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THE NECESSITY OF MATHEMATICS

Juhani Yli-Vakkuri
University of Tartu

John Hawthorne
University of Southern California

> 'As soon as these questions were squarely faced, a wide range of new phenomena were discovered, including quite simple ones that had passed unnoticed.'
> —Noam Chomsky, *Knowledge of Language* (1986), p. 7.[1]

It is a commonplace that statements of pure mathematics are necessarily true if true at all. But why should we think this? A cursory investigation of the practice of mathematics itself presents something of a puzzle here. Mathematicians do not appear to make use of the language of metaphysical necessity and possibility in their own investigations. Of course they do use the modal words 'might' and 'must' and their cognates. However, their use of these words does not provide much evidence that metaphysical modality is in play in any serious way. On the one hand, many of their uses seem to be metaphorical. As Wilfrid Hodges points out, when a mathematician says, for example, that one system 'can be embedded' in another, this is little more than a colorful way of saying that there is an embedding of one into the other. What the modal 'can' adds is

> a certain human colouring, by suggesting that part of the mathematics is carried out by a human being. This adds nothing to the mathematical content, but somehow it helps the readability (Hodges 2013: 6).

On the other hand, many uses of modals in mathematics express epistemic modality. For example, when mathematicians say at some point in their investigations, 'Various answers might be correct', they are not giving voice to a perceived metaphysical contingency in mathematical reality, but signaling that which answer is correct is an open question at the relevant stage in the process of mathematical discovery. And similarly, when they say, 'Only one answer can be correct', they are talking about what has been established at the relevant stage, not about what is and is not metaphysically necessary: if it turns out that two answers are epistemically live at the time of speaking or writing, then the 'can' claim will be reckoned false. Also similarly, when a mathematician says that 'Given that A, it must be that B', arguably the 'must' again expresses a kind of epistemic modality.[2] (One of us has explored elsewhere

[1] This is also the epigraph to Hodges (2013). We did not, however, choose it as an allusion to that paper, but simply because we could think of no better epigraph for our own paper.
[2] We will not defend this take over Hodges' own gloss on these 'must's as 'formatting to guide the reader through the structure of the reasoning' (Hodges 2007: 12).

the behavior of epistemic modals embedded in logically complex sentences.[3]) And it is far from clear that there is any modality left once we set aside the metaphorical and epistemic uses of modals in mathematical texts.[4] *Prima facie*, then, it seems that mathematical practice is silent on the question of the status of mathematical truths *vis-à-vis* metaphysical modality.

It is tempting to take these observations to support the view that the doctrine that mathematics is necessary is something that has been added by philosophers to the body of knowledge provided by mathematics itself, which can proceed just fine without taking on any metaphysically modal commitments. This view has prominent defenders. For example, in 'Modality in Mathematics' Wilfrid Hodges tells us:

> Mathematicians are pleased to know that
>
> (1) Every finite field is commutative.
>
> or that
>
> (2)
>
> $$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}.$$
>
> The fact that these statements are necessarily true might attract the attention of a philosopher of mathematics, and some mathematicians dream about such things in idle moments. But adding 'Necessarily' to either (1) or (2) would introduce nothing of any mathematical significance (Hodges 2013: 1–2).

Hodges begins another paper, 'Necessity in Mathematics', by prominently displaying the following 'fact':

> FACT A: Mathematics contains no modal notions (Hodges 2007: 1).

He continues:

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[3] See Dorr and Hawthorne (2013).

[4] As Timothy Williamson pointed out in conversation, Church's thesis, which asserts the equivalence of the intuitive notion of computability with the formal ones, is an interesting test case here. The key issue is whether the modality expressed by the suffix '-able' in 'computable', in its intuitive sense, is some kind of objective modality (in the sense of Williamson 2017*a*). (There are various inequivalent glosses on the intuitive notion. Turing [1939: 8], for example, says he will 'use the expression 'computable function' to mean a function calculable by a machine'. Another kind of gloss, common in textbooks, speaks of computability by a human or some other kind of agent. For example, Boolos, Burgess, and Jeffrey [2007: 23] say that a function is computable iff 'there are definite, explicit rules by following which one could in principle compute its value for any given argument'.) Certainly the modality in play is not epistemic, but is it metaphysical or otherwise objective? One reason for being cautious here is that mathematicians tend to make free use of Church's thesis without ever worrying about whether it is objectively possible to build a certain kind of machine or for a human or other agent to perform certain kinds of tasks. (Hence the frequently occurring weasel words 'in principle'. More things may be 'in principle possible' than are possible *simpliciter*. 'In principle' does not appear to be factive.) This suggests to us that the mathematicians' intuitive notion of 'computable' might be similar to their intuitive notion of 'provable', in that what makes something computable in the relevant sense is simply that *there is* a certain kind of procedure for computing it, and this is unrelated to the difficult question of whether it is objectively possible for anything to implement that procedure. In his comments on a draft of this paper, Williamson responded to this last suggestion by pointing out that 'the claim that there is a mechanical procedure for computing *f* may itself be derived from the judgment that a proper specification of such a procedure *could* be constructed in line with some impressionistic sketch provided'. We will leave further exploration of these issues to others.

Of course mathematics is full of necessary truths, for example this theorem of analysis:

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)=\frac{1}{2}$$

But only philosophers are interested in the fact that this theorem is necessarily true. Mathematicians are content if they can show that it is true (*ibid.*).

Having gotten this far, one might think that, insofar as we are justified in thinking that mathematics is necessary at all, such justification will proceed not from the practices of mathematicians but from the practices of armchair philosophers. Stories about how such practices confer justification or knowledge are many and varied. Most crudely, one might posit cognitive phenomenology that forcefully presents the necessity of certain propositions, including those of pure mathematics.[5] On the heels of this picture, one might tack on the epistemological principle, popular in some circles, that one is '*prima facie* justified' in believing any proposition of which one has an 'intuition' or an 'intellectual seeming'. And one might then hope to run the gauntlet of candidate 'defeaters' in order to emerge with justification *simpliciter*. Alternatively, one might take a conventionalist approach. Perhaps, one might think, there is no 'joint in reality' that is picked out by the idioms of metaphysical modality, and the truths of mathematics get to be necessary simply on account of our having decided in a quasi-stipulative way that they belong to a special 'list'.[6] Or alternatively, and perhaps most intriguingly, one might claim that mathematics is necessary on the basis of its purported reducibility to logic in combination with the necessity of logic itself. This is the so-called (neo-)logicist program adapted to the role of proving the necessity of mathematics.[7] (The other main approaches to the foundations of mathematics—intuitionism and formalism—are less obviously well-equipped to provide any compelling story about the necessity of mathematics.) The picture-thinking is clear enough: since it is not that mysterious that logic is necessary, by reducing mathematics to logic we also render the necessity of the mathematics unmysterious.[8]

For what it's worth, we find the neologicist approach to our question more promising than the other two mentioned in the previous paragraph.[9] (And we are not alone. When we asked a variety of philosophers why they thought we should think that mathematics is necessary, some variant of 'logicism' was by far the most common answer.) Here is how that approach would work. Neologicists maintain that some decent-sized axiomatizable mathematical theory—typically, the fragment of arithmetic characterized by the Peano axioms—is reducible to logic on account of its axioms being derivable from an abstraction

[5] See Bealer (2002).
[6] See Sider (2011: ch. 12).
[7] See Hale and Wright (2001).
[8] Yet another route to the necessity of mathematics, in some non-epistemic sense of 'necessity', goes via modal interpretations of mathematical language, such as Putnam (1967) and Hellman's (1993) modal structuralism, and Fine (2006), Linnebo (2010), and Studd's (2013) account of 'indefinite extensibility' phenomena. It is unclear, however, how the modalities these authors discuss are related to metaphysical modality.
[9] Just as this paper was about to go to press, Hannes Leitgeb shared with us his intriguing argument for the necessity of mathematics (Leitgeb 2018). His idea is that the necessity of all truths of pure mathematics can be derived from the rigidity of set membership (which we accept) together with the putative fact that all statements of pure mathematics can be 'reformulated' in the language of set theory (this is the thesis of 'set-theoretic foundationalism', which we find less obvious). Although we prefer our own approach, we note that Leitgeb's approach shares one advantage with ours over the alternatives: as he points out, 'at least some form of set-theoretic foundationalism is widely presupposed in contemporary mathematics' (§1). Thus, like us, he is trying to justify the thesis of the necessity of mathematics (at least in part) using materials found within mathematics itself.

principle in some axiomatic system of second-order logic. By supplementing their favorite system of second-order logic with the 'necessitation' rule

$$\frac{A}{\Box A},$$

which ensures that $\Box A$ is a theorem whenever A is, and the standard **K** axiom

$$\Box(A \to B) \to (\Box A \to \Box B),$$

they will obtain a system S that can be shown to be sound (but not complete) on a natural generalization of standard possible worlds semantics for modal logic to the second-order case.[10] Then, for any theorem T of the mathematical theory whose axioms are provable from the abstraction principle α in the original system of non-modal second-order logic, $\Box\alpha \to \Box T$ will be a theorem of S. Thus, given that the abstraction principle is necessary, and given that the system is (informally) sound, it follows that each theorem of the reduced mathematical theory is also necessary.

Yet an appeal to neologicism as a general answer to our question still does not seem very promising. We have no objection to the use of the necessitation rule, which enables the neologicists to prove the necessity of all of the truths of second-order logic they have axiomatized. It will be widely accepted that assuming logic to be necessary is an acceptable starting point.[11] Yet, even granting the necessity of logic, the neologicist strategy has certain inherent limitations. It can, at best, establish only the necessity of those mathematical truths that are provable in whatever axiomatic system it uses. By Gödel's first incompleteness theorem, we know that these cannot even include all truths of first-order arithmetic. (That result was, after all, the downfall of the original logicist program of Frege and Russell; hence the prefix 'neo'. Neologicists are content to reduce some but not all of mathematics to logic.) We also have two more general philosophical qualms about the approach. First, neologicists cannot even establish the necessity of the axiomatizable fragments of mathematics they target unless they can establish the necessity of the abstraction principles they assume. But, insofar as there is any neologicist story about why the abstraction principles are necessary, it tends to proceed via the claim that they are 'analytic' or that they are 'conceptual truths'—ideology that we find problematic for broadly Williamsonian reasons.[12],[13] Second, many neologicists seem to be motivated by a commitment to the view that logic is in some sense metaphysically neutral or innocent (since they often write as if a reduction to logic would purge mathematics of metaphysical tendentiousness).[14] But logic isn't metaphysically neutral.[15] What makes logic logic is not its neutrality but its generality. If one wants to play it safe from an ontological point

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[10] For example, Gallin's (1975) semantics for the system ML$_P$ will do: see Williamson (2013*a*: §5.5) for discussion.

[11] See, however, Clarke-Doane (2017) for a notable exception.

[12] See Williamson (2007: chs. 3–4).

[13] A particularly radical and technically untaxing form of the view that all truths of mathematics follow by logic from analytic truths is the view that all truths of mathematics are analytic. Timothy Williamson gestured at this view in conversation, and pointed out that at least it bypasses worries concerning truths of mathematics that are not provable from standard axioms. (Of course, as an analyticity-skeptic, Williamson himself does not endorse any version of the view that mathematics is analytic.) A proponent of this view cannot think of analytic truths as truths that one immediately assents to upon understanding them, but arguably even ordinary neologicists have to distance themselves from any such conception of analyticity.

[14] Hale and Wright are our paradigms. See the Introduction to Hale and Wright (2001), and see Raatikainen (forthcoming) for discussion.

[15] See Williamson (2013*b*).

of view, sticking to logic as the foundation of both (some) mathematics and the source of its necessity is not a good game plan. Of course we don't expect these brief remarks to convince die-hard neologicists, but we put them forward in the hope that they will clue the reader into our own orientation in the philosophy of logic and mathematics. The original contribution of this paper is the alternative picture it develops and not its critique of extant accounts of the necessity of mathematics.

In our view, the supposition that mathematics is silent on questions of metaphysical modality is completely wrong-headed. Hodges' claim that mathematics does not directly deploy idioms of metaphysical necessity and possibility is certainly plausible. However, we will argue, mathematics makes use of the counterfactual conditional, which in both ordinary and mathematical English is paradigmatically expressed by the subjunctive conditional construction

If . . . (then) - - - would _ _ _.[16]

The use of counterfactual conditionals is by no means a marginal feature of mathematical discourse. (We will later explain why it is not dispensable.) In fact, we will argue, the pattern of their deployment encodes a commitment to the necessity of all mathematical truths. These aspects of mathematical practice put the thesis of the necessity of mathematics on a firmer footing: if our story is correct, then challenging that thesis requires challenging the practice of mathematics itself.

Some foundational questions will remain open even if we are right. Grounding-lovers will still wish to inquire after what grounds the necessity of mathematics. We will not undertake to defend the claim that the necessity of mathematics is grounded in counterfactual facts. And epistemologists may wish to inquire after how mathematicians are justified in thinking and saying the things that, we argue, commit them to the necessity of mathematics. We will not address these further questions. Our ambitions are more modest.

In §1 we introduce our assumptions about the language of mathematics. In §2 we argue that mathematical practice is committed to the necessity of all mathematical truths in virtue of its commitment to the acceptability of a certain inference pattern involving counterfactuals. In §3 we argue that the modal commitments of mathematics extend even further than we found in §2: mathematics, it turns out, is committed to all theorems of the modal system **S5** that are expressible in mathematical language—**S5** being the system that is widely thought to capture the logic of metaphysical modality. In §4 we anticipate and respond to four objections.

1. The language of mathematics

By the 'language of mathematics' we mean the language of pure (i.e., not applied) mathematics that one finds in textbooks and professional journals. In what follows we are going to make some fairly modest assumptions about the logical constants that are present in that language. First, we will assume that the language has at least the standard truth-functional connectives, including $\bot$. ($\bot$ is the 0-place connective—i.e., sentence constant—that is a truth-functional contradiction.) Second, we assume that the language has the counterfactual conditional connective $\Box\!\!\rightarrow$.

Admittedly, our assumption about the presence of the standard truth-functional connectives in the language of mathematics is a little idealized. For example, the use of a primitive contradiction symbol is far from a pervasive feature of mathematical texts. Yet the

[16] Arguably the distinctive hallmark in English is the occurrence in the antecedent of the conditional of 'fake past tense', a layer of tense that has nothing to do with temporal past. See Iatridou (2000).

idealization is harmless enough. For example, one could define ⊥ as an abbreviation for some paradigmatic truth-functional contradiction. [17]

Similarly, while the language of mathematics doesn't contain a single expression with the logical type and meaning of □→, it does contain the resources for expressing everything that can be expressed by □→. Mathematics is rife with counterfactual conditionals, although these are often not in the standard form 'If … (then) - - - would _ _ _'. For example, in the canonical contemporary text on mathematical logic, we encounter the following sentence early on:

> Suppose there were a machine computing *t*. It would have some number *k* of states (Boolos, Burgess, and Jeffrey 2007: 41).[18]

A casual survey of indirect proofs in virtually any classic mathematical text yields many more examples. To pick an example virtually at random, here is one from a classic text on computability:

> THEOREM 6.1. *The set of all Gödel numbers of Turing machines Z, for which* $\Psi_Z(x)$ *is total, is not recursively enumerable.*
>
> PROOF. Let us designate the set of all such Gödel numbers by *R*, and let us suppose that *R* is recursively enumerable. Then, since $R \neq \varnothing$, there would be a recursive function $f(n)$ whose range is *R*.
>
> The function $U(\min_y T(f(n), x, y))$ would be total, and hence recursive. Hence $U(\min_y T(f(n), x, y)) + 1$ would be recursive. Hence, by the very definition of $f(n)$, there would be a number n_0 such that
>
> $$U(\min_y T(f(n), x, y)) + 1 = U(\min_y T(f(n_0), x, y)).$$
>
> Setting $x = n_0$ yields a contradiction (Davis 1958: Ch. 5, p 78).

Here are a few more examples lest the reader suspects that we have confined our search to mathematical logic and computability theory:

> **B** transforms the $s = \mathbf{B}{:}N_p$ Sylow *p*-groups conjugate to **B** under **G** among themselves, and as above it follows that $s \equiv 1\ (p)$. If there were another system of conjugate Sylow *p*-groups, then its members would be transformed into each other by **B** in systems of transitivity whose degree would be divisible by *p*. The system would therefore contain a number s_1, divisible by *p*, of Sylow *p*-groups; on the other hand we conclude for s_1, just as we did for *s*, that $s_1 \equiv 1\ (p)$ (Zassenhaus 1958: §IV.1).

> Conversely, assume that every Cauchy sequence in *S* has its limit in *S*. If *S* were not closed then its complement would not be open. Hence there would be a point $t \in \mathbb{R} \setminus S$ with the property that no interval $(t-\varepsilon, t+\varepsilon)$ lies in $\mathbb{R} \setminus S$. (Krantz 2016: §4.1).

> Clearly, *S* is subanalytic; if *S* were semianalytic, then there would be some real analytic function $f(x, y, z)$ defined near (0, 0, 0), not identical to zero, which vanishes on *S* (Krantz and Parks 2002: 181).

We assume that the common construction 'Suppose … . (Then) - - - would _ _ _' is simply a reader-friendly way of expressing a counterfactual conditional. Mathematical writing

[17] We note in passing that '0 = 1' is standardly used as the *falsum* in intuitionistic mathematics.

[18] Kratzer's (1986) analysis of this discourse would make 'would' a restricted modal operator, where the restriction is supplied by the preceding sentence together with the 'modal base' supplied by the context of speech. We find her treatment of the underlying logical forms of natural language counterfactuals plausible, but we are not going to rely on it.

often splits conditionals, both subjunctive and indicative, into two or more sentences, which makes it easier to parse conditionals with logically complex consequents and antecedents. ('Suppose *A*. Then *B*' is a standard way of stating a theorem. The theorem so stated is just this: if *A* then *B*.) One also often encounters in mathematical texts sentences whose main verb is 'would' with no relevant 'suppose' preceding it. In such cases, at least typically, there is either a preceding sentence that is meant to be understood as the antecedent of a counterfactual or some preceding sentences whose conjunction is meant to be so understood.

We take it that our assumptions about the language of mathematics are not especially tendentious. What has been overlooked in discussions of the modal status of mathematics is that these assumptions entail that that language has the resources for asserting the necessity of every proposition expressible in it. After all, the following definition of the metaphysical necessity operator $\Box$ falls out of both of the two standard semantics for counterfactuals (Stalnaker 1968 and Lewis 1973).

Definition 1. $\Box A =_{df} \neg A \,\Box\!\!\rightarrow \bot$

While the semantics of Lewis and Stalnaker provide a powerful motivation for Definition 1, a perhaps even more powerful motivation for it is supplied by a proof-theoretic observation due to Timothy Williamson. Williamson (2007: 155-58) observes that the material equivalence of the two sides of Definition 1 is derivable from the following two principles in an extremely weak modal logic.

NECESSITY: $\Box(A \rightarrow B) \rightarrow (A \,\Box\!\!\rightarrow B)$

POSSIBILITY: $(A \,\Box\!\!\rightarrow B) \rightarrow (\Diamond A \rightarrow \Diamond B)$

These principles are pretty compelling. NECESSITY says that strict implication implies counterfactual implication—or, in other words, that, whenever it is necessary that if *A* then *B*, it is also the case that, if it had been the case that *A*, then it would have been the case that *B*. Or, equivalently, using the suppositional idiom:

> Suppose that it is necessary that if *A* then *B*. Then, if it had been the case that *A*, it would have been the case that *B*.

POSSIBILITY says that anything counterfactually implied by a possible proposition is also possible—or, equivalently, in suppositional language:

> Suppose that if it were the case that *A* then it would be the case that *B*. Then it is possible that *B* only if it is possible that *A*.

The derivation of $\Box A \leftrightarrow (\neg A \,\Box\!\!\rightarrow \bot)$ from NECESSITY and POSSIBILITY requires nothing more than the weakest normal modal logic **K**. Given the validity of both NECESSITY and POSSIBILITY and the soundness of **K**, it follows that $\Box A$ and $(\neg A \,\Box\!\!\rightarrow \bot)$ are logically equivalent, wherefore we may treat the former as an abbreviation for the latter, as we do in Definition 1.

With NECESSITY and POSSIBILITY in place, the language of mathematics can thus be used to assert the necessity of any proposition it can express. (In §4, we will consider the status of each of NECESSITY and POSSIBILITY in more detail.) But it remains to be argued that mathematical practice is committed to the necessity of all mathematical truths. That is the task of the next two sections.

2. The necessity of mathematics

In this section we will argue that mathematics is committed to the necessity of every mathematical truth, in the sense that it is *provable in mathematics* that every mathematical statement is necessarily true if true. The notion of provability that we are working with is what philosophers of mathematics call 'informal provability' (or 'absolute provability'),[19] as opposed to the system-relative notion of 'formal provability', or provability in a given formal system. The basic idea is that a statement is informally provable just in case there is a proof of it in the sense of 'proof' operative in actual mathematical practice, as opposed to philosophers' formalizations of mathematical theories. We will use the symbol '$\vdash$' to express the relation of informal provability. Thus '$\Gamma \vdash A$' says that A is informally provable from the set Γ of statements; '$\vdash A$' abbreviates '$\varnothing \vdash A$' and says that A is informally provable from the empty set, i.e., informally provable *simpliciter*. Below we omit the set brackets, writing, e.g., '$A_1, \ldots, A_n \vdash B$' instead of '$\{A_1, \ldots, A_n\} \vdash B$'.

Two features of informal provability are important for our discussion. First, informal provability differs from formal provability in that it always preserves truth. As a special case, a statement that is informally provable *simpliciter* is true. In contrast, there are formal systems—ones with false axioms or unsound rules of inference—in which provability does not preserve truth or in which falsehoods are provable. (The system of Frege's *Grundgesetze der Arithmetik*, in which everything turned out to be provable, is a famous example.) Second, the informal provability of B from $A_1, \ldots, A_n$ does not imply that B is a logical consequence of $A_1, \ldots, A_n$. As a special case, the informal provability of A does not imply that A is a logical truth. (Of course, a neologicist may wish to claim that the informal provability of B from $A_1, \ldots, A_n$ implies that B is a logical consequence of $A_1, \ldots, A_n$ together with some analytic or conceptual truths.) An informal proof of a mathematical truth B typically consists in a logically valid argument for B from some assumptions $A_1, \ldots, A_n$, each of which is informally provable. When such a proof is given, we say that $\vdash B$, and not merely that $A_1, \ldots, A_n \vdash B$. Indeed, the mere existence of such a proof, whether or not anyone ever actually gives it, implies that $\vdash B$.[20] In what follows, we will for the most part use the word 'provability' and its cognates for informal provability and related notions, since no other varieties of provability will be at issue.[21]

The notion of informal provability is unclear in various ways that we'll leave for others to sort out. An especially serious source of unclarity is the relationship between our capacity to know certain mathematical truths and their informal provability. According to one perspective, we can know facts of pure mathematics that are not provable from its axioms, and

[19] See Gödel (1951) and Myhill (1960) for the classic discussions, and Leitgeb (2009) and Williamson (2016*a*) for two recent ones.

[20] Of course, 'provability' sounds modal, and one might wonder what kind of modality is in play. Related questions arise for such locutions as 'provable in principle' or 'possible in principle to prove'. The issues here are similar to those that arise in note 4. We will not assume any particular modal gloss or even take a stand on whether the modal idiom as used in this context is a throw-away.

[21] The referee was concerned that this notion of provability seems very different from the one operative in mathematics. One natural source of concern is the thought that mathematicians wouldn't apply 'provable' to their axioms. We are not inclined to take this as decisive evidence that their notion is different. Granted, mathematicians may routinely deny that any axiom is provable. However, it is not obvious how willing they would be, upon careful reflection, to give up the principle that anything provable from the axioms (which includes, obviously, the axioms themselves) is provable. Relatedly, we are highly confident that mathematicians would, on reflection want to say that anything provable from something provable is itself provable (this is a special case of Cut: see below). But if they say this and they allow conjunctions of axioms and theorems to be provable (or conjunctions of axioms to be), then they must also count axioms as provable. The reader who suspects that we are deploying a notion of provability different from that found in mathematical practice should read our '$\vdash A$' as 'A is either an axiom or is provable from the axioms', where 'axiom' and 'provable' mean whatever they mean as used by classical mathematicians.

so knowledge of a mathematical fact doesn't entail its provability. For example, if we know that ZFC is true, we also know that it is consistent, but the consistency of ZFC is not provable from any established axioms. According to another perspective, there is nothing epistemologically deep about the label 'axiom', and the best regimentation of informal provability is along the lines of 'there is a proof of *A* from mathematical premises that can be known'.[22] The latter tack is arguably a bit revisionary, but we're not going to take a position on it here. Another tack would be to say that both 'informal provability' and 'axiom' are context-dependent, with 'axiom' applying in each context to the truths of mathematics that the contextually relevant community of mathematicians treats as axioms. Fortunately, the structural principles of informal provability that will be assumed in our arguments do not depend on these issues, and so we will not pursue them further.[23]

We will now argue that mathematics is committed to the necessity of all mathematical truths. To establish this, we need only make some fairly modest assumptions about provability. First, we will assume that provability is classical in the following sense.

CLASSICALITY. (i) (Classical Consequence) $\Gamma \vdash A$ whenever A follows from Γ by classical logic.

(ii) (Deduction Theorem) If $\Gamma, A \vdash B$, then $\Gamma \vdash A \rightarrow B$.

(iii) (Modus Ponens) $\Gamma, A \Rightarrow B, A \vdash B$, where $\Rightarrow$ is any conditional.

(iv) (Cut) If $\Gamma \vdash A_1, \ldots, A_n$ and $\Pi, A_1, \ldots, A_n \vdash B$, then $\Gamma, \Pi \vdash B$.

In fact, we are making a stronger assumption than we need for our argument for mathematics' commitment to its own necessity: it will only use clauses (i) and (ii) of CLASSICALITY. (iii) and (iv) will only be used in the discussion of **S5** in §3.

We are aware, of course, that some mathematicians, operating within an intuitionistic framework, are explicitly committed to denying CLASSICALITY(i) (but not (ii), (iii), or (iv)). But it is classical mathematics that concerns us here, and we will not be discussing which of our results carry over to an intuitionistic setting.

Second, we will assume the counterfactual analogue of the Deduction Theorem:

COUNTERFACTUAL DEDUCTION.

If $\Gamma, A \vdash B$, then $\Gamma \vdash A \mathrel{\Box\!\!\rightarrow} B$.

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[22] See Williamson (2016*a*) for discussion.

[23] Another tricky issue that we'll leave for others to sort out concerns self-referential sentences. If the diagonal lemma holds for informal provability, as it perhaps will if we express informal provability, as we have, by a metalinguistic predicate, we face some difficult decisions: a rich enough theory that includes, for some predicate *P*, all instances of $P(\ulcorner A \urcorner) \rightarrow A$ and is closed under the rule

$$\frac{A}{P(\ulcorner A \urcorner)}$$

is inconsistent. (See Löb 1955 and Myhill 1960: 463; see Montague 1963 for more general results.) $\vdash$ had better not be such a predicate in our theory of informal provability, then. The simplest solution here is to follow Montague's suggestion along with the tradition of provability logic going back to the ur-text (Gödel 1933) and *not* express informal provability by a metalinguistic predicate, opting for an 'it is provable that' operator instead, but this is awkward because the bearers of provability are more naturally thought of as sentences (as in Myhill 1960) than their contents. Those who prefer the predicate approach may wish to adapt Tarski's approach to the liar paradox by introducing a hierarchy of provability predicates, or they may deny that $\vdash$ belongs to the language of mathematics at all, or reject the rule $A / \vdash \ulcorner A \urcorner$, among other options.

To argue for COUNTERFACTUAL DEDUCTION we cannot argue that it is a principle of *logic*, because it isn't. (The principle does not hold when '$\vdash$' is interpreted as expressing provability in any standard system of counterfactual logic.) Rather, to argue for COUNTERFACTUAL DEDUCTION we must argue that mathematical practice displays a commitment to it. But there is plenty of evidence of such a commitment. In mathematics it is commonplace, having supposed A_n, in addition to any assumptions $A_1, \ldots, A_{n-1}$ one has previously made, to conclude that B would be true (if A_n were true) on the basis of a recognition that B is provable from $A_1, \ldots, A_n$. In such cases, the antecedent of the counterfactual is often left implicit. Consider this example from the passage we already quoted in Davis's *Computability and Unsolvability*:

> Let us designate the set of all such Gödel numbers by R, and let us suppose that R is recursively enumerable. Then, since $R \neq \varnothing$, there would exist a recursive function $f(n)$ whose range is R. (Davis 1958: Ch. 5, p. 78).

Here Davis supposes that a certain set R is recursively enumerable, recognizes that the existence of a recursive function whose range is R is provable from that supposition and his earlier assumptions, and concludes that there would be such a function. The implicit antecedent is that R is recursively enumerable.

Moreover, the practice of using counterfactuals in proofs by *reductio* is hard to make sense of without attributing a commitment to COUNTERFACTUAL DEDUCTION to mathematicians. Consider, for example, the standard (Euclid-inspired) style of *reductio* of the hypothesis that there is a largest prime number.[24] It is natural to put this in counterfactual terms, and indeed it is often put that way both in informal presentations and in textbooks: One begins by supposing that there is a largest prime number, and one deduces further claims from that claim along with certain established truths, eventually concluding that, if there were a largest prime number, then the successor of the product of all of the primes would be both prime and composite (and thus not prime). The reasoning cannot easily be reconstructed without fitting it into the mould of COUNTERFACTUAL DEDUCTION, so that Γ is the set of the established axioms of number theory, A is the hypothesis that there is a largest prime, and B the claim that the successor of the product of all of the primes would be both prime and composite.

Of course, like any complex social practice, the practice of mathematical proof and exposition is not completely uniform, and we can find within it some sub-practices that, when considered in isolation, will seem incompatible with a commitment to COUNTERFACTUAL DEDUCTION. However, we think that those sub-practices are both sufficiently marginal and sufficiently easy to explain by appeal to general conditional heuristics that, on balance, they do not justify withdrawing the attribution of a commitment to COUNTERFACTUAL DEDUCTION to mathematical practice: we will discuss this issue further in §4.4.

Here is our argument. First, by Classical Consequence,

$$A, \neg A \vdash \bot.$$

By COUNTERFACTUAL DEDUCTION, then

$$A \vdash \neg A \mathrel{\Box\!\!\rightarrow} \bot,$$

which, by Definition 1, is none other than

[24] Williamson (2017*b*) places special emphasis on the role of counterfactuals in proofs by *reductio* using this example.

$A \vdash \Box A$.

By the Deduction Theorem, then

($\Box$) $\vdash A \rightarrow \Box A$.

($\Box$) implies that all mathematical truths are necessarily true if true, so it implies that all mathematical truths are necessary. But ($\Box$) says something even stronger than that: it says that it is *provable* that all mathematical truths are necessary. (Because provability implies truth, the provability of the necessity of all mathematical truths implies the necessity of all mathematical truths.)

It is significant that our conclusion is that $\vdash A \rightarrow \Box A$ and not merely that $A \rightarrow \Box A$. The conclusion is that mathematics is committed to the necessity of its truths in the precise sense that the necessity of mathematical truths is itself provable *in* mathematics. There is no epistemic division of labor, then. Philosophers may still wish to debate the modal status of mathematical statements, but in doing so they are calling into question a commitment of mathematics itself.

3. Metaphysical modal logic within mathematics

How deep do the modal commitments of mathematics run? So far we have argued against views that posit an epistemic division of labor in which mathematics supplies the truths and philosophy supplies their necessity. We have argued that the necessity of all mathematical truths is provable within mathematics itself, by the standard of proof operative in mathematics. This still leaves room for a kind of epistemic division of labor: one might think that, while mathematics supplies both the truths and their necessity, it stops short of telling us anything further about necessity. In the new division of epistemic labor, it is up to philosophers to supply any further truths about the modal status of mathematical statements, such as that any mathematical statement that is necessarily true is necessarily necessarily true. According to this picture, to put it in a slogan, mathematics supplies both the truths and their necessity, and philosophy supplies the logic of their necessity.

In fact, it already follows from what has been said that this picture cannot be correct. If we are right, mathematical practice turns out to be highly opinionated about the application of principles of modal logic to mathematics. By replacing 'A' in ($\Box$) with '$\Box A$' and with '$\Diamond A$' we get

4: $\vdash \Box A \rightarrow \Box\Box A$

and

5: $\vdash \Diamond A \rightarrow \Box\Diamond A$.

Classical Consequence, the Deduction Theorem, Modus Ponens, and Cut imply that $\vdash (\neg A \Box\!\!\rightarrow \bot) \rightarrow A$,[25] so, by Definition 1,

T: $\vdash \Box A \rightarrow A$.

[25] By Modus Ponens, $\neg A \Box\!\!\rightarrow \bot, \neg A \vdash \bot$. By the Deduction Theorem, $\neg A \Box\!\!\rightarrow \bot, \vdash \neg A \rightarrow \bot$. By Classical Consequence and Cut, then, $\neg A \Box\!\!\rightarrow \bot, \vdash A$, and by the Deduction Theorem, $\vdash (\neg A \Box\!\!\rightarrow \bot) \rightarrow A$.

It should come as no surprise that the **K** axiom $\Box(A \to B) \to (\Box A \to \Box B)$ is also provable in mathematics.[26] Finally, provability in mathematics obeys the principle of *Necessitation*:

If $\vdash A$ then $\vdash \Box A$.

After all, by Cut, whatever is provable from something provable is itself provable, and on the way to proving ($\Box$) we proved that $A \vdash \Box A$.

The previous paragraph's observations amount to just this:

*Every theorem of the modal system **S5** that is expressible in the language of mathematics is provable in mathematics*.[27]

In effect, mathematics contains within itself the system **S5**. Since philosophical consensus holds that the logic of metaphysical modality is **S5**, this means that mathematics is at least as opinionated as philosophical consensus about the application of principles of modal logic to mathematics. Indeed, mathematics has locked on to the characteristic feature of metaphysical modality, which is the collapse of all iterated modalities: in **S5**, $\Box A$ is equivalent to $(\Box/\Diamond)\Box A$ and $\Diamond A$ is equivalent to $(\Box/\Diamond)\Diamond A$, where $(\Box/\Diamond)$ is any finite string of boxes and diamonds. From ($\Box$) and the above observations it follows that:

$\vdash A \to (\Box/\Diamond)A$.

That is to say, it is provable in mathematics that every mathematical truth is necessarily necessarily true, necessarily possibly true, necessarily necessarily necessarily true, and so on for all finite sequences of 'necessarily's and 'possibly's. There is no epistemic division of labor even of the modest kind envisaged at the beginning of this section.

Might mathematics be *more* modally opinionated than philosophical consensus? Recall our key finding:

($\Box$) $\vdash A \to \Box A$.

The result of adding ($\Box$) to **S5** is the modal system **TRIV**, in which *all* modal distinctions collapse: A is equivalent to $(\Box/\Diamond)A$. At first this might look like bad news for our project, but it is actually the result we want. Recall that '$\vdash$' expresses provability in mathematics—by which we mean *pure mathematics*. $\Gamma \vdash A$ only if both A and all of the statements in Γ are pure mathematical statements. All modal distinctions do collapse for non-contingent statements, and all mathematical statements are non-contingent. Since philosophical consensus holds that all mathematical statements are non-contingent, mathematics is exactly as opinionated as philosophical consensus on the application of principles of modal logic to mathematics. How do we know that mathematics is exactly as opinionated and not *more* opinionated than philosophical consensus on this matter? Because every system of propositional modal logic stronger than **TRIV** is inconsistent,[28] and mathematics is not inconsistent.

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[26] By Classical Consequence,

$(A \to B) \leftrightarrow \Box(A \to B), A \leftrightarrow \Box A, B \leftrightarrow \Box B \vdash \Box(A \to B) \to (\Box A \to \Box B)$

By **T**, ($\Box$), Classical Consequence, and Cut, we have $\vdash (A \to B) \leftrightarrow \Box(A \to B)$, $A \leftrightarrow \Box A$, $B \leftrightarrow \Box B$, so, by Cut, $\vdash \Box(A \to B) \to (\Box A \to \Box B)$.

[27] **4** is a redundant step on the way to this conclusion. It is a theorem, not an axiom, of standard axiomatizations of **S5**.

[28] See Hughes and Cresswell (1996: 67).

Here it may be worth re-emphasizing that $\vdash A$ does not imply that A is a *logical* truth.[29] That all **TRIV** theorems that are expressible in the language of mathematics are provable in mathematics does not indicate a mathematical commitment to **TRIV** as a logic, but only to the truth of those theorems.

4. Objections and replies

We shall now consider four objections to the foregoing.

4.1. False counterpossibles?

The first objection concerns Definition 1, which entails that all counterfactuals with metaphysically impossible antecedents—so-called *counterpossibles*—are true.[30] Many philosophers maintain that there are false counterpossibles,[31] and they will accordingly reject Definition 1. Naturally they will also reject Williamson's argument for the logical equivalence of the two sides of Definition 1. We ourselves think that there are good reasons to think that all counterpossibles are true.[32] However, we need not assume either that principle or Definition 1, which entails it, in order to establish (□). Our arguments for (□) relied on only one direction of the putative definitional equivalence of $\Box A$ and ($\neg A$ □→ $\bot$): the entailment of $\Box A$ by ($\neg A$ □→ $\bot$). And that direction is typically not questioned by philosophers who think there are false counterpossibles. Such philosophers typically reject NECESSITY while accepting POSSIBILITY, from which ($\neg A$ □→ $\bot$) $\rightarrow \Box A$ is derivable in **K**.[33] To make vivid why rejecting the entailment of $\Box A$ by ($\neg A$ □→ $\bot$) would be a bad idea, consider the issue in terms of possible worlds semantics. In any standard 'possible worlds' semantics, the existence of a counterexample to the entailment of $\Box A$ by $\neg A$ □→ $\bot$ would require some impossible world to be closer to the

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[29] Even on a neologicist approach, not everything provable in mathematics is a logical truth, since neologicists draw a distinction between logical and merely 'analytic' or 'conceptual' truth.

[30] Here is a proof. Suppose for a *reductio* that there is a false counterpossible. Then, for some A and B, $\Box\neg A$ is true and A □→ B is false. Then, by Definition 1, and Classical Substitution, A □→ $\bot$ is true. But if A □→ $\bot$ is true then so is A □→ C, for any C. (Everything is a truth-functional consequence of $\bot$, and any truth-functional consequence of a proposition counterfactually implied by A is also counterfactually implied by A.) So, in particular, A □→ B is true, contrary to hypothesis.

[31] E.g., Lowe (2012), Brogaard and Salerno (2013), and Berto et al. (2018).

[32] See Williamson (2010, 2016*b*, 2017*b*).

[33] See Strohminger and Yli-Vakkuri (2017: §4) for review. One recent representative example is Berto et al.'s (2018) development a logic that allows counterpossibles to be false and validates POSSIBILITY. Similarly, Lowe (2012) maintains that there are false counterpossibles and accepts POSSIBILITY, with, however, less promising results (see Strohminger and Yli-Vakkuri 2017: §4 for criticisms).

actual world than some possible world. But it's extremely plausible that every possible world is closer to the actual world than any impossible world.[34],[35],[36]

4.2. Are mathematical counterfactuals dispensable?

The second objection proceeds from a thesis we'll call DISPENSABILITY. DISPENSABILITY says that any counterfactuals in pure mathematics are dispensable in that they contribute nothing to the *content* of mathematics, in the sense that what mathematics contributes to our total body of knowledge it would still contribute (albeit perhaps in a less reader-friendly way) if its counterfactuals were replaced by indicative or (if these are different) material conditionals.[37] DISPENSABILITY is consistent with, and can be motivated by, a variety of views about the role of counterfactual conditionals in mathematics. For example, an advocate of DISPENSABILITY might think that there are, contrary to appearances, no counterfactuals in mathematical texts, and that the subjunctive conditional construction has a non-standard semantics in mathematical contexts. Or he might think that mathematicians are simply being careless when they use subjunctive conditionals, and that what they really mean to express by these conditionals, when push comes to shove, are the corresponding indicative or material conditionals. Or, rather more plausibly, she might concede that, as in the case of epistemic modals, there are good reasons for the occurrence of counterfactuals in mathematical texts, and that they are even indispensable for some (e.g., pragmatic) purposes, but, like epistemic modals, counterfactuals are nevertheless dispensable to what mathematics contributes to our knowledge. According to this perspective, we are making the same kind of mistake that would be made by someone who takes pure mathematics to be committed to various epistemic claims on account of the ubiquity of epistemic modals in mathematical texts. Clearly that would be a mistake: whatever role

[34] Clarke-Doane (2017) is a notable exception to the mainstream assumption that every possible world is closer to the actual world than any impossible world. (He thus, in effect, questions POSSIBILITY: Suppose A is possible and B is not, and that the closest A-world is an impossible world where A and B are true. Then, on the standard semantics, $A \mathrel{\Box\!\!\rightarrow} B$ is true but $\Diamond A \rightarrow \Diamond B$ is not.) One of his main examples is 'If Obama had been named 'Alice', then Obama would not have been female'. He thinks there are contexts in which this is false, even on the assumption that Obama is necessarily not female. If he is right in his judgment about this counterfactual, then it would seem that we need some impossible world to do the job of witnessing the falsity of the counterfactual at the relevant contexts, which on both the standard semantics (Stalnaker's and Lewis's) would require it to be closer to the actual world than the possible worlds in which Obama is not female. We think the example is not optimal because it is far from clear that Obama is necessarily non-female. Counterfactuals like 'If 'Boston' had referred to Obama, he would not have been a city' and 'If 'The Empire State Building' had referred to Obama, then he would not have been a skyscraper' seem like better test cases to us, and our judgment is that they are clearly true in every context. However, considerations of space prevent us from engaging in full with Clarke-Doane here. (See Nolan (1997) for further discussion.)

[35] A subtle complication arises when we retreat to versions of our arguments that do not rely on Definition 1, for then we can no longer assume that the language of mathematics can express the necessity of any mathematical statement: without Definition 1, we have no argument that $\Box A$ is in the language of mathematics whenever A is. To derive ($\Box$) we must either consider a mathematical language enriched with $\Box$ or interpret '$A_1, \ldots, A_n \vdash B$' to mean something like 'For some $C_1, \ldots, C_n$, each of $C_1, \ldots, C_n$ is informally provable from $A_1, \ldots, A_n$ and B is formally provable in **K** from $C_1, \ldots, C_n$ and POSSIBILITY'. The second option strikes us as particularly attractive. It still delivers a result that displays the commitment of mathematics to the necessity of provable mathematical truths. Since **K** is sound and POSSIBILITY is valid, the result becomes that mathematics is committed to the necessity of its truths in the sense that, whatever mathematical statement A may be, $A \rightarrow \Box A$ logically follows from statements provable in mathematics.

[36] Granted, certain proponents of the view that there are false counterpossibles may take the further radical step of rejecting COUNTERFACTUAL DEDUCTION. We are not going to engage with such radicalism, except to remind the reader that we have argued that COUNTERFACTUAL DEDUCTION is a part of the practice of mathematics, and so, if we are right, the radical step is tantamount to revisionism about that practice.

[37] Quine was a prominent defender of this view. In fact, he held the even stronger view that counterfactuals were dispensable to all of science. See, e.g., Quine (1994: 149-50).

epistemic modals play in those texts, mathematics is not in the business of teaching us anything about knowledge: the actual *axioms*, *proofs*, and *results* of mathematics, when strictly and literally stated, do not concern knowledge, no matter what epistemic language one finds in standard mathematical texts. Similarly, the objection goes, those same axioms, proofs, and results, strictly and literally stated, do not concern counterfactual matters,[38] no matter what counterfactual-sounding language one finds in mathematical texts.

A great deal could be said about possible motivations for DISPENSABILITY (we find the hypotheses of carelessness and special semantics to be especially implausible[39]), but there is no need to engage with its motivations when we can attack DISPENSABILITY directly, and we will.

In fact, counterfactuals are absolutely indispensable to what mathematics contributes to our total body of knowledge—which is to say, DISPENSABILITY is false. Note first that myriad applications of mathematics to the hustle and bustle of both everyday life and engineering require our knowing that mathematical truths would remain true even if things had gone differently in various ways. For example, in justifying a particular engineering solution, one often appeals to mathematical truths in reasoning about how things would have gone if one had opted for an alternative solution. In doing so one assumes—and if one is successful, one knows—that those mathematical truths would have been true even if one had opted for the alternative solution. Note second that, as the queen of the sciences, mathematics is primed for application in any area of objective inquiry, whether it be the science of electromagnetism, the theory of rook and pawn endings, or natural language semantics. Each of these disciplines deploys its own counterfactuals, and in applying mathematics to them one must know that the truths of mathematics would remain true also under their counterfactual suppositions. Notably such counterfactual suppositions often include ones that are *nomologically* impossible and yet are not treated as ones from which anything whatsoever follows. For example, when doing physics, we are perfectly happy to hold the truths of mathematics fixed when suppositionally reasoning about the behavior of particles under various permutations of the standard model, and our comfort level does not at all diminish when reasoning about models that we take to be nomologically impossible. To get a good feel for the contrast between mathematics and physics here, consider the theory of Turing machines or the theory of infinite games. When one engages in counterfactual suppositional reasoning in these areas, one does not hold the laws of physics fixed under one's suppositions—here it is simply irrelevant whether infinite tapes or eternal games of chess are physically possible—but one does hold the truths of mathematics fixed.[40] The success of our practice of applying mathematics to anything whatsoever requires that we know that mathematical truths would remain true under any counterfactual suppositions whatsoever[41]—which is to say, it requires that we know that mathematical truths would remain true no matter what, or, equivalently, that they are necessary truths.

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[38] As in the case of knowledge, one might think that this claim is undermined by the existence of counterfactual logic as a *bona fide* area of mathematical inquiry, but, in any case, it would be hard to argue from mathematical commitment to some kind of counterfactual logic to the claims we wish to argue for. After all, our claims concern informal provability, and our premises include claims about counterfactuals that are not principles of any reasonable counterfactual logic.

[39] The short story is that the hypothesis of carelessness ascribes implausible deficits in semantic processing to sophisticated mathematicians, and the hypothesis of special semantics is supported by no linguistic data. And it is no more plausible to suggest that counterfactuals in mathematics are being used non-literally.

[40] Of course we could decide to not hold some particular mathematical facts fixed in some of our mathematical counterfactual reasoning, and plausibly we sometimes do when we are, e.g., trying to determine whether a certain axiom is needed for proving a certain theorem. But such contexts are special, and the unexpected uses of counterfactuals in them can be explained as applications of a general conditional heuristic: for more, see §4.4.

[41] This sentence contains our reply to Gideon Rosen's (2002) parable of the two tribes who disagree about whether it is metaphysically contingent that there are numbers. Rosen finds it hard to tell which tribe is right. Our view is that the 'modally deviant' tribe (to use his term) would have a hard time applying mathematics as widely as we

4.3. Are mathematical counterfactuals restricted?

The third objection turns on the point that, for any restricted objective necessity □′, one can define an associated 'restricted counterfactual' □→′ that obeys the principle

$$\Box' A \leftrightarrow (\neg A \;\Box\!\!\rightarrow' \bot).$$

Even granting that mathematics deploys counterfactual discourse, it may be suggested that the counterfactuals in play are restricted in some way, and consequently mathematics is not committed to the necessity *simpliciter* of mathematical truths, but only to their necessity in a correspondingly restricted sense. By analogy, suppose you are playing craps and you throw a red die and a green die, and the first comes up six and the second three. It is then natural for you to say: 'If the green die had come up six, I would have had boxcars'. It's pretty clear here that you are somehow restricting the domain of the possibilities your counterfactual generalizes over to ones in which the red die comes up six. (Without this restriction, what you say may well be false: perhaps, for all you know, if the green die had come up six, it would have done so by bumping into the red die and causing it to come up one.) One might suspect that mathematical counterfactuals involve a similar implicit restriction to (e.g.) possibilities in which the actual truths of mathematics are true. And if that is right, then mathematical counterfactuals be unimpugned by possibilities in which the truths of mathematics are different from what they actually are, and facts like (□) would not, after all, manifest a mathematical commitment to the necessity of mathematical truths, but only to their truth. (One can imagine various other versions of the view that mathematical counterfactuals are restricted; any of them would undermine our argument. We only offer this one as an illustration.)

One important concern about this proposal is that employing systematically restricted counterfactuals can severely hinder inquiry. To take an extreme example, consider a community that lives in a world in which, as a matter of dumb luck, no explosions ever occur, even though, on many occasions, an explosion could easily have occurred. (For example, on many occasions, matches get lit near barrels of gunpowder but, by sheer luck, the gunpowder never ignites.) Suppose that the community systematically restricts its counterfactuals to explosion-free worlds. Thus (roughly) a counterfactual asserted by a member of this community is true just in case its consequent is true in all of the explosion-free worlds closest to their world in which its antecedent is true. Owing to this quirky restriction, a community member speaks the truth in asserting: 'If I were to set fire to that gunpowder, nothing bad would happen'. This way of using counterfactuals is hardly conducive to survival, as it is insensitive to all sorts of clear and present dangers. It achieves truth at the cost of blindness to danger. Now suppose that we live in a world in which mathematics is contingent, but we systematically restrict our counterfactuals to worlds in which the actual truths of mathematics hold. Even if this doesn't make us quite as reckless with gunpowder as the imagined community, certain other kinds of blindness to danger may threaten. For example, suppose that God's decision to create numbers is contingent on the amount of evil in the world, and suppose that we only just came in under the bar required for numerical creation. Thanks to the way our counterfactuals are restricted, a member of our community speaks the truth in asserting: 'If I had murdered someone, there would have still been numbers' (since, in the restricted domain, the closest worlds where the speaker murders someone are ones where there is a compensating absence of evil). But from God's point of view this restriction makes us blind to an important structural

do. And if they did apply mathematics as widely, they would be in the awkward position of not knowing that various of their applications were correct.

feature of reality, namely that the existence of numbers (unrestrictedly) counterfactually depends on the amount of evil in the world.

We find it completely implausible that, when we routinely and successfully deploy mathematics in everyday life, engineering, and physics and other fields of objective inquiry, we are hampered by a blindness of the sort just described. By our lights, by far the most natural explanation of the success of applied mathematics in promoting engineering and objective inquiry is that the counterfactuals of pure mathematics are completely unrestricted.

Of course this is not the only conceivable explanation. There are various theoretical posits that would allow the various applications of mathematics to proceed profitably and smoothly even if mathematical counterfactuals were restricted.[42] Perhaps least implausibly, one might posit that, while mathematics is contingent, the worlds in which different mathematical truths obtain are so distant (in the counterfactual sense of 'distant') from the actual world that restricting our mathematical counterfactuals to worlds in which the actual mathematical truths obtain doesn't blind us to any possibilities that matter in everyday life, engineering, physics, theology, and so on.[43]

We agree that it is natural enough for a believer in the contingency of mathematics to conjecture that the worlds where mathematics is different are very distant from the actual world. However, there is a further problem. It is still poor form to restrict one's counterfactuals so that in one's chosen theoretical language that conjecture cannot be tested against reality. By analogy, it is a plausible conjecture that all humans are under ten feet tall. Suppose now that we adopt a language in which we define 'human' so as to ensure that nothing at least ten feet tall belongs to its extension. In the new language it simply doesn't matter whether the conjecture expressed in the old language is true: in the new language 'All humans are under ten feet tall' comes out true regardless of the truth value of the hypothesis it expressed in the old language. (Similarly, 'All humans are under ten feet tall' will get positive epistemic status—knowledge, justification, and what have you—on the cheap. Suppose that the conjecture expressed by the sentence in the old language was true but not known. In the new language the sentence expresses a known fact, and the fact that the original conjecture was not known is now hidden from view.) It is not a good idea to gerrymander one's language in this way. Other things being equal, it is better to stick with the old language and let the conjecture actually expressed by 'All humans are under ten feet tall' run the gauntlet of empirical fortune.

Similar remarks hold for the counterfactual case. Consider the counterfactual

(Chippy) If we had bought fish and chips, the axiom of choice would have still been true.

Consider now the conjecture that mathematics is contingent but mathematics is different only in very distant worlds. Assuming that conjecture and the truth of the axiom of choice, (Chippy) will be true—mathematical truths are not counterfactually dependent on what we choose to have for dinner. Still, it would be poor form to restrict our counterfactuals in such a

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[42] We note that there is a minority of philosophers who think that all objective modal operators are inevitably restricted. These are philosophers who think, for example, that for any objective necessity there is an even broader one (see, notably, Clarke-Doane 2017). A potentially useful analogy is with those who disavow absolutely unrestricted quantification and argue that, for any universal quantifier, there is an even broader universal quantifier. The analogy is particularly apt if we think of objective modal operators as quantifiers over domains of worlds. It is beyond the scope of this paper to engage with modal applications of the idea of indefinite extensibility, but see Williamson (2003) for an overview of the problems with the view as applied to standard quantifiers. The idea that metaphysical necessity is indefinitely extensible does not by itself entail the contingency of mathematics (mathematical truths might remain true no matter how far we broaden our domain of worlds), but Clarke-Doane's development of it is favorable to a contingentist approach to mathematics.

[43] See Clarke-Doane (2017) for a discussion of some of the relevant issues.

way that (Chippy) cannot be tested against reality, and in particular in such a way that the truth of (Chippy) is independent of the truth of the conjecture expressed in the original language. (Similarly, such a restriction ensures that (Chippy) is known—so long as we are appropriately sensitive to the restriction in play—whether or not the hypothesis (Chippy) expressed in the original language was known.) Once again, it is best to let the counterfactual run the gauntlet of empirical fortune.

One possible reaction to this is to opt for a divided treatment of pure and applied mathematics. In particular, one might opt for an approach according to which the counterfactuals of pure mathematics are *de jure* restricted to worlds in which the actual mathematical truths are true while the counterfactuals of applied mathematics are unrestricted. Applied mathematics would then run the gauntlet of empirical fortune in the required way. But the problem with the divided treatment of counterfactuals is that it makes a mystery of how knowledge of pure mathematics can readily be parlayed into knowledge of applied mathematics. To illustrate, here is a useful counterfactual of pure mathematics:

> If it were the case that, for some plane separated into contiguous regions, there is no function f from the regions to a four-membered pure set such that, for all pairs x, y of contiguous regions $f(x) \neq f(y)$, then it would be the case that $\bot$.

(More briefly put: if the four color theorem were false then it would be the case that $\bot$.) On the proposal under consideration, the counterfactual is true whether or not mathematics varies at close worlds. But then, on that proposal, the inference from that counterfactual to: 'If we were to draw a map, then the four color theorem would be true' would be invalid, as would various other routine inferences from counterfactuals of pure mathematics to counterfactuals of applied mathematics. Taken at face value, such inferences would equivocate on the counterfactual, which occurs restricted in the premises and unrestricted in the conclusion. In order to account for our ability to extend our knowledge by such inferences, we would have to reconstruct them as enthymemes, and it is unclear what the missing premises would be. Somehow, for example, we would need to reconstruct the inference from a pure and restricted counterfactual $\neg A \;\Box\!\!\rightarrow' \bot$ to an applied and unrestricted counterfactual $B \;\Box\!\!\rightarrow A$ as valid when the closest worlds where B is true are not very distant and as invalid when they are. While this view may not be completely disastrous, it does seem to be out of step with how we naturally think of the relationship between pure and applied mathematics. Inferences from pure to applied mathematics appear to be much more straightforward than they are on the view under consideration. On the view that the counterfactuals of pure mathematics are completely unrestricted, those inferences are very straightforward. On the split treatment under consideration, they are very far from straightforward. We submit that this counts substantially in favor of our approach.

In summary, there are two problems for restricting counterfactuals. One is that it potentially blinds us to possibilities that matter for various applications of mathematics. This worry can be dealt with by claiming that the worlds where mathematics is different are very distant. But there is a second worry, one that applies whether or not one thinks such worlds are distant: it makes one's counterfactuals strangely independent of the truth of such conjectures as that variant mathematical worlds are distant, since the distance of such worlds has no bearing on the truth conditions of counterfactuals in the restricted language. One might try to address this concern by a mixed strategy that treats only pure mathematics as relevantly restricted. But this considerably complicates the relationship between pure and applied mathematics.

4.4. Denials of counterfactuals

The fourth objection needs especially careful attention.[44] It is time to admit that the use of counterfactuals in mathematics is not quite as uniform as we have been pretending. As a point of comparison, it may be helpful to first get clear on which patterns of assertions and denials of counterfactuals we would expect to find in a community of mathematicians—call them the *Boxers*—that explicitly treats it as common ground that all mathematical truths are necessary and explicitly adopts and reasons in accordance with a logic that classifies all counterpossibles as true. We will then see that actual mathematicians' patterns of assertions and denials of counterfactuals differ somewhat from the patterns we should expect to find in the Boxer community.

Let us contrast two kinds of conversational contexts for the use of mathematical counterfactuals. In a *consensus context* the relevant axioms are taken for granted, it is common ground that they are being taken for granted, and no one is interested in challenging any of the axioms or in exploring the ramifications of giving up some but not all of the axioms, as a means, for example, for determining how strong a system of axioms one needs to prove a certain theorem. A *non-consensus* context is a context that is not a consensus context. In a non-consensus context one is not entitled to assume that all of the axioms are true and hence also not entitled to assume that they are provable, since provability entails truth.[45]

Consider a counterfactual $A \mathrel{\Box\!\!\rightarrow} B$ in a consensus context in which A is known to be disprovable from the axioms. In such a context, a Boxer would assert $A \mathrel{\Box\!\!\rightarrow} B$ only for pedagogical purposes, so that the counterfactual either is or is asserted as step towards establishing $A \mathrel{\Box\!\!\rightarrow} \bot$ (or another counterfactual with an absurd consequent), and thus $\neg A$. For example, if A is the claim that there is a largest prime number, the point, if any, of a Boxer's assertion of $A \mathrel{\Box\!\!\rightarrow} B$ will be to contribute to an explanation of why there is no largest prime number. In developing the counterfactual supposition that there is a largest prime number, a Boxer, believing that they hold under any counterfactual supposition whatever, never suspends judgment in the Peano axioms (from which the fundamental theorem of arithmetic is provable) but makes full use of them in order to extract absurd conclusions. A Boxer would not be tempted by the following line of thought.

> 'Granted, the Peano axioms are all true, but perhaps they are only contingently true, and perhaps, if there had been a largest prime, only some of the Peano axioms would have been true, and the fundamental theorem of arithmetic would have been false.'

Now consider a counterfactual $\neg A \mathrel{\Box\!\!\rightarrow} \bot$ in a non-consensus context in which A is one of the contested axioms—the axiom of choice (AC), for example. In such a context, a Boxer would refrain from asserting $\neg A \mathrel{\Box\!\!\rightarrow} \bot$, even if he accepted A (and therefore, being a Boxer, accepted $\neg A \mathrel{\Box\!\!\rightarrow} \bot$ too). But the Boxer would not be equally reluctant to assert every counterfactual whose antecedent is $\neg A$. After all, $\neg A$, together with the axioms on which there is a consensus, can still be counterfactually developed to yield a variety of conclusions. For example, the Boxer might still assert:

[44] We thank an anonymous referee for bringing this objection to our attention.

[45] The referee suggests that, as a general rule, we hold the axioms fixed when we counterfactually suppose the negation of a theorem. We don't think this is a general rule; rather, whether we hold axioms fixed when we counterfactually suppose the negation of a theorem depends on whether we are in a consensus or a non-consensus context. For example, in a consensus context, a counterfactual that begins 'If all sets of reals were Lebesgue measurable, …' would naturally occur at the beginning of a *reductio* in which one holds AC fixed in order to arrive at a contradiction. On the other hand, in a non-consensus context the same antecedent might figure in a counterfactual along the lines: 'If all sets of reals were Lebesgue measurable, then AC would be false'.

(TB) If AC were false, then the Tarski-Banach theorem would not be provable from the truths of set theory.

Similar points hold for other antecedents that the Boxers know to be false on the supposition of the falsehood of AC. For example, a Boxer who suspects that AC is true and knows that, if AC is true, then it is not the case that all sets of reals are Lebesgue measurable, will also suspect that 'If all sets of reals were Lebesgue measurable, then ⊥' is true, but such a Boxer will not assert this counterfactual because AC is not part of the common ground. However, the Boxer would not be at all reluctant to assert:

(LM) If all sets of reals were Lebesgue measurable, then AC would be false (and we would be living in a Solovay model).[46]

The Boxer can reason as follows: 'There are two subcases. If AC is true (and thus necessarily true), then (LM) has an impossible antecedent and gets to be true on the cheap. And if AC is false, then (LM)'s antecedent is possible and indeed the closest world where its antecedent is true is the actual world, where its consequent is true. Either way, (LM) is true.' The fact that, for all the Boxer knows, the counterfactuals

(TB′) If AC were false, then the Tarski-Banach theorem would be provable from the truths of set theory

(LM′) If all sets of reals were Lebesgue measurable, then AC would be true

are also true does not in any way hinder the assertion of (TB) or (LM). Now, of course, if the Boxer accepts AC, he will also accept the (TB′) and (LM′), but, this being a non-consensus context, he will not assert either (TB′) or (LM′). What the boxer won't do is *deny* any mathematical counterfactual whose antecedent is, for all he knows, false. This is because, any such counterfactual, for all he knows, has an impossible antecedent and is vacuously true.

Let us now turn to actual mathematicians. As we see it, they are a pretty good approximation to the Boxers, but with one significant exception. When it comes to consensus contexts, it seems to us that actual mathematicians proceed just like the Boxers in their use of counterfactuals. For example, they manifest little temptation to reject claims like 'If there were a largest prime, then the successor of the product of all primes would be both prime and composite'—a temptation that one would have if one tacitly regarded the truths of number theory as contingent. Like the Boxers, in consensus contexts they only assert counterfactuals that begin 'If there were a largest prime…' for pedagogical purposes. In non-consensus contexts, however, we see a departure from the Boxer practice. The departure concerns neither those counterfactuals that the Boxers would assert nor those that they would not assert. In non-consensus contexts mathematicians will, for example, assert (TB) and (LM), just like the Boxers, and will refrain from asserting (TB′) and (LM′), just like the Boxers. But unlike the Boxers, mathematicians will tend to *deny* both (TB′) and (LM′) in non-consensus contexts.[47] This aspect of mathematical practice may lend significant encouragement to those, like Clarke-

[46] Thanks to the referee for this example.

[47] Our evidence for this comes from conversations with mathematicians. We have not been able to find any instances of negated or denied counterfactuals in mathematics texts. (In this connection, it's worth noting that none of the examples of negated mathematical counterfactuals in Jenny (2016) are drawn from the writing of actual mathematicians.) What goes on, it seems, is that when mathematicians find themselves denying a counterfactual, they simply refrain from using it in articles or textbooks.

Doane (2017), who deny the necessity of mathematics. After all, at least if one assumes orthodoxy about counterfactuals, it is not immediately clear how to make sense of the practice of denying counterfactuals like (TB′) and (LM′) except by attributing to mathematicians—at least in the contexts in which they do deny them—a tacit commitment to the contingency of some mathematical truths. Relatedly, such denials would also seem to signal ambivalence about COUNTERFACTUAL DEDUCTION. After all, if, for all one knows, AC is true, then, by COUNTERFACTUAL DEDUCTION, for all one knows, $\neg$AC $\Box\!\!\rightarrow$ (AC $\wedge$ $\neg$AC) is true. And if $\neg$AC $\Box\!\!\rightarrow$ (AC $\wedge$ $\neg$AC) is true for all one knows, one has no business denying anything of the form $\neg$AC $\Box\!\!\rightarrow A$. (Here we are assuming the standard principle Lewis (1973) calls 'deduction within conditionals', according to which $A \Box\!\!\rightarrow B$ entails $A \Box\!\!\rightarrow C$ whenever B entails C—or, in other words, that the counterfactual consequences of a statement are closed under logical consequence.)

There is thus a *prima facie* tension within mathematical practice. On the one hand, it is hard to make sense of the practice of counterfactual *reductio* except by attributing to mathematicians a tacit commitment both to the necessity of all mathematical truths and a logic according to which all counterpossibles are true. These commitments would also make good sense of assertions of counterfactuals concerning contested axioms. On the other hand, the practice of denying counterfactuals like (TB′) and (LM′) seems contrary to those commitments. As a prelude to our own account of this apparent tension, it bears emphasis that the same tension is likely to be found in the use of counterfactuals by classical logicians. Their use of counterfactual *reductio* arguments and the counterfactuals they affirm regarding the consequences of alternative logics seem to be best explained by a tacit commitment both to the necessity of all logical truths and to a logic according to which all counterpossibles are true. On the other hand, it's pretty clear that classical logicians will also tend to deny various counterfactuals concerning alternative logics. For example, there would be nothing extraordinary about a classical logician affirming 'If the paraconsistent *LP* truth tables were correct and there was a true contradiction, then *modus ponens* would be invalid' while denying 'If the *LP* truth tables were correct and there was a true contradiction, then everything would be true'. Actual classical logicians thus seem to depart from classical Boxer logicians in just the same way in which actual mathematicians depart from the Boxers. It seems immensely plausible that the correct story about the apparent tension in each case should be the same. Let us now consider three stories: two according to which the tension in each case is merely apparent, and one—which we favor—according to which the tension is real.

Here is the first story: Mathematics is contingent. The tension is merely apparent, because mathematical proofs by counterfactual *reductio* use restricted counterfactuals, while counterfactuals that explore the consequences of alternative mathematical axioms are either unrestricted or in any case less restricted than the ones used in *reductios*. It should be clear why we are less than happy with this diagnosis. We won't repeat our arguments from §4.3 against views that posit contextual restrictions in mathematical counterfactuals. Furthermore, telling the same story about the apparent tension in classical logicians' use of counterfactuals would involve adopting the extremely radical view that logic is contingent.

Here is the second story: All counterpossibles are true. The tension is merely apparent because mathematical counterfactuals used in *reductios* are restricted to the possible worlds, whereas counterfactuals that explore the consequences of alternative axioms generalize over larger domains of worlds that include some impossible worlds. This proposal, too, we find objectionable for the reasons given in §4.3 and because telling the same story about the apparent tension in classical logicians' use of counterfactuals would involve adopting an extremely radical view—in this case about the logic of counterfactuals. In telling the same story about logic, we must give up the view that all counterlogicals (counterfactuals whose antecedents are logical falsehoods) are true, and this, in turn, would require giving up vast swathes of the standard logic of counterfactuals. Most obviously, it would require giving up the principle that the counterfactual consequences of a proposition are closed under logical consequence ($A \mathrel{\Box\!\!\rightarrow} B$ implies $A \mathrel{\Box\!\!\rightarrow} C$ when B implies C), as well as, provided that reflexivity ($A \mathrel{\Box\!\!\rightarrow} A$) is not given up, the principle that logical entailment is at least as strong as counterfactual entailment ($A \mathrel{\Box\!\!\rightarrow} B$, when A implies B). Furthermore, if there are impossible worlds in which a conjunction fails to be true even though each conjunct is true, the standard principle of finite agglomeration ($((A \mathrel{\Box\!\!\rightarrow} B) \wedge (A \mathrel{\Box\!\!\rightarrow} C)) \rightarrow (A \mathrel{\Box\!\!\rightarrow} (B \wedge C))$) will also have to go. Such logical revisionism should not be taken lightly.

Here is a third story, which we think is correct: mathematics and logic are both necessary, and the tension is real: mathematicians and classical logicians are making a kind of mistake that is ubiquitous in both ordinary and scientific counterfactual thinking when they deny counterpossibles. One very basic heuristic that we apply to both indicative and counterfactual conditionals is: *if you accept* $A \Rightarrow \neg B$*, reject* $A \Rightarrow B$ (where $\Rightarrow$ is either kind of conditional). This heuristic serves us well much of the time, but it also trips us up in many cases. Timothy Williamson (2018) has suggested a highly plausible explanation of this heuristic, as subsumed under a more general heuristic: our default way of deciding whether to take an attitude X towards a conditional $A \Rightarrow B$ is to check whether, under the supposition of A, we are led to take X towards B; if we are, then we take X towards $A \Rightarrow B$. (The rules for developing a supposition are, of course, different for indicative and counterfactual conditionals, but the abstract heuristic is the same in each case.) That is, if one, under the counterfactual supposition that A, affirms/denies/adopts a credence of r towards/… B, then by default one will affirm/deny/adopt a credence of r towards/… $A \Rightarrow B$. Thus, for example, if, having counterfactually supposed that one tosses a coin, one is 50% confident of tails eventuating, one will tend to be 50% confident that, if one were to toss a coin, it would land tails. Or if, having counterfactually supposed that one lives in a dialetheic world, one denies that every proposition is true, one will tend to deny: 'If we lived in a dialetheic world, then every proposition would be true'. It's a matter of debate in exactly which cases this general conditional heuristic gets us into trouble, but there is no question that in some cases it does. A well-known case involves probabilities: as Lewis (1976) observed, there is no conditional $\Rightarrow$ for which $P(A \Rightarrow B) = P(B|A)$. But even when we take on board this observation, we tend to continue to deploy the heuristic in cases where it doesn't obviously get us into trouble. Yet it may very well be that in many of those cases too the heuristic is leading us into error. The situation is quite similar when it comes to applying the heuristic to yield denials of counterfactuals. In some cases this will obviously lead to trouble. Pretty much everyone accepts reflexivity. So, pretty much anyone, having accepted $\neg A$ under the supposition of A in performing a counterfactual *reductio*, will resist applying the heuristic, since that would yield a denial of $A \mathrel{\Box\!\!\rightarrow} A$.[48] But, as in other cases, people in all walks of life tend to continue to use the heuristic to arrive at denials of counterfactuals in cases where it doesn't lead to any obvious error. Of course, by our lights, many such applications do lead to error. It is clear that, if the standard view that all counterpossibles are true is correct, then the heuristic seriously misfires when applied to them.

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[48] In fact, as Timothy Williamson points out in his comments, the heuristic gets in trouble with reflexivity even when applied to acceptance: suppose that you accept the following instance of reflexivity.

$$(A \wedge \neg A) \mathrel{\Box\!\!\rightarrow} (A \wedge \neg A)$$

And suppose that then, applying the converse of finite agglomeration (when a conjunction follows counterfactually something, so does each conjunct), you accept

$$((A \wedge \neg A) \mathrel{\Box\!\!\rightarrow} A) \wedge ((A \wedge \neg A) \mathrel{\Box\!\!\rightarrow} \neg A).$$

Given that you accept the second conjunct, the heuristic directs you to accept the negation of the first conjunct, and thus to accept an inconsistent pair of claims.

After all, when applied to the attitude of accepting the negation, the heuristic directs us to accept the negation of $A \Box\!\!\rightarrow B$ when we accept the negation of B under the supposition of A. So, when A is impossible and we correctly accept $\neg B$ under the supposition of A, the heuristic directs us to accept $\neg(A \Box\!\!\rightarrow B)$, which is false if all counterpossibles are true. Since, according to the story we prefer, all counterpossibles are true, the general conditional heuristic, together with the plausible assumption that mathematicians and classical logicians are not relevantly more immune than philosophers or linguists to being led to error by that heuristic, yields a satisfying explanation of the genuine tension in the use of counterfactuals by both mathematicians and classical logicians.

It bears emphasis that, if our picture is correct, then these mistakes of counterfactual reasoning are confined to a fairly small part of the practice of counterfactual reasoning by mathematicians and classical logicians. (Our sense of confinement is reinforced further by the fact that these mistakes don't even seem to make it into published work by mathematicians: see note 47.) The widespread practices of counterfactual *reductio* and of making (and refraining from making) positive counterfactual claims about contested mathematical axioms or alternative logics are completely unproblematic from our point of view. There is no need for any kind of special pleading regarding contextual restrictions or departures from entrenched principles of counterfactual logic to account for these. On the other hand, in the one area where we do find mistakes, they are explained by a familiar heuristic of counterfactual reasoning that is widely known to produce mistakes in other areas and that clearly does produce mistakes when applied to counterpossibles, on the standard assumption that all counterpossibles are true. In our view, this picture of the use of counterfactuals in mathematics and logic enjoys extremely powerful abductive support. We concede that our picture is not the only game in town, but, as we see it, the other games are considerably less elegant.[49]

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