A Categorical Characterization of Accessible Domains

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Chapter 1

Introduction

This project attempts to answer a question posed by Wilfried Sieg: "Can one give a category theoretic characterization of accessible domains?" [78, p. 372] [86, p. 245] To properly answer this question and attribute any significance to such an answer, we must first discuss both the mathematical history and philosophical issues that motivate Sieg's question. Of course, we will also have to say what 'accessible domains' are and why their category theoretic characterization is desirable. This introduction will outline the significance of this question.

There are two related roots to Sieg's considerations that are particularly salient from the perspective of this dissertation. The first is the emergence, especially in the late 19th century and early 20th century, of *structural axiomatics* in mathematics. This idea shaped the way mathematicians conceived of their subject, especially in the context of the burgeoning logic of that time. Concepts were given structural definitions and reasoning with these definitions was increasingly regulated in formal and quasi-formal ways. Examples of such structural definitions include the axiomatizations of a group, ring, and Dedekind's formulation of *simply infinite systems*. Because the definitions are 'structural', they are indifferent on whether such a system exists at all, which we will discuss shortly. David Hilbert's view built on the introduction of structural definitions, combining the axiomatics characteristic of his *Grundlagen der Geometrie* with the later exploration of logic via *Principia Mathematica* to arrive gradually at *formal axiomatics* and his finitist consistency program. The second root of Sieg's considerations builds on this synthesis of structural mathematics and formal logic in Hilbert's thought and is expressed by an analysis of reductive proof theory as a partial fulfillment of Hilbert's finitist program.

The emergence of structural axiomatics gave mathematicians a way to deal with systems of mathematical objects in a rigorous way, making structural features precise and analogies between various structures systematic. Some examples of structural concepts are those of group, ring, and field, but also topological space and Euclidean geometry (in the Hilbert style). We see a particularly significant shift in Dedekind's thought towards this view in *Was sind und was sollen die Zahlen?* where he introduces the structural concept "simply infinite system" to characterize the natural numbers. However, this was not the introduction of particular objects to be studied, but a concept under which many different systems of 'natural numbers' may fall. Then, in this case, Dedekind proved a metamathematical result: all simply infinite systems are isomorphic. Indeed, they are *canonically isomorphic*, as the isomorphism is based purely on the build-up of elements in the simply infinite systems. This is not an incidental feature—it will play a significant role in what follows; more will be said about this below.

When we introduce concepts, like that of a simply infinite system, the question arises whether there is any system falling under it. That is, is the concept even coherent, allowing the exemplification of each aspect of the concept in a single system? The notion of consistency via 'models' was to provide the answer to this question. To show a concept is coherent, you exhibit a system falling under it. If your concept contains internal contradictions, the thinking goes, you will not be able to find or construct such a model, as no actual system can have contradictory features.

The system that Dedekind gives is, for him, purely logical. That is, it rests only on the laws of thought. However, it becomes quickly apparent that this system is not obtained by a standard mathematical argument and that we may have quite justified worries about its specification. This very problem can be seen also in Kronecker's criticism of the introduction of the general concept of irrational numbers. In that case, the mathematical 'model' Dedekind gives, that of cuts, requires strong set-theoretic principles that may be inadmissible from certain constructive perspectives and mediation between such perspectives then becomes a relevant question. How are we to know when these perhaps 'problematic' principles have been used and how are we to know what counts as a proof of consistency when the process of giving models follows no strict mathematical criterion? Indeed, Sieg states that at "the heart of the difference between these foundational positions is the freedom of introducing abstract concepts — given by *structural definitions*"[81, p. 11].

The 'semantic' notion of consistency (e.g. Dedekind's 'logical' model of infinite systems and Hilbert's arithmetical model of Euclidean geometry) was revised in Hilbert's Heidelberg lecture [100, p. 130] to be syntactic in kind. A major step towards the finitist consistency program was the recognition, by Hilbert and Bernays, that logic too could be axiomatized, as in the pivotal lectures given in 1917/18 [35, p. 35]. The joining of the contemporary logic of *Principia* and the contemporary structural mathematics is expressed in Hilbert's articulation of *formal axiomatics* that calls for precise concepts and regulated inference. In describing Hilbert's view, Bernays [24] suggests a refined joining of the *logical*, with its strict formalization of inference, and the *constructive*, replacing the 'transcendental' assumption of mathematical existence with constructions grounding proof theoretic investigations. Hilbert and Bernays gradually arrived at the finitist consistency program that was to "achieve a *structural reduction* for ever more encompassing parts of mathematics to a fixed, elementary, and meaningful part of itself" [81, p. 17]. However, with the publication of Gödel's incompleteness theorems, specifically the second theorem, this program was seen to be untenable as stated. There is no fixed, elementary (weak) system containing enough mathematics for metamathematical work that can be used to prove the consistency of theories like number theory or analysis.

But the proof theory that originated in Hilbert's finitist program remained and indeed flourished after the so-called demise of this program. Giving up the unsustainable notion of "absolute consistency", i.e. consistency relative to a single, philosophically distinguished framework, proof theorists proved important and interesting results of relative consistency between various systems. The interesting results that Sieg points to in the context of our motivating question include the reductions of PA to HA and $(\Pi^1_1 - \mathrm{CA})_0$ to $\mathrm{ID}^i_{<\omega}(\mathcal{O})$ [27].[1] These results show a mathematically precise relationship between parts of classical mathematics and parts of constructively acceptable mathematics. Sieg gives a decisive account of the value of these reductive results saying that they

> are to relate two aspects of mathematical experience—namely, the impression that mathematics has to do with abstract objects arranged in structures that are independent of us, and the conviction that the principles for some structures are evident, because we can grasp the build-up of their elements. [84, p. 274]

The importance of such a relation between two important aspects of mathematical experience is plain: it subverts an extremal view of the philosophy of mathematics. This extreme view would take the constructive and classical as diametrically opposed and irreconcilable. But as Sieg argues, the classical and constructive views emphasize different and related aspects of mathematical experience. This can be seen as a generalization of Hilbert's program. We don't privilege any single part of mathematics (i.e. the finitist part), but instead look to ones with a *generally constructive* character. To put it another way, we replace the restriction to a single *part* of mathematics to the restriction to a certain *sort* of mathematics. In that open-ended way, Sieg argues, we gain "a deepened understanding of what is characteristic of and possibly problematic in classical mathematics and what is characteristic of and taken for granted as convincing in constructive mathematics" [79, p. 4]. That is, we gain a sharper understanding of the relation between two broad conceptions of mathematics, which are clearly both important parts of mathematical experience.

Let us turn to examining what is meant when we said that the generalization of Hilbert's finitist program argues for privileging a certain 'sort' of mathematics. The sort of mathematics intended has a generally constructive, or generative, character. In the second example of a philosophically interesting reduction, that of $(\Pi^1_1 - CA)_0$ to $ID^i_{<\omega}(\mathcal{O})$, the system $ID^i_{<\omega}(\mathcal{O})$ has a particular shape: it is a theory of inductive definitions, of specially generated structures, namely the finite constructive number classes. Only (iterated) inductively defined classes of

[1] See for example the detailed discussions in [75] and [14]. These sorts of reductive results are discussed in detail in [80] and there the continuity of such investigations with Hilbert's finitist program is highlighted. That is, the interest in isolating the principles used and needed for branches such as classical analysis and the discovery of constructive principles that suffice for much of this classical field [82, p. 160].

natural numbers are studied. One classical way to consider such classes comes from Dedekind in his definition of a *chain of a system* a that is defined 'from above' by intersection:

$$\bigcap\{x \mid a \subseteq x \text{ and } f[x] \subseteq x\}$$

with respect to some mapping f. We think of the chain of a as the smallest set containing a that is closed under the function f. Another classical way to consider these inductively defined classes is 'from below', approximating the class using unions over the natural numbers. As an example, the chain of a can be considered as having been obtained by all finite iterations of f on the set a, illustrated by the cumulative sequence

$$a \qquad a \cup f[a] \qquad a \cup f[a] \cup f[f[a]] \quad \cdots$$

or more compactly, the chain of a can be shown to be equal to

$$\bigcup_n a \cup f^n[a].$$

We see in [38] how the i.d. classes of natural numbers are easy to work with because they admit of proof by induction and definition by recursion. These are not mere conveniences, but epistemologically significant features of the i.d. classes and will support a generalization of finitist mathematics. However, a generalization will only succeed if the special epistemological status of finitist mathematics is analyzed. From comments made by Bernays (e.g. [26]) with the analysis and expansion due to Sieg in [84], we are interested in moving from inductively defined classes of natural numbers to more general structures, while still retaining the epistemologically significant parts of natural numbers. We find an extremely broad view of i.d. classes in [1].

Parallel to the above, an i.d. class can be viewed from several perspectives. Most basically, we think of them as the smallest set closed under some rules.

Definition 1.1. A *rule* is a pair (X, x) where X is a set, called the set of *premises*, and x is the *conclusion*. If Φ is a set of rules, called a *rule set*, then a set A is called Φ-closed if each rule in Φ whose premises are in A also has its conclusions in A. If Φ is a rule set, then $I(\Phi)$ is called the set *inductively defined by* Φ and is given by

$$I(\Phi) = \bigcap\{A \mid A \text{ is } \Phi\text{-closed}\}.$$

We can see by this definition that $I(\Phi)$ is also Φ-closed.

As suggested in [38, p. 19], the elements of Φ are not schematic rules, but instances of what we would understand as rules. For example, if we wanted a rule set Φ that allowed us to infer the successor of a natural number, then Φ would be infinite:

$$(\{0\}, 1), (\{1\}, 2), (\{2\}, 3), (\{3\}, 4), \ldots$$

We can think more schematically if we instead consider the rule set as a monotonic function from $\mathcal{P}(\mathbb{N})$ to $\mathcal{P}(\mathbb{N})$.

Rule sets Φ in general can be considered equivalent to monotone operators Γ (i.e. monotone functions on $\mathcal{P}(\mathbb{N})$). If we call a set X *closed under* Γ when $\Gamma(X) \subseteq X$, we can assert the existence of the smallest Γ-closed set

$$I(\Gamma) = \bigcap\{A \mid A \text{ is } \Gamma\text{-closed}\}.$$

And for each rule set Φ, we can associate a monotone operator Γ and vice versa and such that $I(\Phi) = I(\Gamma)$. More precisely, given a rule set Φ, the monotone operator Γ would be defined as

$$\Gamma(X) = \{y \mid (\exists Y)(Y \subseteq X \ \& \ (Y, y) \in \Phi)\}.$$

Using the notion of monotone operators, we can see that the set inductively defined by a rule set Φ is simply the smallest fixed point of the corresponding Γ. Indeed, such a smallest fixed point will always exist, by the well-known Knaster-Tarski fixed point theorem.

Accessible domains are particular i.d. classes where the rule set is *deterministic*. A rule set Φ is deterministic if any two rules (X, x) and (Y, x) of Φ implies $X = Y$. This allows the backwards tracing of inductive generations since given a conclusion x, there is at most one set of premises that could have 'derived' that conclusion. Here is how Sieg describes the introduction of accessible domains:

> Accessible domains comprise elements that are inductively and uniquely generated. They are most familiar from mathematics and logic: the natural numbers, the formulas of first order logic, the constructive ordinals, and the sets in segments of the cumulative hierarchy are generated in this way and form accessible domains. The generating procedures allow us in all these cases to grasp the build-up of the objects and to recognize mathematical principles for the domains constituted by just them. For it is the case, I suppose, that the definition and proof principles for such domains follow directly from the comprehended build-up. [81, p. 338]

The notion of "uniquely generated" used here is the same as 'deterministic'.

The idea behind Sieg's introduction of accessible domains is to give an analysis of 'constructive' generating procedures that is neither too narrow nor too general; we want it to be narrow enough to be a fruitful analysis while still giving us the flexibility to compare different systems. That is, we want to give an abstract description of these inductively defined classes and the notion of accessible domain is to serve that role. This way, we can "compare and explicate the difficulties (in our understanding) of generating procedures" by considering all accessible domains, in their full diversity.

To use a motivating notion from Bernays, accessible domains provide us with a systematic way to investigate *methodological frames* ("methodische Rahmen").[2] For example, we can see the Bourbaki project as being in a particular methodological frame which takes for granted segments of the cumulative hierarchy. In contrast, Kronecker may be viewed as preferring a more restricted frame of natural numbers (at least for pure mathematics). The role of accessible domains in foundational discussions now becomes clearer. They allow us to differentiate foundational perspectives, the methodological frames, based on which accessible domains they find admissible. Accessible domains provide a sliding scale of justification that we can view as having two directions. One direction emphasizes the concrete and intuitively given. We will analyze this as relating to the finite nature of the operations. The other direction grows closer to classical mathematics, such as including a classical powerset operation.

Accessible domains are not merely particular inductively defined classes, but have properties that give serious credence to the idea that they can adequately serve the purposes just described. The two most important properties of accessible domains, hinted at in the quote from Sieg above, is that their deterministic build-up of elements gives intuitive justification for the principles of definition by recursion and proof by induction. Each element can be uniquely associated (or identified) with its build-up, giving us an understanding of the objects of such structures. Thus, laws (proven by induction) and operations (defined by recursion) are immediately graspable from the understanding of such internal structure of the objects of accessible domains.

Let us summarize the description of accessible domains. They are inductively defined classes where each element of the class has a unique construction, given by deterministic rules. This justifies two principles:

1. The proof principle of induction, since the class consists only of elements generated by the inductive rules.

2. The definition principle of recursion, since each object has only one build-up from inductive rules; a function defined on these rules will give a unique output.

And these accessible domains, so defined, will be unique up to canonical isomorphisms, which can be thought of as a more concrete sort of categoricity; the structures are really the same and in the same way.

When discussing Dedekind's *WZ*, we noted that every simply infinite system is *canonically* isomorphic *because* we can grasp the build-up of all the elements of any such system.[3] This is not the case for another system that Dedekind considered: complete ordered fields. We have no such understanding of each

[2] See Sieg's paper on methodological frames [83], which expresses more extensively than our current discussion how he thinks about structural axiomatics, accessible domains and proof theory today.

[3] We should remark that "canonical" is not the same as "unique". This implies that concepts that are unique up to unique isomorphism are not necessarily unique up to 'canonical' isomorphism in the sense used here. 'Canonical' for us means that the isomorphism is defined by recursion, i.e. based on the build-up of elements. This comes directly from Sieg's analysis of Dedekind, e.g. in [87].

element of the complete ordered field and are thus without the concrete understanding necessary for any notion of induction or recursion to come into our considerations. And yet, complete ordered fields are all isomorphic. But they are not *canonically* isomorphic since we do not construct isomorphisms between complete ordered fields exploiting any build-up of elements which we have for simply infinite systems or accessible domains more generally. The proof that complete ordered fields are isomorphic to each other proceeds from the topological conditions of the structure, and not the internal structure of each element as identified by the unique construction sequence for those elements.

This description of accessible domains argues for their importance but does not give us a mathematically rigorous way to define the general notion. My project is to do exactly this. We use the language of category theory to do so, partly for pragmatic reasons but also because category theory itself was introduced to see the connections between abstract structures.[4] The characterization given here is rather simple: accessible domains are initial algebras of functors. But giving a category theoretic characterization of accessible domains is only part of the story. To see how accessible domains, so characterized, can serve as a way to compare methodological frames, we need a mathematically precise way to compare these initial algebras. We achieve this by classifying the functors with respect to which initial algebras are defined. Some accessible domains are given as initial algebras for relatively basic and elementary functors, and some are given by functors akin to the classical powerset operation. The final question we will attempt to answer after giving the characterization is this: What mathematical or conceptual features of category were most important to the project?

We finish with a rather open-ended discussion of features of category theory that help make this characterization successful. We discuss how concepts are given an 'externalized' form that differs from their 'internalized' form. And because it is externalized, these concepts can be applied uniformly to different contexts. One way of changing concepts like this is to see them as dependent on other concepts in novel ways; the primary example is that the notion of powerset will be dependent on a notion of 'smallness' when we discuss capturing segments of the cumulative hierarchy.

[4]Eilenberg and Mac Lane explicitly put their project in the context of Klein's "Erlanger Programm" [34] and both category theory and the Erlanger program are exemplars of structural axiomatics.

Chapter 2

Accessible Domains as Initial Algebras

This chapter will provide us with the concepts needed to characterize accessible domains. The central concept is that of an *initial algebra for an endofunctor*. Informally, we can think of initial algebras as the 'free' algebra (of an endofunctor), the smallest fixed point for the endofunctor, or the object inductively defined by the endofunctor. They were introduced by Lambek in [51] in the context of fixed points for endofunctors and have been studied vigorously by e.g. Jiří Adámek and collaborators starting in [6].

2.1 Initial Algebras

The idea that initial algebras capture the structures that admit induction and recursion is not original to me. Similar claims have appeared for instance in [11]:

> *Initial Algebra Semantics*, studied since the 1970's, uses the tools of category theory to unify recursion and induction at the appropriate abstract conceptual level. [11, p. 3, original emphasis]

The literature is rife with hints of this idea, and only after I had settled on this characterization of accessible domains did I find such a concise expression already stated.[1] Let's see what is involved in this interesting concept.

First we note that the definition of a category, functor, and initial object are taken for granted in the following.[2]

Definition 2.1. An *endofunctor* is a functor $F : \mathbb{C} \to \mathbb{C}$. That is, a functor where the domain and codomain are the same category.

[1] And indeed Orlin Vakarelov [95] has already given a different categorical characterization of accessible domains, that focuses more on the logical features.

[2] We refer to Appendix A and [16] for the background category theory.

Definition 2.2. Let $\mathbb{C}$ be a category and $F : \mathbb{C} \to \mathbb{C}$ an endofunctor. An *algebra of* F, or an F-algebra, is an object X in $\mathbb{C}$ and a morphism $\alpha : F(X) \to X$. We denote the algebra as the pair $\langle X, \alpha \rangle$. We call X the *carrier* and α the *structure map* of the algebra.

An intuition, made perhaps most concrete in Chapter 4, is that F represents a collection of operations and constants. For example, for a category $\mathbb{C}$ with the endofunctor $F(X) = X \times X$, a structure map is a map $b : X \times X \to X$. So an F-algebra in this example is an object X equipped with a binary operation b.

We can make the category of algebras of a functor F by defining what a homomorphism between algebras is and by defining a composition relation between the homomorphisms.

Definition 2.3. A *homomorphism between two algebras* $\langle X, \alpha \rangle$ and $\langle Y, \beta \rangle$ of an endofunctor $F : \mathbb{C} \to \mathbb{C}$ is a morphism $m : X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(m)"] \arrow[d, "\alpha"'] & F(Y) \arrow[d, "\beta"] \\ X \arrow[r, "m"'] & Y \end{tikzcd}$$

Composition of such homomorphisms of algebras is given by composition of the underlying morphisms in $\mathbb{C}$. That is, given homomorphisms $m : X \to Y$ and $n : Y \to Z$, the composition $n \circ m : X \to Z$ is a homomorphism as well.

$$\begin{tikzcd} F(X) \arrow[r, "F(m)"] \arrow[d, "\alpha"'] & F(Y) \arrow[r, "F(n)"] \arrow[d, "\beta"] & F(Z) \arrow[d, "\delta"] \\ X \arrow[r, "m"'] & Y \arrow[r, "n"'] & Z \end{tikzcd}$$

Indeed, since F is a functor, we have that $F(n \circ m) = F(n) \circ F(m)$ and so we get the algebra homomorphism diagram:

$$\begin{tikzcd} F(X) \arrow[r, "F(n \circ m)"] \arrow[d, "\alpha"'] & F(Z) \arrow[d, "\delta"] \\ X \arrow[r, "n \circ m"'] & Z \end{tikzcd}$$

The identity map on X, denoted id_X, will be the identity homomorphism for $\langle X, \alpha \rangle$:

$$\begin{tikzcd} F(X) \arrow[r, "F(id_X)"] \arrow[d, "\alpha"'] & F(X) \arrow[d, "\alpha"] \\ X \arrow[r, "id_X"'] & X \end{tikzcd}$$

Thus we create the category of F-algebras.

Definition 2.4. The *category of F-algebras* in a category $\mathbb{C}$, denoted $Alg_{\mathbb{C}}(F)$, has F-algebras as objects and algebra homomorphisms as arrows.

We will focus on a particular object in the category $Alg_{\mathbb{C}}(F)$.

Definition 2.5. An *initial algebra for an endofunctor F in a category* $\mathbb{C}$ is an initial object in $Alg_{\mathbb{C}}(F)$.

In other words, an initial algebra for a functor F is an F-algebra $\langle X, \alpha\rangle$ such that for any F-algebra $\langle Y, \beta\rangle$, there exists a unique F-algebra homomorphism $h : \langle X, \alpha\rangle \to \langle Y, \beta\rangle$.

A very intuitive case of an initial algebra is the natural numbers. But for that example, it is helpful to know how natural numbers are described in category theory independently of initial algebras.

Definition 2.6 ([55])**.** An object N in a category $\mathbb{C}$ is a *natural number object* if it comes equipped with two arrows $1 \xrightarrow{e} N \xrightarrow{f} N$ and for any object A with $1 \xrightarrow{a} A \xrightarrow{g} A$, there is a unique map $h : N \to A$ such that the following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r, "e"] \arrow[dr, "a"'] & N \arrow[r, "f"] \arrow[d, "h"] & N \arrow[d, "h"] \\
& A \arrow[r, "g"'] & A
\end{tikzcd} .$$

Example 2.1. Consider the functor $S : \mathsf{Sets} \to \mathsf{Sets}$ with $S : X \mapsto 1 + X$. Here we use $+$ to denote the coproduct, or disjoint union in Sets, and 1 to be the terminal object, which can be thought of as an ambiguous singleton $\{*\}$ (cf. Definition A.4). By definition, an S-algebra is an object X of Sets equipped with a morphism

$$[a, g] : 1 + X \to X$$

or equivalently with arrows $a : 1 \to X$ and $g : X \to X$. We will often use this more expanded view of the structure map for clarity.

Let us state what it would mean for an S-algebra, $\langle X, [a, g]\rangle$, to be initial. For $\langle X, [a, g]\rangle$ to be initial means that for any $[a_Y, g_Y] : 1 + Y \to Y$, there is a unique algebra homomorphism $f : X \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r, "a"] \arrow[d] & X \arrow[r, "g"] \arrow[d, "f"] & X \arrow[d, "f"] \\
1 \arrow[r, "a_Y"'] & Y \arrow[r, "g_Y"'] & Y
\end{tikzcd}$$

And since 1 is terminal, the arrow $1 \to 1$ is the identity arrow, so we have

$$\begin{tikzcd}
1 \arrow[r, "a"] \arrow[dr, "a_Y"'] & X \arrow[r, "g"] \arrow[d, "f"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g_Y"'] & Y
\end{tikzcd}$$

which is precisely what it means for X to be a natural number object. This way, we see that the definition of initial S-algebra for this particular S matches the definition of natural number object exactly. We could have even taken 'initial algebra for the functor $X \mapsto 1 + X$' as definitional for natural number objects.

With a definition of initial algebra, and a paradigm example in the natural numbers (via natural number objects), we can better understand where we want to go from here. The main claim we will justify is:

ACCESSIBLE DOMAINS ARE INITIAL ALGEBRAS FOR ENDOFUNCTORS.

Several things are worth noting about the definition of an initial algebra in the context of characterizing accessible domains. First, the definition is what I am calling a *relational* definition, which tells us how the defined object (an algebra for an endofunctor) relates to others of the same type (algebras for the same endofunctor). In this case, initiality says that there is a unique algebra morphism from this defined object to all others of the same type. Although we can understand the initial algebra by reference to some inductive build-up of its elements (since we will argue that they are accessible domains), the definition of an initial algebra requires no such understanding. Instead, the definition 'captures' the right notion by referring only to the structure arrows (i.e. algebra homomorphisms). This is characteristic of category-theoretic definitions that rely on universal properties, of which initiality is one instance.

Secondly, this definition of initial algebra, because of its relational character, is fundamentally contextual. Since the definition relies only on the structure of arrows in a category, initial objects cannot be considered in a vacuum; they must always be considered in a categorical context. This amounts to much the same as the relational character of the definition, but deserves special attention. If a set is an initial algebra, for example, it being so is generally *not* invariant under contexts (i.e. categories), where invariant here means that the same set will always be an initial algebra no matter what category and endofunctor you consider. This aspect is particularly important since it highlights the mathematical structuralism that is used throughout this dissertation. Mathematical structuralism, in this context, is the method of considering mathematical objects only in their contexts. A standard example, that can be used to highlight some possibly challenging intuitions inherent in structuralism, is that the natural number 1 is not identical to the rational number 1 which is not identical to the real number 1. This example can indeed be problematic, but here I just want to emphasize that objects are considered in context and we have no way of determining, under our characterization of accessible domains, whether an object (set, structure, etc.) is an accessible domain without also specifying the categorical context.

The contextual necessity included in our characterization is not to be seen as a drawback, but as an important benefit to choosing category theory for our characterization. It may seem that having a context-free definition of accessible domains would be required to show the true essence of these sorts of mathematical structures. While I would never say having such an independent definition would be useless, characterizing accessible domains through a context-dependent notion, like initial algebras for endofunctors, allows us to see how deeply related the notion of 'inductive build-up' is to the mathematical context in which we are working. The relationship between accessible domains and their contexts is particularly important when we think of them as methodological frameworks, allowing the fruitful comparison of various philosophical, logical, and mathematical perspectives. For example, knowing exactly how the accessible domains of the constructive number classes relate to the notion of 'constructive' is absolutely crucial for understanding the *significance* of these number classes, not just what they 'look like'. It is this more methodological purpose to which we put our characterization.[3]

Let us expand on the import of the example of the natural numbers in the context of our characterization. As we know, the natural number structure is not a single unique structure, but can be usefully defined in a number of ways. For example, it is an arbitrary choice whether we consider the natural numbers as starting with 0 or 1. But more importantly, as Dedekind showed, there are elementary axioms that characterize the *type* of structure we call the natural numbers. With these structural axioms, under which many systems may fall, we can recognize the impressive conceptual leap from considering natural numbers as a particular and given set to considering them abstractly characterized by 'characteristic conditions'.[4] Dedekind's notion of a *simply infinite system* characterizes the natural numbers:

> If in the consideration of a simply infinite system N set in order by a transformation ϕ we entirely neglect the special character of the elements; simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting transformation ϕ, then are these elements called *natural numbers* [30, p. 33]

With this characterization of the natural numbers, Dedekind is free to prove theorems that were usually considered as given principles of the natural numbers, such as the definition principle of recursion and the proof principle of induction. Both of which, it should be emphasized, are absolutely crucial in the general considerations of accessible domains! A modified version of Dedekind's Theorem §126 in *Was sind und was sollen die Zahlen?* would be as follows:

Theorem 2.7 (Dedekind's Recursion Theorem)**.** *Let $a \in A$ and $g : A \to A$ be a (set-theoretic) function from A to A and N a simply infinite system. Then*

[3] Recall that one of the purposes of Sieg's discussion of reductive proof theory and our characterization of accessible domains is to gain "a deepened understanding of what is characteristic of and possibly problematic in classical mathematics and what is characteristic of and taken for granted as convincing in constructive mathematics". [79, p. 4]

[4] It is impressive indeed when we also consider that Dedekind made such a conceptual move only in the penultimate draft of *Was sind und was sollen die Zahlen?*. See [87, p. 268–9] for the illuminating analysis of this detailed development as well as the thorough [89] for the development of Dedekind's theory of numbers.

there exists a unique function $f : N \to A$ such that

$$f(0) = a$$
$$f(s(n)) = g(f(n)).$$

The fact that we can iterate a function g on a set A (i.e. define a function by recursion) using any simply infinite system—neglecting the special character of the elements—is important. A similar result allows us to prove statements about all elements of a simply infinite system through proofs by induction. Indeed, the proof principle of induction doesn't depend on the axioms for simply infinite systems per se, but instead comes directly from the properties of chains of a system.[5]

Our development of initial algebras as a characterization of accessible domains is parallel to, and indeed generalizes, Dedekind's characterization of natural number structures as simply infinite systems. Just as Dedekind proves the principle of induction and definition by recursion for simply infinite systems, in the next two sections, we show how the generalizations of such principles can be derived from the definition of initial algebra for an endofunctor. Dedekind's definition of simply infinite system is the not the same as our initial algebras, for it focuses on elements of N and not purely on the relations between N and other sets.[6] However, as we will see, Dedekind's definition is provable from our characteriation; Dedekind's definition is our theorem and his Recursion theorem is our definition, as we see in the next section.

Let us now see how our proposed characterization fares in recapturing the central notions of definition by recursion and proof by induction. We will handle each in a section of its own.

Recursion comes from the unique homomorphism that is guaranteed by the definition of initial algebra and induction comes from a theorem (Theorem 2.16) that any 'sub-algebra' of an initial algebra is actually isomorphic to that initial algebra. In fact, these results come rather directly from the definition, but we take the time to go into the details for it helps us see the interesting way this captures the notion of accessible domain.

2.2 Recursion

The principle of definition by recursion requires deterministic inductive clauses and this amounts to requiring a unique algebra homomorphism from the inductively defined class to any other. Definition by recursion is thus an application of the property of initiality in the context of algebras for endofunctors.

[5]Simply infinite systems are defined as the chains of certain systems under certain mappings. This distinction between 'where' the proof principle of induction comes from is not central or particularly sharp, but it highlights the parallels between Dedekind's development in *WZ* and this dissertation. As we'll see, a generalized notion of induction comes from the initiality of accessible domains and recursion comes from the fact that it's initial with respect to algebra homomorphims, informally speaking.

[6]In our case, the relations are algebras homomorphisms for the endofunctor $1 + X$.

We can best show how to recover recursion by keeping with the example of the natural numbers, defined as the initial algebra for the 'successor functor' $S : \mathsf{Sets} \to \mathsf{Sets}$ we used in Example 2.1. We will first show how simple recursion and then recursion with parameters is captured.

Example 2.2 (Simple Recursion). Consider again the successor functor $S : \mathsf{Sets} \to \mathsf{Sets}$ taking a set X to the set $1 + X$. An initial algebra for this functor is $\langle N, [0, s]\rangle$, where $s : N \to N$ is the successor function and $0 : 1 \to N$ takes the element $*$ to $0 \in N$. So N is equipped with the structure map

$$[0, s] : 1 + N \to N$$

and is initial with respect to other maps $1 + N \to N$.

Now, consider the principle of definition by recursion on the natural numbers. Given any set A together with a distinguished element $a \in A$ and a map $g : A \to A$, there is a unique map $f : N \to A$ such that

$$\begin{aligned} f(0) &= a \\ f(s(n)) &= g(f(n)). \end{aligned}$$

We can rephrase the specification of g and a as defining a map

$$[a, g] : 1 + A \to A$$

since elements are points of $a : 1 \to A$ (Remark A.6). Note that $[a, g]$ is therefore a structure map of the S-algebra $\langle A, [a, g]\rangle$. The two equations that define f express exactly that we require the following diagram to commute:

$$\begin{tikzcd} 1 + N \arrow[r, "S(f)"] \arrow[d, "{[0,s]}"'] & 1 + A \arrow[d, "{[a,g]}"] \\ N \arrow[r, "f"'] & A \end{tikzcd} .$$

To the see the equivalence of the commutative diagram and the recursion equations, we simply write out what commuting means in this case. The diagram states that $f \circ [0, s] = [a, g] \circ S(f)$. Breaking up the co-paired arrows, we get the two equations

$$\begin{aligned} f \circ 0 &= a \circ S(f) \\ f \circ s &= g \circ S(f)\ . \end{aligned}$$

Since $S(f) = id_1 + f$, we can rephrase these equations:

$$\begin{aligned} f \circ 0 &= a \circ id_1 \\ f \circ s &= g \circ f \end{aligned}$$

which is to say, adding arguments to the functions whose domain is not 1,

$$
\begin{aligned}
f(0) &= a \\
f(s(x)) &= g(f(x))
\end{aligned}
$$

as desired. The existence of such a map f that will make the diagram commute is given by the unique S-algebra homomorphism guaranteed by $\langle N, [s, 0]\rangle$ being initial. Thus, for any set A and $[a, g] : 1 + A \to A$, we have a unique function f that satisfies the recursion equations.

Example 2.3 (Parameterized Recursion). We can expand definitions by recursion to include parameters. Namely, we want to show that given two functions

$$
\begin{aligned}
g &: X \to A \\
h &: A \times X \times N \to A
\end{aligned}
$$

there exists a unique $f : N \times X \to A$ such that

$$f(0, x) = g(x) \tag{2.1}$$

$$f(s(n), x) = h(f(n, x), n, x). \tag{2.2}$$

If we define a map $\bar{f} : N \to A^A$ by simple recursion (Example 2.2) we can then transpose[7] to get the desired $f : N \times A \to A$. To use definition by recursion, we must give a structure map $\alpha : A^A + 1 \to A^A$ such that the unique homomorphism $N \to A^A$ is the $\bar{f}$ we want.

Define α by cases as

$$
\begin{aligned}
\alpha(*) &= g : A \to A \\
\alpha(k) &= \lambda a.h(k(a), a) \text{ for all } k \in A^A.
\end{aligned}
$$

Thinking of g as our distinguished element of A^A and $\lambda a.h(k(a), a)$ as our function that takes elements of A^A to elements of A^A, we see that we can use α for a definition by recursion on N.[8] Thus there is a unique $\bar{f}$ such that

$$
\begin{aligned}
\bar{f}(0) &= g \\
\bar{f}(s(n)) &= \lambda a.h(\bar{f}(n)(a), a)
\end{aligned}
$$

Now we can see that the transpose of $\bar{f}$ satisfies 2.1 and 2.2 as desired. Similarly, we can broaden this to show that there is a unique function $f : N \times A \to A$ such that

$$
\begin{aligned}
f(0, x) &= g(x) \\
f(s(n), x) &= h(f(n, x), n, x).
\end{aligned}
$$

[7] This is the operation that uniquely associates arrows $A \to C^B$ with arrows $A \times B \to C$.

[8] The λ notation is shorthand for the perhaps more familiar $a \mapsto h(k(a), a)$ for $a \in A$.

Now that we have seen the relationship between the definition of the natural numbers as the initial S-algebra and the principle of definition by recursion, let us attempt to look at the general picture. The initiality of N was what ensured the existence and the uniqueness of functions that satisfy the recursion equations. Towards generalizing these observations, let us reconstruct, in steps, the recursion principle with more suggestive and general language.

First we have the traditional principle:

Proposition 1. *Given any set A together with a distinguished element $a \in A$ and a map $g : A \to A$, there is a unique map $f : N \to A$ such that*

$$\begin{aligned} f(0) &= a \\ f(s(n)) &= g(f(n)). \end{aligned}$$

And now we can see that the f whose existence is asserted in this principle is really an S-algebra homomorphism, since it must preserve the two 'constructors' of S, namely take the distinguished element 0 to the distinguished element a and the successor s commutes with the map g. This is what the recursion equations amount to: f preserves the 'successor-algebra' structure, that is, preserves the information contained in the successor functor (i.e. distinguished element and successor operation). Preservation of this structure is precisely what it means to be a homomorphism of S-algebras. This suggests the reformulation in the following way:

Proposition 2. *Given any set A together with a distinguished element $a \in A$ and a map $g : A \to A$, there is a unique S-algebra homomorphism $f : N \to A$.*

And as we noticed in Example 2.2, we can reformulate the specification of a and g into one map $[a, g] : 1 + A \to A$. This is an S-algebra structure map for the S-algebra $\langle A, [a, g]\rangle$. Thus, we have the following reformulation of the recursion principle for natural numbers:

Proposition 3. *Given any S-algebra $\langle A, [a, g]\rangle$, there is a unique S-algebra homomorphism $f : N \to A$.*

So finally we can see that this Proposition is justified by the fact that we consider $\langle N, [0, s]\rangle$ as the name for the initial S-algebra. The intuitive picture of this correspondence is compelling: if any object of a category is equipped with a distinguished element and some arrow into itself (note that there are no constraints on g), then you can uniquely embed N into that object, taking distinguished element to distinguished element and 'commuting' with successor. That N is the 'generic' S-algebra is perhaps too informal, but this is why we call N the *free S-algebra*.

Let us define a function on the natural numbers by recursion, using the category-theoretic tools we now have available. First, let us note what the parameterized form of recursion looks like diagrammatically speaking.

Theorem 2.8 (Prop. 2.5.2 in [45]). *Let $(N, 0, s)$ be a natural numbers object in a cartesian closed category. Then, given morphisms $g : X \to A$ and h :*

$A \times N \times X \to A$, there exists a unique morphism $f : N \times X \to A$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "0\times 1_X"] \arrow[dr, "g"'] & N \times X \arrow[d, "f"] & N \times X \arrow[l, "s\times 1_X"'] \arrow[d, "\langle f, 1_{N\times X}\rangle"] \\ & A & A \times N \times X \arrow[l, "h"] \end{tikzcd}$$

commutes.

We will not prove that this proposition follows from the definition of N as an initial S-algebra (equivalently, a natural numbers object). Suffice it to say that the proof uses the same transposition trick we used when deriving the parameterized recursion above in Example 2.3. This is why the arrows seem flipped from the original definition, it comes from using transposition and introducing products into the diagram.

This proposition provides the categorical way of understanding the general method of giving (primitive) recursive definitions for functions on N. Let us see how this is, by deriving the recursion equations from the diagram. First, the base case. For this, let us just consider the left triangle:

$$\begin{tikzcd} X \arrow[r, "0\times 1_X"] \arrow[dr, "g"'] & N \times X \arrow[d, "f"] \\ & A\,. \end{tikzcd}$$

In this diagram, we have hidden the equation of $X \cong 1 \times X$ and so, to make better sense of the arrows, we have the square

$$\begin{tikzcd} 1 \times X \arrow[r, "0\times 1_X"] \arrow[d, equal] & N \times X \arrow[d, "f"] \\ X \arrow[r, "g"'] & A\,. \end{tikzcd}$$

Let us trace an element $(*, x) \in 1 \times X$ around this diagram to see how it gives us the equations for parameterized recursion. Going along the bottom, we associate $(*, x)$ with x and then apply g to get $g(x)$. Along the top, we apply $0 \times 1_X$ to $(*, x)$ to get $(0, x)$, where this latter 0 is $0(*)$, the standard way we've been representing the distinguished element of N. Then f applied to that gives us simply $f(0, x)$. The diagram commuting means that the two results are equal:

$f(0,x) = g(x)$. Diagrammatically, we can see this as

$$\begin{tikzcd}
(*,x) \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & (0,x) \arrow[dd, mapsto] \\
& 1\times X \arrow[r, "0\times 1_X"] \arrow[d, equal] & N\times X \arrow[d, "f"] & \\
& X \arrow[r, "g"'] & A & f(0,x) \arrow[dl, equal] \\
x \arrow[rr, mapsto] & & g(x) &
\end{tikzcd} .$$

Thus we have the base case

$$f(0,x) = g(x).$$

For the successor case, we focus on the right square of diagram in Theorem 2.8.

$$\begin{tikzcd}
N\times X \arrow[d, "f"'] & N\times X \arrow[l, "s\times 1_X"'] \arrow[d, "\langle f,1_{N\times X}\rangle"] \\
A & A\times N\times X \arrow[l, "h"]
\end{tikzcd}$$

Now we take an element of the top *right* object and track what happens to it. This time, sticking to the diagrammatic way of thinking, we have

$$\begin{tikzcd}
(s(n),x) \arrow[dd, mapsto] & & & (n,x) \arrow[lll, mapsto] \arrow[ddd, mapsto] \\
& N\times X \arrow[d, "f"'] & N\times X \arrow[l, "s\times 1_X"'] \arrow[d, "\langle f,1_{N\times X}\rangle"] & \\
f(s(n),x) \arrow[dr, equal] & A & A\times N\times X \arrow[l, "h"] & \\
& h(f(n,x),n,x) & & (f(n,x),n,x) \arrow[ll, mapsto]
\end{tikzcd} .$$

Thus we have the successor case

$$f(s(n),x) = h(f(n,x),n,x).$$

Let us see this in action with two examples: addition and multiplication.

Example 2.4 (Addition). In this case, $A = X = N$, which will be true for all functions of elementary arithmetic. Setting $g : N \to N$ to be the identity 1_N and $h : N\times N\times N \to N$ to be $s\circ\pi_1$, we get the following diagram:

$$\begin{tikzcd}
N \arrow[r, "0\times 1_N"] \arrow[dr, "1_N"'] & N\times N \arrow[d, "+"] & N\times N \arrow[l, "s\times 1_N"'] \arrow[d, "\langle +,1_{N\times N}\rangle"] \\
& N & N\times N\times N \arrow[l, "s\circ\pi_1"]
\end{tikzcd}$$

which means, by our above diagram-to-equation analysis that for all $n \in N$,

$$\begin{aligned} 0 + n &= n \\ s(m) + n &= s(m + n). \end{aligned}$$

Example 2.5 (Multiplication). Again, we set $A = X = N$. We exploit the fact that $+ : N \times N \to N$ is already defined by the previous example to get the following diagram:

$$\begin{tikzcd} N \arrow[r, "0\times 1_N"] \arrow[dr, "0\circ !"'] & N \times N \arrow[d, "\times"] & N \times N \arrow[l, "s\times 1_N"'] \arrow[d, "\langle \times, 1_{N\times N}\rangle"] \\ & N & N \times N \times N \arrow[l, "+(\pi_1,\pi_3)"] \end{tikzcd}$$

which yields the desired equations

$$\begin{aligned} 0 \times m &= 0 \\ s(n) \times m &= (n \times m) + m \ . \end{aligned}$$

For our 'g', we used $0 \circ ! : N \to 1 \to N$ which collapses all natural numbers to the single element of 1 and then uses the 0 morphism to pick out 0. This gives the idea on how we may use this method to define functions whose base case is constant.

It is perhaps remarkable that in the definition of the natural numbers object N, we did not require $s : N \to N$ to be injective, or that 0 to be outside the image of s. The notion of 'freeness' or initiality handles these features automatically, so to speak. The natural numbers object, N, being initial for S requires that the elements are not identified with each other. If $s(k) = s(i)$ with $k \neq i$, then N would not be the initial S-algebra. We will now prove this fact: given the characterization of natural number object N as the initial S-algebras in Sets, we have that s is injective and 0 is not in the image of s.

First, we prove a theorem due to Freyd [40] included in [45, pp. 111–112]. This theorem can be understood as stating that the successor s is injective by showing it is a coproduct inclusion. Coproduct inclusions are the dual of product projections, and are part of the definition of coproducts: they are the arrows included with every coproduct diagram. By showing that the successor is a coproduct inclusion, it will follow that the successor is a monomorphism in Sets and therefore an injection.

Along similar but perhaps unfamiliar lines, we also show that $0 \neq s(n)$ for every $n \in N$ by showing that 0 is a coproduct inclusion with s as the other arrow. In the right categories (like Sets), we know that the coproduct acts as a *disjoint* union. Thus, $0 : 1 \to N$ takes the single element $* \in 1$ to $0 \in N$, and since N will be shown to be the disjoint union with respect to s and 0, we know that the image of 0 and s are themselves disjoint.

Theorem 2.9. *Let $(N, 0, s)$ be a natural numbers object in a category $\mathbb{C}$. If $\mathbb{C}$ has binary products and coproducts, then*

$$1 \xrightarrow{0} N \xleftarrow{s} N$$

is a coproduct diagram.

Proof. If $\mathbb{C}$ has binary coproducts, then the natural number object in $\mathbb{C}$ is the same thing as the initial S-algebra for $S : \mathbb{C} \to \mathbb{C}$ with $S(X) = 1 + X$. We need binary coproducts to be able to define such a functor, but if it can be defined, we have the agreement between initial S-algebras and natural number objects.

To show that the diagram is a coproduct diagram, assume we have arrows indicated in the following diagram:

$$\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr, "g_1"'] & N & N \arrow[l, "s"'] \arrow[dl, "g_2"] \\
& X &
\end{tikzcd}.$$

We need to show that there exists a unique arrow $k : N \to X$ such that $k \circ 0 = g_1$ and $k \circ s = g_2$. The plan is to define k by recursion, which is to say by the initial algebra diagram in Definition 2.6. We will do this by constructing another S-algebra.

We define two arrows by using the definition of Cartesian Product, which the category must have in order for this result to hold. The intuition is that our definition of k by recursion will depend on parameterized recursion which relies on the existence of certain Cartesian Products. First, the product arrow of $\langle 0, g_1 \rangle$:

$$\begin{tikzcd}
& 1 \arrow[dl, "0"'] \arrow[d, dashed, "\langle 0, g_1 \rangle"] \arrow[dr, "g_1"] & \\
N & N \times X \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & X
\end{tikzcd}$$

and then the product arrow $\langle s, g_2 \rangle$:

$$\begin{tikzcd}
& N \arrow[dl, "s"'] \arrow[d, dashed, "\langle s, g_2 \rangle"] \arrow[dr, "g_2"] & \\
N & N \times X \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & X
\end{tikzcd}.$$

Note that for these two product diagrams, the respective projections π_1 and π_2 are identical.

With the arrows $\langle 0, g_1 \rangle$ and $\langle s, g_2 \rangle$, we can define the arrows

$$1 \xrightarrow{\langle 0, g_1 \rangle} N \times X \xrightarrow{\pi_1} N \xrightarrow{\langle s, g_2 \rangle} N \times X$$

or more compactly as

$$1 \xrightarrow{\langle 0, g_1\rangle} N \times X \xrightarrow{\langle s, g_2\rangle \circ \pi_1} N \times X\ .$$

By the definition of N as a natural number object, we have a unique map $\langle h, k\rangle : N \to N \times X$ such that the following diagram commutes:

$$\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr, "\langle 0, g_1\rangle"'] & N \arrow[r, "s"] \arrow[d, "\langle h,k\rangle"] & N \arrow[d, "\langle h,k\rangle"] \\
& N \times X \arrow[r, "\langle s, g_2\rangle \circ \pi_1"'] & N \times X
\end{tikzcd} .$$

We represent the unique map as a product map, since if we have any $f : N \to N \times X$, we can define $h = \pi_1 \circ f$ and $k = \pi_2 \circ f$ so that $f = \langle h, k\rangle$. To tie this use of the definition of natural numbers object to S-algebras, note that

$$\langle N \times X, [\langle 0, g_1\rangle, \langle s, g_2\rangle \circ \pi_1]\rangle$$

is an S-algebra. We use the natural numbers object definition because it shows the components in a less condensed way.

Recall the intuition of what this natural number object diagram tells us: given the distinguished $\langle 0, g_1\rangle \in N \times X$ and the endomorphism $\langle s, g_2\rangle \circ \pi_1 : N \times X \to N \times X$, we can define a map $\langle h, k\rangle$ by recursion on the initial S-algebra N so that 0 is taken to the distinguished element of $N \times X$ and the successors commute with the endomorphism.

From this diagram, we get that $h \circ 0 = 0$ from the left triangle. In addition, the square in the diagram provides the equation

$$\begin{aligned}
\langle h, k\rangle \circ s &= \langle s, g_2\rangle \circ \pi_1 \circ \langle h, k\rangle \\
&= \langle s, g_2\rangle \circ h
\end{aligned}$$

which implies that $h \circ s = s \circ h$. Since $\langle h, k\rangle$ is unique, and 1_N satisfies these same two equations, we have that $h = 1_N$.

This means that $k : N \to X$ is such that

$$\begin{aligned}
k \circ 0 &= g_1 && \text{(diagram triangle)} \\
k \circ s &= g_2 \circ h = g_2 && \text{(diagram square and } h = 1_N\text{)}
\end{aligned}$$

which is the desired arrow $N \to X$ we need to show that our original diagram is a coproduct diagram:

$$\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr, "g_1"'] & N \arrow[d, dashed, "k"] & N \arrow[l, "s"'] \arrow[dl, "g_2"] \\
& X &
\end{tikzcd} .$$

The uniqueness of k in this regard comes from the uniqueness of $\langle h, k\rangle$ given by N being an initial S-algebra, or equivalently, a natural numbers object. $\square$

The main form of argumentation here is to exploit the definition of natural numbers object to get the unique arrow necessary for the diagram to be a coproduct diagram.[9] If we understand this definition as expressing the principle of definition by recursion, we can see that this proof comes down to a clever use of recursion to show that 0 and s never 'intersect' (i.e. their pullback is initial) because they are part of a coproduct diagram.

An intuition behind coproducts is that of disjoint union. This intuition is sound in many contexts, like toposes such as Sets, and it indicates something like $0 \cap s = \emptyset$ which can be interpreted as the Dedekind-Peano axiom

$$0 \neq s(n) \quad \text{for any number } n.$$

Moreover, s being a coproduct inclusion will give us, in categories like Sets, that s is injective. Toward making this a bit more precise, we state the following lemma, without proof, and important corollaries of Theorem 2.9.

Lemma 2.10. *Let $i_A : A \to A + B$ be a coproduct inclusion in a cartesian closed category with coproducts. Then i_A is monic.*

For our context, this gives us the following.

Corollary 2.11. *The arrows 0 and s of the natural number object N are monic (injective) in a cartesian closed category with coproducts.*

Proof. Both 0 and s are coproduct inclusions by Theorem 2.9 and thus monic by Lemma 2.10. That 0 is monic can also easily follow from the fact that it is an arrow $1 \to N$ which are always monic. $\square$

Corollary 2.12. *In an extensive category (e.g.* Sets*; cf. [29])The pullback (intersection) of 0 and s is an initial object (empty).*

This shows, in part, how we can prove the Peano axioms for the set of natural numbers:

1. $0 \in N$: this comes from the definition of $0 : 1 \to N$.

2. $x \in N \to s(x) \in N$: this comes from the nature of maps $s : N \to N$.

3. $x \in N \to s(x) \neq 0$: this comes from the disjointness of 0 and s in Corollary 2.12.

4. s is injective: this comes from s being a coproduct inclusion as shown in Theorem 2.9 and Corollary 2.11.

5. Principle of Induction: This will be shown in the next section., which analyzes initial algebras for endofunctors as 'inductive'.

And we can see a very similar result using the slightly more condensed Dedekind axioms for *simply infinite systems* N:

[9] We should note that Lambek's Lemma (Theorem 3.1) implies that $N \cong 1 + N$ and so s and 0 form a coproduct diagram with N.

1. $s[N] \subseteq N$: this is given by $s : N \to N$.
2. $0 \notin s[N]$: this comes from the disjointness of 0 and s as in Corollary 2.12.
3. s is similar (injective): this comes from s being a coproduct inclusion as shown in Theorem 2.9 and Corollary 2.11.
4. N is the chain of $\{0\}$ with respect to s : This can be understood simply as a minimality condition, which we will explore more fully in the next section.

It is rather appealing to have these conditions given by the the definition of initial S-algebra. In Dedekind [32], the converse is done, deriving the recursion *theorem* from the definition of N as a simply infinite system: 0 is not in the image of s and the minimality of the set with respect to closure under $[s, 0]$. So if you take the recursion principle as given, you can derive these properties of 0 and s, and vice versa, if you take the properties as given, you can prove the recursion theorem. My view on why the former is preferrable for our purposes is that it more uniformly applies to other sorts of algebras for endofunctors.[10] The properties of the structure map will change depending on the algebra under consideration, but the uniqueness of homomorphisms to other algebras is a condition more uniformly stated.

The conclusion of this section has been that

INITIAL ALGEBRAS ARE RECURSIVE.

2.3 Induction

The definition of an initial algebra asserts the existence of a unique homomorphism from that algebra to all others (for the same functor). In this section, we explain how to understand the relationship between this initiality and a notion of minimality. Recall that in Dedekind's analysis of the natural numbers as an abstracted simply infinite system, the notion of minimality is crucially part of the definition of a chain of a system. We will see that this minimality is reflected in the definition of initial algebra for endofunctors more generally.

The first parallel with Dedekind's development in [32] we will see is as follows. Dedekind defines the chain of a system A to be the intersection of all sets that contain that system and are closed under a mapping f:

$$\bigcap\{X \mid A \subseteq X \;\&\; f[X] \subseteq X\}.$$

In the specification of intersections of families of systems, Dedekind remarks that the intersection of $\{A, B, C, \ldots\}$ is a common part of each A, B, and C, in the sense that the intersection is a subset of each.[11] This definition is impredicative

[10] More will be said about this in Chapter 6, where we discuss uniformity.

[11] Dedekind gives the definition of intersection as Definition 17 in *WZ*. Dedekind did not ascribe meaning to the term $\bigcap\{A, B, C, \ldots\}$ if these sets had no common elements. No intersection can thus be empty, for Dedekind.

in the sense that the set defined is already present in the set over which we are taking the intersection:

$$\bigcap\{X \mid A \subseteq X \;\&\; f[X] \subseteq X\} \in \{X \mid A \subseteq X \;\&\; f[X] \subseteq X\}.$$

The intersection functions as a way to specify the *smallest set containing A and closed under f*, in the sense that it contains A, is closed under f, and is a subset of any other set that contains A and is closed under f. With this, Dedekind proves a set-theoretic version of complete induction (Theorem 59 in *WZ*) and notes that this "forms the scientific basis" for the proof principle of induction for chains of systems. And since later, N is defined as a chain of a system, we can use the proof principles of induction for natural numbers.

As to the development given here, we first note that simply infinite systems being the 'smallest' set satisfying certain conditions is equivalent to saying that they have a unique arrow to all sets also satisfying these conditions. This comes from the fact that arrows in a category can easily interpret relations, in this case the subset relation. When we interpret relations as arrows of a category, we only have one arrow per relation, which means that there is at most one arrow between any two objects—all arrows are unique. So to say that $N \subseteq Z$ for all sets $Z \in \{X \mid 0 \in X \;\&\; s[X] \subseteq X\}$ is to say that for any such Z, there is an unique arrow

$$N \to Z.$$

This is what gets generalized in considering accessible domains as initial algebras for endofunctors, we require a unique algebra homomorphism instead of a subset relation, but the intuition is the same: minimality.

To see this parallel in more detail, we need to understand the generalization of subsets to *subobjects* as well as the notion of *subalgebras*. The idea is to generalize the inclusion maps to the categorical notion of an injection, which are monomorphisms. Let $\mathbb{C}$ be a category and F an endofunctor on $\mathbb{C}$.

Definition 2.13. For an object A of $\mathbb{C}$, we define a *subobject* of A as an equivalence class of monomorphisms into A, denoted $[S \rightarrowtail A]$. Two monomorphisms $m_1 : S_1 \rightarrowtail A$ and $m_2 : S_2 \rightarrowtail A$ are equivalent if there is an isomorphism between their domains, that is $S_1 \cong S_2$ such that the following diagram commutes:

$$\begin{tikzcd} S_1 \arrow[rr, "\cong"] \arrow[dr, "m_1"'] & & S_2 \arrow[dl, "m_2"] \\ & A & \end{tikzcd}$$

It is customary to consider these equivalence classes of monomorphisms through a representative $i : S \rightarrowtail A$. In Sets, the notion of subobject collapses to (equivalence classes of) subsets (with respect to equinumerosity) as expected, but monomorphisms can't always be considered as inclusions. We call subobjects in the category $Alg_{\mathbb{C}}(F)$ subalgebras.

Definition 2.14. We call the subobject $m : \langle S, \sigma\rangle \rightarrowtail \langle A, \alpha\rangle$ in $Alg_{\mathbb{C}}(F)$ a *subalgebra* of $\langle A, \alpha\rangle$ if the underlying $\mathbb{C}$ map $m : S \to A$ is mono in $\mathbb{C}$.

The sense of minimality we want is with respect to subalgebras.

Definition 2.15. An algebra $\langle A, \alpha\rangle$ is *minimal* if for any subalgebra of $\langle A, \alpha\rangle$ $i : \langle R, \rho\rangle \rightarrowtail \langle A, \alpha\rangle$, i is an isomorphism. In other words, $\langle A, \alpha\rangle$ is *minimal* if all subalgebras are isomorphic with $\langle A, \alpha\rangle$.

This definition can also be interpreted as saying that there are no proper subalgebras of $\langle A, \alpha\rangle$.

Theorem 2.16. *Initial algebras are minimal.*

Proof. Let $\langle I, \iota\rangle$ be an initial algebra and $i : \langle R, \rho\rangle \rightarrowtail \langle I, \iota\rangle$ be a subalgebra. Since $\langle I, \iota\rangle$ is initial, there exists a unique arrow $! : \langle I, \iota\rangle \to \langle R, \rho\rangle$:

$$\langle I, \iota\rangle \xrightarrow{\ !\ } \langle R, \rho\rangle \overset{i}{\rightarrowtail} \langle I, \iota\rangle$$

By initiality, there is only one arrow from $\langle I, \iota\rangle$ to $\langle I, \iota\rangle$, namely id_I and since $i \circ ! : \langle I, \iota\rangle \to \langle I, \iota\rangle$, we get that $i \circ ! = id_I$. Since i is mono, and so left-cancellable, we get from the equation

$$i \circ ! \circ i = id_I \circ i = i \circ id_R$$

the desired result

$$! \circ i = id_R.$$

Thus, $\langle R, \rho\rangle \cong \langle I, \iota\rangle$. □

We may call minimal algebras *inductive* to highlight the relationship between initiality and induction.[12] Along this vein, let us show that induction on natural numbers follows from this definition of minimality.

Definition 2.17. Let $\langle A, \alpha\rangle$ be an F-algebra. We say that a subobject $i : P \rightarrowtail A$ of A is *closed under* α if there is a structure map

$$\rho : F(P) \to P$$

such that

$$i : \langle P, \rho\rangle \to \langle A, \alpha\rangle$$

is an algebra homomorphism.

This definition is not strictly necessary, since it is captured by subalgebras more generally. We use it for the following example to bring to light the connection to the usual sense of induction.

[12] Vakarelov does exactly this in [95].

Example 2.6. Consider the initial algebra $\langle N, [0,s]\rangle$ for the successor functor S. We interpret predicates over natural numbers as subsets $P \subseteq N$ equipped with the usual inclusion $i : P \rightarrowtail N$. A predicate P is closed under $[0,s]$ just in case there is a $\rho : 1 + P \to P$ making the following diagram commute:

$$\begin{tikzcd} 1+P \arrow[r, "id_1+i"] \arrow[d, "\rho"'] & 1+N \arrow[d, "{[0,s]}"] \\ P \arrow[r, tail, "i"'] & N \end{tikzcd}$$

which can be expressed in the two equations:

$$\begin{aligned} &i \circ \rho(*) = 0 \\ &i \circ \rho(n) = s(n) \quad \text{for each } n \in P \end{aligned}$$

And so P being closed under $[0,s]$ means that $0 \in P$ as well as that $n \in P$ implies $s(n) \in P$. In other words, P is a subalgebra of N just in case $0 \in P$ and whenever $n \in P$, also $s(n) \in P$. Because the inclusion i is an algebra homomorphism and $\langle N, [0,s]\rangle$ is minimal, Theorem 2.16 gives us that P is isomorphic to N.

In this section, we described the natural numbers as an initial algebra of the successor functor. And we saw that this definition implied the usual principle of induction. In general, given Theorem 2.16, we can state the conclusion of this section as:

Initial Algebras are Inductive.

Chapter 3

Fixed Points and Iteration

An interesting and perhaps illuminating way to analyze initial algebras is via smallest fixed points for the endofunctors. Indeed, this was the original framework in which initial algebras are found in [51]. This mirrors the way we understood accessible domains in the introduction as smallest fixed points for monotone operators that correspond to rule sets. Not every endofunctor has a fixed point, and when there is a initial one, it is often an initial algebra. We use an important theorem due to Lambek to make this correspondence between initial algebras and fixed points precise. For the following, let $\mathbb{C}$ be a category and $F : \mathbb{C} \to \mathbb{C}$ a functor.

Theorem 3.1 (Lambek's Lemma)**.** *If F has an initial algebra $\langle X, \alpha \rangle$, then X is isomorphic to $F(X)$ via α. More concisely, we can say that $\alpha : F(X) \cong X$.*

Proof. Because $\langle X, \alpha \rangle$ is initial, there is a unique homomorphism $! : \langle X, \alpha \rangle \to \langle F(X), F(\alpha) \rangle$. That $\langle F(X), F(\alpha) \rangle$ is an algebra can be seen by noting that $F(\alpha) : F(F(X)) \to F(X)$. In general, structure maps, such as α, are also algebra homomorphisms, as shown in the right square of the diagram below. Intuitively, we can think of algebras as 'closed under F' since you can always get another algebra by iterating F as we did here. But in the case of initial algebras, iterating F gets you nothing new, as we now show.

Claim: α and $!$ are inverses. Since $\alpha \circ ! : X \to X$ is a composition of homomorphisms, it too is a homomorphism. But since $id_X : X \to X$ is also a homomorphism of X to itself, the initiality of $\langle X, \alpha \rangle$ implies that these must be equal, namely that $\alpha \circ ! = id_X$. And since the diagram below, specifically the left square, commutes, we have that

$$\begin{aligned} ! \circ \alpha &= F(\alpha) \circ F(!) \\ &= F(id_X) && \text{(since } \alpha \circ ! = id_X\text{)} \\ &= id_{F(X)}. \end{aligned}$$

$$\begin{tikzcd}
F(X) \arrow[r, "F(!)"] \arrow[d, "\alpha"'] & F(F(X)) \arrow[r, "F(\alpha)"] \arrow[d, "F(\alpha)"] & F(X) \arrow[d, "\alpha"'] \\
X \arrow[r, "!"'] & F(X) \arrow[r, "\alpha"'] & X
\end{tikzcd}$$

□

This shows that initial F-algebras are fixed points of F in the following sense:

Definition 3.2. A *fixed point* for a functor $F : \mathbb{C} \to \mathbb{C}$ is an object X of $\mathbb{C}$ such that $F(X) \cong X$.

In many cases, initial algebras are in fact the 'least' fixed point for the endofunctor, in the sense of being initial relative to all fixed points (Definition 3.6). The above theorem also gives us a way to establish the nonexistence of initial algebras, namely by showing the nonexistence of a fixed point for an endofunctor.

Thus we can see that the powerset functor $\wp : \mathsf{Sets} \to \mathsf{Sets}$ has no initial algebra since Cantor's theorem implies that there is no fixed point for $\wp$. But changing the category over which we take powersets can give us an initial algebra. Consider the category SET of all sets and classes, without supposing the axiom of foundation. We then consider the functor $\wp' : \mathrm{SET} \to \mathrm{SET}$ that takes each class to its class of subsets (subsets, not subclasses). The initial algebra for this functor is the class of well-founded sets. In a related way, if we know that every endofunctor on a category has an initial algebra,[1] then we know that it has no powerset functor.

In general, however, we would like to have sufficient conditions for asserting the existence of initial algebras; this section develops the necessary theory to give such a sufficient condition. Furthermore, we can even construct the initial algebra "from below" by iterating the functor in the appropriate sense, corresponding to a natural way of considering the build-up for an inductively defined class.

The end goal of this section is to characterize what functors, when you iterate them enough on an object (usually an initial object), will give you an initial algebra. The number of times that you need to iterate one of these functors depends on the sort of functor it is and the category it is defined on.

The construction of an initial algebra by iterating the functor is extensively studied in [12, ch. 4]; we will use their exposition as a guide. The fixed-point construction of an initial algebra mirrors the Knaster-Tarski fixed-point construction for complete lattices, which are special cases of categories. The generalization of complete lattices are *chain-cocomplete* categories.

Definition 3.3. For an ordinal α, an α-*chain* in a category $\mathbb{C}$ is a diagram

$$D_0 \longrightarrow D_1 \longrightarrow \ldots \longrightarrow D_i \longrightarrow \ldots$$

for all $i < \alpha$.

[1] Peter Freyd calls such categories *algebriacally complete categories* and studies them in [39].

Definition 3.4. A category $\mathbb{C}$ is *chain-cocomplete* if every chain has a colimit. This includes the empty chain's colimit, which is an initial object of $\mathbb{C}$.

Now we can construct an initial algebra, using transfinite induction. Let $\mathbb{C}$ be a chain-cocomplete category with 0 the initial object. For each functor $F : \mathbb{C} \to \mathbb{C}$ we define the chain

$$0 \longrightarrow F(0) \longrightarrow F(F(0)) \longrightarrow \ldots \longrightarrow F^{\omega}(0) = \operatorname*{colim}_{n<\omega} F^n(0) \longrightarrow F^{\omega+1}(0) \longrightarrow \ldots$$

And we label the arrows $w_{i,j} : F^i(0) \to F^j(0)$ where

$w_{0,1}$ is the unique arrow from 0;

$w_{i+1,j+1} = F(w_{i,j})$;

$w_{n,i}$ for limit ordinal i are the colimit injections from $F^n(0)$ to $F^i(0)$.

Definition 3.5. The initial-algebra construction *stops after k steps* when $w_{k,k+1}$ is an isomorphism.

Proposition 4 ([12] Proposition 2.5, page 163)**.**

1. *If the initial-algebra construction stops after k steps, then each $w_{k,n}$ for $n \geq k$ is an isomorphism.*
2. *Let $w_{n,m}$ be an isomorphism for a pair of ordinals $n < m$. Then the initial-algebra construction stops after m steps.*
3. *For each limit ordinal k, the initial-algebra construction stops after k steps iff F preserves the colimit*

$$F^k(0) = \operatorname*{colim}_{n<k} F^n(0).$$

That is, iff

$$F(\operatorname*{colim}_{n<k} F^n(0)) \cong \operatorname*{colim}_{n<k} F^{n+1}(0)$$

This proposition gives us some purchase on when the initial-algebra construction stops. The idea is that if the construction stops, then you'll continue getting isomorphisms as you apply F and that if you arrive at an isomorphism, the constructions has stopped. Part 3 suggests that preserving the specified colimit is equivalent to the initial-algebra construction stopping. With this proposition, we understand better what it takes for the initial-algebra construction to stop.

But the question really is, does the 'initial-algebra construction' really provide us with initial algebras? The answer is yes, under some conditions. We do know that if the initial-algebra construction stops after k steps, then $F^k(0)$ is least fixed point for the functor F, we can see as follows. It is clear that if the construction stops, $F^k(0)$ is a fixed point, since $F^k(0) \cong F(F^k(0)) = F^{k+1}(0)$. That it's the least such is obvious when we make precise what it means to be the least fixed point of a functor:

Definition 3.6. A fixed point X with isomorphism $x : F(X) \to X$ for a functor $F : \mathbb{C} \to \mathbb{C}$ is called the *least fixed point* if for any fixed point Y with isomorphism $y : F(Y) \to Y$, there exists a unique morphism $f : X \to Y$ such that $F(f) = y^{-1} \circ f \circ x$ as in the following diagram:

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "x"'] & FY \arrow[d, shift right, "y"'] \\ X \arrow[r, "f"'] & Y \arrow[u, shift right, "y^{-1}"'] \end{tikzcd}$$

It is interesting that this definition looks quite like the definition for initial algebra, the significant difference being that the universality of the least fixed point is only guaranteed with respect to other fixed points (i.e. F-algebras whose structure map is an isomorphism), not general F-algebras.

So far we've seen that the initial F-algebra will be a fixed point (Theorem 3.1) and it is clear that it will be the least such, given its unique algebra homomorphism to every F-algebra, including other fixed points. But the question remains, does the existence of a least fixed point guarantee the existence of an initial algebra? In general, the answer is no. It turns out that we need a few more reasonable assumptions on the category and the functor to ensure the convergence of these two notions. The first notion is just a specialization of Definition 3.4.

Definition 3.7. A class C of morphisms is said to be *chain-cocomplete* if for every ordinal γ and every chain of C-morphisms

$$w_{i,j} : F^i(0) \to F^j(0) \text{ in } C \quad (i \leq j < \gamma),$$

there is a colimit $F^\gamma(0)$ and $w_{i,\gamma}$ such that

1. $w_{i,\gamma} \in C$ for each $i < \gamma$
2. for every compatible family $U_i : F^i(0) \to U$ in C, the factorizing morphism $h : F^\gamma(0) \to U$ is in C too.

This is all to say roughly that C is closed under the morphisms for colimits of chains for any ordinal. The class of all monomorphisms in the categories of Sets, complete posets, topological spaces, and varieties of finitary algebras are all examples of chain-cocomplete classes of morphisms.

Then we have the following important theorem.

Theorem 3.8 ([12] p. 181)**.** *Let M be a chain-cocomplete class of monos in a category $\mathbb{C}$ and let $\mathbb{C}$ be M-well powered. The following conditions are equivalent for any functor $F : \mathbb{C} \to \mathbb{C}$ preserving M (i.e. such that $m \in M$ implies $F(m) \in M$):*

1. *F has a fixed point;*
2. *F has a least-fixed point;*

3. *the initial F-algebra exists;*

4. *the initial-algebra construction stops.*

The proof is not important here because it relies on the careful construction of a compatible family of M-monos, which gives little intuition about why the conditions are sufficient. An intuition we can gather, however, is from the condition on the functor F. The functor, in this theorem, has to preserve M, and M is assumed to contain all the necessary morphisms for colimits of chains of M-morphisms.

Indeed, some features of any chain-cocomplete class C can help us better understand this theorem. First, $\mathbb{C}$ must have an initial object because the empty chain in C must have a colimit, namely the initial object. Moreover, each $0 \to X$, for $X \in \mathbb{C}$, is in the class C. The class C is also closed under isomorphisms and composition.

All this together can be understood as requiring that M ensures that $\mathbb{C}$ is 'chain-cocomplete enough', and that F works well with M. Once this is assumed (along with the assumption that $\mathbb{C}$ is M-well powered), the existence of fixed points and initial algebras coincide. It can help to think of this in analogy to a weak form of the Tarski-Knaster fixed point theorem.[2]

To see better how the colimit of chains translates into initial algebras (and thus why it's chain-*co*complete), we quote a nice theorem.

Theorem 3.9 ([10, Lemma on p. 201]). *Let $\mathbb{C}$ be a category with an initial object 0 and transfinite composition of length ω, hence colimits of sequences $\omega \to \mathbb{C}$, and suppose $F : \mathbb{C} \to \mathbb{C}$ preserves colimits of ω-sequences. Then the colimit I of the sequence*

$$0 \xrightarrow{i} F(0) \xrightarrow{F(i)} \dots \longrightarrow F^n(0) \xrightarrow{F^n(i)} F^{n+1}(0) \longrightarrow \dots$$

is an initial F-algebra.

Proof. Let
$$I = \operatorname*{colim}_n F^n(0)$$
be the colimit of the ω-sequence

$$0 \longrightarrow F(0) \longrightarrow F^2(0) \longrightarrow \cdots$$

Since F preserves colimits, we have the isomorphism

$$F(I) = F(\operatorname*{colim}_n F^n(0)) \cong \operatorname*{colim}_n F(F^n(0)) =^* \operatorname*{colim}_n F^n 0 = I. \tag{3.1}$$

[2]For example, consider the following weakening: *Let L be a poset with a smallest element and let $f : L \to L$ be an order-preserving function. Also suppose there exists $u \in L$ such that $f(u) \leq u$ and any chain in the subset $\{x \in L \mid x \leq f(x)\ \&\ x \leq u\}$ has a supremum. Then f has a least fixed point.*

We get the equality marked with * by using the properties of the initial object 0. To show this, define $I' = \operatorname*{colim}_n F(F^n(0))$. More precisely since I' is a colimit, we have the cocone

$$\begin{tikzcd} F(0) \arrow[r] \arrow[dr] & F^2(0) \arrow[r] \arrow[d] & F^3(0) \arrow[r] \arrow[dl] & \cdots \\ & I' & & \end{tikzcd}$$

And we get the requisite arrow $0 \to I'$ simply by initiality of 0:

$$\begin{tikzcd} 0 \arrow[r] \arrow[drr] & F(0) \arrow[r] \arrow[dr] & F^2(0) \arrow[r] \arrow[d] & F^3(0) \arrow[r] \arrow[dl] & \cdots \\ & & I' & & \end{tikzcd}$$

which shows us that I' is a cocone over the full ω-sequence. The fact that this cocone is universal in the sense that I' is the colimit of $F^n(0)$ comes from the fact that arrows from 0 are unique; we only have to show that, for any other cocone C, the diagram

$$\begin{tikzcd} & C \\ 0 \arrow[ur, "!"] \arrow[dr, "!"'] & \\ & I' \arrow[uu] \end{tikzcd}$$

commutes, which is immediate since 0 is initial.

So we know that I is a fixed point of F (3.1) and we now need to show it is actually an initial algebra. The structure map that comes with I is the isomorphism from 3.1 above; denote it $i : F(I) \to I$. So $\langle I, i\rangle$ is an F-algebra.

Now let $\langle X, \alpha\rangle$ be an F-algebra. By initiality of 0, we get the unique map $!_X : 0 \to X$. Applying F to this map results in $F(!_X) : F(0) \to F(X)$ which when composed with α results in

$$\alpha \circ F(!_X) : F(0) \to X.$$

Applying F to this map gives us $F(\alpha \circ F(!_X)) : F^2(0) \to F(X)$ and again composing with alpha gives us an arrow

$$\alpha \circ F(\alpha \circ F(!_X)) : F^2(0) \to X.$$

We can do this for all n and so get the family of arrows $q_n : F^n(0) \to X$, showing that (X, q_n) is a cocone and since I is a colimit, we have an arrow $u : I \to X$ as needed. The uniqueness of this arrow comes from the universal mapping property of colimits. $\square$

It is clear that we did not need that F preserved all colimits of ω-sequences, just the 'initial' such where the least element of $(\omega, \leq)$ gets sent to the initial object 0.

Let us see how this relates to our simplest example of the successor functor $S(X) = 1 + X$.

Example 3.1. In Sets we get the ω-sequence, recalling that $1 + 0 \cong 1$:

$$0 \longrightarrow 1 \longrightarrow 1+1 \longrightarrow 1+1+1 \longrightarrow \dots$$

The colimit of this sequence is the set N, which emphasizes the correspondence between colimits and least upper bounds: we can think of N as the least upper bound for the suggestively denoted objects 1, $1+1$, $1+1+1$, ... This construction of N stops after ω steps after which we have

$$N \;\cong\; 1+N \;\cong\; 1+1+N \;\cong\; \dots$$

The first of these $N \cong 1 + N$ gives us the arrow $1 + N \to N$ which provides the familiar structure, with a successor and a distinguished object. The later isomorphisms merely show that you can fix more than one constant and then proceed to take successors of these, with no successors resulting in one of these constants (see Section 2.2). In other words, these later isomorphisms give us the familiar facts that

$$\begin{aligned}
\mathrm{card}\{0,1,2,\dots\} &= \mathrm{card}\{1,2,3,\dots\}\\
\mathrm{card}\{1,2,3,\dots\} &= \mathrm{card}\{2,3,4,\dots\}\\
\mathrm{card}\{2,3,4,\dots\} &= \mathrm{card}\{3,4,5,\dots\}\\
&\vdots
\end{aligned}$$

The category Sets satisfies the conditions of the above theorem (Theorem 3.9), with the empty set serving the role of 0. But not every functor on Sets preserves colimits of ω-sequences. Are there categories, for which every endofunctor has an initial algebra? To see that this would be a strict requirement, we note that a weaker condition, having fixed points for every endofunctor, results in a relatively simple category.

Theorem 3.10 ([8] Theorem, p. 174)**.** *If a category $\mathbb{C}$ has the fixed point property (i.e. if every functor $F : \mathbb{C} \to \mathbb{C}$ has a fixed point) and has all limits or colimits, then $\mathbb{C}$ is a preordered class.*

We should note that the original theorem is stated in terms of the category $\mathbb{C}$ having all powers or copowers of all objects.

This theorem says that if $\mathbb{C}$ has all limits and colimits, then having the fixed point property ensures there is at most one arrow between any two objects of $\mathbb{C}$. Interestingly, if Sets(α) denotes the category of sets of rank $\leq \alpha$ and functions between them, then for every α, Sets(α) has the fixed point property. But Sets(2^α) then has the fixed point property, but is not a preorder. And this is why the word "all" is present in the theorem: $\mathsf{Sets}(2^\alpha)$ has all limits with diagrams with fewer than α objects. But clearly, it doesn't have the colimit of every object within $\mathsf{Sets}(2^\alpha)$, namely 2^α itself.

The functors that stop after ω steps in the initial algebra construction have been studied in [6] where they are called *algorithmic*. More accurately, they study *varietors*, which focus on 'free' algebras, which are parameterized initial algebras. Briefly, we can define them as follows.

Definition 3.11. A *free F-algebra*, generated by an object I, is an F-algebra (I^*, f) together with a morphism $s : I \to I^*$ (insertion of generators) which is universal in the sense that every diagram

$$\begin{tikzcd} F(I^*) \arrow[r, "f"] & I^* & I \arrow[l, "s"'] \arrow[dl, "h"] \\ F(Q) \arrow[r, "d"] & Q & \end{tikzcd}$$

gives rise to a unique f^* such that the following diagram commutes:

$$\begin{tikzcd} F(I^*) \arrow[r, "f"] \arrow[d, "F(h^*)"'] & I^* \arrow[d, "h^*"'] & I \arrow[l, "s"'] \arrow[dl, "h"] \\ F(Q) \arrow[r, "d"] & Q & \end{tikzcd}$$

Free F-algebras play a large role in the interesting theory of machines in categories by Arbib and Manes [13]. The functor F is called a *varietor* if there is a free algebra (I^*, f) for every object I. It is easy to see that if $I = 0$ is the initial object, then the conditions (I^*, f) must satisfy to be a free algebra collapse to the conditions for an initial algebra. That is, the free algebra generated by the initial object is the initial algebra.

Definition 3.12. Let $F : \mathbb{C} \to \mathbb{C}$ be a functor on a category $\mathbb{C}$. For each object A of $\mathbb{C}$, denote by $F_A : \mathbb{C} \to \mathbb{C}$ the functor with the following form:

$$F_A(X) = F(X) + A$$

$$F_A(f) = F(f) + id_A.$$

Then F is called an *algorithmic varietor* if F is a varietor and the initial algebra construction stops after ω steps.

The functor F_A is just the coproduct of F with the constant A functor. The notion of algorithmic is, in our context, not particularly meaningful as a term. The important idea is that it is finitary in the sense that we can find an initial algebra after at most ω many iterations on the initial object of $\mathbb{C}$.

We now come to an interesting result about how close this sort of 'finitary' matches with the traditional view that a functor is finitary if it preserves filtered colimits in the context of Sets. It turns out to depend on large cardinals axioms of set theory, unlike many of the results using Sets.

Theorem 3.13 ([8] Theorem 18, p. 176). *The following statements are equivalent*

1. *There are no measurable cardinals*
2. *a functor F : $\mathsf{Sets} \to \mathsf{Sets}$ is an algorithmic varietor iff it is finitary (i.e. preserves filtered colimits).*

This is interesting for several reasons. First, that measurable cardinals come into play at all is quite surprising. In the many uses to which we put the category Sets, we often don't particularly care which version of set theory you use. That is, most of the theorems of category theory that talk about sets are presumed to be invariant under "reasonable" interpretations of "sets and functions". The second reason this theorem is interesting is that it answers a natural question: Do finitary functors achieve initial algebras after finite iteration on the initial object? The answer is—a common answer—that it depends. The initial algebra construction always stops after ω steps for finitary functors only if you assume there are no measurable cardinals. And, it should be noted, the nonexistence of measurable cardinals is consistent with ZFC.

Although the notion of a filtered colimit is described in the Appendix, I will briefly note here what those are. Many of easiest examples of limits and colimits to understand come from diagrams indexed by directed sets. Directed sets are preordered sets such that every finite subset has an upper bound. So a directed set, considered as a category, is such that all finite diagrams admit of a cocone. Any chain is a directed set, since the upper bound will simply be the latest (i.e. the maximum) in the chain of any subset you choose.

This result shows that we cannot divide up the world of functors into those that have initial algebras and those that don't simply by noting the finiteness of the functor. For a more nuanced classification of functors, we turn now to *polynomial functors* which will serve as the major divide between how 'complex' the functor is that produces the initial algebra, if any.

Chapter 4

Polynomial Functors

We have seen general features of initial algebras and how they combine the principles of recursion and induction characteristic of accessible domains. In this chapter, we will be more explicit about the surrounding category and the sorts of functors that have initial algebras. The salient distinction here is between endofunctors that are *polynomial* and those that are not. This chapter summarizes the theory of polynomial functors, with an eye toward interesting features of their (initial) algebras. An especially important feature of polynomial endofunctors is that they give us initial algebras through a notion of iteration, though they are not unique in this sense. The smallest fixed points for polynomial endofunctors are actually initial algebras for those same endofunctors. Moreover, we can say how much iteration is required in some general ways. Much of the following theory comes from Joachim Kock's notes on polynomial functors [49].

The examples from Chapter 2 (e.g. 2.1) are based on the successor functor $X \mapsto 1 + X$, which gives us one nullary and one unary operation which we may think of as the constant 0 and the successor, respectively. We can generalize this to any *signature* consisting of finitely many finitary operations on a set. Here "finitary" refers to operations of finite arity (number of arguments). This allows us to capture part of universal algebra, without equations, as an instance of algebras for endofunctors on Sets, for instance.

Example 4.1. A group is a set G equipped with one binary (multiplication), one unary (inverse), and one nullary (identity element) operation as indicated in the following diagram:

$$\begin{tikzcd} G \times G \arrow[r, "m"] & G & G \arrow[l, "i"'] \\ & 1 \arrow[u, "u"] & \end{tikzcd}$$

or more compactly as an arrow

$$1 + G + G \times G \xrightarrow{[u,i,m]} G,$$

that satisfies the usual equations. And so we see that group *signatures* can be captured by algebras of the endofunctor $F(X) = 1+X+X\times X$ on Sets. But note that this is only the signature; we didn't specify any relations (e.g. equality) between these operations such as how the identity interacts with inverse and multiplication.

Example 4.2. [16, p. 267] More generally, we can consider an endofunctor on Sets that looks like a polynomial:

$$F(X) = C_0 + C_1 \times X + C_2 \times X^2 + \cdots + C_n \times X^n \tag{4.1}$$

to represent C_0 many nullary operations (i.e. constants), C_1 many unary operations, and so on. The C_k are natural numbers and are shorthand, so that $2 \times X^3$ is really $X^3 + X^3$. These functors look very much like the polynomial functions we are used to, with multiplication being replaced by Cartesian product, addition by coproduct, and exponents replaced by exponential objects (X^2 represents the function space $\{0,1\} \to X$ or simply $X \times X$). These polynomial functors allow us to represent sets with any number of finitary operations.

Let us make this more precise. A *signature* Λ is a set of function symbols together with associated finite arities. We write $f^{(n)}$ to indicate that f is a function symbol of arity n. If the arity of a function symbol c is 0, then we call $c^{(0)}$ a *constant symbol*.

A Λ-*algebra* is a pair

$$\mathcal{S} = \langle S, \{f_{\mathcal{S}}^{(n)} : S^n \to S \mid f^{(n)} \in \Lambda\}\rangle,$$

where S is a set, the *carrier of the algebra*. Already we see a similarity to algebras for endofunctors, with the place of a structure map being instead a family of functions. A constant symbol is a function symbol $S^0 \to S$ which is an element of S, considering $S^0 \cong 1$. Given two Λ algebras $\mathcal{S}$ and $\mathcal{T}$, we say that a set function $\phi : S \to T$ is a Λ-homomorphism if, for every function symbol $f^{(n)}$ in Λ, and every $s_1, \ldots, s_n \in S$,

$$f_{\mathcal{T}}^{(n)}(\phi(s_1), \ldots, \phi(s_n)) = \phi(f_{\mathcal{S}}^{(n)}(s_1, \ldots, s_n)).$$

For every constant symbol $c^{(0)}$, this means that $\phi(c_{\mathcal{S}}^{(0)}) = c_{\mathcal{T}}^{(0)}$.

But Λ-algebras correspond to algebras for endofunctors, where we replace the family of functions with the coproduct of the family. Given a signature Λ, consider the (polynomial) functor $\mathcal{P} : \mathsf{Sets} \to \mathsf{Sets}$ given by

$$\mathcal{P}(S) = \sum_{f^{(n)} \in \Lambda} S^n.$$

Note that this is just another way to express the functor F above in Equation 4.1. For each Λ-algebra $\mathcal{S}$ and each $f^{(n)} \in \Lambda$, we have the interpretation of $f^{(n)}$ in $\mathcal{S}$,

$$f_{\mathcal{S}}^{(n)} : S^n \to S.$$

By the universal property of coproducts, we get a unique map $\lambda : \mathcal{P}(S) \to S$ making the following diagram commute:

$$\begin{tikzcd} S^n \arrow[r, tail] \arrow[dr, "f_{\mathcal{S}}^{(n)}"'] & \displaystyle\sum_{f^{(n)} \in \Lambda} S^n \arrow[d, dashed, "\lambda"] \\ & S \end{tikzcd}$$

This means that $\lambda : \mathcal{P}(S) \to S$ is a $\mathcal{P}$-algebra structure map. So each Λ-algebra $\mathcal{S}$ gives rise to the $\mathcal{P}$-algebra $\langle S, \lambda \rangle$.

Conversely, given any $\mathcal{P}$-algebra $\langle S, \lambda \rangle$, we have a Λ-algebra with $f_{\mathcal{S}}^{(n)}$ given by the composition

$$S^n \rightarrowtail \sum_{f^{(n)} \in \Lambda} S^n \xrightarrow{\lambda} S \, .$$

Coupled with the fact that Λ-homomorphisms are $\mathcal{P}$-homomorphisms and vice versa, we get that the category of Λ-algebras is isomorphic to the category of $\mathcal{P}$-algebras.

This example showed how algebras for functors generalize the sorts of algebras that universal algebra studies. So the signatures for groups, rings, monads, etc. are all instances of algebras for polynomial functors in a straightforward manner. Again, these functors deal with the signatures, not the explicit algebraic nature of these structures; they do not include identities like associativity.

In the above example, our definition of signature was general enough to allow for infinitely many function symbols, and so operations. Likewise, the definition of the corresponding polynomial functor $\mathcal{P}$ was also general enough to include infinite coproducts. These examples give rise to a precise definition of polynomial functor in one variable. The generalization of functors like those of the form (4.1) above is achieved by moving from natural number exponents to generalized families of objects (usually sets).

We will consider an A-indexed family $(B_a : a \in A)$ of sets to be given by the map $f : B \to A$ with $B_a = f^{-1}(a)$. So each B_a is the collection of elements in B that f sends to $a \in A$. We will be exploiting this fact often and will thus introduce notation for this relationship.

Definition 4.1. Let $f : B \to A$ be a function in Sets. This gives rise to the family $(B_a : a \in A)$. We say that f *represents the polynomial functor* $P_f : \mathsf{Sets} \to \mathsf{Sets}$ defined as

$$P_f(X) = \sum_{a \in A} X^{B_a}$$

on objects and for arrows $h : X \to Y$,

$$P_f(h) = \sum_{a \in A} h^{B_a} : \sum_{a \in A} X^{B_a} \to \sum_{a \in A} Y^{B_a}$$
$$(B_a \to X) \mapsto (B_a \to X \xrightarrow{h} Y)$$

determined termwise.

We will also say that f *gives rise* to P_f.

This allows us to define what a polynomial functor is, for one variable X.

Definition 4.2. A functor $P : \mathsf{Sets} \to \mathsf{Sets}$ is a (one-variable) *polynomial functor* if it is isomorphic to P_f for some $f : B \to A$ in Sets.

Example 4.3. Let us see how this works for the simple polynomial functor

$$P(X) = X + 2X^2.$$

First, we expand to $P(X) = X + X^2 + X^2$. So we know that we want the exponents to be $1, 2, 2$. Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Then we can define the function $f : B \to A$ as follows

$$\begin{tikzcd}
B \arrow[r, "f"] & A \\
b_1 \arrow[r] & a_1 \\
b_2 \arrow[r] & a_2 \\
b_3 \arrow[ur] & \\
b_4 \arrow[r] & a_3 \\
b_5 \arrow[ur] &
\end{tikzcd}$$

So we get $B_{a_1} = \{b_1\}, B_{a_2} = \{b_2, b_3\}$, and $B_{a_3} = \{b_4, b_5\}$. In Sets, all sets of the same cardinality are isomorphic, since isomorphism is bijection. This means that $\{b_4, b_5\}$, for example, is isomorphic to the ordinal 2. Therefore

$$\begin{aligned}
\sum_{a \in A} X^{B_a} &= X^{\{b_1\}} + X^{\{b_2,b_3\}} + X^{\{b_4,b_5\}} \\
&\cong X^1 + X^2 + X^2 \\
&\cong X + X^2 + X^2
\end{aligned}$$

as desired.

We can thus see that the function f gives all the information needed for our polynomial functor.

Some examples of this relationship between polynomial functors and their representing arrows will help us gain some intuition.

Example 4.4. Let $f : B \to 1$ be a set function into the terminal set $\{*\}$. Then there is only one fiber, namely $B_* = B$. Thus f represents the polynomial functor

$$\begin{aligned} P_f(X) &= \sum_{a \in \{*\}} X^{B_a} \\ &= X^{B_*} \\ &= X^B \end{aligned}$$

This shows us how to get 'monomial' functors, a special case of polynomial functors that are represented by functions into the singletons (i.e. terminal objects of Sets).

Example 4.5. Consider now the map $f : \emptyset \to A$ and its corresponding polynomial functor

$$P_f(X) = \sum_{a \in A} X^{\emptyset_a} = \sum_{a \in A} X^{\emptyset} = \sum_{a \in A} 1 = A.$$

This shows that taking the polynomial functor represented by the empty function gives us *constant* functors. Note that any category with initial and terminal objects will have similar constant functors since generally $X^0 \cong 1$. We note the special cases that for $A = \emptyset$, the polynomial functor $P_f(X) = \emptyset$ is the constant empty functor. Similarly, when $A = 1$, the terminal object, we get the polynomial functor $P_f(X) = 1$ the constant singleton functor. This way, we can represent any set A as a constant polynomial functor given by the family $f : \emptyset \to A$.

Example 4.6. Given a polynomial functor $P_f : \mathsf{Sets} \to \mathsf{Sets}$ for some $f : B \to A$, we can evaluate it on the empty set, for instance:

$$P_f(\emptyset) = \sum_{a \in A} \emptyset^{B_a}$$

But notice that $\emptyset^{B_a}$ is empty unless B_a is empty. If B_a is indeed empty, then the set of maps $B_a \to \emptyset$ has exactly one map. So then

$$P_f(\emptyset) = \{a \in A \mid B_a = \emptyset\}.$$

That is, $\mathcal{P}_f(\emptyset)$ is the set of elements of A that are not in the image of f.

Example 4.7. Of course we can evaluate on singletons as well:

$$P_f(1) = \sum_{a \in A} 1^{B_a} = \sum_{a \in A} 1 = A.$$

Example 4.8. Let $f : B \to A$ be a bijection. This means that each fiber will be a singleton:

$$P_f(X) = \sum_{a \in A} X^1 = A \times X.$$

We may call these *linear functors*; P_f consists entirely of unary operations.

Example 4.9. Now let $f : B \to A$ be only an injection. We can thus partition A into those elements that are in the image of f and those that aren't. Let $A_0 = \{a \in A \mid B_a = \emptyset\}$, those elements with empty fibers, and $A_1 = A \setminus A_0$. Note that every element of A_1 will have a singleton fiber, since f is injective. Thus the polynomial functor represented by f is

$$\begin{aligned} P_f(X) &= \sum_{a \in A_0} X^{B_a} + \sum_{a \in A_1} X^{B_a} \\ &= \sum_{a \in A_0} X^{\emptyset} + \sum_{a \in A_1} X^1 \\ &= A_0 + A_1 \times X \end{aligned}$$

We call these *affine functors* and they are really the sum of a constant and a linear functor. Thus we can think of this functor as having a nullary operation for each element of A_0 and a unary operation for each element of A_1. The successor functor is one example of an affine functor, where A_0 and A_1 are singletons.

Example 4.10. If $f : B \to A$ is a surjection, then we know

$$A_0 = \{a \in A \mid B_a = \emptyset\} = \emptyset.$$

This shows us why injections (that represent affine functors) may have a constant term A_0 and yet bijections (representing linear functors) do not.

4.1 Initial algebras of Polynomial Functors

Given this theory of polynomial functors let us turn to considering their initial algebras, when they have them. It is common to refer to initial algebras for polynomial endofunctors as W-types, which comes from Martin-Löf's original specification in [62].

Definition 4.3. The initial algebra of a polynomial functor P_f, if it exists, is called the *extensional W-type* for the map f and is denoted $W(f)$.

W-types are quite broad and "can be used to provide a constructive counterpart of the classical notion of a well-ordering and to uniformly define a variety of inductive types" [91]. This section will cover W-types, or initial algebras of polynomial endofunctors.

To use some of the examples from the last chapter, we can start to see how initial algebras work for these relatively well-understood functors.

Example 4.11 (Based on Example 4.8). If $f : B \to A$ is a bijection in Sets, then $P_f(X) = A \times X$. So, a P_f-algebra homomorphism is an h that makes the following diagram commute:

$$\begin{tikzcd} A \times X \arrow[r, "id_A \times h"] \arrow[d] & A \times C \arrow[d] \\ X \arrow[r, "h"'] & C \end{tikzcd}$$

It is easy to see that $\emptyset$ constitutes an initial P_f-algebra. First, it's obvious that $\emptyset$ is a fixed point of P_f since $P(\emptyset) = A \times \emptyset \cong \emptyset$. And since $\emptyset$ is an initial object of Sets, it's equally obvious that it has no proper sub-algebras (i.e. it is the *least* fixed point for P_f).

Example 4.12 (Based on Example 4.10). If $f : B \to A$ is merely surjective, then again we have that $\emptyset$ is an initial algebra for P_f. To see this in a different way than the previous example, we note that

$$P_f(\emptyset) = \sum_{a \in A} \emptyset^{B_a},$$

but $B_a = f^{-1}(a)$ will always be nonempty, since f is a surjection. Thus, for each summand, the function space $B_a \to \emptyset$ is empty, which gives us $P_f(\emptyset) \cong \emptyset$. Another way to say this is that if P_f has no nullary operations (i.e. constants), then the 'smallest' world with all the operations is an empty one.

Example 4.13 (Based on Example 4.5). For $f : \emptyset \to A$, we have the constant polynomial functor $P_f(X) = A$. Then a P_f-algebra is a set X with a map from A:

$$P_f(X) = A \longrightarrow X.$$

This actually implies that the category of P_f algebras $Alg_{\mathsf{Sets}}(P_f)$ is the coslice category A/Sets. That the arrows between coslices are the same as P_f homomorphisms is easy to see from just writing what a P_f homomorphism h would be:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d] & A \arrow[d] \\ X \arrow[r, "h"'] & Y \end{tikzcd} \quad \text{which is} \quad \begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "h"'] & & Y. \end{tikzcd}$$

The initial P_f-algebra for this particular f is just $\langle A, id_A \rangle$. This can be seen by the following diagram for any P_f algebra $\langle X, h : A \to X \rangle$:

$$\begin{tikzcd} & A \arrow[dl, "id_A"'] \arrow[dr, "h"] & \\ A \arrow[rr, dashed, "h"'] & & X. \end{tikzcd}$$

Clearly h itself is the unique homomorphism from the initial P_f-algebra making the diagram commute.

4.2 Iterating Polynomial Functors

In Chapter 3, we discussed many aspects of iterating a functor on an initial object to arrive, possibly, at fixed points and initial algebras for that functor. Now that we are restricting ourselves to polynomial endofunctors, we can do a little better.I would like to thank Joachim Kock (personal correspondence) for help on formulating the following proposition.

Proposition 5. *For a polynomial functor $P_f : \mathsf{Sets} \to \mathsf{Sets}$ with $f : B \to A$ and a regular cardinal κ, the following are equivalent*

1. *P_f preserves κ-filtered colimits.*

2. *The set B_a is smaller than κ for all $a \in A$.*

3. *The initial algebra construction for P_f stops after κ steps.*

Proof. (2) $\implies$ (1) Since P_f is the sum of functors of the form X^{B_a}, and sums commute with colimits, it is enough that we prove the functor $X \mapsto X^C$ preserves κ-filtered colimits if C is smaller than κ. We use C since we won't require any facts about B_a being the fibers of f in this proof. It is sufficient to show that κ-filtered colimits commute with κ limits since X^C for $C \prec \kappa$ is a κ limit. For the case when $\kappa = \omega$, there is the well-known result from [57, Theorem 1 in IX.2, p. 215] that says colimits of (ω-)filtered diagrams commute with finite (ω) limits. The generalization to κ-filtered colimits and κ limits comes from [59, Theorem 1.2.1] and whose proof is essentially the same as that of the special case of ω.

(1) $\implies$ (2) For this direction, it is sufficient to consider the case when $C \not\prec \kappa$ for some C and show that κ-filtered colimits are not preserved by $X \mapsto X^C$. Since we are working with Sets, allow me to use a relatively informal example. For this proof sketch, we will consider the case when $\kappa = \omega$. Let C be infinite (i.e. of cardinality $\geq \omega$). Now consider the infinite 'triangle' matrix with 1 below the diagonal and 0 above.

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 1 & 0 & 0 & \dots & 0 & \dots \\ 1 & 1 & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & & \vdots & \\ 1 & 1 & 1 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & & & \end{bmatrix}$$

The colimit of every row is 1 and the product of all of these colimits will thus be 1. However, the product of every column is 0 and the colimit of all those 0's will again be 0. Thus we have a counterexample to an infinite C giving rise to a functor that preserves (ω-)filtered colimits.

(1) $\iff$ (3)
This follows from (3) of Proposition 4. $\square$

One of the things this Proposition tells us is that for polynomial endofunctors that have finite exponents, the initial algebra construction will stop after ω steps. Thus, for the endofunctor $P_f(X) = 1 + X$, we have that N is isomorphic to the result of iterating P_f at least ω-many times. More generally, the initial algebra construction stops after κ steps for the smallest regular cardinal κ that is an upper bound to B_a for all $a \in A$.

Polynomial endofunctors are used in modeling W-types from type theory, and these types are intended to capture constructive notions of well-founded trees. In [19], it is remarked that "W-types can be seen informally as the free algebras for signatures with operations of possibly infinite arity, but no equations." (p. 8) Similarly, in [70, p. 194], it is claimed that "A W-type is a direct generalization of a free term algebra from finite arities to arbitrary arities (specified by a signature), and is thus an algebra of possibly infinite wellfounded trees".

For the extensional W-types, we have the following formation and introduction rules:

- W-formation rule.

$$\frac{A : \mathtt{type} \qquad x : A \vdash B(x) : \mathtt{type}}{(\mathsf{W}x : A)B(x) : \mathtt{type}}$$

- W-introduction rule.

$$\frac{a : A \qquad t : B(a) \to (\mathsf{W}x : A)B(x)}{\mathtt{sup}(a,t) : (\mathsf{W}x : A)B(x)}$$

Although the full rules are important, it is easier to see the connections in the following if we write $(\mathsf{W}x : A)B(x)$ simply as W. For example, the W-introduction rule can be rephrased as

$$\frac{a : A \qquad t : B(a) \to \mathsf{W}}{\mathtt{sup}(a,t) : \mathsf{W}} \quad .$$

Which says that for every element a of A and element t of $W^{B(a)}$ we have an element of W, called $\mathtt{sup}(a,t)$. This introduction rule therefore determines a functor

$$P(X) = \sum_{a:A} X^{B(a)}$$

which is just a type-theoretic expression of our familiar polynomial functor

$$P^*(X) = \sum_{a \in A} X^{B_a}.$$

This functor is on the category of types as objects and terms of the type theory as arrows.[1]

Moreover, if we take rules for Π and Σ types into account, the introduction rule gives us an arrow

$$s_{\mathsf{W}} : P(\mathsf{W}) \to \mathsf{W}$$

with $\mathsf{W} = (\mathsf{W}x : A)B(x)$ given by the formation rule. And so we have a P-algebra $\langle \mathsf{W}, s_{\mathsf{W}} \rangle$. The authors go on to show that $\langle \mathsf{W}, s_{\mathsf{W}} \rangle$ is initial, in the sense that we have been using the term. In general, to have an initial algebra for

[1] This is the category $\mathcal{H}$ in [19, § 1.2] that is an intensional type theory.

the polynomial endofunctor is the same thing as having a type W satisfying the introduction, elimination, and computation rules they set down for extensional W-types.

This correspondence between initial algebras and the type-theoretic rules is then extended to a correspondence between intensional W-types and 'homotopy-initial' algebras. This weakening of the initiality is very interesting for inductive types in homotopy type theory, but we leave the topic here.

Example 4.14 ([70, Example 3.5(a)] and [62, p. 82]). In Example 2.1 we saw how a natural number object can be characterized as the initial algebra to the functor S. Let us see now how the same definition can be given as a polynomial endofunctor, as in Definition 4.1 and 4.2. A natural number object is isomorphic to $W(f)$ where

$$f : \{1\} \hookrightarrow \{0, 1\}.$$

To see this, we only need to notice how f gives the data for $S = P_f$.

Given f as the inclusion, we note the two fibers:

$$\begin{aligned} f^{-1}(0) &= \emptyset \\ f^{-1}(1) &= \{1\}. \end{aligned}$$

Informally, it is easiest to think of this as saying that there is one nullary operation and one unary operation, respectively. However, the full description of P_f is simple enough:

$$\begin{aligned} P_f(X) &= X^{\emptyset} + X^{\{1\}} \\ &= 1 + X \end{aligned}$$

which is the same as the successor functor S. And with $W(f)$ and N both as initial algebras for $S = P_f$, we have that $W(f) \cong N$.

Example 4.15. An interesting example is that of the *Brouwer ordinals* $\mathcal{O}$. These ordinals are also called the *second number class* and the description used here is given in [62, pp. 82–85]. The elements of $\mathcal{O}$ are defined inductively very similarly to the natural numbers, but with one additional inductive rule:

$$0 \in \mathcal{O} \qquad \frac{x \in \mathcal{O}}{s(x) \in \mathcal{O}} \qquad \frac{f : \mathbb{N} \to \mathcal{O}}{\sup(f) \in \mathcal{O}}\,.$$

It is also traditionally (for the constructive perspective, for example) assumed that f above falls under some notion of 'effective'. We could just implicitly require that all functions are 'effective' to achieve this, or have some test of effectiveness in place, either in the system itself or externally. Thus we have a nullary operation 0, a unary operation s, and an operation with (the cardinality of) N arguments, the supremum operation. The set $\mathcal{O}$ is the initial algebra with such a signature and we can see that by constructing a P_g such that $\mathcal{O} \cong W(g)$. Such a g may look like this:

$$\begin{aligned} g : N &\to \{0, 1, 2\} \\ x &\mapsto 1 + \min(x, 1). \end{aligned}$$

With this specification, we see that

$$\begin{aligned} g^{-1}(0) &= \emptyset \\ g^{-1}(1) &= \{0\} \\ g^{-1}(2) &= \{1, 2, 3, \ldots\} \cong N \end{aligned}$$

giving us the polynomial functor

$$\begin{aligned} P_g(X) &= X^{\emptyset} + X^{\{0\}} + X^N \\ &= 1 + X + X^N \end{aligned}$$

Thus a P_g algebra is a structure map $P_g(X) \to X$ or

$$[0,\, s,\, \mathrm{sup}] : 1 + X + X^N \to X$$

which matches the inductive definition given above, keeping in mind that the initial P_g-algebra, $W(g)$, should be $\mathcal{O}$.

If we consider this definition of $\mathcal{O}$ as the first constructive number class $\mathcal{O}_1$, then we may define the n-th number class $\mathcal{O}_n$ as the W-type for the polynomial endofunctor

$$P(X) = 1 + X + X^N + X^{\mathcal{O}_1} + \ldots + X^{\mathcal{O}_{n-1}}.$$

Thus the n-th number class will be a well-founded tree that can branch on any previous number class or the natural numbers.

4.3 Accessible Domains of Polynomial Endofunctors

In the preceding sections of this chapter, we have seen a mathematical treatment of polynomial endofunctors and their initial algebras. The point was to highlight a particular class of endofunctors whose algebras are well behaved. In addition, I want to suggest that we can use polynomial endofunctors to distinguish different accessible domains. The idea is that we can classify accessible domains by understanding the differences between the underlying endofunctors. This classification is not as precise as giving a quantitative measure of 'how basic' the endofunctor is, but remains relatively open-ended.

Let us collect some facts together.

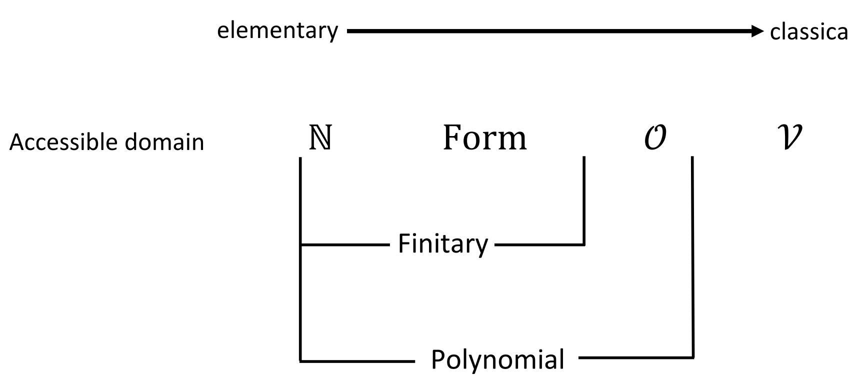
- $\mathbb{N}$ is the initial algebra for the functor $1 + X$ on Sets.
- $\mathcal{O}$ is the initial algebra for the functor $1 + X + X^{\mathbb{N}}$ on a category like Sets but may require all maps to be effective in some sense.
- FORM, the class of formulas for a formal language, is the initial algebra for the functor that reflects the construction rules of the formal language.

- all W-types are initial algebras for some polynomial endofunctor.

As we will see in the next chapter, there are accessible domains that are not captured by polynomial endofunctors, such as the segments of the classical cumulative hierarchy. We can thus distinguish between accessible domains that arise as the initial algebra for a polynomial endofunctor, and those that do not.

We can also distinguish between sorts of polynomial functors by how 'finite' they are. Polynomial functors that only have finite exponents (i.e. the underlying map f has only finite fibers) are more *elementary* than those which have infinitary exponents (i.e. f has some infinite fibers). These gradations of elementariness can be helpful in making sense of the Bernays and Sieg notion of *methodological frame* which sees accessible domains as capturing the ontological commitments of a methodology (e.g. classical, finitist, constructive, predicative, etc.). So if you want a relatively minimalist methodology, that focuses on justifying things only by reference to 'simple' objects, you may restrict yourself to accessible domains that arise as initial algebras for finitary polynomial endofunctors like N. If, however, you would like to work in a broad context such as ZFC you may want to work relative to the objects in segments of the classical cumulative hierarchy $\mathcal{V}$ instead. This focus on the sorts of objects you take for granted does not claim to be an entire characterization of a methodology since the sorts of proof methods or inference rules you take for granted also play a role in a methodological perspective.

Schematically, we can see the gradient of methodological frames as follows:



In this diagram, we have situated four important accessible domains (i.e. $\mathbb{N}$, Form, $\mathcal{O}$, and $\mathcal{V}$) into the categories of polynomial and finitary. The notion of finitary in the context of polynomial endofunctors is quite straightforward (i.e. the exponents are finite), but there may be notions of finitary applied to non-polynomial functors. It may very well be that there are initial algebras for these finitary functors which don't fit into this diagram. This diagram mostly emphasizes the sorts of functors we have already encountered.

But this is a *syntactic* understanding of the endofunctors; it concerns what definitions apply to the functor (e.g. polynomial or not). How the functor can be defined is an interesting question, but it depends on the categorical context—it is not invariant. Functors may *look* different without actually being any different and so a proper classification of endofunctors that give rise to accessible domains must look toward properties of the functors that *are* invariant (e.g. preservation properties, adjunctions). In particular, we've seen how important preservation properties are in finding initial algebras (e.g. Propositions 4, 5, and Theorems 3.8, 3.9).

Importantly, polynomial endofunctors preserve wide pullbacks [49, p. 179]. Indeed, important 'intrinsic properties' of (more general) polynomial endofunctors on Sets are expressed in Kock's notes:

Theorem 4.4 (Theorem 8.6.3 [49])**.** *For a functor $P : \mathsf{Sets}/I \to \mathsf{Sets}/J$, the following conditions are equivalent.*

1. *P is polynomial*
2. *P preserves wide pullbacks (equivalently connected limits, or pullbacks and cofiltered colimits).*
3. *P is a sum of representables.*
4. *The comma category $(\mathsf{Sets}/J) \downarrow P$ is a presheaf topos.*
5. *P is a local right adjoint (i.e. the slices of P are right adjoints).*
6. *P admits strict generic factorizations.*
7. *Every slice of $\mathrm{el}(P)$ has an initial object (Girard's normal-form property).*

Some of these facts come from the properties of Sets in which $\mathsf{Sets}/I \cong \mathsf{Sets}^I$. Analyzing the underlying functors of accessible domains through these sorts of properties would constitute a proper (invariant) way to understand methodological frames in the context of initial algebras.

These properties will not hold in general for the powerclass functor, which preserves κ-filtered colimits for regular cardinal κ. We think that future work should elaborate the properties of functors that we want to use as distinctions between sorts of accessible domains. For example, we might already be able to say that preserving filtered colimits constitutes a relatively elementary endofunctor and that if this endofunctor has an initial algebra, the accessible domain is correspondingly elementary.

Chapter 5

Sets and algebraic hierarchies

This chapter will treat the case of various set-theoretic constructs as accessible domains (initial algebras of endofunctors). To do this, we must get more explicit about the surrounding categorical context than we were in the previous cases of natural numbers, constructive number classes, and other W-types. In those previous examples, we left implicit the sorts of categories we were relying on for the results to hold, such as the important facts that initial algebras are 'inductive' and 'recursive'.

The sort of constructs we will now consider include segments of the cumulative hierarchy, ordinal numbers, and again natural numbers, which all turn out to be deeply related in the following presentation. But each of these will have different versions that depend on the particular categorical context. For example, different categories will produce different (categorical versions of) ordinal numbers: variously resulting in all finite ordinals, all countable ordinals, or all classical ordinals. Similarly, cumulative hierarchies vary with the surrounding categorical context, and we will sketch in what way these variations depend on which categorical assumptions.

The point of this is to show that these constructs are accessible domains, by finding appropriate endofunctors on the (to be) axiomatized categories. Often, the endofunctor will revolve around a 'powerset' or 'powerclass' functor, but these will vary with context as well. To handle all these variations, we will centrally rely on axiomatic presentations, so that removing an axiom, weakening an axiom, or adding an axiom will produce a context in which another accessible domain may be characterized. The hope is that relatively simple changes to the axiomatization of the category, and related structure, will produce important accessible domains. And this in turn will, as far as possible, treat these set-theoretic constructs uniformly.

The set theories that we will focus on are outlined in the first section of this chapter. We rehearse some motivation for the axiomatizations themselves, starting with Zermelo-Fraenkel set theory; namely, the construction of a cumulative hierarchy.[1] We will briefly discuss some variations of this traditional formulation and some consequences of these axioms. Of particular interest will be the intuitionistic and 'constructive' versions of Zermelo-Fraenkel set theory. However, we will also look at weaker theories and any differences that might be interesting for the categorical contexts.

The axiomatization of set theories, using categories, will run along two complementary paths: logical and 'set theoretic'.[2] To axiomatize the logic that is available in a set theory (i.e., intuitionistic or classical), we have to make sure there is enough structure in the category to interpret things like conjunctions, disjunction, negations, etc. Though I will not explain in detail how these axioms really do give us a semantics for logic, I will indicate which axioms are for what purpose. Once we have settled on the 'internal logic' of the category, we consider axioms specific to set theoretic constructions like powerset and separation. This will essentially depend on a key insight of *algebraic set theory* which is to axiomatize a notion of 'smallness'; the intuition behind 'small' as opposed to 'large' sets mirrors the difference between 'sets' and 'proper classes' from Gödel-Bernays set theory, for example. When we have axiomatized what it means to be 'small', we can frame the operations like powerset and union in a way that depends uniformly on the notion of smallness. This has the benefit that if we want to consider set theories with a restricted powerset operation, or that validate only bounded separation, we vary the notion of smallness to do so.

Thus, the axiomatization of smallness will be the mechanism by which we derive different accessible domains. The scheme will be as follows:

- Given an axiomatization of smallness, we get a particular powerclass functor.
- Given that powerclass functor, we get a particular 'universe', or cumulative hierarchy, as the initial powerset algebra.
- The universe *is an accessible domain* (and is a model of a particular set theory).

This will be how we talk about entire set theories, but we shall not prove, for example, that the universe is a model of a particular set theory; we shall leave that to the references.

There are other constructs related to the cumulative hierarchy that require less work to capture the desired notion. Our main example in this respect will

[1] The naming of the cumulative hierarchy and a definitive study of its semi-categoricity can be found in the classical paper of Zermelo [103]. Zermelo showed that we get categorical models of set theory if in addition to the axioms, we fix the size of the class of urelements and the ordinals that are representable in the set theory. By varying these, especially the ordinals, we arrive at ever larger models of set theory. In our presentation, the universe and the ordinals will both be derived from the axiomatization of the category and will both be unique up to isomorphism.

[2] These two paths coincide with the 'synthesis' discussed in the introduction of structural axiomatics (axiomatization of smallness) and formal axiomatics (partially included in class categories).

be that of *ordinal* as seen in Section 5.7. To show, for example, that a particular accessible domain models set theory requires quite a bit of technical work on the logical side, but showing that our familiar notion of ordinal is captured by an appropriate accessible domain requires only showing some properties hold of the object. For this reason, the end of our discussion on accessible domains in set theories will include a substantial presentation of ordinals in algebraic set theory.

5.1 Set Theories

The first axiomatic set theory that we will discuss is Zermelo-Fraenkel **ZF** set theory. The axioms of **ZF** can be interpreted as just those which give the 'iterative' view of the universe of sets: the cumulative hierarchy. The cumulative hierarchy is a standard interpretation of the universe of sets. It consists in the (transfinite) iteration of the powerset operation on the emptyset:

$$V_0 = \emptyset \qquad V_{\alpha+1} = \mathcal{P}(V_\alpha) \qquad V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha \qquad V = \bigcup_{\lambda} V_\lambda.$$

We define V inductively, with a successor case $V_{\alpha+1}$ and a limit case of V_λ for a limit ordinal λ. We then define V as the (smallest) proper class of all steps of such a construction which is indexed by the ordinals. We will show how, when considered in a particular categorical context, V will be the initial algebra for a powerset endofunctor and thus the set-theoretic universe is an accessible domain. Indeed, when we perturb the categorical context, we arrive at set-theoretic universes for set theories other than **ZF** as well as important set-theoretic constructs like the ordinals.

The axioms of **ZF** codify the intuition that the universe of sets contains only extensional, well-founded collections equipped with the operations necessary to define V (i.e. emptyset, powerset, union, replacement). To explain this claim, let us examine the axioms, where we implicitly quantify over free variables and formulas:

Z1 (Extensionality). $a = b \leftrightarrow (\forall x)(x \in a \leftrightarrow x \in b)$

This axiom says that equality between sets is determined by the sets having the same members.

Z2 (Union). $\exists y \, \forall x (x \in y \leftrightarrow (\exists z \in a) \, x \in z)$

This says that to be an element of $\bigcup(a)$ means being an element of an element of a. This axiom allows us to create $\bigcup(a)$ from any set a that consists of all elements of elements of a. This allows us to form the 'limit' case in the construction of V.

Z3 (Powerset). $\exists y \, \forall x (x \in y \leftrightarrow x \subseteq a)$

This axiom asserts the existence of a powerset $\mathcal{P}(a)$ for any set a consisting of all the subsets of a. This gives us the successor case for the construction of V.

Z4 (Infinity). $\exists y\,(\emptyset \in y \;\&\; (\forall x \in y) x \cup \{x\} \in y)$

This axiom asserts that there is an infinite set. Zermelo originally asserted that there was a set containing $\emptyset$ and closed under the singleton operation $x \mapsto \{x\}$, but it is now customary to use the von Neumann successor $x \mapsto x \cup \{x\}$ instead. In the construction of V, we implicitly assumed that λ ranges over ordinals that extend into the infinite and this axiom ensures that at least one infinite set exists. It implies, combined with the powerset and replacement axiom (below), that there is a proper class of ordinals.

Z5 (Replacement). $(\forall y \in a)(\exists! z)\phi(y, z) \to (\exists z)(\forall x)(x \in z \leftrightarrow (\exists y \in a)\phi(y, x))$

This axiom schema says that for any functional formula $\phi(y, z)$ on a set a, the 'image' or 'range' of ϕ is also a set. This is necessary if we want to iterate the powerset operation more than finitely many times. Indeed, Zermelo's original axiomatization did not have Replacement and so could only guarantee the existence of V_ω (or $V_{\omega+\omega}$ if the axiom of infinity is included).[3]

Z6 ($\in$-induction). $\forall x\,((\forall y \in x)\phi(y) \to \phi(x)) \to \forall x\,\phi(x)$

This axiom asserts that universal statements about sets can be proved by induction on the elements: showing that the elements of a set x having the property ϕ guarantees that $\phi(x)$. Just like in the arithmetic case, induction here is equivalent to a well-foundedness or least-element principle.

Definition 5.1. Zermelo-Fraenkel set theory **ZF** is axiomatized by Z1$-$Z6 and uses classical logic. That is, **ZF** consists of Extensionality, Union, Powerset, Infinity, Replacement, and $\in$-induction.

To see how these axioms give us the cumulative hierarchy V, notice that the axioms fall into three groups: fundamental properties of sets, operations on sets, and the existence of particular sets. The first group consists of Extensionality (Z1) and $\in$-induction (Z6) which express that sets are to be extensional and well-founded collections. The second group consists in Union (Z2), Powerset (Z3), and Replacement (Z5) which allow us to construct V. Finally, the existence of an infinite set is given by Infinity (Z4).

Important consequences of the axioms, which we label "axioms" for convenience, as follows:

Axiom 5.1.1 (Emptyset). $\exists y\,\forall x(\neg x \in y)$

This axiom asserts the existence of a set with no members, the first step in constructing V.

[3] The axiom of replacement was independently formulated by Skolem and Fraenkel in 1922 to explicitly address this issue, but there were informal statements as early as Cantor's letter to Dedekind in 1899 [28] cf. [58, p. 489], [100, p.291].

Axiom 5.1.2 (Foundation). $\forall x(x \neq \emptyset \rightarrow (\exists y \in x) y \cap x = \emptyset)$

This axiom says that every nonempty set has an element which is disjoint from it. This prevents self-membership and any infinite 'descending' chain of membership $y \ni x_1 \ni x_2 \ni x_3 \ni \ldots$. The statement used here corresponds to a least-element principle of sorts.

Axiom 5.1.3 (Separation). $\exists y \forall x(x \in y \leftrightarrow (x \in a \wedge \phi(x)))$ *for any formula* ϕ *perhaps with additional parameters, in which* y *does not occur.*

This axiom schema allows us to construct the set $\{x \in a \mid \phi(x)\}$.

Axiom 5.1.4 (Pairing). $\exists y \forall x(x \in y \leftrightarrow (x = a \vee x = b))$

This axiom allows us to create the unordered pair of two sets a and b, denoted by $\{a, b\}$.

With these axioms, much of modern set theory can be carried out. Indeed, many see **ZF** as the core of standard set theory and, descriptively speaking, this seems to be the case.[4] But set theory was formed in the midst of serious methodological and philosophical change in foundational considerations and thus many of the various methodological views can be interpreted in set theory. For example, a constructivist attitude can be reflected in different sorts of set theories, including more or fewer axioms than **ZF**, or changing some of the axioms of **ZF** to better suit the philosophical perspective. The background logic, independently of the axioms themselves, can also conform to philosophical or pragmatic perspectives such as intuitionist logic or even more 'exotic' deviations from the assumed classical logic above.

We shall take seriously these different set theories because some important models, like the constructive cumulative hierarchy, are accessible domains in the same way that V is. The following are descriptions of major set theories that can be easily expressed in terms of their difference to **ZF** as well as some minor variations (e.g. whether atoms or urelements are permitted).

The simplest variation is changing the logic of the axiomatic system to be intuitionistic and not classical.

Definition 5.2. Intuitionistic Zermelo-Fraenkel set theory **IZF** is axiomatized by Z1$-$Z6 and uses intutionistic logic.

A set theory that differs from **ZF** and **IZF** comes from a broadly 'constructivist' or 'predicativist' perspective. It is called *Constructive set theory* or **CZF** [5]. This perspective looks on the powerset axiom and replacement as suspiciously strong: these axioms permit too much. That is not to say that these axioms are forgotten entirely; instead they are replaced by weaker versions, which we will now go over. The following axioms are taken from both [5] and [96].

[4] The inclusion of the axiom of choice is usually also taken for granted in the standard presentations of set theory. That is, **ZFC** is often thought of as standard. We leave considerations of choice out of our current presentation.

Axiom 5.1.5 (Strong Collection)**.** $(\forall x \in a)\,(\exists y)\,\phi(x,y) \rightarrow \exists b\,\mathbf{B}(x \in A, y \in b)\phi$ *where the formula* $\mathbf{B}(x \in A, y \in b)\phi$ *abbreviates*

$$(\forall x \in a)\,(\exists y \in b)\,\phi \wedge (\forall y \in b)\,(\exists x \in a)\,\phi.$$

Strong Collection is a strengthening of an axiom schema called, not surprisingly, Collection. The intuition for both versions of Collection are similar to the schema of Replacement and indeed, in the context of **CZF**, they have the same strength [73]. The only apparent difference between Strong Collection and Replacement is that Strong Collection does not require the formula ϕ to be single-valued (or functional). But collection only requires the existence of a superset of the image of ϕ, not necessarily the exact image.

Axiom 5.1.6 (Collection)**.** $(\forall x)(\exists y)\phi(x,y) \rightarrow \forall a \exists b(\forall x \in a)(\exists y \in b)\phi(x,y)$ *with neither* a *nor* b *free in* ϕ.

In the context of the other axioms of **ZF**, Collection is equivalent to Replacement, and when we provide universes of **IZF** and **ZF** below , we will be including Collection as an axiom instead of replacement, for it is simpler.

Axiom 5.1.7 (Subset Collection)**.**

$$\exists c\,\forall z\,((\forall x \in a)(\exists y \in b)\phi(x,y,z) \rightarrow (\exists d \in c)\,\mathbf{B}(x \in a, y \in d)\phi(x,y,z))$$

Subset collection is a modification of the Powerset axiom (Z3) which is classically equivalent, but constructively weaker. It can be taken to assert the existence of 'enough' total relations between every set a and b. The notion of 'enough' can be understood by the notion of *fullness*, which is equivalent, given the other (to be specified) axioms of **CZF**.

Definition 5.3 (Fullness)**.** A set c is *full in* a *and* b if

1. every element of c is a total relation on a and b.
2. for every total relation $r \subseteq a \times b$, there exists a total relation $s \in c$ such that $s \subseteq r$.

Subset collection can be thought of as asserting that for every a and b, there is a c that is full in a and b. The key here is to recognize that in intuitionistic contexts, the assertion of function sets is strictly weaker than the assertion of subset collection (or fullness), which in turn is strictly weaker than an axiom of powerset.

The theory **CZF** also requires that we allow separation only in restricted contexts, namely to bounded formulas:

Axiom 5.1.8 (Bounded Separation)**.** $\exists y\,\forall x(x \in y \leftrightarrow (x \in a \,\wedge\, \phi(x)))$ *for any bounded formula in which* y *does not occur free.*

The use of Separation for any formula whatsoever is thought to be impredicative, and therefore not constructive under this interpretation. For that reason, this restricted form only allows you to define properties (for the purpose of separation) based on 'previously' generated sets, which are the bounds for the bounded quantifiers.

Definition 5.4. Constructive Zermelo-Fraenkel set theory **CZF** uses intuitionistic logic and is axiomatized by

Z1: Extensionality

Z2: Union

Z4: Infinity

Z6: $\in$-induction

Axiom 5.1.5: Strong Collection

Axiom 5.1.7: Subset Collection

Axiom 5.1.8: Bounded Separation

All of the above set theories (**ZF**, **IZF**, **CZF**) assumed that the quantifiers ranged over a domain of only sets. But set theories that contain non-sets, that can be members of sets, but can have no members, have been long considered.[5] We will call these non-sets *atoms* and introduce the set-hood predicate $S(x)$ to assert that x is a set, and not an atom.

If we consider the language of the above set theories as extended by this predicate, we can formulate axioms regarding S and atoms:

Axiom 5.1.9 (Universal Sethood)**.** $\forall x\, S(x)$

This asserts that everything is a set. We may also say that only sets have elements:

Axiom 5.1.10 (Membership)**.** $x \in a \to S(a)$

Of course, if we want there to exists an empty set, we cannot make 'having members' and 'being a set' equivalent. Being a set is necessary to have members, but this way, not all sets must have members.

For each of the set theories we have so far considered, we can extend them to include axioms that allow for atoms to exist. How many atoms there are is a different question which will not influence our development here; we will pick this thread up when talking about *algebraic set theory* in Section 5.2. For notation, we will refer to **ZF** with atoms as **ZFA** and likewise **IZFA** and **CZFA** as set theories with atoms. One of the major differences when atoms are introduced is that the cumulative hierarchy starts with the set of atoms, not the emptyset; this will be an easy modification in the categorical setting and so we leave the topic for now.

[5]Zermelo's original axiomatization in [102] uses atoms.

5.1.1 Weaker Theories

Theories weaker than **CZF** are also considered in the context of finding interesting category-theoretic models of sets. We consider two studies of these weaker systems as examples.

Example 5.1 (**BIST**). In the thorough study of [21], the authors consider a theory called **BIST** which stands for Basic Intuitionist Set Theory. The theory **BIST** includes a unary predicate S for "set", where sets are closed under "various useful operations on sets, all familiar from mathematical practice." [21, p. 432] Notably, **BIST** includes a powerset axiom and implies full replacement in the sense of

$$(\forall y \in a)(\exists! z)\phi(y, z) \to (\exists z)(\forall x)(x \in z \leftrightarrow (\exists y \in a)\phi(y, x)). \tag{5.1}$$

It also includes a rather strange "Equality" axiom:

$$\exists y\, \forall x(x \in y \leftrightarrow (x = a \wedge x = b))$$

which is necessary given the weak context.

In the context of **BIST**, full replacement implies that only *restricted* properties can be used in an instance of separation. This notion of "restricted" is not the same as "bounded" so as to be equivalent to Bounded Separation. Instead, a property ϕ is "restricted" when it does not include the atomic formula $S(x)$ (asserting set-hood of x) and where every quantifier is bounded.[6] Simply adding the axiom that $S(x)$ is 'restricted' allows Bounded Separation for **BIST**. Adding some assumptions on how "restriction" works with arbitrary quantifiers allows for the validation of full separation.

The nature of infinity in **BIST** is somewhat subtle. The axiom of infinity in **BIST** simply states

$$(\exists I)(\exists 0 \in I)(\exists s : I \to I)(\forall x \in I\, s(x) \neq 0) \wedge (\forall x, y \in I\, s(x) = s(y) \to x = y).$$

But even were we to strengthen **BIST** by a collection axiom, we can't prove the infinity axiom where we use the von-Neumann successor (here modified to ignore $S(x)$):

$$\exists I(\emptyset \in I \wedge (\forall x \in I)\ x \cup \{x\} \in I).$$

Additional interesting features (regarding e.g. the law of excluded middle, notions of infinity, and consistency) and modifications of **BIST** can be found in [21].

Example 5.2 (**BCST** and **CST**). Michael Warren in [101] and Awodey and Warren in [18] discuss constructive and 'predicative' set theories that are closely related to **BIST** from Example 5.1. In those papers, the authors focus on three constructive set theories, denoted by **BCST** for Basic Constructive Set Theory,

[6] This is actually Corollary 2.3 in [21, p. 434] and not the definition of "restricted".

CST for Constructive Set Theory, and **CZF** for Constructive Zermelo-Fraenkel set theory. The relationship between them can be stated simply using the table included on [101, p. 24]:

Axioms	**BCST**	**CST**	**CZF**
Membership	$\bullet$	$\bullet$	$\circ$
Extensionality, Pairing, Union	$\bullet$	$\bullet$	$\bullet$
Emptyset	$\bullet$	$\bullet$	$\circ$
Binary Intersection	$\bullet$	$\bullet$	$\circ$
Replacement	$\bullet$	$\bullet$	$\circ$
Bounded Separation	$\circ$	$\circ$	$\bullet$
Exponentiation		$\bullet$	$\circ$
Infinity			$\bullet$
$\in$-induction			$\bullet$
Strong Collection			$\bullet$
Subset Collection			$\bullet$
Universal Sethood			$\bullet$

The filled in circles $\bullet$ indicate axioms of the set theory and hollow circles $\circ$ indicate a consequence of the axioms. Note that this formulation of **CZF** is slightly different then the above formulation; in that it explicitly includes the universal sethood axiom. For completeness, let me include the axioms that we have not seen yet.

Axiom 5.1.11 (Binary Intersection). $\exists y \forall x(x \in y \leftrightarrow (x \in a \wedge x \in b))$ *for sets* a *and* b.

This axiom ensures the existence of binary intersections, as the name suggests.

Axiom 5.1.12 (Exponentiation). *If* $S(a)$ *and* $S(b)$ *then*

$$\exists y \forall x(x \in y \leftrightarrow x \textit{ is a function from } a \textit{ to } b).$$

This axiom asserts the existence of function spaces b^a for sets a and b. In the categorical context, this allows us to leverage W-types; more on that below.

Some major differences to note:

- **CST** has exponentiation, which is important for W-types to exist, whereas **BCST** does not;
- **CZF** is quite a bit stronger than the other two, containing an axiom of infinity and $\in$-induction. It also cannot contain urelements, in this formulation, given the Universal Sethood axiom.

5.2 Algebraic Set Theory

In this section, we will introduce *algebraic set theory* to incorporate hierarchies and universes of sets into the framework of accessible domains as initial algebras for endofunctors. It should not be too surprising that this is possible, since the iterative conception of set emphasizes that the universe of sets is inductively defined in a "very similar" way to that of the natural numbers. What *is* perhaps surprising is how uniform we can treat multiple sorts of set theories: e.g., Zermelo-Fraenkel set theory in its classical (**ZF**), intuitionistic (**IZF**) and constructive or 'predicative' (**CZF**) versions. To do this, we axiomatize set theories in an algebraic way (using categories), and some of these axioms can be weakened or left out to accommodate various perspectives or preferences for set theories. Notions of definability, length of iteration, and background logic are all dimensions that algebraic set theory can vary within its framework. Thus, the use of algebraic set theory for this dissertation is clear, since our mission is to provide abstract methods for relating differing methodological frames.

Algebraic set theory can be thought to start with the book *Algebraic Set Theory* by Andre Joyal and Ieke Moerdijk (1995), though they had published results already in [46]. They were motivated to perceive the hierarchy of sets as an algebraic structure of a simple kind, very similar to the functor algebras we have been considering. However, in our earlier discussions of initial algebras, we treated notions like the category of sets and functions (Sets) without much detail. For example, the proof that finitary polynomial functors on Sets preserve ω-colimits relied on what you take to be the category Sets. The result does not hold for arbitrary categories! In category theory—especially when giving familiar examples—we allow the reader to pick their favorite set theory, within some bounds, to substitute for that category. But now that we are to focus on sets themselves, it behooves us to take care on how we understand the ambient category.

First, I will describe the general outlines of algebraic set theory, and then I will give parts of the mathematical theory that are relevant for characterizing universes of sets as accessible domains. The end result is important: just as the Dedekind-Peano axioms characterize the initial algebra for the functor S (see section 2.3), the axioms of Zermelo-Fraenkel set theory are just a description of what is called the "initial ZF-algebra". The analogy goes even deeper: ZF-algebras have one unary successor operation[7] but are also equipped with the ability to make indexed unions. However, the key insight of algebraic set theory is summed up by Alex Simpson:

> The crucial idea for obtaining category-theoretic models of set theory is that, rather than axiomatizing the structure of the category of sets on its own, one should instead axiomatize the structure of the category of sets together with that of its surrounding supercategory of classes. [90, p. 15]

For many purposes, the 'surrounding supercategory of classes' can be thought of in a way similar to the Morse-Kelley and Gödel-Bernays set theories: we make distinctions between sets and proper classes. In algebraic set theory, we will use the word "small" to refer to sets, since the distinction is really one of size limitation.

[7] As in the last chapter, you may think of this as the singleton operation $a \mapsto \{a\}$.

Once we have axiomatized the notion of *smallness* in a category of classes (also to be axiomatized), we will construct the relevant accessible domains. And we will see that changing the axiomatization allows us to study multiple interesting structures from a similar vantage point. Speaking informally, if we take the notion of "smallness" to be finiteness, the initial ZF-algebra will be the algebra of heriditarily finite sets. If we take the notion to refer to sethood in the sense of Morse-Kelley, then the initial ZF-algebra will be "essentially the cumulative hierarchy of sets": isomorphic to the familiar von-Neumann hierarchy.

We can go a step further and 'add relations' to the notion of ZF-algebra. A natural set of examples comes from requiring that the 'successor' operation is monotone. Again, by changing the notion of smallness (by varying the axioms that characterize it), we get interesting accessible domains. For example, if our 'classes' consist of all countable sets, then the initial ZF-algebra with monotone successor is the set of natural numbers. If, as before, we take sethood to be the notion of smallness, the initial ZF-algebra with monotone successor is the class of ordinal numbers. As we showed with initial algebras for endofunctors, the transfinite induction for ordinals comes from the algebraic properties of this characterization.

And finally, before getting to the mathematical detail, let me quote Joyal and Moerdijk in describing their framework:

> Our abstract framework thus consists of a suitable category $\mathbb{C}$, with a designated class of arrows in $\mathbb{C}$, which are called small, and satisfy natural axioms. In this general context, it is possible to define algebras L as objects in $\mathbb{C}$ equipped with an operation $s : L \to L$ for successor, and with a partial order on L which is complete in the sense that the supremum exists along any map which is designated small. Such algebras L will be called *Zermelo-Fraenkel algebras* in $\mathbb{C}$. We investigate the structure of the free (initial) ZF-algebra V, and show that it can be viewed as an algebra of small sets, via an explicit isomorphism between V and the object $P_{\mathcal{S}}(V)$ of "small subsets" of V. This free algebra V should be viewed as a cumulative hierarchy of small sets, relative to the ambient category C and its class of small maps. Indeed, [...] under very general conditions this algebra V is a model of the axioms of (intuitionistic) Zermelo-Fraenkel set theory.

The category $\mathbb{C}$ is suitable in the sense of being able to interpret 'enough' logic, including some definability constraints. It is this category $\mathbb{C}$ that will determine the background logic, usually assumed to be at least intuitionistic first-order logic with equality. We can consider categories that interpret classical logic by simply adding a condition to the axiomatization of $\mathbb{C}$. The correspondence between logics and categories is quite interesting and robust. To name particularly important ones, Awodey lists "higher-order logic (elementary topos); Martin-Löf dependent type theory (LCC pretopos); first-order logic (Heyting pretopos)[...] infinitary higher-order logic (cocomplete topos)" [18, p. 4].

To each of these correspondences, algebraic set theory adds a correspondence to a particular set theory. For example, in [18] it is shown that Basic Intuitionistic Set Theory represents exactly the elementary set theory whose models are elementary toposes.[8] In the next two sections, we will give axioms that characterize first a 'class category' that interprets logic and will serve as our ambient category and second the notion of smallness that provides a way to construct universes of sets within the class category.

5.3 Axioms for Class Categories

The categories that will serve as what Simpson calls the "surrounding supercategory of classes" need enough structure to interpret logical constructions such as conjunctions, predicates, etc. We start with the axioms for a *Heyting pretopos*. As with all the axiomatizations in algebraic set theory, we provide the axioms so that we may change them to cover many cases. After all, we are not after one set theory, but many.

We now give the axioms for a Heyting category, indicating logical and mathematical constructions that each allows.

C1. *$\mathbb{C}$ has pullbacks and a terminal object. Hence $\mathbb{C}$ has all finite limits.*

This axiom ensures that we have some basic structure in the category. That is, we have finite products, e.g. $A \times B \times C$, a terminal object 1, pullbacks, and equalizers. Pullbacks are the interpretation of equality, as described in Appendix A.

C2. *$\mathbb{C}$ has finite coproducts, and these are disjoint and stable under pullback.*

This axiom tells us that $\mathbb{C}$ is *extensive*. Like Axiom C1, this adds some structure to the category; in this case, by providing what we may interpret as disjoint unions. In categorical terms disjointness says that the pullback of the coproduct inclusions is an initial object:

$$\begin{tikzcd} 0 \cong & P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "inr"] \\ & A \arrow[r, "inl"'] & A+B. \end{tikzcd}$$

Since we can think of the pullback as a subobject $P \rightarrowtail A + B$ consisting of pairs (a, b) that satisfy the equation $inl(a) = inr(b)$, the fact that P is 'empty' tells us that the images of A and B have no overlap in $A + B$. Without C1, we would have to state C2 in more general terms, based on subobjects.

Stability of the finite coproducts means that for any family $\{f_i : Y_i \to X \mid i = 1, \ldots, n\}$ and any arrow $X' \to X$, the canonical map

$$\coprod_i (X' \times_X Y_i) \to X' \times_X \coprod_i Y_i$$

[8] For the precise sense of this, see [18].

is an isomorphism. More simply put, this says that the coproduct of the pullback is isomorphic to the pullback with the coproduct. In diagrams, that means that we start with a family (Y_i) and maps $f_i : Y_i \to X$ and arrow $X' \to X$:

$$\begin{tikzcd} Y_1 \arrow[drr, "f_1"'] & \cdots & Y_i \arrow[d, "f_i"] & X' \arrow[dl] \\ & & X & \end{tikzcd} .$$

We define the family of pullbacks:

$$\begin{tikzcd} Y_i \times_X X' \arrow[r] \arrow[d] & X' \arrow[d] \\ Y_i \arrow[r] & X. \end{tikzcd}$$

and then we take the coproduct of all the pullbacks and its map to X':

$$\begin{tikzcd} X' \times_X Y_1 \arrow[dr] & X' \times_X Y_2 \arrow[d] & \cdots & X' \times_X Y_n \arrow[dll] \\ & \coprod_i (X' \times_X Y_i) \arrow[rr] & & X'. \end{tikzcd}$$

We can also define the coproduct of the (Y_i) with its canonical morphism to X:

$$\begin{tikzcd} Y_1 \arrow[drr] & \cdots & Y_n \arrow[dr] & & X' \arrow[d] \\ & & & \coprod_i Y_i \arrow[r] & X. \end{tikzcd}$$

Then we have the pullback

$$\begin{tikzcd} Y_1 \arrow[drr] & \cdots & Y_n \arrow[dr] & \coprod_i Y_i \times_X X' \arrow[r] \arrow[d] & X' \arrow[d] \\ & & & \coprod_i Y_i \arrow[r] & X. \end{tikzcd}$$

Stability of coproducts says that $\coprod_i (Y_i \times_X X')$ and $\coprod_i Y_i \times_X X'$ are isomorphic via the map $\coprod_i (Y_i \times_X X') \to \coprod_i Y_i \times_X X'$.

The next axiom provides a way to talk about quotients if equivalence relations.

C3. *For every arrow $f : X \to Y$, the kernel pair k_1, k_2 has a coequalizer $q : X \to Q$.*

$$\begin{tikzcd} K \arrow[r, shift left, "k_1"] \arrow[r, shift right, "k_2"'] & X \arrow[r, two heads, "q"] \arrow[d, "f"] & Q \\ & Y & \end{tikzcd}$$

Further, regular epimorphisms are stable under pullback.

This axiom is somewhat dense so let us unpack it. The first notion that is important is that of a *kernel pair* of an arrow f. This is simply the pullback of f with itself:

$$\begin{tikzcd} K \arrow[r, "k_1"] \arrow[d, "k_2"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

The name is suggestive, as it captures the standard notion of a kernel, as described in Appendix A The coequalizer Q can be thought of as the subobject of Y corresponding to the image of the map f. This comes from taking the quotient of X with respect to the kernel equality; so two elements x_1 and x_2 are identified if their values under f are equal.

That we require these regular epimorphisms to be stable means that for the pullback diagram

$$\begin{tikzcd} X \times_B A \arrow[r] \arrow[d, "f'"'] & A \arrow[d, "f"] \\ X \arrow[r] & B \end{tikzcd}$$

if f is a regular epimorphism, then so is f'.

C4. *For each arrow $f : X \to Y$ in $\mathbb{C}$, the pullback functor $f^* : Sub(Y) \to Sub(X)$ has a right adjoint $\forall_f$.*

As the notation suggests, this helps in getting a universal quantifier; the existential quantifier comes from **C3**.

Now we can summarize this axiomatization of class categories.

Definition 5.5 ([18, p. 7])**.** A *class category* is a locally small category $\mathbb{C}$ that satisfies the axioms

C1. $\mathbb{C}$ has finite limits

C2. $\mathbb{C}$ has finite disjoint colimits that are stable under pullback

C3. Every kernel pair in $\mathbb{C}$ has a coequalizer; regular epimorphisms are stable under pullback.

C4. The pullback functor for any arrow has a right adjoint.

The power of this axiomatization is apparent from the following fact:

Proposition 6. *If $\mathbb{C}$ is a class category, then $\mathbb{C}$ models intuitionistic, first-order logic with equality.*

This axiomatization really aims at simply getting to this point: we can reason with the objects of $\mathbb{C}$ using the familiar intuitionistic, first-order logic with equality. To get to classical logic, we need only add the assumption that every subobject has a complement, since this will validate the law of excluded middle for predicates.

For the remainder of the chapter, unless explicitly stated, we will assume $\mathbb{C}$ is a class category. We now move to axiomatizing the notion of smallness. Recall that the intuition here is to get a category of classes, which we've done, and to then isolate the "sets" from that category.

5.4 Axioms for Small Maps

This section axiomatizates the notion of "smallness" by axiomatizing the class of small maps $\mathcal{S}$. The intuition for smallness comes from considering the fibers of the maps, a topic we discussed in relation to polynomial functors in Chapter 4. Intuitively, a map $f : B \to A$ is small when each fiber '$f^{-1}(a) \subseteq B$' is a set. More formally, the fiber of f is the pullback of $a : 1 \to A$ along f. In this way, we can envision f as a parameterized family of sets $(B_a)_{a \in A}$ where $B_a = f^{-1}(a)$ as before. This particular axiomatization can be found as the main overlap for the axiomatizations given in [47], [69, 96], and [21]. We will add to, and modify, this basic axiomatization to capture the cumulative hierarchies of **CZF**, **IZF**, and **ZF**, in addition to some discussion of other set theories, such as **BIST**, the theory of Awodey et al. in [21]. We will, mostly in the axiomatization for **IZF**, characterize the class of ordinals as a specialized 'cumulative hierarchy' or sorts, in Section 5.7.

Now the axioms of the class of small maps $\mathcal{S}$, based on [47], where we call a map small if it is in $\mathcal{S}$.

S1. *$\mathcal{S} \hookrightarrow \mathbb{C}$ is a subcategory of $\mathbb{C}$ with the same objects as $\mathbb{C}$. In particular, every identity arrow $1_X : X \to X$ is in $\mathcal{S}$. Similarly, $\mathcal{S}$ is closed under composition.*

S2 (Stability under Pullback)**.** *The pullback of a small map is small. That is, in the pullback diagram*

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f'"'] & A \arrow[d, "f"] \\ B \arrow[r] & C \end{tikzcd}$$

if f is in $\mathcal{S}$ then so is f'.

S3 (Descent)**.** *In the pullback diagram*

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f'"'] & A \arrow[d, "f"] \\ B \arrow[r, two heads, "p"] & C \end{tikzcd}$$

if p is an epimorphism and f' belongs to $\mathcal{S}$, then so does f.

S4. *Both $0 \to 1$ and $1 + 1 \to 1$ are in $\mathcal{S}$.*

S5 (Sums)**.** *If $Y \to X$ and $Y' \to X'$ belong to $\mathcal{S}$, the canonical map $Y + Y' \to X + X'$ belongs to $\mathcal{S}$.*

S6 (Quotients)**.** *If $f \circ e$ is in $\mathcal{S}$ and e is an epimorphism, then f is small:*

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[dr, "f\circ e"'] & B \arrow[d, "f"] \\ & C \end{tikzcd}$$

We call a map or arrow f *small* when f is in $\mathcal{S}$ and we also define what it means for an object to be *small*.

Definition 5.6. An object X of $\mathbb{C}$ is small if the unique arrow $X \to 1$ is in $\mathcal{S}$.

This makes sense with our intuitive picture of small maps. Small maps are those with 'set-sized' fibers, and the fiber of the map $X \to 1$ of the only element of 1 is precisely all of X. So saying the map is small is saying that X is a set. But of course, we will reverse the usual order: small maps will be defined axiomatically the smallness of objects will be derived from that axiomatization. In other words, in the axiomatization, we aren't using the notion of set-hood, we're defining it.[9] But for intuition, it can help to remind ourselves of the intended idea.

We will eventually need to know when relations are small, as well.

Definition 5.7. A relation $R \hookrightarrow A \times B$ is *small* if its second projection is a small map:

$$R \hookrightarrow A \times B \to B.$$

5.5 Powerclass Functors

This section will describe the powerclass functors so that we can see how segments of various cumulative hierarchies are initial algebras for these powerclass functors. These powerclass functors will depend essentially on the notion of smallness as axiomatized in Section 5.4. The intuition for powerclass functors $P_{\mathcal{S}}$ is that for any object X of $\mathbb{C}$ they give us the class of all *small* sub-objects of X. If smallness coincides with some notion of set-hood, for example, then $P_{\mathcal{S}}(X)$ is the class of all sub*sets* of the object X. Note that without an explicit axiom, we cannot say whether $P_{\mathcal{S}}(X)$ itself is small, even if it is the collection of small subobjects of X. Such an axiom can have different presentations, which we invoke where necessary.

If, in our intended set theory, powerclasses of small objects are themselves small, then we can construct the cumulative hierarchy of 'all' small objects. This will be done explicitly in the next section (Section 5.6). For now, we will focus on describing the powerclass object and the powerclass functor on which the cumulative hierarchy is based. For the following discussion, assume $\mathbb{C}$ is a

[9]Notably, we are defining it uniformly. This methodology is characteristic to category theory and provides one of the key benefits for this project. More on this will be discussed in Chapter 6.

class category and $\mathcal{S}$ is a class of small maps. In [47], the authors construct the powerclass as the representing object of a functor $\mathcal{P}^{\mathcal{S}}$ as follows. Let X be an object of $\mathbb{C}$. Generally, for any object I, an I-index family of subobjects of X is a subobject $S \rightarrowtail I \times X$.

Definition 5.8. For any objects X and I in $\mathbb{C}$, an *I-indexed family of small subobjects of X* is an I-index family of subobjects of X such that

$$S \rightarrowtail I \times X \xrightarrow{\pi_1} I$$

is small.

Denote by $\mathcal{P}^{\mathcal{S}}(X)(I)$ the set of all I-indexed families of small subobjects of X. Given any map $g : J \to I$ in $\mathbb{C}$, we can define a map

$$g^{\#} : \mathcal{P}^{\mathcal{S}}(X)(I) \to \mathcal{P}^{\mathcal{S}}(X)(J)$$

by pullback. That is, taking the pullback of each arrow in an I-indexed family of small subobjects of X will result in the diagram:

$$\begin{tikzcd}
S' \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, tail] \\
J \times X \arrow[r] \arrow[d, "\pi_1'"'] \arrow[dr, phantom, "\lrcorner", very near start] & I \times X \arrow[d, "\pi_1"] \\
J \arrow[r, "g"'] & I
\end{tikzcd}$$

By Axiom **S2**, π_1' is small because π_1 is, thus showing that $S' \rightarrowtail J \times X \to J$ is a J-indexed family of small subobjects. This makes $\mathcal{P}^{\mathcal{S}}(X)$ into a contravariant functor $\mathbb{C}^{\mathrm{op}} \to \mathsf{Sets}$. Then, we have a theorem about this functor.

Theorem 5.9 ([47, Theorem II.3.1])**.** *Let $\mathbb{C}$ be a class category and $\mathcal{S}$ a class of small maps. If there is a universal small map $\pi : E \to U$ in $\mathcal{S}$ [47, **S2**, p. 9], then the functor $\mathcal{P}^{\mathcal{S}}(X)$ is representable.*

The representing object is denoted by $P_{\mathcal{S}}(X)$. This means that there is a bijection, natural in I, between I-indexed families of small subobjects $S \rightarrowtail I \times X$ and arrows $\chi_S : I \to P_{\mathcal{S}}(X)$. We call χ_S the *characteristic map for S*.

The proof of Theorem 5.9 is complex, and relies on a construction of $P_{\mathcal{S}}(X)$ through the universal small object labeled by X, which is obtained by the universal small map $\pi : E \to U$. We did not include this as one of our axioms for $\mathcal{S}$ because only the presentation of the cumulative hierarchy of **CZF** will need this universal small map (in Section 5.6).

For **IZF** and **ZF**, we may simply require that these families of small subobjects exist, as done in [18].

P1. *For every object C of $\mathbb{C}$, there is an object $P_{\mathcal{S}}C$ with a small relation $\in_C \rightarrowtail C \times P_{\mathcal{S}}C$ such that for any small relation $R \subseteq C \times X$, there is a unique* *"classifying map"* $\chi_R : X \to P_{\mathcal{S}}C$ *such that* R *is the pullback of* $\in_C$ *along* $(1_C \times \chi_R)$*:*

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \in_C \arrow[d, hook] \\ C \times X \arrow[r, "(1_C \times \chi_R)"'] & C \times P_{\mathcal{S}}C. \end{tikzcd}$$

The first axiom is a requirement of $\mathbb{C}$ to have certain power objects, which is similar to the the axiom **S2** in [47, p. 9]. [10] Axiom **P1** states that every object of $\mathbb{C}$ has a powerclass of all small subobjects. Instead of collecting together all the subclasses of C, the powerclass contains all the sub*sets* of C, where "sets" is understood to refer to smallness. The relation $\in_C$ can be understood as the universal relation such that all small relations on C must be pullbacks of $\in_C$.

A further requirement we may have for powerclasses can be stated as follows:

P2. *The internal subset relation* $\subseteq_C \hookrightarrow P_{\mathcal{S}}C \times P_{\mathcal{S}}C$ *is small.*

Recall that $\subseteq_C$ can be intuitively thought of as a relativized subset[11] such that

$$X \subseteq_C Y \leftrightarrow (X \in P_{\mathcal{S}}C \ \&\ Y \in P_{\mathcal{S}}C \ \&\ X \subseteq Y).$$

With this additional assumption, not only does the powerclass object exist for every object of $\mathbb{C}$, this axiom implies that the powerclass of a small object is again small: sets are closed under powerclasses. Since relations are small when their second projection is small (Definition 5.7), this axiom requires that the map

$$\subseteq_C \hookrightarrow P_{\mathcal{S}}C \times P_{\mathcal{S}}C \xrightarrow{\pi_2} P_{\mathcal{S}}C$$

is small. Intuitively, this map is small only if the fibers are sets. And so, this axiom requires that $\pi_2^{-1}(B)$, the set of all subsets of B (which in turn is a subset of C), is a set as well.

This relation of assigning a powerclass to each object is functorial as needed. For each object X of $\mathbb{C}$, there is a powerclass of all small subobjects of X called $P_{\mathcal{S}}(X)$. This extends to arrows through the notion of an *image*. The (direct) image of a map $f : A \to B$ is a subobject $m : \mathrm{im}(f) \hookrightarrow B$ that is universal in the sense that given any factorization of $f = m' \circ e$ there exists a unique map k such that $m = m' \circ k$.

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend left, "f"] \arrow[dr, "e"'] & \mathrm{im}(f) \arrow[r, hook, "m"] \arrow[d, dashed, "k"] & B \\ & X \arrow[ur, "m'"'] & \end{tikzcd}$$

[10] This axiom is a reformulation of our discussions of powersets in Appendix B for example with diagram B.1. The only difference now is the dependency on the notion of smallness: $\in_C$ and R are small.

[11] See Appendix B.

We can show how $P_{\mathcal{S}}$ acts on arrows as follows [47, p. 18]: For a subobject $s : S \hookrightarrow I \times X$ and a map $f : X \to Y$, define $f_!(S)$ to be the image of S under the map $id_I \times f : I \times X \to I \times Y$. That is, $f_!(S)$ is the subobject $\mathrm{im}(id_I \times f \circ s) \hookrightarrow I \times Y$. Now considering indexed families of small subobjects, we note that

$$f_!(S) \hookrightarrow I \times Y \xrightarrow{\pi_1} I$$

is small when

$$S \hookrightarrow I \times X \xrightarrow{\pi_1} I$$

is small because $f_!(S)$ is defined with id_I. This means, that whenever S belongs to $P^{\mathcal{S}}(X)(I)$, we have that $f_!(S)$ belongs to $P^{\mathcal{S}}(Y)(I)$, thus making an operation $f_! : P^{\mathcal{S}}(X)(I) \to P^{\mathcal{S}}(Y)(I)$. This operation is natural in I.

By the Yoneda Lemma, the operation $f_!$ is given by composition with a uniquely determined map

$$P_{\mathcal{S}}(f) : P_{\mathcal{S}}(X) \to P_{\mathcal{S}}(Y).$$

This shows that $P_{\mathcal{S}}$ is a covariant functor on $\mathbb{C}$.

$$\begin{aligned} P_{\mathcal{S}}(f : A \to B) &: P_{\mathcal{S}}A \to P_{\mathcal{S}}B \\ &(s : X \hookrightarrow A) \mapsto (\mathrm{im}(f \circ s) \hookrightarrow B). \end{aligned}$$

So the functor $P_{\mathcal{S}}$ takes arrows and gives the 'image' arrow which takes subsets of A and returns subset of B corresponding to the image of A under f. This is the powerclass functor $P_{\mathcal{S}}$ that will be used to exhibit cumulative hierarchies as accessible domains.

5.6 ZF Algebras and Cumulative Hierarchies

In this section we quote central results about Zermelo-Fraenkel algebras that will aid our understanding of how cumulative hierarchies are instances of accessible domains in the categorical context.[12] For this section, let $\mathbb{C}$ be a Heyting pretopos as defined in Section 5.3 and $\mathcal{S}$ be a class of small maps as axiomatized in Section 5.4.

To set the stage, let us define what it means to have a suprema along a small map, since this will play a significant role in the definition of ZF algebra (Definition 5.12 below). Notice that if L is a poset in $\mathbb{C}$ then the set $\hom_{\mathbb{C}}(A, L)$ of all arrows $A \to L$ is partially ordered for each $A \in \mathbb{C}$. So we can compare $f, g \in \hom_{\mathbb{C}}(A, L)$, which we take advantage of in the following definition.

Definition 5.10 ([47, p. 22])**.** Let $g : B \to A$ and $\lambda : B \to L$ be maps in $\mathbb{C}$. A map $\mu : A \to L$ is said to be the *supremum of* λ *along* g if for any maps

[12] These results all come from [47] unless otherwise noted.

$t : C \to A$ and $\nu : C \to L$ with the pullback square

$$\begin{tikzcd} C \times_A B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "g"] \\ C \arrow[r, "t"'] & A, \end{tikzcd}$$

we have that

$$\mu \circ t \leq \nu \quad \text{iff} \quad \lambda \circ \pi_2 \leq \nu \circ \pi_1.$$

Given both λ and g, we can denote this supremum as

$$\mu = \bigvee_g \lambda : A \to L$$

since it will necessarily be unique. Using the internal logic of $\mathbb{C}$ we can also write

$$\mu(a) = \bigvee_{g(b)=a} \lambda(b)$$

for the supremum. The poset L is called *$\mathcal{S}$-complete* if suprema in L exist along any map in $\mathcal{S}$.

Take the following example as giving the intuition for these suprema. Let $g^{-1}(a) = \{b_1, b_2, b_3\}$. Then $\mu(a) = \bigvee\{\lambda(b_1), \lambda(b_2), \lambda(b_3)\}$ in L. So μ gives us the suprema of λ applied to the fiber $g^{-1}(a)$. The map g gives us the fiber information (like in the case of polynomial functors) and λ brings elements of this fiber into the poset L, and μ in turn finds the supremum in L on these.

Example 5.3. We can construct the binary supremum operation $\vee : L \times L \to L$ as a supremum of $[\pi_1, \pi_2] : (L\times L)+(L\times L) \to L$ along the map $[id, id] : (L\times L)+(L\times L) \to (L\times L)$. This is because we have that $[\pi_1, \pi_2]([id, id]^{-1}(x, y)) = \{x, y\}$ and so

$$\vee(x, y) = \bigvee\{x, y\} = x \vee y.$$

In addition, since $1 + 1 \to 1$ is small (Axiom **S4**) and $\mathcal{S}$ is closed under pullbacks (Axiom **S2**), then $[id, id]$ is small too. Thus, under our axiomatization of Section 5.4, if L is $\mathcal{S}$ complete, then $\vee$ exists.

It turns out that if we consider the powerset $P_{\mathcal{S}}(X)$ as a partially ordered set, ordered by the inclusion of subobjects, then $P_{\mathcal{S}}(X)$ is complete relative to small maps.

Proposition 7. *For any object $X \in \mathbb{C}$, the power object $P_{\mathcal{S}}(X)$ ordered by inclusion of subobjects is $\mathcal{S}$-complete.*

It turns out that the powerset $P_{\mathcal{S}}(X)$ is actually free with respect to being a $\mathcal{S}$-complete sup-lattice on X.

Definition 5.11. We call an object Y the *free $\mathcal{S}$-complete sup-lattice generated by X* if there is a map $X \to Y$ and if any map $g : X \to L$ into a $\mathcal{S}$-complete sup-lattice L can be uniquely extended to a map $\bar{g}$ which preserves suprema along small maps:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d] & L \\ Y. \arrow[ur, dashed, "\bar{g}"'] & \end{tikzcd}$$

Proposition 8 ([47, Proposition 4.2]). *$P_{\mathcal{S}}(X)$ is the free $\mathcal{S}$-complete sup-lattice generated by X with $\{\cdot\} : X \to P_{\mathcal{S}}(X)$.*

Now we use these definitions to define the central notion of a ZF algebra.

Definition 5.12. A *Zermelo-Fraenkel (ZF) algebra* is an $\mathcal{S}$-complete sup-lattice L in $\mathbb{C}$ equipped with a map $s : L \to L$ called the *successor* operation. We denote the algebra as (L, s).

Definition 5.13. A *homormophism between ZF algebras* $(L, s) \to (M, t)$ is a map $f : L \to M$ which preserves suprema along small maps and commutes with the successor.

More concretely, a ZF algebra homomorphism $f : L \to M$ preserves suprema along small maps if $\mu : A \to L$ is a supremum of $\lambda : B \to L$ along the small map $g : B \to A$, then

$$\begin{tikzcd} A \arrow[d, "\mu"'] \arrow[dr, "\mu'"] & \\ L \arrow[r, "f"'] & M \end{tikzcd}$$

commutes, with μ' the supremum of $f \circ \lambda$ along g. Similarly, a ZF algebra homomorphism $f : L \to M$ commutes with successor if the following diagram commutes:

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[d, "s"'] & M \arrow[d, "t"] \\ L \arrow[r, "f"'] & M . \end{tikzcd}$$

We now define what it means to be a free ZF algebra.

Definition 5.14 ([47, p. 22–23]). For an object A in $\mathbb{C}$, a *free ZF algebra* on A is a ZF algebra $(V(A), \eta)$, with the property that for any object B in $\mathbb{C}$ and any ZF algebra (X, s) in $\mathbb{C}/B$, any map $\phi : B^*(A) \to L$ in $\mathbb{C}/B$ can be uniquely extended to a homomorphism of ZF algebras $\overline{\phi} : B^*(V(A)) \to L$ in $\mathbb{C}/B$:

$$\begin{tikzcd} A \times B \arrow[r, "\phi"] \arrow[d, "\eta \times B"'] & L \\ V(A) \times B \arrow[ur, dashed, "\overline{\phi}"'] & \end{tikzcd}$$

The object $V(A)$ is called the *cumulative hierarchy on* A. If $A = 0$ then we simply write V for the cumulative hierarchy. The set A on which $V(A)$ is based is the set of urelements or atoms of the set theory.

We can understand V as an algebra of "small sets", up to isormophism, by showing that V is isomorphic to the object of all small subsets of V, namely $P_{\mathcal{S}}(V)$.

Theorem 5.15 ([47, Theorem II.1.2]). *The map $r : P_{\mathcal{S}}(V) \to V$ defined by the formula*

$$r(E) = \bigvee_{x \in E} s(x)$$

is an isormorphism of $\mathcal{S}$-complete sup-lattices.

Proof. First, we use the singleton map $\{\cdot\} : V \to P_{\mathcal{S}}(V)$ to define a successor operation

$$s' : P_{\mathcal{S}}(V) \to P_{\mathcal{S}}(V)$$

by

$$s'(E) = \{\bigvee_{x \in E} s(x)\}.$$

Remember that with ZF-algebras, we are using sup-lattices so $\bigvee_{x \in E} s(x)$ intuitively is the supremum of the set $\{s(x) \mid x \in E\}$ where E is an element of $P_{\mathcal{S}}(V)$, understood as a subset of V. More precisely, we note that $x \in E$ is categorical short hand for $x : U \to E$ for some U, which means x is properly considered as an arrow.[13] Thus s' takes subsets of V and returns the singleton of the supremum of the successors of elements of that subset.

We know that $P_{\mathcal{S}}$ itself is a $\mathcal{S}$-complete sup-lattice (Proposition 7) and so $(P_{\mathcal{S}}(V), s')$ is a ZF algebra. Since V is the free ZF algebra, there is a unique ZF algebra homomorphism

$$i : V \to P_{\mathcal{S}}(V).$$

But the r given in the statement of the theorem is a map $r : P_{\mathcal{S}}(V) \to V$. It's clear that r is a homomorphism of ZF algebras since it commutes with successor by definition of s' and since it is a supremum itself it preserves them, in particular along small maps.

Then $r \circ i = id_V : V \to V$ by freeness of V. To show that $i \circ r = id_{P_{\mathcal{S}}(V)}$ note that $r \circ i = id_V$ implies that for any $v \in V$

$$v = r \circ i(v) = r(i(v)) = \bigvee_{x \in i(v)} s(x)$$

This implies that

$$s'(i(v)) = \{\bigvee_{x \in i(v)} s(x)\} = \{v\}. \tag{5.2}$$

[13] See Appendix B for more on this.

Then for any $E \in P_{\mathcal{S}}(V)$,

$$\begin{aligned}
i \circ r(E) &= i(\bigvee_{x \in E} s(x)) \\
&= \bigvee_{x \in E} i(s(x)) && (i \text{ preserves suprema}) \\
&= \bigvee_{x \in E} s'(i(x)) && (i \text{ commutes with successors}) \\
&= \bigvee_{x \in E} \{x\} && (\text{Equation 5.2}) \\
&= E
\end{aligned}$$

Thus r and i are mutually inverse ZF algebra homomorphisms. □

We have a related result for cumulative hierarchies on an object A.

Theorem 5.16 (Joyal and Moerdijk Chapter 2, Theorem 1.5). *The map* $r : P_{\mathcal{S}}(A) \times P_{\mathcal{S}}((V(A)) \to V(A)$ *defined by*

$$r(U, E) = \bigvee_{a \in U} \eta(a) \vee \bigvee_{x \in E} s(x)$$

is an isomorphism of $\mathcal{S}$-complete sup-lattices.

Recall that $\eta : A \to V(A)$ here comes from the definition of free ZF algebra (Definition 5.12).

These two theorems allow us to see that V is isomorphic to the small subsets of V and that $V(A)$ is an algebra of pairs of small sets (U, E) where U is a set of a atoms and E is a set of such pairs of small sets. Now, we have to see that these free ZF algebras are in fact accessible domains. We will do this by using what we know of the powerclass functor from Section 5.5 to show that V and $V(A)$ are actually initial algebras for endofunctors.

The main relevant fact shows that the powerclass functor is a monad and that initial algebras for the powerclass functor are equivalent to initial algebras that are very close to how we defined ZF algebras.[14] It is important to note that ZF algebras do not in general coincide with $P_{\mathcal{S}}$ algebras, but that *free* ZF algebras coincide with *free* (or initial) $P_{\mathcal{S}}$ algebras.

Definition 5.17. A *monad* $\mathbf{T} = \langle T, \eta, \mu \rangle$ in a category $\mathbb{C}$ consists of a functor $T : \mathbb{C} \to \mathbb{C}$ and two natural transformations

$$\eta : id_{\mathbb{C}} \implies T, \qquad \mu : T^2 \implies T$$

[14] This fact is used, for example, in [18, p. 5] to replace discussions of initial ZF algebras with free $P_{\mathcal{S}}$-algebras.

which makes the following diagram commute:

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta_T"] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, equal] \\ & T & \end{tikzcd} .$$

From Section 5.5, we know that $P_{\mathcal{S}}$ is a covariant endofunctor on $\mathbb{C}$. We also saw that $P_{\mathcal{S}}$ is free in the sense of Proposition 8 which means we can uniquely extend the identity map on $P_{\mathcal{S}}(X)$:

$$\begin{tikzcd} P_{\mathcal{S}}(X) \arrow[r, "id"] \arrow[d, "\{\cdot\}_{P_{\mathcal{S}}(X)}"'] & P_{\mathcal{S}}(X) \\ P_{\mathcal{S}}P_{\mathcal{S}}(X) \arrow[ur, dashed, "\cup_X"'] & \end{tikzcd} .$$

The maps $\cup_X$ for $X \in \mathbb{C}$ give rise to a natural transformation $\cup : P_{\mathcal{S}}^2 \Rightarrow P_{\mathcal{S}}$ with components $\cup_X : P_{\mathcal{S}}(P_{\mathcal{S}}(X)) \to P_{\mathcal{S}}(X)$. Similarly, the map $\{\cdot\}_X : X \to P_{\mathcal{S}}(X)$ is the component of a natural transformation $\{\cdot\} : id_{\mathbb{C}} \Rightarrow P_{\mathcal{S}}$.

To show that the powerclass functor is indeed a monad, we first consider the associativity of the union operation as captured by the left diagram in Definition 5.17. We want to show that for any X, the following diagram commutes

$$\begin{tikzcd} P_{\mathcal{S}}(P_{\mathcal{S}}(P_{\mathcal{S}}(X))) \arrow[r, "P_{\mathcal{S}}(\cup_X)"] \arrow[d, "\cup_{P_{\mathcal{S}}}"'] & P_{\mathcal{S}}(P_{\mathcal{S}}(X)) \arrow[d, "\cup_X"] \\ P_{\mathcal{S}}(P_{\mathcal{S}}(X)) \arrow[r, "\cup_X"'] & P_{\mathcal{S}}(X) \end{tikzcd}$$

or as an equation, for $A \hookrightarrow P_{\mathcal{S}}(P_{\mathcal{S}}(X))$

$$\cup_X(\cup_{P_{\mathcal{S}}(X)}(A)) = \cup_X(\mathrm{im}(\cup_X)(A))$$

since $P_{\mathcal{S}}(\cup_X)$ is the direct image operation as defined in Section 5.5. This equality states a well-known set-theoretic identity

$$\bigcup \bigcup A = \bigcup \{\cup(a) \mid a \in A\}.$$

We can understand this as stating that taking the union of the union of A is the same as taking the union of all elements of A and then taking the union of that set.

Now we want to show that the singleton operation satisfies the conditions expressed in the diagram

$$\begin{tikzcd} P_{\mathcal{S}} \arrow[r, "\{\cdot\}_{P_{\mathcal{S}}}"] \arrow[dr, equal] & P_{\mathcal{S}}(P_{\mathcal{S}}(X)) \arrow[d, "\cup_X"] & P_{\mathcal{S}}(X) \arrow[l, "P_{\mathcal{S}}\{\cdot\}"'] \arrow[dl, equal] \\ & T & \end{tikzcd} .$$

or in equations

$$\cup_X \circ \{\cdot\}_{P_{\mathcal{S}}(X)} = id_{P_{\mathcal{S}}(X)} = \cup_X \circ P_{\mathcal{S}}(\{\cdot\}_X).$$

The left equation comes from the definition of $\cup_X$ defined as the extension of the identity map $id_{P_{\mathcal{S}}(X)}$ above. The right equation expresses the set-theoretic identity

$$\bigcup\{\{a\} \mid a \in A\} = A.$$

The right diagram is akin to saying that the singleton operation leaves the set 'alone', that the operation is 'similar' to an identity operation.

Now we can say that $\mathbf{P} = (P_{\mathcal{S}}, \{\cdot\}, \cup)$ constitutes a monad.

Definition 5.18. A $\mathbf{P}$-algebra with a successor, a *successor algebra*, is a $P_{\mathcal{S}}$-algebra (X, a) equipped with a map $s : X \to X$.

Theorem 5.19 (Bénabou, Jidbladze [47, Theorem A.5])**.** *Let $\mathbf{P} = (P, \sigma, \mu)$ be a monad on a category $\mathbb{C}$.*

1. *If (V, a, s) is an initial successor algebra, then $(V, a \circ P(s))$ is an initial P-algebra.*

2. *If (V, h) is an initial P-algebra (so that h is an iso by Lambek's Lemma) then $(V, h\mu P(h^{-1}), h \circ \sigma)$ is an initial successor algebra.*

And these two constructions are inverses of each other.

This theorem shows that the initial algebra for the powerclass functor (as given in Section 5.5) coincides with the initial ZF algebra (successor algebra here). The core idea for this theorem is that Lambek's Lemma (Theorem 3.1) tells us that the initial algebra V is isomorphic to $P_{\mathcal{S}}(V)$ as algebras for endofunctors and Theorem 5.15 tells us that they're also isomorphic as $\mathcal{S}$-complete sup-lattices. You can either consider V as the free ZF algebra with no urelements or as the initial algebra for the powerclass functor. This latter characterization is particularly important for this dissertation: it means that the cumulative hierarchy is an accessible domain!

Result 1 (Cumulative hierarchy for *IZF* is an accessible domain)**.** *If $\mathbb{C}$ is a class category, and $\mathcal{S}$ is an axiomatization of smallness that satisfies the powerset axioms (**P1, P2**), then the initial $P_{\mathcal{S}}$ algebra V is the cumulative hierarchy with no atoms and $V(A)$ is the cumulative hierarchy with a set of atoms A.*

But which cumulative hierarchy is V? Is it classical, intuitionistic, constructive, or some entirely different cumulative hierarchy? As given above, it is the cumulative hiearchy for **IZF**. To get the cumulative hierarchy for **ZF**, we only need to add the following axiom to our class categories, to validate classical logic:

Axiom 5.6.1. *[Boolean] For any monomorphism $A \hookrightarrow X$, there is a monomorphism $B \hookrightarrow X$ such that $A \cap B$ is an initial object and $A \cup B = X$.*

In a class category that satisfies this axiom, the law of excluded middle will be satisfied and thus the internal logic will become classical. Without changing anything about the axiomatization of smallness or the development of ZF algebras and initial $P_{\mathcal{S}}$ algebras, we can say

Result 2 (Cumulative hierarchy for **ZF** is an accessible domain)**.** *If $\mathbb{C}$ is a class category that satisfies Axiom **5.6.1**, and $\mathcal{S}$ is an axiomatization of smallness that satisfies the powerset axioms (**P1, P2**), then the initial $P_{\mathcal{S}}$ algebra V is the classical cumulative hierarchy with no atoms and $V(A)$ is the cumulative hierarchy with a set of atoms A.*

Working rather informally, we can even say the following: Let $\kappa > \omega$ be a regular inaccessible cardinal. If $\mathbb{C}$ is a category of sets, relative to a fixed model of set theory, and $\mathcal{S}$ are all functions whose fibers have fibers of cardinality striclty less than κ, then $V = V_\kappa$.[15]

CZF

This section summarizes Moerdijk and Palmgren's [69] and van den Berg's Ph.D. thesis [96] in which they discusses in great detail the modeling of **CZF** by an initial powerset algebra. Many of the notions we have introduced are employed to show this but there are central and subtle additional concepts used in the demonstrations.

The axioms for smallness that [69, 96] use for their models of **CZF** include additional axioms than those provided in Section 5.4, and so we now discuss this expanded axiomatization. We use the label **SC** to denote that this is an axiomatization of **S**mallness that will lead to the cumulative hierarchy of **C**onstructive set theory **CZF**.

Definition 5.20. A class of maps $\mathcal{S}$ in a ΠW-pretopos $\mathbb{C}$ must satisfy Axioms **S1**–**S5** and the following axioms:

SC1. *If $f : X \to Y$ belongs to $\mathcal{S}$, the diagonal $X \to X \times_Y X$ in $\mathbb{C}/Y$ also belongs to $\mathcal{S}$.*

SC2. *If $Y \to X$ and $Z \to X$ are in $\mathcal{S}$, then so is $Y + Z \to X$.*

SC3. *For an exact diagram in $\mathbb{C}/X$,*

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & Y \arrow[r, two heads] \arrow[d] & Y/R \arrow[dl] \\ & X & \end{tikzcd}$$

if $R \to X$ and $Y \to X$ are in $\mathcal{S}$ then so does $Y/R \to X$.

[15] This is one way to articulate the general results found in [103] where κ is thought of as a boundary number that gives rise to a model of set theory (ZF for Zermelo).

SC4. *For any $Y \to X$ and $Z \to X$ in $\mathcal{S}$, their exponent $(Z \to X)^{(Y \to X)}$ in $\mathcal{E}/X$ belongs to $\mathcal{S}$.*

SC5. *For a commutative diagram*

$$\begin{tikzcd} B \arrow[rr, "f"] \arrow[dr] & & A \arrow[dl] \\ & X & \end{tikzcd}$$

with all maps in $\mathcal{S}$, the W-type $W_X(f)$ taken in $\mathcal{E}/X$ belongs to $\mathcal{S}$.

The first three axioms essentially make sure that maps belong to $\mathcal{S}$ in virtue of properties of their fibers. The last two are equivalent to the requirement that if $f \circ g = h$ and $f \in \mathcal{S}$, then $g \in \mathcal{S} \leftrightarrow h \in \mathcal{S}$.

There are several things to note about this axiomatization. First, a ΠW-pretopos is just a Heyting pretopos, axiomatized in Section 5.3, with W-types whose slice categories are all cartesian closed (i.e. has finite products and Y^X for any X and Y). These sorts of pretoposes are common in modeling what is understood as "weakly predicative constructive mathematics" that may not be finitist, which makes it suitable for constructive set theory.

These axioms mainly ensure closure conditions under slicing, relative to small maps. That is, the class of small maps in a slice category $\mathbb{C}/X$, denoted $\mathcal{S}_X$, is closed under finite limits (implied by **SC1**), finite sums (**SC2**), quotients of equivalence relations (**SC3**), exponentials (**SC4**), and W-types (**SC5**).

Finally, we add one more requirement for the class of small maps $\mathcal{S}$: representability. Representability is an important axiom, specifically for constructing a powerset operation without assuming its existence as in Axioms **P1** and **P2**.

SC6. *There is a map $\pi : E \to U$ in $\mathcal{S}$ such that any map $f : B \to A$ in $\mathcal{S}$ fits into a double pullback diagram of the form*

$$\begin{tikzcd} B \arrow[d, "f"'] & B' \arrow[l] \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ A & A' \arrow[l, two heads, "p"] \arrow[r] & U \end{tikzcd} \tag{5.3}$$

where p is an epimorphism.

This definition is desirable in this context as a weak form of assuming the existence of a 'universe', which we could have done in the **ZF** context. Again, we call π a *universal small map* and for a ΠW-pretopos $\mathbb{C}$, we can express this important map using the internal logic of $\mathbb{C}$. Representability means that a map $f : B \to A$ belongs to $\mathcal{S}$ iff the following formula is valid in the internal logic:

$$(\forall a \in A)(\exists u \in U) B_a \cong E_u.$$

This formula says that every fiber of f is isomorphic to a fiber of π for some $u \in U$. This assumption is weaker than having a universe that contains every set, since we only have isomorphic copies of every fiber of small maps.

Equipped with this axiomatization of smallness (i.e. Axioms **S1–5** and **SC1–6**), along with stronger assumptions on the surrounding category of classes (i.e. all ΠW-pretoposes are Heyting, but not vice versa), we can find models of **CZF** as initial algebras for the powerset endofunctors. We now have a choice of an additional axiom in order to get to **CZF**: The Collection Axiom **(CA)** and the Axiom of Multiple Choice **(AMC)**.

(CA) For any map $f : A \to X$ in $\mathcal{S}$ and any epimorphism $C \twoheadrightarrow A$, there exists a quasi-pullback of the form

$$\begin{tikzcd} B \arrow[r] \arrow[d] & C \arrow[r, two heads] & A \arrow[d] \\ Y \arrow[rr, two heads] & & X \end{tikzcd}$$

where $Y \to X$ is an epimorphism and $B \to Y$ belongs to $\mathcal{S}$.

This axiom relates to the axiom of the same name in set theory (Axiom 5.1.5). Informally, we can think of it as saying, that for any small set A and any surjection $C \to A$, there is a surjection $B \to A$ from a small set B which factors through C.

We will now see how these authors model **CZF** using these axiomatizations of smallness, in the class categorical context of ΠW-pretoposes. First, we get the powerclass functor relative to the axiomatization, through the notion of representability given in **SC6**. We construct this from the universal small map, using the internal set-builder notation:

$$P_{\mathcal{S}}(X) = \{R \in \mathcal{P}X \mid (\exists y \in U) R \cong E_u\}.$$

The use of the power object functor $\mathcal{P}$ here indicates that the category is rich enough to have general power-objects, though these are not in general going to be small.

The initial algebra for $P_{\mathcal{S}}$ exists and is the cumulative hierarchy V. To show the initial algebra for $P_{\mathcal{S}}$ exists, [69] construct it through a sort of 'Mostowski Collapse'.[16] They then analyze this new notion of a Mostowski collapse with respect to W-types and collection maps. Taking the Mostowski collapse of the W-type based on the universal small map $\pi : E \to U$, denoted $W(\pi)$, gives us a neater object to work with, called $V(\pi)$. In addition, they prove that $\langle V(\pi), Int\rangle$ carries the structure of a $P_{\mathcal{S}}$ algebra (Proposition 6.2) and indeed the initial such (Proposition 6.3), for a particular map Int.[17] This $V(\pi)$ is the cumulative hierarchy which, since it is an initial algebra, is an accessible domain.

[16] Recall that the traditional Mostowski Collapse lemma ensures that every well-founded model of **ZF** is isomorphic to a unique transitive model.

[17] In [69], they do not actually use $P_{\mathcal{S}}$, since this functor is only covariant if you assume that the quotient of a small object is again small. They define the very similar $\mathcal{P}_\pi$. And $V(\pi)$ is also the initial ZF-algebra, if you replace $P_{\mathcal{S}}$ with $\mathcal{P}_\pi$. We use $P_{\mathcal{S}}$ only to unify the notation and make the uniformity clear. In [96], the usual $P_{\mathcal{S}}$ functor is used.

Result 3 (Cumulative Hierarchy for **CZF** is an accessible domain)**.** *In a* ΠW*-pretopos* $\mathbb{C}$ *and a class of small maps satisfying Axioms* ***S1–5*** *and* ***SC1–6****, the initial powerset algebra* $V(\pi)$ *is the constructive cumulative hierarchy (with no atoms).*

The reason this result is slightly more nuanced is because the notion of 'constructive' in **CZF** limits the powerset operation. So we don't assume power-objects are small in the same way that Axioms **P1** and **P2** assert. We could have used the approach taken here to define the powerclass functor through representable maps, but it made sense to make relatively strong assumptions for **IZF** and **ZF** and leave the more uniform but technical approach to **CZF**.

5.7 The Ordinals

This section will focus on capturing another important sort of set-theoretic structure: the ordinals. The details here, I think, are satisfying and help understand the use of the category-theoretic framework better. Our exposition here will be again from [47] and will skip proofs where necessary.

The definition of the ordinals is similar to that of ZF algebras but with an additional requirement.

Definition 5.21. Let O be an $\mathcal{S}$-complete sup-lattice with a monotone successor map $t : O \to O$, i.e. that satisfies

$$x \leq y \implies t(x) \leq t(y).$$

If O is the initial ZF algebra with a monotone successor t, then we call (O, t) the algebra of ordinal numbers.

Just like we did for the free ZF algebra V, we can prove that O looks like it should, to be the ordinal numbers. First, define a preorder $\preccurlyeq$ on $P_{\mathcal{S}}(O)$ by setting for $E, F \in P_{\mathcal{S}}(O)$,

$$E \preccurlyeq F \quad \text{iff} \quad (\forall x \in E)(\exists y \in F) x \leq y.$$

This preorder defines an equivalence relation on $P_{\mathcal{S}}(O)$, call it $\sim$:

$$E \sim F \quad \text{iff} \quad E \preccurlyeq F \;\&\; F \preccurlyeq E.$$

We now quotient the powerset $P_{\mathcal{S}}(O)$ by this equivalence relation:

$$D_{\mathcal{S}}(O) := P_{\mathcal{S}}(O)/\sim .$$

Thus, $D_{\mathcal{S}}$ is the object of downward closed subclasses of O, which are generated by small sets. To suggest this downward closedness, for every $E \in P_{\mathcal{S}}(O)$, we denote the equivalence class with respect to $\sim$, in $D_{\mathcal{S}}(O)$, as $\downarrow(E)$.

We also have a partial order on $D_{\mathcal{S}}(O)$ based on the preorder $\preccurlyeq$ on $P_{\mathcal{S}}(O)$. With this partial order, $D_{\mathcal{S}}(O)$ is a sup-lattice, with a bottom element $\downarrow(\emptyset)$ and binary supremum given by

$$\downarrow(E) \vee \downarrow(F) = \downarrow(E \cup F).$$

Generally, $D_{\mathcal{S}}(O)$ has small suprema based on unions in $P_{\mathcal{S}}(O)$.

Proposition 9 ([47, Proposition II.2.2])**.** *The sup-lattice $D_{\mathcal{S}}(O)$ is $\mathcal{S}$-complete. For a small family $\{E_i \mid i \in I\} \subseteq P_{\mathcal{S}}(O)$ of small subsets of O, the sup in $D_{\mathcal{S}}(O)$ is computed as*

$$\bigvee_{i \in I} \downarrow(E_i) = \downarrow(\bigcup_{i \in I} E_i).$$

The proof of this proposition uses the collection axiom (Axiom **5.1.5**).

In direct analogy to Theorem 5.15 and 5.16, we have the following theorem for ordinals, where the successor $\bar{t}$ on $D_{\mathcal{S}}(O)$ is defined just as s' is in theorem 5.15:

$$\bar{t}(\downarrow E) = \downarrow\{\bigvee_{x \in E} t(x)\}.$$

Theorem 5.22. *[47, Theorem II.2.3] The map $\theta : (D_{\mathcal{S}}(O), \bar{t}) \to (O, t)$ defined by*

$$\theta(\downarrow(E)) = \bigvee_{x \in E} t(x)$$

is an isomorphism of ZF algebras.

The structure of the proof is parallel to the theorems about ZF algebras.[18]

Because of the deep similarity in how the free ZF algebra $V(A)$ and the free monotone ZF algebra O are defined, we would expect that these two algebras are related. Indeed, they are related just as they are in classical treatments of set theory. Every set in the cumulative hierarchy $V(A)$ (or V if $A = \emptyset$) can be found at a certain ordinal level of the construction, called its rank. Conversely, the cumulative hierarchy can be understood as constructed from the ordinals just as is traditionally done as we will now show, culminating in Theorem 5.23.

The rank mapping comes from the fact that V is a the initial ZF algebra and O is a ZF algebra. That is, there is a unique ZF homomorphism $(V, s) \to (O, t)$ which we may denote by

$$\mathrm{rank} : V \to O.$$

Just by being a ZF algebra homomorphism, we have two facts:

$$\mathrm{rank}(s(x)) = t(\mathrm{rank}(x)), \qquad \mathrm{rank}(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} \mathrm{rank}(x_i)$$

for small suprema; and the rank map is uniquely determined by these identities.

[18]This parallel is because all of these theorems are instances of a more abstract general theorem [47, Theorem A.1].

Now we get to construct V from the ordinals, through an internalized notion of powerset based on the powerclass functor $P_{\mathcal{S}}$. This 'internalized' notion is a map in $\mathbb{C}$ which takes elements of the cumulative hierarchy V and returns the powerset of that set, which is also in V. First, we require that $\mathcal{S}$ satisfies the following powerset axiom:

S7. *For any small map $X \to B$ in $\mathbb{C}$, the power-object $P_{\mathcal{S}}(X \to B)$ is a small object of the slice category $\mathbb{C}/B$.*

This allows us to show the following Proposition.

Proposition 10. *[[47, Proposition II.2.5]] Suppose the class of small maps $\mathcal{S}$ satisfies the powerset axiom (**S7**). Then there exists an 'internal' powerset operation*

$$p : V \to V$$

with the property that for all $x, y \in V$,

$$y \leq x \quad \textit{iff} \quad s(y) \leq p(x)$$

The proof of this proposition uses our knowledge of what V looks like, from Theorem 5.15, namely the two inverse maps r and i.

With this, we can state the theorem that gives the relationship between the cumulative hierarchy V and ordinal numbers O.

Theorem 5.23 ([47, Theorem II.2.7])**.** *Suppose the class of small maps $\mathcal{S}$ satisfies the powerset axiom. Then the map* rank $: V \to O$ *has a right adjoint $V_{(-)} : O \to V$, which means*

1. *$x \leq V_\alpha$ iff $\mathrm{rank}(x) \leq \alpha$ (for any $x \in V$, $\alpha \in O$)*
   which has the following additional properties:
2. *$V_\alpha = \bigvee_{i \in I} V_{\alpha_i}$, for any small $\sup \alpha = \bigvee \alpha_i$ in O,*
3. *$V_{t(\alpha)} = p(V_\alpha)$, for any $\alpha \in O$,*
4. *$\mathrm{rank}(V_\alpha) = \alpha$, for any $\alpha \in O$.*

The condition in (1) expresses the adjunction, and the identities in (3) and (4) express that $V_{(-)}$ is a ZF algebra homomorphism. The equality in (2) gives us the analogy to the limit ordinal case in building the cumulative hierarchy. If we were to rewrite these four parts in traditional notation, we would have for ordinals α,

1. $x \subseteq V_\alpha$ iff $\mathrm{rank}(x) \leq \alpha$
2. $V_\alpha = \bigcup_{\gamma < \alpha} V_\gamma$ for limit ordinal α.
3. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$
4. $\mathrm{rank}(V_\alpha) = \alpha$.

Now that we see that the ordinals look as expected, we can see whether the class O is an accessible domain. Recall that whereas V is the initial ZF algebra, O is the initial ZF algebra with *monotone* successor.[19] This means that O is not the initial $P_{\mathcal{S}}$ algebra, as O is not isomorphic to V. Might there be a functor for which O is the initial algebra? We've already seen it: $D_{\mathcal{S}}$, the result of taking the quotient of $P_{\mathcal{S}}$.

Result 4 (Ordinals are an accessible domain)**.** *For a class category $\mathbb{C}$ and a class of small maps $\mathcal{S}$ that satisfies the powerset axiom (Axiom **S7**), O is the initial $D_{\mathcal{S}}$ algebra.*

Remark 5.24. It is interesting to note that if the class category consists of countable sets, O will be the set of natural numbers, so again we have that natural numbers constitute an accessible domain.

Indeed, since the ordinals here are defined in generality, based on a notion of smallness, Result 4 includes the classical ordinals, the countable ordinals, and the finite ordinals. We should note here that having the ordinals defined in this way, using the quotient of the powerclass functor, we can see that in one sense the ordinals are even more 'classical' than the cumulative hierarchy V. We will return to this in the conclusion when we address the comparison between accessible domains as methodological frames.

[19] We could require that the successor is in addition inflationary ($x \leq s(x)$). This would result in the von Neumann Ordinals as described in [47, §II.3].

Chapter 6

Methodological Reflections

This chapter uses the conceptual motivation of the dissertation to analyze itself. Here, I examine mathematical practice much like in Sieg's analysis of accessible domains but now through the added lens of our categorical characterization. As we've seen, Sieg articulates a core aspect of traditional notions of finitist mathematics that should guide generalization: being an accessible domain. This chapter will articulate aspects of the mathematical framework of this dissertation that have aided in the categorical characterization of accessible domains. More specifically, I will show that the category theory used here and the appropriateness of category theory to the task at hand all stem from a methodology and motivation that can be found at the historical roots of this project.

Unlike Sieg's introduction of accessible domains, however, I will not focus on giving *mathematical* characterizations of the features of practice that I find worthy of articulation. So, whereas Sieg isolates an important class of mathematical structures much as Dedekind conceptualized simply infinite systems, I will prioritize an account of the value and nature of the facets of mathematical practice that turned out to be most useful to the completion of this dissertation. In essence, this chapter will be a discussion about notions such as 'externalizing' concepts but without sharp notions of what this entails. Principles of induction and recursion were picked out as relevant and interesting features of the structures used in proof theory, and I will pick out relevant and interesting features of the theory we used to characterize these domains.

One of the central notions that I think crystalizes some of the most interesting features of category theory—we also see it in Dedekind's treatments of mathematical objects—is the externalization of properties internal to mathematical objects. So instead of asking the question "What is *in* object X", we ask "How does X relate to other objects (like X)?" Externalization occurs when we replace reasoning about the constitution of an object with reasoning about the relation that object has to others. A class of ubiquitous examples from category theory can be found in universal properties such as the universal properties of the product of two objects $A \times B$.

Universal properties are usually stated as follows:

> X has the universal property U when there are maps M such that there is a unique map Q such that the diagram D commutes.

or more elaborately:

> An object-X-together-with-morphisms-f_i has a universal property if and only if for every other object-Y-with-morphisms–g_i from (or to) the same objects as the f_i, there exists a unique $h : X \to Y$ (or $h : Y \to X$) such that the g_i can be obtained by compositions of h and the f_i.

We will use the example of a simple binary product to elucidate. Recall the definition of a binary product in category theory:

> C is the product of A and B if there exists maps $\pi_1 : C \to A$ and $\pi_2 : C \to B$ such that given any X and any pair of maps $f_1 : X \to A$, $f_2 : X \to B$, there is a unique map $f : X \to C$ such that the following diagram commutes.

$$
\begin{tikzcd}
 & X \arrow[dl, "f_1"'] \arrow[d, dashed, "f"] \arrow[dr, "f_2"] & \\
A & C \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}
$$

Now let us compare this with a standard set-theoretic definition of the Cartesian product of A and B:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

Where the ordered pair (a, b) in turn is usually given via the Kuratowski definition:

$$(x, y) = \{\{x\}, \{x, y\}\}$$

Both of these definitions (i.e. Cartesian product and ordered pair) in set theory differ from the definition given in category theory in a rather apparent way: set theory defines the objects by the constituent objects of the defined object and category theory defines the object $A \times B$ in terms of its arrow-theoretic relations to other objects (i.e. existence of projections and unique arrow from any other object of the category that has similar 'projection' arrows). The set-theoretic definition tells us what the ordered pair (x, y) and product $A \times B$ really looks like 'on the inside', using notions such as pairing and union in a standard axiomatic formulation to build up the objects that make up products. Moreover, we are told that the product consists of just these ordered pairs. Thus, these standard definition answers the question "What objects make up a Cartesian product?"

In contrast, the category-theoretic definition answers the question "How does the product relate to other similarly equipped objects?" This replacement of internal queries with external ones is what I mean by the 'externalization of properties internal to mathematical objects'. I do not believe that category theory merely shows us how we may think about these properties as external, which are nonetheless 'really' internal. Instead, the externalization of properties shows that neither the internal nor external is more primitive or in general privileged. They both serve the same purpose, giving an understanding of a mathematical object, but in different ways. There are reasons for preferring one over the other, depending on what purposes we intend to put them to. In general, however, I think the tandem use of both provides the best understanding of the objects in question. I will argue that it is the characteristic generality of category theory that supports the externalized version of such definitions.

To see this, we first note the simplicity—-or at least the self-containment—of the definition of product in category theory. It uses notions that are basic to category theory. No further definition is needed. The definition uses the notions of arrow, object, commuting diagram (equality of arrows), and uniqueness. In contrast, the axiomatic set theory definition requires further analysis of the ordered pairs that constitute the Cartesian product. This of course is not an in-principle problem for the axiomatic set theorist, since definitions are eliminable. In that case, the definition of the Cartesian product, using a generous notion of set-builder notation, would be

$$A \times B = \{\{\{x\}, \{x, y\}\} \mid x \in A \text{ and } y \in B\}.$$

This is formally fine but lacks the benefit of seeing clearly what the construction of a Cartesian product is supposed to grant us, structurally speaking. The point is not that set theory puts an in-principle barrier to a good definition of the Cartesian product, but that the specificity required in such a definition emphasizes the basic building blocks of that product and thus the specific mathematical context in which you are defining it. Definitions must 'drill down' to the parts that make up the object. Category theory, on the other hand, lacks specificity in this sense because it replaces the questions of parts (and parts of parts etc.) with a question of relation. Informally, we can think of the product as the maximal object with projections to A and B.

The generality gained by the category-theoretic definition can be seen to follow from the objects it uses in the definition. The idea here is that all categories can 'make sense' of a product, even if they don't happen to exist in that category. This single definition is actually sufficient to capture the notions of the Cartesian product, product topology, and direct product of groups, for example. The generality is achieved by using the common language of categories and not the normal concepts used in each of these fields (i.e. sets, topology, group theory). No mention is needed of what an open set is, for each topology, or what the multiplication of the original groups happens to be *in order to specify the product in the category.* And so, the definition lifts the concept of product away from individual mathematical contexts into a generality that applies equally well to any categorical context.

The form of the product definition is common to many definitions in category theory; the example of the product is illustrative, but only one case. The construction of objects with universal properties is in large part the business of category theory. The specification of a special object in a category often falls into the format given above for universal properties. Indeed, we have such a sentiment given on a popular resource on category theory, ncatlab:

> To a fair extent, category theory is all about representable functors and the other universal constructions: Kan extensions, adjoint functors, limits, which are all special cases of representable functors—and representable functors are special cases of these.

Here we see that products, a kind of limit, are really a small sliver of the universal constructions and that much of the mathematics in Chapter 2–5 center on limits, adjoint functors, and representable functors, included here as universal constructions.[1]

The similarity between products (a type of limit) and Galois connections (a type of adjunction) isn't easy to see if we focus on the internal constructions of these concepts. Instead, category theory characteristically highlights the relational aspects of the concept. By "relational aspects", I mean the way the concept, in each instance, relates to the objects around it.[2] Saying an object (with associated arrows) has a universal property is to say that it is either initial or terminal in some category of arrows. As we know from the definition of initial and terminal, these objects are specified only by their relation to other objects. Initial objects, we recall, are simply those with a unique map to all other objects of the category. Nothing more is said about them, because nothing more is needed.

Another way to see this phenomenon is to see how the 'internal' perspective is relative to the conceptual context. In the comparison just made between products and Galois connections, we can see that the contexts for set-theoretic Cartesian products and order-theoretic adjunctions dictate what concepts are allowed to constitute such definitions. The definition of Cartesian product, just discussed, relies on notions of set-membership and particular set-theoretic constructions like the Kuratowski ordered pair. In contrast, the definition of Galois connections relies on order-theoretic concepts, namely the relation $\leq$ along with monotone functions that preserve this order. The different concepts available in the specific theory can determine how the definitions are naturally given.

Externalized concepts rely on relations between objects and 'arrows' (an associative, compositional relation satisfying the category axioms). The objects and arrows, are of course, determined by context in the definition of the category. However, the different contexts will all make sense of the same conceptual

[1] Many of these uses were implicit or left relatively unanalyzed. For example, in discussing initial algebras for polynomial functors around Theorem 5, we implicitly used the fact that polynomial endofunctors are coproducts of representable functors. In discussing the categories of set theories, we used the fact that certain adjoints exist in order to model the quantifiers.

[2] This spatial metaphor should be read as "objects in the category". I am not implying that there are meaningful distance neighborhoods and objects are specified by how they relate to these 'close' objects. It is the entire relational situation between the object falling under the concept and all other objects of the category.

repertoire: objects and arrows. The initial object of the natural number sequence 0 is the 'same' as the initial object of the cumulative hierarchy $\emptyset$. This similarity is brought out in the category theoretic definition of initial object that goes beyond the intuitive parallel. This is because the externalized notion, that of initial object of a category, is specified only with "an object that has a unique arrow to all other objects". What it means to "have an arrow" means something different in each case, but this reliance on the general concept of 'arrow' allows this flexibility while maintaining the obvious unity between the notions of 0 and $\emptyset$. In this example, we can see that "having an arrow" can be understood order-theoretically, for the natural numbers, the less-than relation $\leq$ and for sets, the subset relation $\subseteq$.

One way to understand this difference between the internal perspective (e.g. "What objects make up this structure?") and externalized perspective (e.g. "How does this structure relate to others of its kind?") is to note how the conceptual context changes the investigation. As we've just discussed, the internal perspective relies *immediately* on the context in order to understand a concept; we can't simply say that the empty set is the smallest set, without specifying what that set contains (nothing) and what ordering we're referring to (e.g. subset, cardinality), and thus relying on a membership relation. Although this exact question will also be answered in category theory—an externalized theory—it will be delayed since we can use arrows to avoid questions about particularities of the order relation, say $\subseteq$, until it becomes interesting to ask such questions. In effect, category theory, because of its intentional generality, postpones questions of constitution until you have a particular structure or system you want to investigate more thoroughly. Before that level of granularity is reached, and mathematics resumes as usual, we can simply ask whether the empty set is an initial object. This will *imply* facts about where the empty set is situated in the cumulative hierarchy, for example, but the concept under which it falls (i.e. initial object), is not tied to any particular conceptual repertoire at all. And so, we can ask the exact same question of any object in any category, regardless of whether we are talking about manifolds or discrete sets.

This relates to a broader fact of how category theory continues the process of abstracting away from individual objects to systems of objects which we will now discuss. We can think of category theory as conceptually a third-order sort of theory. It does not need third-order quantifiers, of course, but it talks about systems of systems of objects.

First order concepts and definitions refer to individual mathematical objects. For example, the definition of addition on the natural numbers, as it is usually given, refers only to individual natural numbers. Depending on how you look at it, the definition of a simply infinite system is also first order, insofar as mappings and chains are individual mathematical objects.[3]

[3]The word "individual" does not imply that the object is 'primitive' in the sense of having no definition (e.g. the concept of chain is a complex definition) but does imply the object is not a system of other objects. This distinction is not particularly sharp, since you may treat systems of objects *as if* they are simple objects, as we go on to argue that category theory does.

But Dedekind does not just define simply infinite systems, he shows that the notion is categorical, by showing that every simply infinite system is isomorphic to a privileged one (the one where Dedekind himself is the starting point 1). This study of simply infinite systems is second order, since it takes as its object systems of first order objects. It is important to note that essential use is made of the first order definition of a simply infinite system to relate these systems to each other. For example, the Recursion theorem (Theorem 2.7) shows that we may define maps from simply infinite systems N by exploiting both the distinguished object 0 and the map $s : N \to N$. No such theorem is possible without reference to the first order objects.

Just as we move from considering specific objects (first order) to the relations of systems of such objects (second order), we may expect that moving to a third order perspective would simply be adding another layer to these increasingly abstract perspectives. But I think this is not quite the case. Let us take the example of group theory. In analyzing a specific group $(G, *)$, we need to know what properties the multiplication has (e.g. is it commutative). We might suspect that there are other groups with similar properties, but we are concerned just with G at the moment. This constitutes a first order perspective.

If we want to relate G to other groups, we have to have a notion of relation that is specific to the concept of group, that of a group homomorphism. We notice that homomorphisms are, in part, useful to describe properties of each of the related groups. For example, the image of a homomorphism $f : G \to H$ is a subgroup of H and the fiber of the identity $f^{-1}(e_H)$ is a normal subgroup of G, called the *kernel*. This can help us understand what G and H 'look like'.

Now, say, we want to isolate classes of groups that deserve study (perhaps even all groups). These classes would be classes of systems of objects. Category theory takes such a perspective, given that the classes we are studying satisfy certain axioms. But at this level of abstraction, category theory simply has objects and arrows and remains *prima facie* agnostic to the first order construction of such groups. In order to avoid reference to elements of objects in a category (e.g. $e_G \in G$), we must replace the standard definitions with what I have been calling 'externalized' concepts.

For example, in the category of groups and group homomorphisms Grp, we can define the binary product of two objects of the category just as it is defined in every category (i.e. without reference to group multiplication). This binary product, if it exists, will imply facts about first order objects: In Grp, the binary product of two groups $(G, *)$, $(H, +)$ is the direct product of these groups, with the Cartesian product $G \times H$ as the underlying set and the operation $\cdot$ defined as

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 * g_2, h_1 + h_2).$$

This all comes out of the definition of binary product in the category, which makes no mention of elements of G and H.

In this way, category theory separates reasoning about systems of systems from reasoning about the internal construction of these systems. Of course, category theoretic definitions do not prevent the inclusion of facts about internal construction of objects in the category. Indeed, you can recapture many notions of elementhood in category theory, depending on what category you are considering. Many of the examples used throughout this dissertation can be seen as a translation of concepts from the categorical to the 'traditional'. This is an important and essential part of category theory: to externalize concepts to the extent required to talk about them without reference to constituent objects.

This broad characteristic of category theory, though by no means unique in mathematics, is important for this dissertation specifically. The project aims at giving a characterization that applies equally well to structures in different parts of mathematics. The conceptual repertoire of natural number differs from that of sets, and yet, the unification of these two structures is a primary goal of the concept of accessible domains—insofar as these structures are considered as accessible domains. That these structures have different concepts appropriately used in discussions involving them can be seen by noting the widely held belief that it is infelicitous to ask questions such as "Does $1 \in 5$?" *simpliciter.*[4] That is, can we ask such a question without specifying a particular set-theoretic representation of these natural numbers? It seems inappropriate to do so, which goes to show that set-membership just doesn't make sense in the natural numbers, when considering the natural numbers as objects in their own right and not reduced to sets. Of course, mixed 'conceptual repertoires' are employed, especially in representational work, where we consider the natural number 1 to really be the set $\{\emptyset\}$, for example. But these are usually fully within the representing system, here set theory, and definitions are introduced that employ signs from the represented system, but these are 'mere' mnemonics to remember what they are supposed to represent.

The externalized definition of accessible domains was the main contribution of this dissertation, in answering Sieg's question of whether we could give a category-theoretic characterization of this class of structures. When we think of accessible domains as inductively defined structures with deterministic inductive clauses, we are thinking about the internal structure. We have some starting points, some rules for generating new elements, and principles that can be derived from this understanding of the internal build-up of an accessible domain. But the characterization given here, of initial algebras for endofunctors, externalizes this into a relational specification of the whole structure of the accessible domain with other structures. In this framework, the definition asks us what the relation of structure X is to a class of 'similar' structures in order to determine the applicability of the term "accessible domain". We don't look inside the domain to find out whether it is accessible, under this definition. This is not to say that it is the only useful definition! Instead, it is preferred when generality is desired, since it makes the similarities among accessible domains crystal clear: they all have a unique map that preserves the inductive clauses of the structure to any other structure with the same signature. The original

[4]Here we have many discussions centered on Benacerraf's [22]. There, Benacerraf points to a long-known comparison of Zermelo ordinals $(\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots)$ and von Neumann ordinals $(\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots)$ saying that "The two accounts agree in over-all structure. They disagree when it comes to fixing the referents for the terms in question." [22, p. 56]

notion of accessible domain is clear as well, but can only be specified with reference to the particular context that takes for granted notions of inductive rule, for instance.

The process of externalizing traditionally internal concepts takes place in category theory through a number of avenues. Here, I want to discuss a special way that I think category theory uses concept formation that directly supports the externalization process. I will call it parameterization, since it is a special case of what we normally consider parameterization. Perhaps to be more specific we should call it "conceptual parameterization".

First, let us remind ourselves how parameters usually function in mathematics. In some contexts, parameterization serves as a way to express curves, lines, and manifolds in terms of independent variables. That is not quite the connotation of 'parameterization' I intend here. It's a less technical sense of the word that simply refers to the expression of a concept but with new 'inputs': the parameters. We have seen a notion of parameterization already in this dissertation, in the description of how initial algebras for endofunctors recapture the standard notions of recursion (in Chapter 2). There, several versions of 'recursion' were recaptured, each with more parameters than the last.

My notion of conceptual parameterization is less about functions and more about definitions. Let us start with what I take to be a paradigm example of parameterization in this sense. Recall from Chapter 5 how we defined the powerset operation in algebraic set theory. Where traditionally, the operation is supported by powerset axioms (such as $(\forall a)(\exists y)(\forall x)(x \in y \leftrightarrow x \subseteq a)$) the new one depended on concepts not always considered in set theory: a notion of smallness. This new dependency is the new conceptual parameter. It's not a simple variable, but it's a 'free' input that is independent of the rest of the definition of the powerset operation. Changing the parameter (the axiomatization of smallness) changes the powerset operation.

Just like a parameter a can be changed at will in some functional specification $f(a, x)$ without having to change the definition of f, the parameters in this broader conceptual sense are inputs that can be changed without having to consider new concepts. This relates back to the externalization aspect discussed above in that this addition of a parameter to a traditionally unparameterized concept achieves a new sort of generality. But not just any generality, in which the new concept contains as a special case the old concept. A case of basic generality that does not have the qualities of parameterization would be simple considering *operations* as a generalization of the powerset operation. The generality I intend could be more appropriately called "uniformity", since the new concept can treat differing contexts in the same 'uniform' way. Moreover, it makes the analogy between related concepts more precise by considering them all as instances of a single mathematical definition.

Let us return to the example of powersets to see how this works in practice. The classical definition of the powerset operation is to input any set a and return the set of all subsets of a. And since the definition of subset is a simple one that uses the primitive set-membership relation, we see that a main 'conceptual input' for the powerset operation is the relation $\in$. Traditionally, then, we get the following picture:

$$\begin{array}{c} \text{Cumulative Hierarchy} \\ \uparrow{\scriptstyle \text{Builds into}} \\ \text{Concept of Powerset} \\ \uparrow{\scriptstyle \text{Defines}} \\ \text{Subset Relation} \\ \uparrow{\scriptstyle \text{Defines}} \\ \text{Membership Relation} \end{array}$$

I would suggest that with an algebraic set theory perspective, we have the more complex parameterized picture:

$$\begin{tikzcd} & \text{Cumulative Hierarchy} \\ \text{Axiomatization of Smallness} \arrow[r, "\text{determines}"] & \text{Concept of Powerset} \arrow[u, "\text{Builds into}"'] \arrow[d, "\text{Defines}"] \\ & \text{Subset Relation} \end{tikzcd}$$

Note both the addition of the axiomatization of smallness as a parameter of the concept of powerset as well as the changed direction of the arrow between the powerset operation and subset relation. As discussed with Axiom P1 and P2, the interpretation of subset and membership comes from the specification of the powerset $P_{\mathcal{S}}$, which depends on a notion of smallness. Indeed, the subset relation is defined as a relation on the powerset, which reverses the usual definition of powerset in terms of the global subset relation.

Although similar, this notion of parameterization is not simply replacing one set of primitives for another. In some sense, the primitive membership relation is being replaced by a more complex picture of the powerset that includes other inputs, mostly axiomatizing the structure of arrows. It may be tempting to see this as merely replacing the membership relation of set theory with the primitives of object and arrow of category theory. But in concrete category theory, the objects and arrows are *analyzed* further, showing that objects and arrows are not primitives, but abstract names. The reference of these names can be filled in afterwards. This agrees with the considerable period at the beginning of category theory where many considered it only a language, not a theory in its own right. It was to help algebraic topology and homological algebra but had no theorems that mathematicians thought of as specifically category theoretical. So it was not a replacement of any theory, but a way to see natural transformations of structures, which needed no basic primitives, like axiomatic set theory might. Category theory in this sense is more akin to group theory: it does not suppose the existence of things, only their conditional existence based on assumptions that happen to be satisfied in a particular case.

Instead of replacing one primitive with another, category theory considers its object 'algebraically', that is, relationally and not concerned with the internal structure of each object, per se. It is a different treatment of familiar concepts and objects, like sets or even inductively defined structures as in this dissertation.

Let us now compare this with Dedekind's introduction of simply infinite systems. In *Was sind und was sollen die Zahlen?*, henceforce *WZ*, Dedekind develops, step by step, a theory of natural numbers. For his theory, he has two sorts of mathematical objects: systems and mappings. Systems are collections that we may consider from a single perspective—as one 'thing'. Dedekind allowed unrestricted, and therefore contradictory, comprehension in the construction of new systems, so any objects of thought that we may consider as collected together in any way constituted a system. Mappings are the transformations of elements of one system into elements of another. Dedekind did not specify the inferential rules of his system but employed a broad notion of 'logic', which includes all the objects of thought. It was not until Zermelo axiomitized set theory in 1908 that we see the reduction of mappings (functions) to sets of a particular kind. In Dedekind, mappings are their own sortal (cf. [87]) and he creates a theory that includes many elementary facts about sets and functions we would recognize today. For example, a theorem stated in WZ, but only proved in an 1887 manuscript [31, pp.447–449] is equivalent to the Cantor-Bernstein Theorem.

The central concept introduced in this theory is the inductive closure of a set a under a function f, which Dedekind called the *chain* of a given f. Dedekind defined the chain of a given f as the intersection of all f-closed sets containing a as a subset. That is, the chain of a given f is

$$\bigcap \{x \mid f[x] \subseteq x \text{ and } a \subseteq x\}.$$

In particular, this implies that the chain of a is the smallest f-closed set that contains a. If f is in addition injective, then the chain of a given f is in fact an accessible domain.

Armed with this notion of the chain of a system, Dedekind defines a *simply infinite system* as a system N along with an injection $f : N \to N$ such that N is the chain of $\{x\}$ for some $x \in N$ and x is not in the image of f. By our understanding of chains as inductive, we can see that a system is simply infinite when there is an element from which all the other elements of N may be generated by the injective map f:

$$N = \{x, f(x), f(f(x)), f(f(f(x))), f^4(x), \ldots\}$$

where $f^n(x) \neq x$ for all n and $f^n(x) \neq f^m(x)$ for $n \neq m$. Thus, a simply infinite system looks like the natural numbers N where x is 0 (or 1 for Dedekind) and f is the successor function. For Dedekind, the natural numbers, as such, were seen as elements of a simply infinite system where we ignore the special character of the system and just focus on what is included in the definition of a simply infinite system.[5] Of course, we used the natural numbers in our exponents, but Dedekind only needed the notion of a chain to capture this concept; we used them only as convenient signs. In *WZ*, he proved the soundness of the above understanding of N in terms of successive applications of f, but his definition of the natural numbers actually came from the abstract simply infinite system, not vice versa.

Thus, Dedekind characterizes simply infinite systems, and goes on to show that they are all isomorphic in the same (uniform) way. That is, simply infinite systems are *canonically isomorphic*. The isomorphism simply relies on the 'successor' function of each simply infinite system for its recursive definition, built in the same way every time. Here we see two important features of Dedekind's characterization of this sort of structure: 1) canonical isomorphism 2) parameterization of a traditionally given mathematical structure, the natural numbers.

Let us first consider how Dedekind's simply infinite systems parameterized what was then thought of as given mathematical objects: the natural numbers. The natural numbers are a canonical case of a structure taken for granted by mathematicians. It was thought that they were mathematical objects we had epistemological access to, in some sense without mediation. If we couldn't take for granted the counting numbers, what then would we be left with? the thought goes. This historical privileging of natural numbers precluded questions that we have come to expect about mathematical structures, such as "What is the cardinality of this structure?" and "What do similar structures look like?"

Dedekind recognizes the novel way he is treating these familiar numbers.

> But I feel conscious that many a reader will scarcely recognise in the shadowy forms which I bring before him his numbers which all his life long have accompanied him as faithful and familiar friends; he will be frightened by the long series of simple inferences corresponding to our step-by-step understanding, by the matter-of-fact dissection of the chains of reasoning on which the laws of numbers depend, and will become impatient at being compelled to follow out proofs for truths which to his supposed inner consciousness seem at once evident and certain. [30, p. 15]

Dedekind can be understood as replacing intuitive truths of the natural numbers with theorems and characterizations.[6] This is a notion of deepening the

[5][32, § 73]:

> If in the consideration of a simply infinite system N ordered by a mapping ϕ we entirely neglect the special character of the elements, simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the ordering mapping ϕ, then these elements are called *natural numbers* or *ordinal numbers* or simply *numbers*, and the base-element 1 is called the *base-number* of the *number-series* N. With reference to this liberation of the elements from every other content (abstraction) we are justified in calling the numbers a free creation of the human mind.

[6]That is, his project is to replace intuitive evaluations of statements like "1<2" with sys-

foundations[7] where we take the given foundations (e.g. natural numbers) and provide ways to derive these primitives or axioms as definitions and theorems. Dedekind proves the principle of induction and recursion, from his definition of the natural numbers and the ambient logic.

This 'deepening of foundations' can be seen as a sort of conceptual parameterization as discussed above. Just as we can define the membership relation based on an axiomatization of smallness, we can define the natural numbers based on an analysis of systems, mappings, and chains. These notions were not considered usually as relevant to the specification of the natural numbers because the natural numbers were unproblematically given. By adding these conceptual 'parameters' to the traditionally unparameterized notion, Dedekind was able to say something about structures that looked like the natural numbers. In particular, he gave the categoricity result: All simply infinite systems are isomorphic. This came first by providing a particular simply infinite system N and then showing that all simply infinite systems X are isomorphic to N. Then, by composing isomorphisms, he shows that any two simply infinite systems X and Y are isomorphic. These sorts of results simply don't make sense in a context where the natural numbers are taken for granted. This is because the structure is unique, not definable from other more basic concepts, and access to its properties is intuitive, not axiomatic.

In addition, this conceptual parameterization, like many parameterizations of this sort, introduce the question of whether such a system exists. For this dissertation, we might wonder whether any reasonable class of maps satisfied the axioms for smallness given in Section 5.4. It turns out that in a Heyting pretopos, the class of exponentiable maps satisfies **S1-6** and if we consider **SC1-6** but in the effective topos Eff, it turns out that the sub-countable sets are exactly the small sets. This shows how we can instantiate the axioms concretely; it does not by itself imply the existence of an initial algebra for an endofunctor. In a limited sense, we can understand this as parallel to Dedekind's 'proof' that there is an infinite system by providing a construction of one.

tematic proofs.

[7] A notion we get from Hilbert, cf. [85].

Chapter 7

Conclusion

In this dissertation, we have given an abstract, category-theoretic characterization of accessible domains: they are precisely the initial algebras for endofunctors. Accessible domains are inductively defined classes where the inductive clauses are deterministic in the sense that for any inductive rule, the same premises result in the same conclusion. Accessible domains underlie many important methodological approaches to the foundations of mathematics, from finitism, to constructivism and through to classical mathematics. This is because the elements of accessible domains are built-up inductively in such a way as to justify two core mathematical principles:

Induction The proof principle of induction allows us to prove universal claims about the accessible domains by checking that the truth of the claim is preserved by the inductive rules of the accessible domain. The paradigmatic cases are proofs by induction over N.

Recursion The definition principle of recursion provides a way to define functions and operations on accessible domains, by specifying what the functions and operations do relative to the inductive rules of the accessible domain. Structural recursion is a ubiquitous tool in mathematical logic, but transfinite recursion is included in this as well.

These two features are central to the notion of accessible domain and must be preserved in my abstract characterization thereof. We showed that initial algebras are indeed inductive and recursive in Chapter 2. In the case of recursion, we used the example of a natural numbers object N to show that the normal notions are capturable by initial algebras. Although we gave no proof that every initial algebra is in some sense 'recursive', the example was easily generalized. After all, we did show that the recursion equations really just express the commuting of an algebra homomorphism diagram.

For induction, we gave a general treatment. The basic idea is that initial algebras by definition are *minimal*, which means that they have no proper subalgebras. So if you show that some property holds of a sub-algebra of an initial algebra (by respecting the inductive clauses), then it holds of the entire initial algebra as well. This comes from a straightforward understanding of minimality with the 'internal perspective': only elements that are built from the inductive clauses are in the initial algebra. That is, by establishing a claim for the inductively built-up objects, we've done so for all the objects.

A third property of accessible domains was not emphasized in Chapter 2 but can be included here. Accessible domains are unique up to *canonical* isomorphism, through recursion. That is, given any two accessible domains of the same type, we can define an isomorphism between them based on the inductive build-up of elements. For initial algebras, this comes out in the fact that initial algebras are unique up to isomorphism. If there are two initial algebras for the same functor, the unique homomorphisms included in the definition constitutes an isomorphism between them. This isomorphism can be viewed as 'given by recursion' in the sense of being an algebra homomorphism and can be seen as canonical in the sense that it will always be present for initial algebras.

Now that we see that our characterization captures the central features of accessible domains, induction and recursion, we can explore the broader setting of initial algebras. It's all well and good to give an abstract characterization that has the intended two properties, but a stronger characterization is given by seeing in what ways these initial algebras exemplify accessible domains. Moreover, we want the characterization to serve as a framework for methodological frames. For that, we discuss categorical ways of understanding iteration and fixed points: Chapter 3. This helps us see that initial algebras make sense of the intuition of building up the elements of an accessible domain by iterating the inductive clauses (the endofunctor). Indeed, we saw some conditions under which we can know that this build-up will stop and we've arrived at an initial algebra. For example, we may want to understand N as the set of all finite iterations of the successor function on the element 0. We saw that this is a general phenomenon in initial algebras for endofunctors that satisfy some preservation properties.

But then we turned to seeing what could be said about the endofunctor by which the initial algebra is defined. Specifically, do we know what these endofunctors look like? In Chapter 4 we analyzed a class of functors called *polynomial functors* that, in their basic cases, have the form

$$P(X) = \sum_{a \in A} X^{B_a}$$

where the B_a are fibers of a function f. These look like polynomials from other parts of mathematics, but the 'coefficients' are natural numbers. That is, we may have $X + 2X^2$ but we don't have $\frac{1}{2}X + \pi X$.[1]

These polynomial functors provided a class of endofunctors that were well-behaved. Specifically, we know when they have initial algebras based on facts

[1] We could look at polynomial functors with more complex notions of coefficients, and these might reasonably be said to produce initial algebras as well. Indeed, once one introduces such complexity, one can see differentiation and integration in a categorical way. Similarly, multiple variables are relatively easy to incorporate into this framework [41, 42, 50].

about which objects they preserve. Moreover, when a polynomial has an initial algebra, we call it a W-type, which is meant to suggest that the initial algebra contains well-founded trees. Well-founded trees are the abstract way to understand a large class of important accessible domains, such as the natural numbers, the constructive ordinals, and formulas or terms of a formal language. Because we understood polynomial endofunctors as based on an indexing function f, we see that these initial algebras take very little data to set up: a function f that determines the exponents of the polynomial and a categorical context. From this data, we can build a polynomial endofunctor and decide whether it has an initial algebra or not.

The relative simplicity of polynomial endofunctors gives a sense of how elementary the initial algebra is. If the exponents of the polynomial (the fibers of f) are all finite, then the polynomial endofunctor gives rise to an elementary accessible domain. Both the natural numbers and usual classes of formulas and terms of a language fall into this elementary class. A slightly less elementary accessible domain results from a polynomial endofunctor such as constructive number classes $\mathcal{O}$ which have infinite exponents such as N. But since $\mathcal{O}$ is still based on a polynomial endofunctor with a finite coproduct, it can be seen as more elementary than well-founded trees with an infinite number of operations. All these are still in the class of initial algebras of polynomial endofunctors, however, which means that even if they are complex, they are limitedly so.

Not all accessible domains are W-types, however. Segments of the cumulative hierarchy are important examples of classes of objects relative to which foundational work is done but that aren't initial algebras for polynomial endofunctors. We considered various cumulative hierarchies in Chapter 5 to fold them into the characterization of accessible domains as initial algebras. We had to spend a good deal of effort specifying the categorical context of each of these hierarchies, in terms of what a 'set' is in the different approaches (e.g. predicative, intuitionistic, classical). We saw that cumulative hierarchies arise as initial algebras for powerset functors. We did not, however, focus on how iterating the powerset functor would result in the cumulative hierarchy, although we mentioned the relationship between this chapter and Zermelo's study of cumulative hierachies in [103]. Instead we focused on the fact that these initial algebras contain all the 'small' sets of the theory axiomatized by the category and notion of smallness.

We used the same framework of ZF algebras to define the class of ordinals, which requires a monotone successor. The relationship between ordinals and the initial ZF algebra was elucidated by showing how the maps $rank$ and $V_{(-)}$ were adjoints, giving us the standard properties of the class of ordinals.

Now that we have a categorical version of various cumulative hierarchies and a notion of ordinal number, we can expand our comparison of accessible domains as methodological frames. First, we note that in general, there can be many sorts of powerset operation as given in Section 5.5. This is because we can change the notion of smallness to be more or less restrictive. If we axiomatized smallness so as to include only maps that had fibers of no greater cardinality than 2, we would produce a much 'smaller' structure than segments of the classical cumulative hierarchy. In this case, in a category like a Heyting pretopos, we would expect the powerset P_2 to give us all the subclasses that contain 0, 1, or 2 elements. If an initial algebra existed for this very limited powerset operation, it would not model a traditional set theory, since if any infinite set X existed, we would not validate Cantor's fact of $card(X) < card(P_2(X))$.[2]

It is in general not true that the powerset functor will be 'less elementary' than polynomial endofunctors because they are defined so broadly. In fact, as Remark 5.24 suggests, initial powerset algebras (or their quotients) can coincide with initial algebras for polynomial endofunctors. So our diagram that compares methodological frames can now become Figure 7.1, where we use Ord to refer to the classical ordinals.

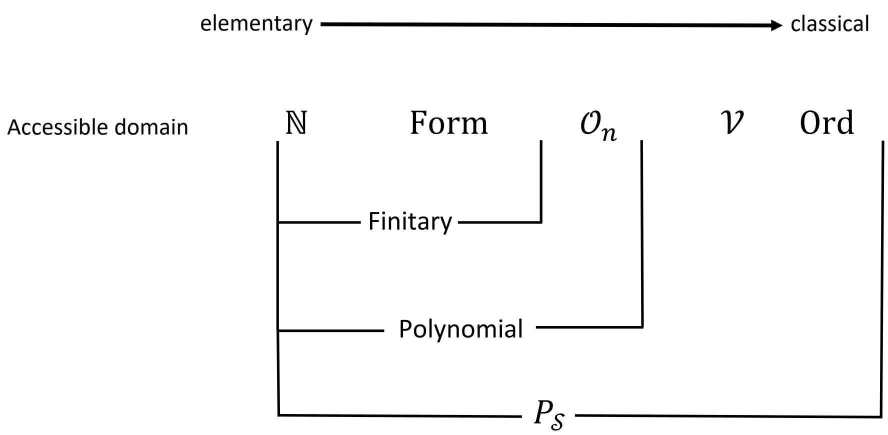


Figure 7.1: Diagram suggesting the ordering of accessible domains by the type of their underlying endofunctor.

This diagram is not a precise one. In particular, we are not claiming that $P_\mathcal{S}$ can capture every accessible domain simply by axiomatizing smallness correctly. Not only is it unclear whether this would be in principle possible, extremely divergent axiomatizations would remove the sense of what the axiomatization is supposed to capture, which are the 'well-defined and not too big' sets.

The main point in Figure 7.1 is that different accessible domain are the result of different sorts of endofunctors. While we do not give a precise ordering to these sorts of endofunctors, we do give a comparative framework for them. Finite polynomial endofunctors are particularly elementary in their expression and their initial algebras. After all, ω many iterations of the endofunctor will produce any initial algebra of this sort. This is true for accessible domains with one inductive rule (like the successor in $\mathbb{N}$) or many (like the grammatical rules of a formal language in Form).

[2]To see this, note that $\mathbb{N} \cong \mathcal{P}_2(\mathbb{N})$ by an argument much like how one would prove that $\mathbb{N}$ is bijective with the rational numbers $\mathbb{N} \cong \mathbb{Q}$. In general (and without choice), the set of all n-length sequences of natural numbers is countable for a finite number n.

Some polynomial endofunctors have infinitary operations, such as the constructive number classes $\mathcal{O}_n$. The first constructive number class $\mathcal{O}_1$ is the initial algebra for an endofunctor that uses the natural number object as an exponent $1+X+X^{\mathbb{N}}$. Higher number classes build on this one and use previous number classes as exponents, making each number class a bit less elementary. And yet, these number classes are all given as initial algebras for polynomial endofunctors. This can be viewed as saying that these accessible domains are still relatively 'constructive' in the sense discussed in Chapter 1.

But some accessible domains are not based on polynomial endofunctors; they are 'exponential'. If we use the analogy of polynomials over reals to polynomial endofunctors, we can also analogize powerset functors to exponential functions 2^X. These exponential functions are of a 'stronger' sort than polynomials; they grow faster.[3] In our setting, powerset functors are so generalized that they may not be in general 'exponential' in the same sense. In the case of **CZF**, we may allow a notion of arbitrary function spaces (exponential objects), but not a notion of arbitrary subsets.[4] We did not spend much time delineating between powerset conceptions in Section 5.1, but it suffices to note that the powerset operation given in **IZF** and **ZF** in Section 5.5 implies subset collection, but the reverse does not hold.[5]

In our axiomatization of smallness for **CZF** we included an explicit mention of exponential objects (Axiom **SC4**). So, for the cumulative hierarchies of **CZF**, **IZF**, and **ZF** all required some operation beyond polynomial. It is not crucial that there be a hard notion of 'beyond' used here.

Setting aside any informal notions of complexity notice that in our setup in Figure 7.1, accessible domains can be compared in a broad sense, by considering their underlying endofunctor. We may shy away from saying that this or that endofunctor is more elementary, but we can at least say that this endofunctor needs these categorical assumptions rather than those categorical assumptions.

All of this is to say that the properties of the category that are necessary to produce the accessible domain can be compared. As mentioned in Section 4.3, the syntactic presentation of these functors do no constitute intrinsic properties of the functor and so future work should consider invariant properties of the functors underlying accessible domains as the proper way to capture the notion of methodological frame.

Even this syntactic formulation is not a pure order, since natural number objects can be thought of as the collection of finite ordinals (as in Remark 5.24) or as initial algebras for the functor $1+X$. The assumptions on the category are different for these two conceptions. To say that N is thus elementary ignores the multiple avenues by which we may define it as an initial algebra. Intuitively it may seem that definition through $1+X$ is more elementary since we didn't

[3]We may even analogize with good reason to the computational complexity of these functions. Big-O analyses of computational complexity puts exponential time as significantly 'worse', in the limit, than polynomial time. That is, algorithms that take only polynomial time are quicker than those that take exponential time.

[4]In particular, Myhill [71, p. 354] expresses this preference.

[5]For a fuller discussion, see [5, Ch. 3, 5].

axiomatize the surrounding category. But this was a product of our presentation, not of the mathematical fact. To truly arrive at the existence of a natural number object, plenty must be assumed about the category. However, it is clear that whereas the ordinals were presented in a class category that supported an internal logic, the natural numbers were not. This again, is more a matter of presentation and emphasis than outright complexity of the necessary categorical assumptions.

After giving the mathematical elements to the characterization, we discussed in Chapter 6 the way that category theory fit the task at hand and the historical precedent for the features we picked out. I believe that the ability to externalize features of properties that are traditionally thought of as internal to an object or structure is the key affordance of category theory for this project. It is not just that category theory is 'abstract' by providing a sort of third-order perspective, but it does so in a particularly interesting and effective way. Externalizing these properties supports the uniform treatment of multiple mathematical subjects and makes precise the analogies between these disparate corners of mathematics.

Thus, the goal for the dissertation is achieved. I have characterized accessible domains as initial algebras for endofunctors and shown that initial algebras capture the relevant features and examples of accessible domains. In addition, I have shown that more can be said about comparing different accessible domains by comparing the endofunctors on which the initial algebra is based. Some functors are more elementary than others and so we have an open-ended way to consider the sorts of commitments that different methodological frames may make. On one end, we have very simple endofunctors like $1+X$ which gives rise to the natural numbers. On the other end, we have the classical powerset functor $P_{\mathcal{S}}$ which gives rise to the cumulative hierarchy $\mathcal{V}$. With this characterization we can say they all share some key features, as accessible domains, but differ in what methodological frame they express.

Appendix A

Category Theory Preliminaries

A category is an extremely useful sort of mathematical object to consider. The theory of categories provides a flexible and wide-ranging framework in which to describe abstract structures. It is sometimes contrasted with set-theory as a putative foundation or as a language for mathematics [36] but we will leave that topic for another time. Instead, let us develop the elementary theory of categories a bit so that we can think about accessible domains in a broad, abstract way. We pull most of the definitions and useful lemmas from [16].

The definition of category is somewhat complicated at first sight:

Definition A.1. A *category* consists of the following data:

- Objects: $A, B, C, \ldots$
- Arrows: $f, g, h, \ldots$
- For each arrow f, there are objects given as

$$dom(f) \text{ and } cod(f)$$

  called the *domain* and *codomain* of f. We write $f : A \to B$ to indicate that $A = dom(f)$ and $B = cod(f)$.
- Given arrows $f : A \to B$ and $g : B \to C$, there is an arrow $g \circ f : A \to C$ called the *composite* of f and g.
- For each object A, there is an arrow $id_A : A \to A$ called the *identity arrow* of A.
- Associativity:

$$h \circ (g \circ f) = (h \circ g) \circ f$$

  for all $f : A \to B, g : B \to C, h : C \to D$.

- Unit:
$$f \circ id_A = f = id_B \circ f$$
for all $f : A \to B$.

We usually denote categories by 'blackboard bold' fonts:

$$\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}, \ldots$$

These are the 'axioms' of category theory in the same sense that the group axioms are for groups; anything that satisfies this definition is a category. Perhaps the most familiar is the category Sets where the objects are sets and the arrows are all functions between sets. Of course, functions have a domain and codomain and their composition is associative. The identity arrows are the identity functions and act as the unit correctly.

We point out that arrows need not be functions, as such. A partially ordered set $(P, \leq)$ is category where the elements of P are the objects and there is an arrow $f : a \to b$ when and only when $a \leq b$. We will always specify the objects and arrows of a category when necessary, often relying on the reader to see why composition is associative and that there are appropriate identity arrows.

There is a notion of homomorphism between categories in the sense that they are mappings that preserve the key structure of categories. These mappings are called *functors*:

Definition A.2. A *functor*
$$F : \mathbb{C} \to \mathbb{D}$$
between categories $\mathbb{C}$ and $\mathbb{D}$ is a mapping of objects to objects — $F(A)$ is an object of $\mathbb{D}$ when A is an object of $\mathbb{C}$ — and arrows to arrows such that, for all objects A, B and arrows g, f of $\mathbb{C}$:

1. $F(f : A \to B) = F(f) : F(A) \to F(B)$
2. $F(id_A) = id_{F(A)}$
3. $F(g \circ f) = F(g) \circ F(f)$

In our discussion of accessible domains, we will be concerned mostly with *endofunctors* which mean functors on a category to itself

$$F : \mathbb{C} \to \mathbb{C}.$$

For accessible domains in general, we will not restrict the type of these endofunctors, but for many of the cases, we can usefully restrict attention to endofunctors that are called *polynomial* endofunctors. These are a well-behaved subclass of endofunctors that will give us additional insights into accessible domains.

However, before we get to that, we must explain some central constructions in category theory called *limits* and *colimits*. These are 'universal' objects in the sense that they are described analogously to 'the smallest object such that...' or 'the object with these properties that contains all other objects with those properties'. They play an important role in our investigations since they give us something of a generalization of lowest upper bound or minimal structure satisfying certain conditions. Our focus will be on colimits, but since limits are dual, we define them as well. First, let us define two dual notions that will be exploited heavily in both the discussion of limits and colimits on the one hand and algebras over endofunctors (see Chapter 2) on the other.

Definition A.3. An *initial* object of a category $\mathbb{C}$ is an object 0 of $\mathbb{C}$ such that there is an unique arrow

$$0 \to X$$

for every object X of $\mathbb{C}$.

This notation and definition is reminiscent of minimal elements in order theory and indeed corresponds if the category is a poset since the definition would then say that for all objects X, we have $0 \leq X$. A feature of this definition is that the identity arrow on 0 is the only arrow $0 \to 0$ since each arrow with domain 0 is the only one to a given codomain. In Sets, the empty set $\emptyset$ is an initial object since for every set X, there is exactly one function from $\emptyset \to X$, namely the empty function.

The dual notion, with the arrow simply flipped (and change the notation for the objects since they do not always coincide) in the definition of initial objects, is that of terminal objects.

Definition A.4. A *terminal* object of a category $\mathbb{C}$ is an object 1 of $\mathbb{C}$ such that there is an unique arrow

$$X \to 1$$

for every object X of $\mathbb{C}$.

Dual to the case of initial objects, these correspond to maximal elements in posets since the definition would then say that for all objects X, we have $X \leq 1$. A terminal object in Sets must be nonempty, since it must act as the codomain to functions from *any* set. But these functions must be unique and so we are led to conclude that the terminal objects in Sets are the singletons. For brevity, whenever we need a terminal object of Sets, we will denote it by either 1 or the ambiguous singleton $\{*\}$ that you may replace with any singleton you like; it will never matter since:

Fact A.5. *Initial and terminal objects are unique up to isomorphism.*

That means that if A and B are both initial (terminal) objects in a category $\mathbb{C}$, then A is isomorphic to B. Isomorphisms, of course, depend on the category in question; in Sets isomorphisms are bijections.

Remark A.6. It is useful to remember the case of singletons being the (isomophic) terminal objects in Sets. Particularly, the fact that arrows $1 \to X$ for a set X is a function taking the single element of 1 to a single element of X. In this way, the elements of X are in correspondence with arrows $1 \to X$.

Now we build up the theory of limits and colimits by first introducing the notion of a *diagram*.

Definition A.7. Let $\mathbb{I}$ and $\mathbb{C}$ be categories. A *diagram of type* $\mathbb{I}$ *in* $\mathbb{C}$ is a functor

$$D : \mathbb{I} \to \mathbb{C}.$$

The intuition for diagrams is having a picture of $\mathbb{I}$ in $\mathbb{C}$. Often, $\mathbb{I}$ will be relatively simple, with finite objects and finite arrows. We write objects in the *index category* $\mathbb{J}$ lower case, $a, b, c, \ldots i, \ldots$ and the values of the functor $D : \mathbb{J} \to \mathbb{C}$ in the form D_a and D_b instead of $D(a)$ and $D(b)$.

Definition A.8. A *cone* to a diagram D consists of an object C in $\mathbb{C}$ and a family of arrows in $\mathbb{C}$,

$$c_i : C \to D_i$$

one for each object $i \in \mathbb{I}$, such that for each arrow $\alpha : a \to b$ in $\mathbb{I}$, the following triangle commutes:

$$\begin{tikzcd} C \arrow[r, "c_b"] \arrow[d, "c_a"'] & D_b \\ D_a \arrow[ur, "D_\alpha"'] & \end{tikzcd}$$

This last condition will be referred to as "c_i respects arrows in the diagram".

The intuition behind the notion of cone is that the object C has arrows to each object of the diagram, so the arrows out of C emanate and look like a cone, such as in the heuristic diagram

$$\begin{tikzcd} & & C \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\ D_a & D_b & D_c & D_d & D_e \end{tikzcd}$$

suggests. This diagram shows none of the arrows between the D_i, should there be any, so it remains heuristic. We denote this cone as (C, c_i). We define a *cocone* dually, being an object C and a family of arrows $c_i : D_i \to C$ and denote it the same way, namely (C, c_i).

Example A.1. Take $\mathbb{I}$ to be the discrete category with two objects and no non-identity arrows:

$$\begin{tikzcd} a \arrow[loop left, "\mathrm{id}"] & & & b \arrow[loop right, "\mathrm{id}"] \end{tikzcd}$$

For this index category, a diagram $D : \mathbb{I} \to \mathbb{C}$ would simply be a pair of objects D_a, D_b in $\mathbb{C}$ corresponding to the two objects of $\mathbb{I}$. A cone on D would therefore be an object C of $\mathbb{C}$ equipped with an arrow *to* each object of the diagram:

$$\begin{tikzcd} & C \arrow[dl, "c_a"'] \arrow[dr, "c_b"] & \\ D_a & & D_b \end{tikzcd}$$

A cocone on D, on the other hand, would be an object B of $\mathbb{C}$ equipped with an arrow *from* each object of the diagram:

$$\begin{tikzcd} & B & \\ D_a \arrow[ur, "c_a"] & & D_b \arrow[ul, "c_b"'] \end{tikzcd}$$

Example A.2. Take $\mathbb{J}$ to the following category, with implicit identity arrows:

$$\begin{tikzcd} & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

For this index category, a diagram $D : \mathbb{J} \to \mathbb{C}$ would be three objects A, B, and C in $\mathbb{C}$ with the shape

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

A cone on D would therefore be an object D of $\mathbb{C}$ equipped with an arrow to each object of the diagram:

$$\begin{tikzcd} D \arrow[r] \arrow[d] \arrow[dr] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

Since cones must respect the arrows in the diagram, the arrow $D \to C$ need not be explicit. So we get the cone:

$$\begin{tikzcd} D \arrow[r] \arrow[d] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

Diagrams can be quite varied, but one sort of diagram that will prove useful are ordinals, and later, filtered or directed categories. For any ordinal, we can create the diagram with that many objects and arrows showing the partial order. Really, we are using the ordinals to gives us a poset for our index category. Thus, the ordinal 1 gives us the diagram

$$\cdot$$

and the ordinal 3 gives us the diagram

$$\cdot \longrightarrow \cdot \longrightarrow \cdot$$

and so on. We can therefore have a diagram of type ω

$$\cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \dots$$

We can think of diagrams $D : \omega \to \mathbb{C}$ as sequences in $\mathbb{C}$ and since we will use these later, let's define them.

Definition A.9. For a category $\mathbb{C}$, a diagram of type ω in $\mathbb{C}$, for the ordinal category ω, is called a *ω-sequence*.

To get to a notion of limits and colimits, it is convenient to define the arrows between cones. A *morphism of cones*

$$\theta : (C, c_j) \to (C', c'_j)$$

is an arrow $\theta : C \to C'$ in $\mathbb{C}$ making each triangle

$$\begin{tikzcd} C \arrow[r, "\theta"] \arrow[dr, "c_j"'] & C' \arrow[d, "c'_j"] \\ & D_j \end{tikzcd}$$

commute. That is, such that $c_j = c'_j \circ \theta$ for all $j \in \mathbb{J}$. We can thus collect the cones to a diagram D into a category denoted $\mathbf{Cone}(D)$. The terminal and initial cones to D are central in expressing universal mapping properties in category theory.

Definition A.10. A *limit* for a diagram $D : \mathbb{J} \to \mathbb{C}$ is a terminal object in $\mathbf{Cone}(D)$.

Definition A.11. A *colimit* for a diagram $D : \mathbb{J} \to \mathbb{C}$ is an initial object in $\mathbf{Cone}(D)$.

Note that thus limits and colimits are themselves cones and cocones, respectively. We denote limits with their attendant arrows as

$$p_i : \varprojlim D_j \longrightarrow D_i$$

with $\varprojlim D_j$ often called "the limit". We denote colimits dually as

$$q_i : D_i \longrightarrow \varinjlim D_j$$

with $\varinjlim D_j$ often called "the colimit".

Although we will describe these examples in greater detail if we need them, it may be helpful to have the basic examples of limits and colimits at hand.

Example A.3. We can see that initial and terminal objects are examples themselves of colimits and limits, respectively. The way to show this is to define the index category $\mathbb{I}$ and then show that the initial (terminal) cone is the initial (terminal) object. The diagram is the same in both cases, speaking to the duality of these notions: the empty diagram.

Take $\mathbb{I}$ to be a category with no objects and thus no arrows. A cone of the diagram $D : \mathbb{I} \to \mathbb{C}$ in a category $\mathbb{C}$ is thus an object C with arrows to each object D_i — but there are none. That is, the diagram is simply an object in $\mathbb{C}$. Now, the initial cone must have an unique cone morphism to all other cones of this diagram. This is equivalent to asking for a unique arrow in $\mathbb{C}$ to all objects of $\mathbb{C}$, since objects of $\mathbb{C}$ are all cones of D. Similarly for terminal objects.

Example A.4. The dual notions of product and coproduct use a nontrivial category, but it remains rather simple. Take $\mathbb{I}$ to be the category from Example A.1 but let us drop the names of objects in $\mathbb{I}$ to make the following more perspicuous:

$$\begin{tikzcd} \cdot \arrow[loop left, "\mathrm{id}"] & & \cdot \arrow[loop right, "\mathrm{id}"] \end{tikzcd}$$

So a diagram $D : \mathbb{I} \to \mathbb{C}$ would simply be a pair of objects A, B in $\mathbb{C}$ corresponding to the two objects of $\mathbb{I}$. A cone on D would therefore be an object C of $\mathbb{C}$ equipped with arrows, call them π_1 and π_2, to each object of the diagram:

$$A \xleftarrow{\pi_1} C \xrightarrow{\pi_2} B$$

Now, a terminal cone of this diagram is first of all a cone, with an object we call $A \times B$:

$$A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$$

and is the 'maximal' such cone, which means that for any other cone (C, f, g), there is a unique arrow from C to $A \times B$ such that the following diagram commutes:

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g \rangle"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

where we use the name $\langle f, g \rangle$ for the new arrow, denoted by a dashed line to indicate its unique existence. As the notation suggests, we call the limit of the diagram with two objects, which is the terminal cone we just constructed, the *binary product* of A and B. In the category of sets, this is (up to isomorphism) the set of ordered pairs with first elements in A and second elements in B. It should be noted that not all categories have binary products for all objects. We often suppose their existence when we study classes of categories that have certain limits and colimits so that these constructions are guaranteed. In each concrete case, however, we do have to show their existence, if we want to use them.

Example A.5. This example is the dual of the previous. Instead of products, we construct coproducts using the same diagram, but constructing the *initial* cone on that diagram. A cocone on D would be an object B of $\mathbb{C}$ equipped with an arrow *from* each object of the diagram:

$$\begin{tikzcd} & B & \\ D_a \arrow[ur, "c_a"] & & D_b \arrow[ul, "c_b"'] \end{tikzcd}$$

And since we are just dualizing, let us merely switch the arrows of the diagrams of the last example around but call the object of the colimit (initial cone) $A+B$ for *coproduct*. So the colimit is a cocone:

$$A \xrightarrow[inr]{} A+B \xleftarrow[inl]{} B$$

Being the minimal (initial) such cocone, means that for any cocone (B, f, g), there exists a unique map from $A+B$ to C such that the following diagram commutes:

$$\begin{tikzcd} & B & \\ A \arrow[ur, "g"] \arrow[r, "inr"'] & A+B \arrow[u, dashed, "{[f,g]}"'] & B \arrow[ul, "f"'] \arrow[l, "inl"] \end{tikzcd}$$

and we call this unique arrow $[f, g]$ as indicated.

Definition A.12. A functor $F : \mathbb{C} \to \mathbb{D}$ is said to *preserve colimits of type* $\mathbb{J}$ if, whenever $p_j : D_j \to C$ is a colimit for a diagram $D : \mathbb{J} \to \mathbb{C}$; the cone $F(p_j) : F(D_j) \to F(C)$ is then a colimit for the diagram $F(D) : \mathbb{J} \to \mathbb{D}$. Briefly,

$$F(\varinjlim D_j) \cong \varinjlim F(D_j).$$

Preserving limits is defined dually. In this definition, the cone $F(p_j) : F(D_j) \to F(C)$ is just the image of the colimit cone.

Appendix B

Categorical Set Theory

For the following analysis of algebraic set theory to succeed, let us take the time to understand the way set-theoretic notions, like those of Section 5.1, get translated into category-theoretic terms. In my experience, the notions of category theory are best explained by using familiar examples of sets and members to generalize to the language of objects and arrows. We will restrict ourselves here to talking about the set-theoretic notions like *set* and *member* and set-theoretic operations like *intersection* and *powerset* in the categorical context.

In much of traditional set theory, we think of the membership relation $\in$ as a global relation on objects, which in turn are all considered to be sets.[1] This provides a rich structure that can interpret things like pairs, functions, groups, topological spaces, etc.. As many have noted, having only one 'type', that of 'set', provides a satisfying simplification to the varied world of mathematics. And yet, a global relation can be tricky, as evidenced by e.g. Russell's paradox; self-application becomes a question only with these sorts of universal relations. Similarly, it allows the formation of other less pernicious, but nonetheless unnatural, questions such as "what in our model and implementation is the set-theoretic intersection of the real number π and Cantor space?" (Trimble2011)

One response to these paradoxes is to create axioms that avoid them by refining our notion of set to include well-foundedness and restricted comprehension. This is not to say the refinement was purely motivated by avoiding paradox, but that we realized there was more conceptual work to be done in analyzing the notion of set. Another attractive limitation on the notion of set is to introduce *types* of some sort so that sets must be labeled and these labels enter into definiability assumptions. This provides differentiation among the various sets, creating more limitations on the comparisons between mathematical objects, thus eliminating questions such as the above question of Trimble's.

From the category-theoretic perspective, the typed set theory is arguably

[1] Some set theories do have atoms, or urelements, but that seems to have been less popular in mainstream set theory these days.

the more natural. I do want to emphasize that I am not claiming that this means we should therefore prefer something other than a global relation of $\in$. However, for this dissertation to be successful, we must see what the abstract theory of categories affords us in understanding accessible domains, including universes of sets.

One relatively easy way to see how category theory 'localizes' the membership relation—as opposed to the global relation—is with the intersection of two sets A and B. Usually, we think of the intersection simply as

$$x \in A \cap B \iff (x \in A \wedge x \in B).$$

In contrast, a localized membership relation asks us to find the intersection of A and B relative to a set X that contains both A and B. Another way to put this last requirement is to ask which set A and B live in. So the operation of intersection is on subsets of some X, giving us an intersection operation for every set X. This is not an overly strict requirement, since if we have binary unions available, we may consider $X = A \cup B$, which clearly contains both A and B as subsets.

But before we can properly define intersections in terms of subsets, we must know how subsets themselves are to work in this localized frame. The intuition behind subsets in category theory centers on thinking of them as inclusion functions: a subset $A \subseteq B$ is an inclusion map $A \hookrightarrow B$. So then operations like intersection act on maps, not objects. In category theory, inclusions are generalized to *monomorphisms*, but in categories of sets, monomorphisms are the same as injections. We might wonder now how we can have 'localized' $A \subseteq B$, even if we include a superset X that contains both A and B (i.e. categorically $A \hookrightarrow X$ and $B \hookrightarrow X$). This is done by the following definition

Definition B.1. Given monomorphisms $a : A \hookrightarrow X$ and $b : B \hookrightarrow X$, we write $A \subseteq B$, or $A \subseteq_X B$, if there is a morphism $i : A \to B$ such that $a = b \circ i$, which is to say that the following diagram commutes:

$$\begin{tikzcd} A \arrow[rr, dashed, "i"] \arrow[dr, hook, "a"'] & & B \arrow[dl, hook', "b"] \\ & X & \end{tikzcd} .$$

We also employ something I call *structural ambiguity* as to what the letter A refers to.[2] Often in the category-theoretic literature, we intentionally blur the distinction between the monomorphism $A \hookrightarrow X$ and the subset A as an object. Indeed, we see that the arrow and domain objects can both be denoted by A; thus the above definition could have used $A : A \hookrightarrow X$ and $B : B \hookrightarrow X$ instead, though this would have been very confusing. This free interchange between objects and arrows in these contexts actually helps keep things straight by allowing us to pick the easiest representation for whatever we need. Instead

[2]The sense of ambiguity I employing is not one of uncertainty, but of a sign bearing multiple meanings, disambiguated by context.

of saying that the arrow is the subset, we like to say that the domain A is the subset. But then composition, which gives us the transitivity of subsets, doesn't make sense with the object as much as the arrow $a : A \hookrightarrow X$ that corresponds to the subset in question. So strictly speaking, the definition stands as stated, but we should feel free to exploit the correspondence between the arrow a and the object A when the context is clear.

This notion of ambiguity is to be separated from any negative connotations. Indeed, I take it to be a crucial aspect of mathematics that it can systematically handle the case where one symbol refers to two or more nonidentical things. The definition of subset is not ambiguous, it is our use of the expression "A" that is. Some would call this use an 'abuse of notation', but this suggests the exact opposite of what I think is happening. It is not that we are contradicting the intentions, i.e.'abusing', the notation; we are building notation to be usefully ambiguous.

In addition to the relation $A \subseteq B$, we also say that two objects A and B are isomorphic 'as subsets' when i is an isomorphism. This is an equivalence relation and gives us an easier way to refer to subsets.

Definition B.2. A *subobject* of X is an equivalence class of monomorphisms $A \hookrightarrow X$, with respect to isomorphisms. We usually denote a subobject with either a representative of the equivalence class $a : A \hookrightarrow X$, or with brackets $[A \hookrightarrow X]$.

With this definition, we can even say that two maps define the same subset when they are isomorphic. It isn't hard to see that this means that maps a, b define the same subset if and only if $A \subseteq B$ and $B \subseteq A$. This in turn constitutes a partial order on subobjects, which corresponds to the partial order of subsets traditionally conceived.

What about elements of these subsets? Elements of sets in the categorical way of thinking are again 'localized' so that an element is always associated with a set; it's never independent. The basic notion of element may be thought of in the following way:

$$x \in A \iff x : 1 \to A.$$

So we may think of elements of an object A as *points* of A, which is to say arrows from the terminal object to A. If we think of arrows as functions (as we do in the case of the category Sets), then this is reversing the traditional priority of functions and elements. Traditionally, functions are defined on elements with values. But in this conception, elements are determined by certain functions that take the single element $*$ of the terminal object 1 to something in A, namely (what we think of as) x!

Again I want to point the intentional ambiguity between the arrow $x : 1 \to A$ and the 'element' $x \in A$. The element is an arrow, strictly speaking, but it corresponds to an element 'inside' A and so we choose which description is best depending on which intuition we want to use.

Perhaps counter-intuitively, when we localize elements in this way, we actually get a natural generalization: *generalized elements*. This is just to remove the restricted attention to arrows $1 \to X$ and consider any arrow $Y \to X$ as a generalized element of X, which in the case of Sets is just an ordinary function from Y to X. This may not feel natural from a traditional perspective, and that's fine. But something to keep in mind is that the Yoneda Lemma implies that with the above conception, a set is determined uniquely, up to a specified isomorphism, by its generalized elements. Sometimes we may denote a generalized element $x : A \to B$ by $x \in_A B$ since it's really the codomain that 'contains' x as an element (though it 'varies over A'). For me, this is an expression of the structural way of thinking. A set is determined by the structure of functions into it, which then leads us to define 'elements' only in reference to these relational (functional) facts. I will make this clearer in Chapter 6.

To summarize the conceptual translations thus far, we have

$x \in A$	$x : 1 \to A$
$x \in_B A$	$x : B \to A$
$A \subseteq_X B$	$a : A \hookrightarrow X, b : B \hookrightarrow X$ and there is a morphism $i : A \to B$ such that $a = b \circ i$

Now, we can talk more concretely about more advanced operations like intersection and powerset. It's important to remember the shift from thinking about operations on objects A and B to operations on certain arrows, for example $A \hookrightarrow X$ and $B \hookrightarrow X$. Indeed, the intersection of A and B, relative to X is just the pullback of any two representive inclusions:

$$\begin{tikzcd} A \cap B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, hook] \\ A \arrow[r, hook] & X \end{tikzcd}$$

The powerset operation can also be 'categorified' in a similar manner: by including information that the powerset is relative to. One product of this relativization of the powerset will be that different conceptions of the powerset (e.g. definability restrictions, limitations of size, predicativity) are instances of the same notion.

The basic idea is that for every set X, there is a set of subsets of X denoted by $P(X)$. This set has some structure that is specified by what we call a 'universal' relation $\in_X \subseteq X \times P(X)$ such that for any relation $R \subseteq X \times A$, there is a unique "classifying map" $\chi_R : A \to P(X)$ whereby, under $(1_X \times \chi_R) : X \times A \to X \times P(X)$, we have

$$R = (1_X \times \chi_R)^{-1}(\in_X).$$

In diagrammatic notation, this requires that R is the pullback of $\in_X$ along

$(1_X \times \chi_R)$:

$$\begin{tikzcd} R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \in_X \arrow[d, hook] \\ X \times A \arrow[r, "(1_X \times \chi_R)"'] & X \times P(X). \end{tikzcd} \tag{B.1}$$

In categorical set theory notation, this means that R can be expressed as

$$R = \{\langle x, a\rangle \in X \times A \mid \; x \in_X \chi_R(a)\}.$$

Or in yet another formulation, $\langle x, a\rangle$ belongs to R if and only if $\langle x, \chi_R(a)\rangle$ belongs to $\in_X$.

This universal mapping property of powersets (technically called *power objects*) means there are natural bijections

$$\frac{R \subseteq X \times A}{A \to P(X)} \qquad \frac{R \subseteq X \times A}{X \to P(A)}$$

between relations and classifying maps. The subset corresponding to $\phi : A \to P(X)$ is denoted $[\phi] \subseteq X \times A$ or $[\phi] \subseteq A \times X$ and is called the extension of ϕ.

The set $P(1)$, the powerset of the initial object, plays a particularly important role; it is called the "subset classifier" because subsets $A \subseteq X$ are in natural bijection with functions $\chi : X \to P(1)$. In ordinary set theory, the role of $P(1)$ is played by the 2-element set $\{f, t\}$. Here subsets $A \subseteq X$ are classified by their characteristic functions $\chi_A : X \to \{f, t\}$, defined by $\chi_A(x) := t$ iff $x \in A$. We thus have $A = \chi_A^{-1}(t)$; the elementhood relation $\in_1 \hookrightarrow 1 \times P(1)$ boils down to $t : 1 \to P(1)$.

Continuing the pattern of reversing the traditional order of things, we can consider defining a membership relation a different way, one that works best with *algebraic set theory.* To do this, we first use algebraic notions, like that of a complete lattice, to define the central concept in [47].

Definition B.3. A *ZF-algebra* is a complete sup-lattice A that comes equipped with a successor operation $s : A \to A$. We denote it as (A, s).

Technically, we will relativize the 'completeness' of the sup-lattice to be complete relative to a class of 'small maps', but we don't need that quite yet. We may also define what it means to be a homomorphism between ZF-algebras; again we will return in more detail to these definitions, but the intuition is what is important at this stage.

Definition B.4. A *homomorphism of ZF-algebras* $(A, s) \to (B, t)$ is a map $f : A \to B$ that preserves suprema (along 'small maps') and commutes with successor.

For a motivating and concrete example, consider the class of all sets with the subset relation as the order, unions as joins, and the successor is the singleton operation $a \mapsto \{a\}$. It turns out that this is the *initial* ZF-algebra, which we will show later. And now for the point of this detour: *defining* the membership relation.

The way to define the membership relation on sets given the algebraic structure of a ZF-algebra is as follows:

$$a \mathrel{\varepsilon} b \text{ iff } s(a) \leq b.$$

In traditional settings, this could be expressed as a theorem

$$a \in b \text{ iff } \{a\} \subseteq b$$

where the concepts on the right are defined in terms of primitive $\in$ relation. Hence, we use a different epsilon symbol in the case of ZF-algebras to emphasize that it is a defined, not primitive, notion. It turns out that defining things 'in this order' allows greater flexibility in what we consider to be a set.

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