. But this is impossible by Lemma A1, since $\vdash_S \neg CJ(\{H\neg(P\phi \wedge \psi),P(P\phi \wedge \psi)\})$.

(2) $q' <_{\Sigma_{m+n+1}} q'$. This is analogous to the preceding case.

(3) q is q'. By Lemma A3 (since $\neg\chi \in \Sigma_{m+n+1,2}(\langle p,p'\rangle)$), $H\neg(\phi \wedge \psi) \in \Sigma_{m+n+1,2}(\langle p,p'\rangle)$.

We will show that $\phi \wedge \psi \in \Sigma_{m+n+1,2}(\langle q',p'\rangle)$. Suppose that $\phi \wedge \psi \notin \Sigma_{m+n+1,2}(\langle q',p'\rangle)$; by Lemma A2, $\neg(\phi \wedge \psi) \in \Sigma_{m+n+1,2}(\langle q',p'\rangle)$. By D47,

$lev(CJ(\{\phi,\psi,\neg(\phi \wedge \psi)\},1,\Sigma_{m+n+1},\langle q',p'\rangle)$. But this is impossible by Lemma A1, since $\vdash_S \neg CJ(\{\phi,\psi,\neg(\phi \wedge \psi)\})$.

Since $\phi \wedge \psi \in \Sigma_{m+n+1,2}(\langle q',p'\rangle)$, $lev(CJ(\{H\neg(\phi \wedge \psi),P(\phi \wedge \psi)\}),1,\Sigma_{m+n+1},\langle p,p'\rangle)$. But this is impossible by Lemma A1, since $\vdash_S \neg CJ(\{H\neg(\phi \wedge \psi),P(\phi \wedge \psi)\})$.

This completes the proof.

T82. (a) $\vdash_S PG\phi \rightarrow G\phi$

(b) $\vdash_S FH\phi \rightarrow H\phi$

Proof:

(a)	(i)	$\vdash_S F\neg\phi \rightarrow HFF\neg\phi$	GA14
	(ii)	$\vdash_S G\neg\neg\phi \rightarrow G\neg\neg G\neg\neg\phi$	GA12,T38,T43
	(iii)	$\vdash_S FF\neg\phi \rightarrow F\neg\phi$	(ii),T39b
	(iv)	$\vdash_S F\neg\phi \rightarrow HF\neg\phi$	(i),(iii),T44a
	(v)	$\vdash_S \neg H\neg G\neg\neg\phi \rightarrow \neg\neg G\neg\neg\phi$	(iv),T39b
	(vi)	$\vdash_S PG\phi \rightarrow G\phi$	T38,T43
(b)	Similar to (a)		

T83. (a) $\vdash_S FP\phi \rightarrow M\phi$

(b) $\vdash_S PF\phi \rightarrow M\phi$

Proof:

(a)	(i)	$\vdash_S L\neg\phi \rightarrow GPL\neg\phi$	GA13
	(ii)	$\vdash_S GPL\neg\phi \wedge FP\phi \rightarrow F(PL\neg\phi \wedge P\phi)$	T48g
	(iii)	$\vdash_S L\neg\phi \wedge FP\phi \rightarrow F(PL\neg\phi \wedge P\phi)$	(i),(ii),T39b
	(iv)	$\vdash_S PL\neg\phi \wedge P\phi \rightarrow P(L\neg\phi \wedge \phi)$	
		$P(PL\neg\phi \wedge \phi) \vee P(L\neg\phi \wedge P\phi)$	T81a

	(v)	$\vdash_S \neg P(L\neg\phi \wedge \phi)$	T38,T35d
	(vi)	$\vdash_S \neg H\neg G\neg\phi \rightarrow \neg\phi$	GA14
	(vii)	$\vdash_S PG\neg\phi \rightarrow \neg\phi$	(vi)
	(viii)	$\vdash_S PL\neg\phi \rightarrow \neg\phi$	(vii),T44
	(ix)	$\vdash_S \neg P(PL\neg\phi \wedge \phi)$	(viii),T35d
	(x)	$\vdash_S \neg P(L\neg\phi \wedge P\phi)$	T38,T35d
	(xi)	$\vdash_S \neg(PL\neg\phi \wedge P\phi)$	(iv),(v),(ix),(x)
	(xii)	$\vdash_S \neg F(PL\neg\phi \wedge P\phi)$	(xi),T35d
	(xiii)	$\vdash_S \neg(L\neg\phi \wedge FP\phi)$	(iii),(xii)
	(xiv)	$\vdash_S FP\phi \rightarrow M\phi$	(xiii),T39b

(b) Similar to (a)

T84. (a) $\vdash_S MM\phi \rightarrow M\phi$

(b) $\vdash_S L\phi \rightarrow LL\phi$

Proof:

(a)	(i)	$\vdash_S PP\phi \rightarrow M\phi$	GA11,T43,T48e
	(ii)	$\vdash_S FF\phi \rightarrow M\phi$	GA12,T43,T48e
	(iii)	$\vdash_S P\phi \vee F\phi \rightarrow M\phi$	T48e
	(iv)	$\vdash_S PF\phi \vee FP\phi \rightarrow M\phi$	T83a,T83b
	(v)	$\vdash_S PP\phi \vee P\phi \vee PF\phi \vee FP\phi \vee F\phi \vee FF\phi \rightarrow M\phi$	(i),(ii),(iii),(iv)
	(vi)	$\vdash_S P(P\phi \vee \phi \vee F\phi) \vee F(P\phi \vee \phi \vee F\phi) \rightarrow M\phi$	T49a,T49b,T43
	(vii)	$\vdash_S PM\phi \vee M\phi \vee FM\phi \rightarrow M\phi$	T48e,T43
	(viii)	$\vdash_S MM\phi \rightarrow M\phi$	(vii),T48e
(b)	(i)	$\vdash_S \neg L\neg\neg L\neg\neg\phi \rightarrow \neg L\neg\neg\phi$	(a)

(ii) $\vdash_S \neg LL\phi \rightarrow \neg L\phi$ T37,T43

(iii) $\vdash_S L\phi \rightarrow LL\phi$ (ii)

T85. (a) $\vdash_S PL\phi \rightarrow L\phi$

(b) $\vdash_S FL\phi \rightarrow L\phi$

(c) $\vdash_S ML\phi \rightarrow L\phi$

(d) $\vdash_S M\phi \rightarrow LM\phi$

Proof:

(a) (i) $\vdash_S \neg\phi \rightarrow HF\neg\phi$ GA14

(ii) $\vdash_S \neg\phi \rightarrow \neg PG\phi$ T43

(iii) $\vdash_S \neg\phi \rightarrow \neg P(H\phi \wedge \phi \wedge G\phi)$ T38,T44b

(iv) $\vdash_S \neg P(\neg\phi \wedge P(H\phi \wedge \phi \wedge G\phi))$ T39b,T35d

(v) $\vdash_S P\neg\phi \wedge P(H\phi \wedge \phi \wedge G\phi) \rightarrow$
$P(\neg\phi \wedge H\phi \wedge \phi \wedge G\phi) \vee P(P\neg\phi \wedge H\phi \wedge \phi \wedge G\phi) \vee$
$P(\neg\phi \wedge P(H\phi \wedge \phi \wedge G\phi))$ T81

(vi) $\vdash_S \neg P(\neg\phi \wedge H\phi \wedge \phi \wedge G\phi)$ T38,T35d

(vii) $\vdash_S \neg(P\neg\phi \wedge H\phi)$ T38,T43

(viii) $\vdash_S \neg P(P\neg\phi \wedge H\phi \wedge \phi \wedge G\phi)$ (vii),T44b,T35d

(ix) $\vdash_S \neg(P\neg\phi \wedge P(H\phi \wedge \phi \wedge G\phi))$ (iv),(vi),(viii),(v)

(x) $\vdash_S PL\phi \rightarrow H\phi$ (ix)

(xi) $\vdash_S PL\phi \rightarrow \phi$ (iii)

(xii) $\vdash_S PL\phi \rightarrow G\phi$ T82a,T44b

(xiii) $\vdash_S PL\phi \rightarrow L\phi$ (x),(xi),(xii)

(b) Similar to (a)

(c) An immediate consequence of (a), (b), and T48e.

(d) (i) $\vdash_S ML\neg\phi \rightarrow L\neg\phi$ (c)

(ii) $\vdash_S \neg L\neg\phi \rightarrow \neg ML\neg\phi$

(iii) $\vdash_S M\phi \rightarrow \neg\neg L\neg L\neg\phi$

(iv) $\vdash_S M\phi \rightarrow LM\phi$ T43

T86. If τ is a generalizer that does not contain $\langle \underline{k} \rangle$ or $\langle \underline{q} \rangle$ then $\vdash_S L\phi \rightarrow \tau\phi$.

Proof: First define $L^n\phi$ as follows:

(1) $L^0\phi$ is ϕ

(2) For each $n \in \omega$, $L^{n+1}\phi$ is $LL^n\phi$

Let N be the set of $n \in \omega$ such that, for any n-level generalizer τ that does not contain $\langle \underline{k} \rangle$ or $\langle \underline{q} \rangle$, and any $m \in \omega$, $\vdash_S L\phi \rightarrow L^m\tau\phi$.

We will prove by induction that N is ω. We have $\vdash_S L\phi \rightarrow L^m\phi$ by repeated application of T84b, so $0 \in N$.

Now suppose that, for any n-level generalizer τ that does not contain $\langle \underline{k} \rangle$ or $\langle \underline{q} \rangle$, and any $m \in \omega$, $\vdash_S L\phi \rightarrow L^m\tau\phi$.

Suppose also that τ is an n+1-level generalizer that does not contain $\langle \underline{k} \rangle$ or $\langle \underline{q} \rangle$, and suppose $m \in \omega$. Let τ' be that generalizer such that τ is $G\tau'$ or τ is $H\tau'$. Then τ' is an n-level generalizer. By the inductive hypothesis, $\vdash_S L\phi \rightarrow L^{m+1}\tau'\phi$. By T38 and T44a, $\vdash_S L^{m+1}\tau'\phi \rightarrow L^m\tau\phi$. By T39b, $\vdash_S L\phi \rightarrow L^m\tau\phi$. This completes the induction.

In order to prove the theorem itself, suppose that τ is a generalizer that does not contain $\langle \underline{k} \rangle$ or $\langle \underline{q} \rangle$. By the above induction, $\vdash_S L\phi \rightarrow L^0\tau\phi$. But $L^0\tau\phi$ is $\tau\phi$.

13. Theorems about K and R

T87. $\vdash_S \Lambda\alpha K\phi \leftrightarrow K\Lambda\alpha\phi$

Proof:

(i)	$\vdash_S \Lambda\alpha\phi \to \phi$	T54
(ii)	$\vdash_S K\Lambda\alpha\phi \to K\phi$	(i),T35d,GA17
(iii)	$\vdash_S \Lambda\alpha K\Lambda\alpha\phi \to \Lambda\alpha K\phi$	(ii),T35d,GA4
(iv)	$\vdash_S K\Lambda\alpha\phi \to \Lambda\alpha K\Lambda\alpha\phi$	GA5
(v)	$\vdash_S K\Lambda\alpha\phi \to \Lambda\alpha K\phi$	(iii),(iv)
(vi)	$\vdash_S \Lambda\alpha K\phi \leftrightarrow K\Lambda\alpha\phi$	(v),GA18

T88. $\vdash \phi \leftrightarrow K\phi$

Proof:

(i)	$\vdash_S \phi \to K\phi$	RA1
(ii)	$\vdash_S \neg\phi \to K\neg\phi$	RA1
(iii)	$\vdash_S \neg\phi \to \neg K\phi$	(ii),GA16
(iv)	$\vdash_S \phi \leftrightarrow K\phi$	(i),(iii)

T89. (a) If $\vdash LKL\phi$, then $\vdash_S \phi$.

(b) If $\vdash_S KL\phi$, then $\vdash_S \phi$.

Proof:

(a) Assume $\vdash LKL\phi$, and also assume that τ is a generalizer. It is sufficient to show that $\vdash \tau\phi$. We take two cases:

(1) $\langle \underline{k} \rangle$ does not occur in τ. Let τ', τ'' be the universal part of τ and the tense part of τ respectively. By T86, $\vdash_S L\phi \to \tau''\phi$. By T35d, $\vdash_S \tau'(L\phi \to \tau''\phi)$. By T36, $\vdash \tau' L\phi \to \tau\phi$. By T88

(since $\vdash LKL\phi$), $\vdash L\phi$. By T80, $\vdash \tau' L\phi$ and hence $\vdash \tau\phi$.

(2) $\langle \underline{k} \rangle$ occurs in τ. Let ξ, τ' be those generalizers such that τ is $\xi K \tau'$ and τ' does not contain $\langle \underline{k} \rangle$. Let ξ' be the result of dropping all occurrences of $\langle \underline{k} \rangle$ from ξ. By repeated applications of GA15 and T44b, $\vdash_S \xi' K \tau' \phi \rightarrow \tau\phi$. Let $\sigma, \sigma', \rho, \rho'$ be the universal part of ξ', the tense part of ξ', the universal part of τ', and the tense part of τ', respectively. By repeated applications of T55a, T55b, T87, and T43, $\vdash_S \sigma\rho\sigma' K \rho' \phi \rightarrow \xi' K \tau' \phi$. Hence it is sufficient to show $\vdash \sigma\rho\sigma' K \rho' \phi$. By T80, it is sufficient to show $\vdash \sigma' K \rho' \phi$. By T86, $\vdash_S L\phi \rightarrow \rho'\phi$. By T35d and GA17, $\vdash_S KL\phi \rightarrow K\rho'\phi$. By T35d again, $\vdash_S \sigma'(KL\phi \rightarrow K\rho'\phi)$. By T36, $\vdash \sigma' KL\phi \rightarrow \sigma' K \rho' \phi$. By T86, $\vdash LKL\phi \rightarrow \sigma' KL\phi$. Hence $\vdash LKL\phi \rightarrow \sigma' K \rho' \phi$. Since $\vdash LKL\phi$, $\vdash \sigma' K \rho' \phi$, which completes the proof.

(b) Follows from (a) and T42a

T90. (a) $\vdash_S (K\phi \rightarrow K\psi) \leftrightarrow K(\phi \rightarrow \psi)$

(b) $\vdash_S K\phi \wedge K\psi \leftrightarrow K(\phi \wedge \psi)$

Proof:

(a)	(i)	$\vdash_S K\psi \rightarrow K(\phi \rightarrow \psi)$	T38,T35d,GA17
	(ii)	$\vdash_S K\neg\phi \rightarrow K(\phi \rightarrow \psi)$	T38,T35d,GA17
	(iii)	$\vdash_S \neg K\phi \rightarrow K(\phi \rightarrow \psi)$	(ii),GA16

	(iv)	$\vdash_S (K\phi \to K\psi) \to K(\phi \to \psi)$	(i),(iii)
	(v)	$\vdash_S (K\phi \to K\psi) \leftrightarrow K(\phi \to \psi)$	(iv),GA17
(b)	(i)	$\vdash_S K\phi \to (K\psi \to K(\phi \wedge \psi))$	T38,T35d,GA17
	(ii)	$\vdash_S K\phi \wedge K\psi \to K(\phi \wedge \psi)$	(i)
	(iii)	$\vdash_S K(\phi \wedge \psi) \to K\phi$	T38,T35d,GA17
	(iv)	$\vdash_S K(\phi \wedge \psi) \to K\psi$	T38,T35d,GA17
	(v)	$\vdash_S K\phi \wedge K\psi \leftrightarrow K(\phi \wedge \psi)$	(ii),(iii),(iv)

T91. $\vdash_S K(\phi \leftrightarrow K\phi)$

Proof:

(i)	$\vdash_S K\phi \leftrightarrow KK\phi$	GA15
(ii)	$\vdash_S K(\phi \to K\phi) \wedge K(K\phi \to \phi)$	(i),T90a
(iii)	$\vdash_S K(\phi \leftrightarrow K\phi)$	(ii),T90b,T43

T92. $\vdash_S R(\phi \leftrightarrow K\phi)$

Proof:

(i)	$\vdash_S K((\phi \leftrightarrow K\phi) \leftrightarrow R(\phi \leftrightarrow K\phi))$	GA19
(ii)	$\vdash_S K(\phi \leftrightarrow K\phi) \to KR(\phi \leftrightarrow K\phi)$	(i),T44a,GA17
(iii)	$\vdash_S KR(\phi \leftrightarrow K\phi)$	(ii),T91
(iv)	$\vdash_S KLR(\phi \leftrightarrow K\phi)$	(iii),GA22,T44a
(v)	$\vdash_S R(\phi \leftrightarrow K\phi)$	(iv),T89b

T93. $\vdash_S \neg R\phi \leftrightarrow R\neg\phi$

Proof:[9]

(i)	$\vdash_S K(\phi \leftrightarrow R\phi)$	GA19
(ii)	$\vdash_S K(\neg\phi \leftrightarrow \neg R\phi)$	(i),T43
(iii)	$\vdash_S K(\neg\phi \leftrightarrow R\neg\phi)$	GA19

(iv)	$\vdash_S$	$K(\neg R\phi \leftrightarrow R\neg\phi)$	(ii),(iii),T90b,T44a
(v)	$\vdash_S$	$K(\neg LR\phi \leftrightarrow LR\neg\phi)$	(iv),T38,GA22,T43
(vi)	$\vdash_S$	$K(M\neg R\phi \leftrightarrow LR\neg\phi)$	(v),T43
(vii)	$\vdash_S$	$M\neg R\phi \wedge LR\neg\phi \rightarrow L(M\neg R\phi \wedge LR\neg\phi)$	T85c,T84b,T49f
(viii)	$\vdash_S$	$M\neg R\phi \wedge LR\neg\phi \rightarrow L(M\neg R\phi \leftrightarrow LR\neg\phi)$	(vii),T44a
(ix)	$\vdash_S$	$LR\phi \wedge M\neg R\neg\phi \rightarrow L(LR\phi \wedge M\neg R\neg\phi)$	T84b,T85c,T49f
(x)	$\vdash_S$	$\neg M\neg R\phi \wedge \neg LR\neg\phi \rightarrow L(\neg M\neg R\phi \wedge \neg LR\neg\phi)$	(ix),T43
(xi)	$\vdash_S$	$\neg M\neg R\phi \wedge \neg LR\neg\phi \rightarrow L(M\neg R\phi \leftrightarrow LR\neg\phi)$	(x),T44a
(xii)	$\vdash_S$	$(M\neg R\phi \leftrightarrow LR\neg\phi) \rightarrow L(M\neg R\phi \leftrightarrow LR\neg\phi)$	(vii),(xi)
(xiii)	$\vdash_S$	$KL(M\neg R\phi \leftrightarrow LR\neg\phi)$	(vi),(xii),T44a
(xiv)	$\vdash_S$	$M\neg R\phi \leftrightarrow LR\neg\phi$	T89b
(xv)	$\vdash_S$	$\neg LR\phi \leftrightarrow LR\neg\phi$	(xiv),T43
(xvi)	$\vdash_S$	$\neg R\phi \leftrightarrow R\neg\phi$	(xv),T38,GA22,T43

T94. $\vdash_S R(\phi \rightarrow \psi) \leftrightarrow (R\phi \rightarrow R\psi)$

Proof:

(i)	$\vdash_S$	$R(\phi \rightarrow \psi) \rightarrow (R\phi \rightarrow R\psi)$	GA20
(ii)	$\vdash_S$	$R(\psi \rightarrow (\phi \rightarrow \psi))$	T38,T42b
(iii)	$\vdash_S$	$R\psi \rightarrow R(\phi \rightarrow \psi)$	(ii),GA20
(iv)	$\vdash_S$	$R(\neg\phi \rightarrow (\phi \rightarrow \psi))$	T38,T42b
(v)	$\vdash_S$	$R\neg\phi \rightarrow R(\phi \rightarrow \psi)$	(iv),GA20
(vi)	$\vdash_S$	$\neg R\phi \rightarrow R(\phi \rightarrow \psi)$	(v),T93,T43
(vii)	$\vdash_S$	$R(\phi \rightarrow \psi) \leftrightarrow (R\phi \rightarrow R\psi)$	(i),(iii),(vi)

T95. $\vdash_S R(\phi \leftrightarrow R\phi)$

Proof:

(i)	$\vdash_S R(R\phi \to KR\phi)$	T92,T44a
(ii)	$\vdash_S RR\phi \to RKR\phi$	(i),GA20
(iii)	$\vdash_S KR\phi \to K\phi$	GA19,T44a,GA17
(iv)	$\vdash_S RKR\phi \to RK\phi$	(iii),T42b,GA20
(v)	$\vdash_S RK\phi \to R\phi$	T92,T44a,T42b,GA20
(vi)	$\vdash_S RR\phi \to R\phi$	(ii),(iv),(v)
(vii)	$\vdash_S R\phi \to RR\phi$	GA22,GA21,T44a
(viii)	$\vdash_S (\phi \to R\phi) \to ((R\phi \to \phi) \to (\phi \leftrightarrow R\phi))$	T38
(ix)	$\vdash_S R(\phi \to R\phi) \to (R(R\phi \to \phi) \to R(\phi \leftrightarrow R\phi))$	T42b,GA20,T44a
(x)	$\vdash_S (R\phi \to RR\phi) \to R(\phi \to R\phi)$	T94
(xi)	$\vdash_S (RR\phi \to R\phi) \to R(R\phi \to \phi)$	T94
(xii)	$\vdash_S R(\phi \leftrightarrow R\phi)$	(vi),(vii),(x) (xi),(ix)

T96. $\vdash_S \Lambda\alpha R\phi \leftrightarrow R\Lambda\alpha\phi$

Proof:

(i)	$\vdash_S K(\Lambda\alpha\phi \leftrightarrow R\Lambda\alpha\phi)$	GA19
(ii)	$\vdash_S \Lambda\alpha K(\phi \leftrightarrow R\phi)$	GA19,T35d
(iii)	$\vdash_S K\Lambda\alpha(\phi \leftrightarrow R\phi)$	(ii),T87
(iv)	$\vdash_S \Lambda\alpha(\phi \leftrightarrow R\phi) \to (\Lambda\alpha\phi \leftrightarrow \Lambda\alpha R\phi)$	T44a,GA4
(v)	$\vdash_S K(\Lambda\alpha\phi \leftrightarrow \Lambda\alpha R\phi)$	(iii),(iv),T44a
(vi)	$\vdash_S K(\Lambda\alpha R\phi \leftrightarrow R\Lambda\alpha\phi)$	(i),(v),T90b,T44a
(vii)	$\vdash_S K(\Lambda\alpha LR\phi \leftrightarrow LR\Lambda\alpha\phi)$	GA22,T38,T43
(viii)	$\vdash_S K(L\Lambda\alpha R\phi \leftrightarrow LR\Lambda\alpha\phi)$	T55c,T43
(ix)	$\vdash_S L\Lambda\alpha R\phi \wedge LR\Lambda\alpha\phi \to L(L\Lambda\alpha R\phi \wedge LR\Lambda\alpha\phi)$	T84f,T49f

(x)	$\vdash_S$	$L\Lambda\alpha R\phi \wedge LR\Lambda\alpha\phi \rightarrow L(L\Lambda\alpha R\phi \leftrightarrow LR\Lambda\alpha\phi)$	(ix),T44a
(xi)	$\vdash_S$	$M\neg\Lambda\alpha R\phi \wedge M\neg R\Lambda\alpha\phi \rightarrow L(M\neg\Lambda\alpha R\phi \wedge M\neg R\Lambda\alpha\phi)$	T85c,T49f
(xii)	$\vdash_S$	$\neg L\Lambda\alpha R\phi \wedge \neg LR\Lambda\alpha\phi \rightarrow L(\neg L\Lambda\alpha R\phi \wedge \neg LR\Lambda\alpha\phi)$	(xi),T43
(xiii)	$\vdash_S$	$\neg L\Lambda\alpha R\phi \wedge \neg LR\Lambda\alpha\phi \rightarrow L(L\Lambda\alpha R\phi \leftrightarrow LR\Lambda\alpha\phi)$	(xii),T44a
(xiv)	$\vdash_S$	$(L\Lambda\alpha R\phi \leftrightarrow LR\Lambda\alpha\phi) \rightarrow L(L\Lambda\alpha R\phi \leftrightarrow LR\Lambda\alpha\phi)$	(x),(xii)
(xv)	$\vdash_S$	$KL(L\Lambda\alpha R\phi \leftrightarrow LR\Lambda\alpha\phi)$	(xiv),(viii),T44a
(xvi)	$\vdash_S$	$L\Lambda\alpha R\phi \leftrightarrow LR\Lambda\alpha\phi$	(xv),T89b
(xvii)	$\vdash_S$	$\Lambda\alpha LR\phi \leftrightarrow LR\Lambda\alpha\phi$	(xvi),T55c,T43
(xviii)	$\vdash_S$	$\Lambda\alpha LR\phi \leftrightarrow \Lambda\alpha R\phi$	T38,GA22,T43
(xix)	$\vdash_S$	$LR\Lambda\alpha\phi \leftrightarrow R\Lambda\alpha\phi$	GA22,T38
(xx)	$\vdash_S$	$\Lambda\alpha R\phi \leftrightarrow R\Lambda\alpha\phi$	(xvii),(xviii),(xix)

CHAPTER IV

THE COMPLETENESS PROOF

This chapter contains the final proof of completeness and the theorems leading up to it.

1. Infinite Minimal Extension Sequences

D59. If Σ is an ω-place minimal extension sequence with respect to Γ and Δ, then the arrangement corresponding to Σ is the triple $\langle j,R,F\rangle$, where

(a) j is $\bigcup_{k \in \omega} \Sigma_{k,0}$

(b) R is $\bigcup_{k \in \omega} \Sigma_{k,1}$

(c) F is that function with domain $j \times j$ such that, for each $i \in j \times j$, $F(i)$ is $\bigcup_{k \in \omega} \Sigma_{k,2}(i)$.

T97. If B is a finite arrangement, Σ is an ω-place minimal extension sequence with respect to Γ and Δ, and B is part of the arrangement corresponding to Σ, then there is a $k \in \omega$ such that B is part of Σ_k.

Proof: Assume the hypothesis. Let j,R,F be B_0,B_1,B_2 respectively. For each $n \in j$, there is an $m \in \omega$ such that $n \in \Sigma_{m,0}$. For each $n \in j$, let $f(n)$ be the least $m \in \omega$ such that $n \in \Sigma_{m,0}$. Let m be the largest number in Rng(f). By T27, $j \subseteq \Sigma_{m,0}$.

We will show that $R \subseteq \Sigma_{m,1}$. Suppose that $\langle p,p'\rangle \in R$. Then $p,p' \in j$, and $p,p' \in \Sigma_{m,0}$. By the hypothesis, there is a $k \in \omega$ such that $\langle p,p'\rangle \in \Sigma_{k,1}$. If $k \leq m$, then $\langle p,p'\rangle \in \Sigma_{m,1}$, by T27. Suppose, then, that $m < k$ and $\langle p,p'\rangle \notin \Sigma_{m,1}$. Then p is not p', because $\Sigma_{m,1}$ is reflexive, and $\langle p',p\rangle \in \Sigma_{m,1}$. By T27, $\langle p',p\rangle \in \Sigma_{k,1}$. But this is impossible, since $\langle p,p'\rangle \in \Sigma_{k,1}$ and p is not p'.

Let g be that function whose domain is the set of pairs $\langle i,\phi\rangle$ such that $i \in Dom(F)$ and $\phi \in F(i)$, and such that, for each $\langle i,\phi\rangle \in Dom(g)$, $g(\langle i,\phi\rangle)$ is the first $k \in \omega$ such that $i \in \Sigma_{k,0}$ and $\phi \in \Sigma_{k,2}(i)$. Let m' be the largest number in Rng(g). Note that $m \leq m'$ and hence, by T27, $j \subseteq \Sigma_{m',0}$ and $R \subseteq \Sigma_{m',1}$.

It remains only to show that, for each $i \in Dom(F)$, $F(i) \subseteq \Sigma_{m',2}(i)$. Suppose $i \in Dom(F)$ and $\phi \in F(i)$. Let q be $g(\langle i,\phi\rangle)$. Then $i \in \Sigma_{q,0}$ and $\phi \in \Sigma_{q,2}(i)$. Since $q \leq m'$, (by T27) $\phi \in \Sigma_{m',2}(i)$.

T98. If Σ is an ω-place minimal extension sequence with respect to Γ and Δ, then the language of the arrangement corresponding to Σ is the language of Γ.

Proof: Assume the hypothesis. Let A be the arrangement corresponding to Σ. Let L be the language of A, and let L' be the language of Γ.

First, suppose that $\pi \in L$; then there must be some $n \in \omega$ such that π is in the language of Σ_n. Since $Rng(\Sigma_{0,2})$

is 0, n is not 0. Therefore Σ_n is a minimal extension of Σ_{n-1} with respect to $\Gamma*n$ and Δ. But, by D46, π must occur in some formula of Γ.

On the other hand, suppose that $\pi \in L'$. Then there is a formula $\phi \in \Gamma$ such that π occurs in ϕ. For some $n \in \omega$, ϕ is the $n^{\underline{th}}$ formula in Γ, and hence $\phi \in \Gamma*n$. Since Σ_n is an extension of Σ_{n-1} with respect to $\Gamma*n$ and Δ, either $\phi \in \Sigma_{n,2}(\langle 0,0\rangle)$ or $\neg\phi \in \Sigma_{n,2}(\langle 0,0\rangle)$. In either case, $\pi \in L$.

T99. If Σ is an ω-place minimal extension sequence with respect to Γ and Δ and $k \in \omega$, then Σ_k is part of the arrangement corresponding to Σ.

Proof: A trivial consequence of D59

T100. If L is a language, Γ is the set of formulas of L, and Σ is an ω-place minimal extension sequence with respect to Γ and Δ, then the arrangement corresponding to Σ is a complete arrangement.

Proof: Assume the hypothesis, and let A be the arrangement corresponding to Σ. Let j,R,F be those objects such that A is $\langle j,R,F\rangle$. In order to show that A is complete, it is sufficient to show that clauses (3), (4), and (5) of D44 hold.

For clause (3), suppose that $\langle m,n\rangle \in j\times j$ and $\phi \in \Gamma$. Then for some $k \in \omega$, $\phi \in \Gamma*k$. Let p be the first $p \geq k$ such that $m,n \in \Sigma_{p,0}$. Then Σ_{p+1} is a minimal extension of Σ_p

with respect to $\Gamma^*(p+1)$ and Δ. Then either $\phi \in \Sigma_{p+1,2}(\langle m,n\rangle)$ or $\neg\phi \in \Sigma_{p+1,2}(\langle m,n\rangle)$. By T99, either $\phi \in F(\langle m,n\rangle)$ or $\neg\phi \in F(\langle m,n\rangle)$.

Clauses (4) and (5) also hold, by similar arguments.

2, Construction of a Complete Arrangement

D60. Γ is an acceptable set of formulas if and only if the set of variables that do not occur free in any formula in Γ is denumerable.

T101. If Γ is a set of formulas, Γ' is an acceptable, consistent set of formulas, Δ is the set of variables that do not occur in any formula in Γ', and $n \in \omega$, then there is a Σ such that Σ is an n+1-place minimal extension sequence with respect to Γ and Δ, and Γ' is consistent with $CH(\Sigma_n,n)$.

Proof: Assume the hypothesis. Then $\vdash_S CH^*(\Gamma,\Delta,n,n)$. By T63, Γ' is consistent with $CH^*(\Gamma,\Delta,n,n)$. Hence, there is a disjunct ϕ of $CH^*(\Gamma,\Delta,n,n)$ such that Γ' is consistent with ϕ; that is, there is an n+1-place minimal extension sequence with respect to Γ and Δ such that $\Gamma' \cup \{\forall\alpha_0 \ldots \forall\alpha_{k-1} CH(\Sigma_n,n)\}$ is consistent, where $\alpha_0,\ldots,\alpha_{k-1}$ are (in order) the variables in Δ that occur free in $CH(\Sigma_n,n)$. By k applications of T62, $\Gamma' \cup \{CH(\Sigma_n,n)\}$ is consistent.

T102. If Γ' is acceptable and consistent and Δ is the set of variables that do not occur free in any formula in Γ, then there is an infinite sequence Σ such that

(1) Σ is a minimal extension sequence with respect to Γ and Δ.

(2) For each $n \in \omega$, $CH(\Sigma_n, n)$ is consistent with Γ'.

Proof: Assume the hypothesis. Let T be the set of sequences Σ such that for some p, Σ is a p-place minimal extension sequence with respect to Γ and Δ, and Γ' is consistent with $CH(\Sigma_n, n)$, for each $n \in p$.

Let Σ be that infinite sequence such that, for each $k \in \omega$, Σ_k is the first arrangement A such that for any $j > k$, there is a j-place sequence $\Sigma' \in T$ such that $\Sigma 1 k^\frown\langle A\rangle \subseteq \Sigma'$. (We take ourselves to have defined a standard ordering of the arrangements; by T21, this can be done.) It is sufficient to show that $\Sigma \in T$.

Lemma A. For each $k \in \omega$, there is an arrangement A such that for any $j > k$, there is a j-place sequence $\Sigma' \in T$ such that $\Sigma 1 k^\frown\langle A\rangle \subseteq \Sigma'$.

Proof: Suppose that the lemma does not hold, and let k be the first $k \in \omega \sim \{0\}$ such that there is no arrangement A such that for any $j > k$, there is a j-place sequence $\Sigma' \in T$ such that $\Sigma 1 k^\frown\langle A\rangle \subseteq \Sigma'$.

By T101, and since all minimal extension sequences have the same first term, k is not 1. Hence, there is an arrangement A such that for each $j > k-1$ there is a j-place sequence $\Sigma' \in T$ such that $\Sigma 1 (k-1)^\frown\langle A\rangle \subseteq \Sigma'$. It follows from the derinition of Σ that, for each $j \geq k$, there is a j-place sequence $\Sigma' \in T$ such that $\Sigma\upharpoonright k \leq \Sigma'$.

Let Δ be the set of arrangements A such that $\Sigma\upharpoonright k^\frown\langle A\rangle \in T$; by the preceding sentence, Δ is not 0. By T24, Δ is finite. Then for each $A \in \Delta$, there is some $j > k$ such that there is no j-place sequence $\Sigma' \in T$ such that $\Sigma\upharpoonright k^\frown\langle A\rangle \in T$. Let f be that function such that Dom(f) is Δ and for each $A \in \Delta$, f(A) is the least $j > k$ such that there is no j-place sequence $\Sigma' \in T$ such that $\Sigma\upharpoonright k^\frown\langle A\rangle \leq \Sigma'$. Let j be the largest number in Rng(f). Then there is a j-place sequence $\Sigma' \in T$ such that $\Sigma\upharpoonright k \leq \Sigma'$. Let A be Σ'_k. Then $A \in \Delta$. By the definition of f there is no f(A)-place sequence $\Sigma'' \in T$ such that $\Sigma\upharpoonright k^\frown\langle A\rangle \leq \Sigma''$. But this is a contradiction; since $f(A) \leq j$, $\Sigma'\upharpoonright(f(A))$ is such a sequence.

It is an immediate consequence of Lemma A that for any $k \in \omega$, $\Sigma\upharpoonright(k+1) \in T$; and therefore (by the definition of T) that $\Sigma \in T$.

T103. If Σ is an ω-place minimal extension sequence with respect to Γ and Δ, A is the arrangement corresponding to Σ, and lev(ϕ,k,A,$\langle 0,0\rangle$), then there is an $n \in \omega$ such that $\vdash CH(\Sigma_n,n) \rightarrow \phi$.

Proof: Assume the hypothesis. By T26, there is a finite arrangement B such that B is part of A and lev(ϕ,k,B,$\langle 0,0\rangle$). By T97, there is an $m \in \omega$ such that B is part of Σ_m. Let n be the maximum of k and m. By T27, Σ_m is part of Σ_n, so B is part of Σ_n. By T25c, lev(ϕ,n,B,$\langle 0,0\rangle$)

By T25b, $lev(\phi, n, \Sigma_n, \langle 0,0 \rangle)$. By T77, $\vdash CH(\Sigma_n, n) \rightarrow \phi$.

T104. If Σ is an ω-place minimal extension sequence with respect to Γ and Δ, then the arrangement corresponding to Σ is consistent with Γ' if and only if, for each $n \in \omega$, $\Gamma' \cup \{CH(\Sigma_n, n)\}$ is consistent.

Proof: Assume that Σ is an ω-place minimal extension sequence with respect to Γ and Δ and let A be the arrangement corresponding to Σ.

Suppose first that A is consistent with Γ', $n \in \omega$ and $\Gamma' \cup \{CH(\Sigma_n, n)\}$ is inconsistent. By T25a, $lev(CH(\Sigma_n, n), n, \Sigma_n, \langle 0,0 \rangle)$. By T99 and T25b, $lev(CH(\Sigma_n, n), n, A, \langle 0,0 \rangle)$; but this contradicts the hypothesis.

Secondly, suppose that A is not consistent with Γ' but that, for each $n \in \omega$, $\Gamma' \cup \{CH(\Sigma_n, n)\}$ is consistent. There is a formula ϕ and a $k \in \omega$ such that $lev(\phi, k, A, \langle 0,0 \rangle)$ and $\Gamma' \cup \{\phi\}$ is not consistent. By T103, there is an $m \in \omega$ such that $\vdash CH(\Sigma_m, m) \rightarrow \phi$. By T63, $\Gamma' \cup \{CH(\Sigma_m, m)\}$ is not consistent, contradicting the hypothesis.

T105 is the key theorem for the completeness proof. Any arrangement satisfying the conditions of T105 specifies a model for Γ', constructed from the formulas in Γ' in the general manner of Henkin [3].

T105. If Γ' is acceptable and consistent, then there is a complete arrangement A such that the arrangement A is consistent with Γ' and the language of A is the language of Γ'.

Proof: Assume the hypothesis. Let Γ be the set of formulas of the language of Γ' and let Δ be the set of variables that do not occur free in any formula in Γ'. By T102, there is an infinite sequence Σ such that (1) Σ is a minimal extension sequence with respect to Γ and Δ, and (2) For each $n \in \omega$, $CH(\Sigma_n, n)$ is consistent with Γ'.

Let A be the arrangement corresponding to Σ. By T100, A is complete. By T104, A is consistent with Γ'. By T98, the language of A is the language of Γ'.

3. The Expansion of a Set of Formulas

D61. If ξ is a term or a formula, then ex(ξ), or the expansion of ξ, is rep(ξ,f), where f is that function with domain Iv such that for each $k \in \omega$, $f(v_k)$ is v_{2k}.

D62. EX(Γ), or the expansion of Γ, is the set of formulas ex(ϕ), for $\phi \in \Gamma$.

T106. EX(Γ) is an acceptable set of formulas.

Proof: Obvious, since none of the variables v_k (where k is odd) occur in any formula of EX(Γ).

In the final completeness proof, we replace the consistent set of formulas Γ by EX(Γ) in order to obtain an acceptable set of formulas. Theorems 107 and 108 guarantee the properness of this procedure.

T107. Γ is satisfiable if and only if EX(Γ) is satisfiable.

Proof: First, suppose that Γ is satisfiable. Then there is an interpretation $\langle T, \leq, U, G \rangle$, a $t \in T$, and an $x \in U^{\omega}$ such that for each $\phi \in \Gamma$, $x \in Int_{\mathfrak{a}}(\phi)(\langle t,t \rangle)$. Let x' be that sequence such that for each $k \in \omega$, x'_{2k} is x_k and x'_{2k+1} is x_0. By an easy induction, it may be shown that for each formula ϕ of the language of Γ, $x \in Int_{\mathfrak{a}}(\phi)(\langle t,t \rangle)$ if and only if $x' \in Int_{\mathfrak{a}}(ex(\phi))(\langle t,t' \rangle)$, and hence that EX($\Gamma$) is satisfiable.

Conversely, suppose that EX(Γ) is satisfiable and let x be a sequence that satisfies EX(Γ) in some interpretation. Let x' be the sequence such that for each $k \in \omega$, x'_k is x_{2k}. Then x' satisfies Γ in the same interpretation.

T108. Γ is consistent if and only if EX(Γ) is consistent.

Proof: Follows immediately from T64

4. The Completeness Theorem

T109. If Γ is consistent, then Γ is satisfiable.

Proof: Suppose that Γ is consistent. Then, by T106 and T108, EX(Γ) is acceptable and consistent. Let L be the language of EX(Γ). By T105, there is a complete arrangement A such that the arrangement A is consistent with EX(Γ) and the language of A is L.

Let $T, \leq, F$ be those objects such that A is $\langle T, \leq, F\rangle$.

For each $t \in T$, let h_t be that function whose domain is the set of terms of L and such that for each $\eta \in \mathrm{Dom}(h_t)$, $h_t(\eta)$ is the first variable β such that $\beta=\eta \in F(\langle t,0\rangle)$. By GA7, T68a, and the completeness of A, there always is such a variable. Hence,

Lemma A. For each $t \in T$ and each term η of L, $h_t(\eta)=\eta \in F(\langle t,0\rangle)$.

Lemma B. For each $t,t' \in T$ and each terms ζ,η of L, $\zeta=\eta \in F(\langle t,t'\rangle)$ if and only if $h_t(\zeta)$ is $h_t(\eta)$.

Proof: In order to prove Lemma B, suppose that $t,t' \in T$ and $\zeta,\eta \in Tm_L$. Also suppose first that $\zeta=\eta \in F(\langle t,t'\rangle)$. By T69b, $\zeta=\eta \in F(\langle t,0\rangle)$. By Lemma A, $h_t(\zeta)=\zeta \in F(\langle t,0\rangle)$ and $h_t(\eta)=\eta \in F(\langle t,0\rangle)$. By T69c, $h_t(\zeta)=h_t(\eta) \in F(\langle t,0\rangle)$. Now suppose that $h_t(\zeta)$ is not $h_t(\eta)$; then there is a variable β that precedes $h_t(\eta)$ such that $\beta=h_t(\zeta) \in F(\langle t,0\rangle)$. But then by T69c, $\beta=\eta \in F(\langle t,0\rangle)$, which is impossible, since $h_t(\eta)$ is the first variable β such that $\beta=\eta \in F(\langle t,0\rangle)$.

Conversely, suppose that $h_t(\zeta)$ is $h_t(\eta)$. By Lemma A, $h_t(\zeta)=\zeta \in F(\langle t,0\rangle)$. Then $h_t(\eta)=\zeta \in F(\langle t,0\rangle)$. By T45 and T69c, $\zeta=\eta \in F(\langle t,0\rangle)$. By T69b, $\zeta=\eta \in F(\langle t,t'\rangle)$.

We will now begin to construct the model in which $EX(\Gamma)$ is satisfiable. The set of moments of the model will simply be T and the 'earler than' relation will be the $\leq$ of the arrangement. Let U (which is to be the universe of the model) be $\mathrm{Rng}(h_0)$. We will show:

Lemma C. For each $t, t' \in T$ and each term η of L, there is exactly one $\alpha \in U$ such that $\alpha = \eta \in F(\langle t, t' \rangle)$.

Proof: Assume that $t, t' \in T$ and η is a term of L; then by Lemma A, $h_t(\eta) = \eta \in F(\langle t, 0 \rangle)$. Suppose that there is an $\alpha \in U$ such that α is not $h_t(\eta)$ and $\alpha = \eta \in F(\langle t, t' \rangle)$. By T69b, $\alpha = \eta \in F(\langle t, 0 \rangle)$. Since $h_t(\eta)$ is the first variable β such that $\beta = \eta \in F(\langle t, t' \rangle)$, $h_t(\eta)$ precedes α. By T45 and T69c, $\alpha = h_t(\eta) \in F(\langle t, t' \rangle)$. By T69b, $\alpha = h_t(\eta) \in F(\langle t, 0 \rangle)$. By T71b, $\alpha = h_t(\eta) \in F(\langle 0, 0 \rangle)$. Since $\alpha \in U$, there is a term ζ of L such that α is $h_0(\zeta)$. Then α is the first variable α such that $\alpha = \zeta \in F(\langle 0, 0 \rangle)$. By T69c, $h_t(\eta) = \zeta \in F(\langle 0, 0 \rangle)$, but this is impossible, since $h_t(\eta)$ precedes α.

Lemma D. For each variable α and each $t, t' \in T$, $h_t(\alpha)$ is $h_{t'}(\alpha)$.

Proof: Assume the hypothesis, and suppose that $h_t(\alpha)$ is not $h_{t'}(\alpha)$. By Lemma A, $h_t(\alpha) = \alpha \in F(\langle t, 0 \rangle)$ and $h_{t'}(\alpha) = \alpha \in F(\langle t', 0 \rangle)$ and hence, by T71b, $h_{t'}(\alpha) = \alpha \in F(\langle t, 0 \rangle)$. Since $h_t(\alpha)$ is the first variable β such that $\beta = \alpha \in F(\langle t, 0 \rangle)$, $h_t(\alpha)$ must precede $h_{t'}(\alpha)$. But by a symmetrical argument, $h_{t'}(\alpha)$ must precede $h_t(\alpha)$, which is a contradiction.

Lemma E. For each $\beta \in U$ and $t \in T$, $h_t(\beta)$ is β.

Proof: Suppose that $\beta \in U$ and $t \in T$. Then there is a term η of L such that β is $h_0(\eta)$. Suppose also that $h_t(\beta)$ is not β. Since, by Lemma A, $h_t(\beta)=\beta \in F(\langle t,0\rangle)$, there must be a variable α such that α precedes β and $\alpha=\beta \in F(\langle t,0\rangle)$. By T71b, $\alpha=\beta \in F(\langle 0,0\rangle)$. But then, by T45 and T69c, $\alpha=\eta \in F(\langle 0,0\rangle)$. But this is impossible, since β is the first variable β such that $\beta=\zeta \in F(\langle 0,0\rangle)$.

As the final step in the construction of the model, let G be that function whose domein is L and such that

(a) For each k-place predicate letter $\pi \in L$, $G(\pi)$ is that function with domain T such that for each $t \in T$, $G(\pi)(t)$ is the set of sequences $\alpha \in U^k$ such that $\pi\alpha_0 \ldots \alpha_{k-1} \in F(\langle t,0\rangle)$.

(b) For each k-place operation letter $\delta \in L$, $G(\delta)$ is that function with domain T such that for each $t \in T$, $G(\delta)(t)$ is itself that function f such that Dom(f) is U^k and for each $\alpha \in U^k$, $f(\alpha)$ is that $\beta \in U$ such that $\delta\alpha_0 \ldots \alpha_{k-1}=\beta \in F(\langle t,0\rangle)$.

Let $\mathfrak{A}$ be $\langle T,\leq,U,G\rangle$ and let x be that infinite sequence such that for each $n \in \omega$, x_n is $h_0(v_n)$.

Then $\mathfrak{A}$ is an interpretation for L. We will now show that for each formula $\phi \in EX(\Gamma)$, $x \in Int_{\mathfrak{A}}(\phi)(\langle 0,0\rangle)$, and hence that $EX(\Gamma)$ is satisfiable. To do this, we need two preliminary lemmas:

Lemma F. For each term ζ of L and each $t \in T$, $Ext_{t,\mathfrak{A}}(\zeta)(x)$

is $h_t(\zeta)$.

Proof: The proof is by induction, using T2.

(1) If $k \in \omega$, then $\mathrm{Ext}_{t,\mathfrak{a}}(v_k)(x)$ is x_k is $h_0(v_k)$ is (by Lemma D) $h_t(v_k)$.

(2) Suppose δ is a k-place operation letter, $\eta_0,\ldots,\eta_{k-1}$ are terms of L, and for each $i < k$, $\mathrm{Ext}_{t,\mathfrak{a}}(\eta_i)(x)$ is $h_t(\eta_i)$. Then $\mathrm{Ext}_{t,\mathfrak{a}}(\delta\eta_0\ldots\eta_{k-1})(x)$ is $G(\delta)(t)(\langle \mathrm{Ext}_{t,\mathfrak{a}}(\eta_0)(x),\ldots,\mathrm{Ext}_{t,\mathfrak{a}}(\eta_{k-1})(x)\rangle)$ is $G(\delta)(t)(\langle h_t(\eta_0),\ldots,h_t(\eta_{k-1})\rangle)$ is that $\beta \in U$ such that $\delta h_t(\eta_0)\ldots h_t(\eta_{k-1})=\beta \in F(\langle t,0\rangle)$ is (by Lemma A, T47a, and T69c) that $\beta \in U$ such that $\delta\eta_0\ldots\eta_{k-1}=\beta \in F(\langle t,0\rangle)$ is $h_t(\delta\eta_0\ldots\eta_{k-1})$.

Lemma G. For each formula ϕ of L and $t,t' \in T$, $x \in \mathrm{Int}_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$ if and only if $\phi \in F(\langle t,t'\rangle)$.

Proof: The proof proceeds by induction on the rank of ϕ. Let Δ be the set of formulas of the language of L such that for each $t,t' \in T$, $x \in \mathrm{Int}_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$ if and only if $\phi \in F(\langle t,t'\rangle)$.

(1) Suppose that ϕ is an atomic formula of L. Then there are two cases.

(a) ϕ is $\zeta=\eta$, for some terms ζ,η of L. Then $x \in \mathrm{Int}_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$ if and only if $\mathrm{Ext}_{t,\mathfrak{a}}(\zeta)(x)$ is $\mathrm{Ext}_{t,\mathfrak{a}}(\eta)(x)$ if and only if (by Lemma F) $h_t(\zeta)$ is $h_t(\eta)$ if and only if (by Lemma B) $\zeta=\eta \in F(\langle t,0\rangle)$ if and only if (by T69b) $\zeta=\eta \in F(\langle t,t'\rangle)$.

(b) ϕ is $\pi\eta_0 \ldots \eta_{k-1}$, where π is a k-place predicate letter in L and $\eta_0, \ldots, \eta_{k-1}$ are terms of L. Then $x \in Int_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$ if and only if $\langle Ext_{t,\mathfrak{a}}(\eta_0)(x), \ldots, Ext_{t,\mathfrak{a}}(\eta_{k-1})(x)\rangle \in G(\pi)(t)$ if and only if (by Lemma F) $\langle h_t(\eta_0), \ldots, h_t(\eta_{k-1})\rangle \in G(\pi)(t)$ if and only if (by the definition of G) $\pi h_t(\eta_0) \ldots h_t(\eta_{k-1}) \in F(\langle t,0\rangle)$ if and only if (by Lemma A, T47b, and T69c) $\pi\eta_0 \ldots \eta_{k-1} \in F(\langle t,0\rangle)$ if and only if (by T69b) $\phi \in F(\langle t,t'\rangle)$.

(2) Suppose that $\phi \in \Delta$; then $x \in Int_{\mathfrak{a}}(\neg\phi)(\langle t,t'\rangle)$ if and only if $x \notin Int_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$ if and only if (since $\phi \in \Delta$) $\phi \notin F(\langle t,t'\rangle)$ if and only if (by T68b and the completeness of A) $\neg\phi \in F(\langle t,t'\rangle)$.

(3) Suppose that $\phi, \psi \in \Delta$; then $x \in Int_{\mathfrak{a}}(\phi \rightarrow \psi)(\langle t,t'\rangle)$ if and only if either $x \notin Int_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$ or $x \in Int_{\mathfrak{a}}(\psi)(\langle t,t'\rangle)$, if and only if (since $\phi, \psi \in \Delta$) either $\phi \notin F(\langle t,t'\rangle)$ or $\psi \in F(\langle t,t'\rangle)$ if and only if (by T68b, T69c, the completeness of A, and T69a) $\phi \rightarrow \psi \in F(\langle t,t'\rangle)$.

(4) Suppose that $\phi \in \Delta$ (We may suppose also that every formula ψ such that $rk(\psi)=rk(\phi)$ is in Δ.) and $k \in \omega$; then $x \in Int_{\mathfrak{a}}(\wedge v_k\phi)(\langle t,t'\rangle)$ if and only if for each $\beta \in U$, $x^k_\beta \in Int_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$ if and only if (since U is $\{x_i : i \in \omega\}$) for each $m \in \omega$, $x^k_{x_m} \in Int_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$ if and only if (by T18) for each $m \in \omega$, $x \in Int_{\mathfrak{a}}(ps(v_m, v_k, \phi))(\langle t,t'\rangle)$ if and only if (by the

inductive hypothesis and T8b) for each variable β, $ps(\beta, v_k, \phi) \in F(\langle t,t'\rangle)$ if and only if (by T68b and the completeness of A) for each variable β, $\neg ps(\beta, v_k, \phi) \notin F(\langle t,t'\rangle)$ if and only if (by the completeness of A) $\Lambda v_k \phi \in F(\langle t,t'\rangle)$.

(5) Suppose that $\phi \in \Delta$; then $x \in Int_{\mathfrak{a}}(H\phi)(\langle t,t'\rangle)$ if and only if for each t" such that $t'' <_{\mathfrak{a}} t$, $x \in Int_{\mathfrak{a}}(\phi)(\langle t,t'\rangle)$ if and only if (since $\phi \in \Delta$) for each $t'' <_{\mathfrak{a}} t$, $\phi \in F(\langle t'',t\rangle)$ if and only if for each $t'' <_A t$, $\phi \in F(\langle t'',t'\rangle)$ if and only if (by T68b and the completeness of A) for each $t'' <_A t$, $\neg\phi \notin F(\langle t'',t'\rangle)$ if and only if (by the completeness of A) $H\phi \in F(\langle t,t'\rangle)$.

(6) If $\phi \in \Delta$, then $G\phi \in \Delta$ by an argument analogous to the one for case (5).

(7) Suppose that $\phi \in \Delta$; then $x \in Int_{\mathfrak{a}}(K\phi)(\langle t,t'\rangle)$ if and only if $x \in Int_{\mathfrak{a}}(\phi)(\langle t,t\rangle)$ if and only if (since $\phi \in \Delta$) $\phi \in F(\langle t,t\rangle)$ if and only if (by T70b) $K\phi \in F(\langle t,t\rangle)$ if and only if (by T69b) $K\phi \in F(\langle t,t'\rangle)$.

(8) Suppose that $\phi \in \Delta$; then $x \in Int_{\mathfrak{a}}(R\phi)(\langle t,t'\rangle)$ if and only if $x \in Int_{\mathfrak{a}}(\phi)(\langle t',t'\rangle)$ if and only if (since $\phi \in \Delta$) $\phi \in F(\langle t',t'\rangle)$ if and only if (by T71c) $R\phi \in F(\langle t,t'\rangle)$.

This completes the proof of Lemma G.

By T65, $EX(\Gamma) \in F(\langle 0,0\rangle)$. Hence, by Lemma G, for each $\phi \in EX(\Gamma)$, $x \in Int_{\mathfrak{a}}(\phi)(\langle 0,0\rangle)$. So $EX(\Gamma)$ is satisfiable and (by T107) Γ is satisfiable.

We can now state the final completeness theorems.

T110. Γ is consistent if and only if Γ is satisfiable.

Proof: If Γ is consistent, then Γ is satisfiable, by T109.

Conversely, suppose that Γ is satisfiable and inconsistent. Then by T60, there is a finite set of formulas Γ' such that $\Gamma' \subseteq \Gamma$ and Γ' is inconsistent; by T61, $\vdash \neg\phi$, where ϕ is the conjunction (in order) or the formulas in Γ'. By T33, $\neg\phi$ is logically valid and therefore $\{\phi\}$ is not satisfiable. But then Γ' is not satisfiable and neither, therefore, is Γ. This contradicts the original hypothesis.

T111. $\vdash \phi$ if and only if ϕ is logically valid.

Proof: If $\vdash \phi$ then ϕ is logically valid, by T33.

Suppose ϕ is logically valid. Then $\{\neg\phi\}$ is unsatisfiable and (by T110) $\{\neg\phi\}$ is inconsistent. By T60, $\vdash \neg\neg\phi$; and, by T39a, $\vdash \phi$.

NOTES

1. The logical system set forward in this thesis and its intuitive interpretation were first set forward in full in a paper, "'Now' and 'Then'", read at a meeting of the Australasian Association for Logic, Sydney, August, 1970. The completeness proof was not discovered until 1972.

2. Of course 'is going to' is not a precise translation of 'F'. We use 1 to explain the point of the formal system because it seems to be something like the closest possible natural English translation of 2.

3. This semantical notion of past tense does not correspond very closely to any ordinary grammatical notion. The idea could also be stated as follows: ψ is a past tense of ϕ just in case an utterance of ψ at a given moment would be true if and only if some earlier utterance of ϕ would be true.

4. This seems obvious but is not entirely trivial to prove. A proof can be given that there is no past tense of 2 within the N system (i.e., that 5 is not expressible within the N system) along the lines of the proof in Kamp [5] that there are sentences expressible with N that are not expressible without N.

5. Kamp evaluates formulas with respect to ordered pairs of moments in Kamp [5], but (as he points out) this is unnecessary in the N system; formulas in the N system can be evaluated with respect to single moments if one moment is distinguished as the 'present' one. In the N system the evaluation of a formula at a point of reference $\langle t,t\rangle$ can only involve other points of reference whose second term is t. In the system with K and R this is no longer the case since K changes the second term of the relevant point of reference. For K and R it is essential that the points of reference form a square matrix.

6. This is not strictly correct, since 8 does not say that the moment at which the lights turn out to belong to the house precedes the moment of utterance. We ignore

that point for purposes of simplicity. A more strictly correct symbolization would be $PKP(Q \wedge \bigvee x(L(x) \wedge S(x)) \wedge \bigvee y(R\ H(y) \wedge \bigwedge x((L(x) \wedge S(x)) \rightarrow R\ T(xy))))$

7. 'Intension' is not exactly the right word. We use it because of the correspondence with modal logic and in particular Montague [7].

8. Cocchiarella does not set things up precisely as described here. We put it this way in order to stress the fundamental similarities.

9. The main idea of this proof is taken from Prior [9].

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APPENDIX

A. Stronger Systems

There are other ways of doing the sort of thing that the system of this paper is supposed to do and many systems that are stronger than the present one. The present system was chosen for its relative simplicity, and because it seems sufficient to handle most actual English examples that would naturally be expressed without the use of expressions that refer explicitly to times, like 'the first moment'.

However, the present system is clearly limited in that it is possible to refer back from any temporal context to only one previously established temporal context. As a first illustration of this, suppose we introduce into the system with K and R an additional operator R', which is a true 'now' operator in that it refers back always to the moment of utterance. Then the new system would be more expressive than the system with just K and R because it would be possible sometimes to refer back from the same point to two previously established temporal contexts. For example, the sentence

(1) $FKF(\bigwedge x(A(x) \leftrightarrow RA(x)) \wedge \bigwedge x(B(x) \leftrightarrow R'B(x)))$

could not be expressed without R'. For such a system we would naturally take 3-tuples as points of reference, because it would be necessary to keep track of two (rather than one) previously established moments.

The system with R' would be like the system without it in that there would be sentences with R' (e.g. 1) that would have no past tense (in the sense of page 3) in the system with R'. As in Chapter II, we could again strengthen this system by introducing an additional index operator K'; the operator R', analogously to R, would always refer back to the moment established by K'. In the absence of K' it would refer back to the moment of utterance. This last system would be closed under past and future tense, but would have no true 'now' operator.

For any $n \in \omega$, we could construct in this way a system with n paired 'K' operators $K_0, \ldots, K_{n-1}$ and 'R' operators $R_0, \ldots, R_{n-1}$. The n+1th system would always be stronger than the nth system because it would allow back-reference to one more previously established context.

The limit of this process would be the system that contains operators K_n and R_n for each $n \in \omega$. The key points about the semantics for this system would be as follows:

An interpretation would be as before, but a point of reference would be an ordered pair of an infinite sequence of moments and a natural number.

Suppose $\mathfrak{A}$ is an interpretation, $\mathfrak{A}$ is $\langle T, \leq, U, G \rangle$, $x \in T^{\omega}$, and ϕ is a sentence. Then:

$H\phi$ is true at $\langle x,k\rangle$ if and only if for each $t \leq_{\mathfrak{a}} x_k$, ϕ is true at $\langle x_t^k,k\rangle$.

$K_j\phi$ is true at $\langle x,k\rangle$ if and only if ϕ is true at $\langle x_{x_k}^j,k\rangle$.

$R_j\phi$ is true at $\langle x,k\rangle$ if and only if ϕ is true at $\langle x,j\rangle$.

We would say that ϕ is logically valid if and only if ϕ is true in each interpretation at each point of reference $\langle x,0\rangle$ where all the terms of x are the same. This reflects the idea that for each $j \in \omega$, R_j refers back to the moment of utterance in the absence of K_j.

There is also another way of achieving the same strength as the system just described. We could introduce into Cocchiarella's system just the one operator R, and construct the semantics for R in such a way that R refers back always to the immediately preceding temporal context. We could iterate R's to refer back to contexts further back than the immediately preceding one. Thus, the sentence (1) could be expressed as

(2) $FF(\Lambda x(A(x)\leftrightarrow RA(x)) \wedge \Lambda x(B(x)\leftrightarrow RRB(x)))$

The semantics for this system would be set up as follows:

An interpretation would be as before, but a point of reference would be any finite, non-empty sequence of moments.

Suppose $\mathfrak{a}$ is an interpretation, $\mathfrak{a}$ is $\langle T,\leq,U,G\rangle$, x is a finite, non-empty sequence of moments, and ϕ is a sentence.

Then:

$H\phi$ is true at x if and only if for each $t <_{a} x_{lh(x)-1}$, ϕ is true at $x^{\frown}\langle t\rangle$.

$R\phi$ is true at x if and only if ϕ is true at $x \upharpoonright (lh(x)-1)$, if $1 < lh(x)$; otherwise, $R\phi$ is true at x if and only if ϕ is true at x.

We would say that a formula ϕ is logically valid if and only if ϕ is true in every interpretation at every 1-place point of reference.

B. The Propositional Part of the System with K and R

The operators K and R are superfluous in the propositional part of our system in the sense that, for any propositional formula ϕ, there is a logically equivalent propositional formula ψ such that neither K nor R occur in ψ. We will now make this claim precise and prove it informally.

We define the set of <u>propositional formulas</u> as the smallest set Γ that includes all the propositional constants and such that if $\phi, \psi \in \Gamma$, then $\neg\phi$, $\phi \rightarrow \psi$, $H\phi$, $G\phi$, $K\phi$, $R\phi \in \Gamma$.

Two formulas ϕ, ψ are <u>logically equivalent</u> if and only if $\phi \leftrightarrow \psi$ is logically valid, and <u>strongly logically equivalent</u> if and only if $\phi \leftrightarrow \psi$ is strongly logically valid.

We define a <u>GH-formula</u> as a propositional formula ϕ such that neither $\langle \underline{k} \rangle$ nor $\langle \underline{r} \rangle$ occurs in ϕ.

The claim is simply that every propositional formula is logically equivalent to a GH-formula.

We define a *basic disjunction* as a formula $\phi \vee R\psi$ where ϕ, ψ are GH-formulas, and a *basic conjunction* as a formula $\phi \wedge R\psi$ where ϕ, ψ are GH-formulas.

We define CNF (the set of formulas in *conjunctive normal form*) as the set of formulas which are conjunctions of one or more basic disjunctions, and we define DNF (the set of formulas in *disjunctive normal form*) as the set of formulas which are disjunctions of one or more basic conjunctions.

We need two lemmas in order to establish our claim.

Lemma A. Every formula in CNF is strongly logically equivalent to a formula in DNF, and vice versa.

Proof: The theorem corresponds to the similar theorem in the propositional calculus and can be established by a simple argument involving truth tables. We simply display as an example the logically valid formula $(\phi \vee R\psi) \wedge (\chi \vee R\xi) \leftrightarrow ((\phi \wedge \chi) \wedge R(\phi \vee \neg\phi)) \vee (\phi \wedge R\xi) \vee ((\phi \vee \neg\phi) \wedge R(\psi \wedge \xi)) \vee (\chi \wedge K\psi)$, where ϕ, ψ, χ, ξ are any GH-formulas.

Lemma B. Every propositional formula is strongly logically equivalent to a formula in CNF.

Proof: We prove the Lemma by an induction on the set of propositional formulas. Let Γ be the set of propositional formulas that are strongly logically equivalent to a formula in CNF.

(1) If P is a propositional constant, then P is strongly logically equivalent to $P \vee R(P \wedge \neg P)$, so $P \in \Gamma$.

(2) Suppose $\phi \in \Gamma$. By Lemma A there is a formula ψ in DNF such that ϕ is strongly logically equivalent to ψ. Then there are GH-formulas $\chi_0, \ldots, \chi_n$, $\xi_0, \ldots, \xi_n$ such that ψ is $(\chi_0 \wedge R\xi_0) \vee \ldots \vee (\chi_n \wedge R\xi_n)$. Then $\neg\phi$ is strongly logically equivalent to $\neg((\chi_0 \wedge R\xi_0) \vee \ldots \vee (\chi_n \wedge R\xi_n))$ which is in turn strongly logically equivalent to $\neg(\chi_0 \wedge R\xi_0) \wedge \ldots \wedge \neg(\chi_n \wedge R\xi_n)$, $(\neg\chi_0 \vee \neg R\xi_0) \wedge \ldots \wedge (\neg\chi_n \vee \neg R\xi_n)$, and (by T93) $(\neg\chi_0 \vee R\neg\xi_0) \wedge \ldots \wedge (\neg\chi_n \vee R\neg\xi_n)$. The last is in CNF, so $\neg\phi \in \Gamma$.

(3) Suppose $\phi, \psi \in \Gamma$. By an argument similar to the preceding, there is a formula χ in DNF such that χ is strongly logically equivalent to $\neg\phi$. By Lemma A, there is a formula ξ in DNF such that ψ is strongly logically equivalent to ξ. Then $\phi \rightarrow \psi$ is strongly logically equivalent to $\neg\phi \vee \psi$, which is strongly logically equivalent to $\chi \vee \xi$, which is in DNF. So $\phi \rightarrow \psi$ is strongly logically equivalent to a formula in DNF, and by Lemma A it is also strongly logically equivalent to a formula in CNF.

(4) Suppose $\phi \in \Gamma$. Then there are GH-formulas $\psi_0, \ldots, \psi_n$, $\chi_0, \ldots, \chi_n$ such that ϕ is strongly logically equivalent to $(\psi_0 \vee R\chi_0) \wedge \ldots \wedge (\psi_n \vee R\chi_n)$. Then $H\phi$ is strongly logically logically equivalent to $H(\psi_0 \vee R\chi_0) \wedge \ldots \wedge H(\psi_n \vee R\chi_n)$. But the latter is strongly logically

equivalent to $(H\psi_0 \vee R\chi_0) \wedge \ldots \wedge (H\psi_n \vee R\chi_n)$ which is in CNF, so $H\phi$ is in Γ.

(5) If $\phi \in \Gamma$, then $G\phi \in \Gamma$ as in case (4).

(6) Suppose $\phi \in \Gamma$. Then there is a formula ψ in CNF such that ϕ is strongly logically equivalent to ψ, so $K\phi$ is strongly logically equivalent to $K\psi$. Let ψ' be the result of erasing all the R's from ψ. Then (by GA19) $K\psi$ is strongly logically equivalent to $K\psi'$ which is strongly logically equivalent (by GA15) to $\psi' \vee K\psi'$, which is in CNF, since ψ' is a GH-formula. But $K\phi$ is strongly logically equivalent to $\psi' \vee K\psi'$, so $K\phi \in \Gamma$.

(7) Suppose $\phi \in \Gamma$. Then there is a formula ψ in CNF such that ϕ is strongly logically equivalent to ψ. Then $R\phi$ is strongly logically equivalent to $R\psi$. Let ψ' be the result of erasing all the R's from ψ. Then (T95) $R\psi$ is strongly logically equivalent to $R\psi'$, which is strongly logically equivalent to $(\psi' \wedge \neg\psi') \vee R\psi'$. Since the latter is in CNF, $R\phi \in \Gamma$.

This completes the proof of Lemma B.

To prove the original claim, suppose that ϕ is a propositional formula. Then by Lemma B there is a formula ψ in CNF such that ϕ is strongly logically equivalent to ψ. Let ψ' be the result of erasing all the R's in ψ. Then ϕ is logically equivalent to ψ', and ψ' is a GH-formula.

Although we have not stated it previously, it is clear from the above proof that an effective function can be defined which assigns to each propositional formula a logically equivalent GH-formula, so that decidability results about the propositional part of Cocchiarella's system can be carried over to the propositional part of the system with K and R.