Leibnizian and Nonstandard Analysis: Philosophical Problematization of an Alleged Continuity

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Abstract

In the present paper the philosophical and mathematical continuity alleged by A. Robinson in *Nonstandard Analysis* (1966) between his theory and Leibniz's calculus is investigated. In Section 1, after a brief overview of the history of analysis, we expose the historical, mathematical and philosophical aspects of Leibniz's calculus. In Section 2 the main technical aspects of nonstandard analysis are presented, and Robinson's philosophy is discussed. In Section 2.1 we claim the absence of a complete and direct continuity and the only possibility of a conceptual similarity between Leibniz's and Robinson's theories, both at the philosophical and at the mathematical level.

Summary

1 History and Philosophy of Mathematical Analysis from Leibniz to Robinson

1.1 Fundamental Steps in History of Analysis

Some of the main problems of that discipline today known as 'mathematical analysis' - synthetically definable with the words of Hans N. Jahnke as «the study of *dependencies among variable quantities*»[1] - were already present in ancient Greek mathematics and philosophy: the definition and computation of lengths, areas, volumes, involved curves and tangents to curves, the problems of infinite and of infinitely small, etc. Archimedes, Euclid, Eudoxus, alleged inventor of the famous methods of exhaustion, Zeno, Aristoteles with their important ideas and paradoxes about the infinite are some of the most significant examples of Greek thinkers whose mathematical theories and discoveries will be very important for the upcoming developments in mathematical analysis.

These problems continued to be handled throughout the whole Middle and Early Modern Age, within an approach not so different from the ancient one, that is, by means of geometrical concepts and methods. The Greek and in particular Euclidian 'material' ontology of mathematics still applied and characterized the notions of mathematical concept or mathematical truth in a very empirical way, which sometimes represented also a limiting factor for the development of new mathematical theories or the solution of mathematical problems[2].

At about the time of the Scientific Revolution a remarkable connection between all the old problems of analysis[3] as well as new methods for treating them were discovered, in particular by Isaac Newton and Gottfried Wilhelm von Leibniz, with whom modern analysis undoubtedly began, as we all know from our mathematics textbooks. Newton and Leibniz simultaneously and independently developed ideas which, albeit expressed in different forms, built up the core of the *infinitesimal Calculus* - or *Analyse des Infiniment Petits*, as called by Leibniz's pupil Guillaume de l'Hôpital[4] - the science of the infinitely small forerunner of today's mathematical analysis.

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[1] Jahnke H.N. (ed.), *A History of Analysis*, American Mathematical Society and London Mathematical Society, Providence 2003, p. vii, emphasis in the original.

[2] The case of the problem of consonance in mathematical music theory is exemplary in this respect, cf. Borzacchini L., *Incommensurability, Music and Continuum: A Cognitive Approach*, in *Archive for History of Exact Sciences*, 61 (2007), pp. 273–302.

[3] Especially between the problems of the quadrature of curves and of tangents to curves, cf. Jahnke H.N., op. cit., par. 2.4; Barrow I., *Lectiones geometricae*, Godbid G., London 1674, rpt. Olms G. Verlag, Hildesheim - New York 1976; Bottazzini U., *Storia della mathematica moderna e contemporanea*, UTET, Torino 1990, ch. 1.

[4] De l'Hôpital G., *Analyse des infiniment petits pour l'intelligence des lignes courbes*, F. Montalant, Paris 1715.

In comparison to the past, *modern calculus* did no longer use traditional geometrical methods. The old geometrical researches and achievements was certainly important both to Newton and Leibniz[5], but it was especially the new analytic geometry of Descartes and Fermat which offered the novel conceptual framework within which the calculus could develop in a totally different methodological direction[6]. Geometrical entities could be now represented by algebraic equations and thus geometrical problems became of pure algebraic nature. So, metaphorically speaking, square and compass were replaced by algebraic differential equations in dealing with the traditional problems of analysis.

Newton's core ideas on calculus developed during his *anni mirabiles* (1665-1667) along with many other ideas and discoveries in physics and mechanics. The first work about the topic was *De analysi per aequationes numero terminorum infinitas* (1669, published in 1711), containing the fundamental principles of his calculus but still influenced by Cavalieri's and Wallis' geometrical approach of the method of indivisibles. Aside from some references in the *Philosophiae Naturalis Principia Mathematica* (1687), the main works on calculus by Newton can be considered *De methodis serierum et fluxionum* (1671, published in 1736) and *De quadratura curvarum* (1676, published in 1704), in which he thoroughly exposed his method of fluents, fluxions and moments for dealing algebraically with analytical problems. So, if x and y were *fluents*, i.e. quantities varying over time, $\dot{x}$ and $\dot{y}$ represented their *fluxions*, i.e. their instantaneous speeds; Finally, the *moments* were their infinitely small variations within an infinitely small interval of time o. In considering these infinitely small variations, Newton referred in his works sometimes to infinitesimals, sometimes to limiting processes or physical and kinetic intuitions. Thus, he leaved the foundational problem of his calculus practically unsolved, which soon led to strong criticisms by contemporary thinkers like Georg Berkeley (see Section 1.5).

The delayed publication of Newton's discoveries about mathematical analysis led to a controversy that reached the proportions of a nationalistic antagonism between English and Continental mathematicians, lasted at any rate until the beginnings of the 19$^{\text{th}}$ century. In fact, during the same period of Newton's mathematical researches,

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5 For example, Newton fluxional calculus owes much to the geometrical techniques used by the medieval scientist Nicole Oresme in representing the behavior of a quality over time, cf. Sherry D., *The Wake Of Berkeley's Analyst: Rigor Mathematicae?*, in *Studies in History and Philosophy of Science*, 18 (4) (1987), pp. 455-480, par. III.

6 Cf. Jahnke H.N., op. cit., ch. 2, in pt. pp. 90-91; Bos, H.J.M., *Newton, Leibniz and the Leibnizian tradition*, in Grattan-Guinness I., *From the Calculus to Set Theory, 1630-1910: An Introductory History*, Duckworth, London 1980, pp. 49-93., see in pt. pp. 64-65; Sherry D., op. cit., p. 474.

Leibniz was independently developing his own version of calculus, published in 1684[7] (see Section 1.3). He was accused of plagiarism by Newton and the members of the Royal Society, with the effect that English mathematicians did never accept Leibniz's theory of calculus and that Newton's one found no spread on the continent. Leibniz's ideas were indeed more intuitive and technically handier than the complex and controversial system of fluents and fluxions of Newton. Therefore, the isolated 'English calculus' developed more slowly than the 'continental' Leibnizian one, which generally influenced the history of calculus to a greater extent, as also confirmed by the present use of the Leibnizian mathematical notation for differential, integrals, etc.[8] Nevertheless, both Newton and Leibniz are today acknowledged as the official founders of modern calculus.

Despite the success of Leibniz's calculus, evident also in its application to mechanics and physics, many of the foundational problems connected with it still required an explanation, above all that concerning the nature and the logical role of infinitesimals (see Section 1.4), a problem already present in Newton. This brought out the necessity, from the end of the 18th century, to give once and for all a solid and *rigorous* foundation to the calculus. Efforts in this direction were made especially by d'Alambert, Lagrange, Bolzano, Cauchy and Weierstrass, who gradually eliminated any reference to the controversial and problematic notions of infinitesimals and infinite quantities and introduced the concept of limit, which still constitutes the base of today's standard analysis[9]. The work of Cantor and Dedekind on set theory and in particular on the fields of reals numbers finally contributed to define a consistent foundational framework for mathematical analysis[10].

If mathematicians tried as much as possible to avoid involving in theorems and definitions problematic notions like those of infinitesimals and infinite quantities, the interest in these letters never decreased and assumed in some cases an almost 'exoteric' dimension. Thus, David Hilbert said about Cantor - the controversial genius who had finally 'dominated' the mathematical infinite with his transfinite set theory

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7 Cf. Leibniz G., *Nova methodus pro maximis et minimis...,* in *Acta Eruditorum*, October 1684, in Gerhardt C.I. (ed.), *Leibnizens mathematische Schriften*, Eidmann, Berlin - Halle 1850-1863 v. 5, pp. 220-226.

8 Cf. Geymonat L., *Storia e filosofia dell'analisi infinitesimale*, Levrotto e Bella, Torino 1947, rpt. Bollati Borinighieri, Torino 2008, p. 140; Castelnuovo G., *Le origini del calcolo infinitesimale nell'era moderna*, Feltrinelli, 1962, p. 115.; Loria G., *Storia delle matematiche*, STEN, Torino 1933, 2, pp. 564-582; Hankel H., *Die Entwickelung der Mathematik in den letzten Jahrhunderten*, Fues, Tübingen 1885, p. 12; Jahnke H.N., op. cit., ph. 3.5; Kline M., *Mathematics in Western Culture*, Oxford University Press 1953, ch. 15;

9 Cf. Jahnke H.N., op. cit., ch. 6.

10 Cf. Jahnke H.N., op. cit., ch. 10.

- he created a real *paradise* for every mathematician[11]. The hope that such a mathematical conquest could take place for infinitesimals too was however very weak after the *rigorization* of analysis in 19th century, if not completely undesired by the major part of the mathematicians (among which Cantor himself, Peano, Russell, etc.). Nevertheless, in the 1960s the Jewish-German (later naturalized American) mathematician Abraham Robinson realized this hope, warmly expressed also by his mentor Abraham Fraenkel[12]. In fact, he managed to give to infinitesimals a solid and consistent arithmetical foundation and to make them usable both in mathematics and empirical sciences. He was really that «*zweiter Cantor*»[13] - albeit at the opposite extreme of mathematical infinite - which Fraenkel dreamed of[14].

Researches on nonstandard models of arithmetic and analysis, which refused for example the Weierstrassian, standard, epsilontic, limit-based approach to analysis or the Archimedean, Dedekind-Cantorian continuum were carried out even from the 19th century. Many contributions for example by Paul du Bois-Reymond (1871), Otto Stolz (1883), Giuseppe Veronese (1889), Rodolfo Bettazzi (1890), Tullio Levi-Civita (1893), less known those by Charles Sanders Peirce (1881) and Charles L. Dodgson (1885)[15] as well as the late ones by Kurt Geissler (1904), Paul Natorp (1923), Curt Otto Schmieden (1958) and Detlef Laugwitz (1961) attempted to develop consistent theories in geometry, analysis or arithmetic based on infinitesimals and non-Archimedean fields[16]. It was Robinson, however, who offered in 1966[17] (developed since 1960) a first nonstandard model of analysis which was completely consistent from the logical point of view and that would soon demonstrate a great applicative power both inside and outside mathematics[18]. Decisive were to Robinson's formulation of nonstandard

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[11] Cf. Dauben J.W., *Georg Cantor: His Mathematics and Philosophy of the Infinite*, Princeton University Press, Princeton (NJ) 1990.

[12] Fraenkel A., *Einleitung in die Mengenlehre*, Springer, Berlin 1923, p. 163, emphasis added.

[13] Ibidem.

[14] Cf. Robinson A., *Non-Standard Analysis*, North-Holland Publishing Co., Amsterdam 1966, p. 279; Dauben J.W., *Abraham Robinson: The Creation of Nonstandard Analysis, a Personal and Mathematical Odyssey*, Princeton University Press, Princeton (NJ) 1995, pp. 353-354.

[15] Cf. Abeles F.F., *The Enigma of the Infinitesimal: Toward Charles L. Dodgson's Theory of Infinitesimals*, in *Modern Logic*, 8 (3), pp. 7–19.

[16] Cf. Ehrlich, P., *The rise of non-Archimedean mathematics and the roots of a misconception. I. The emergence of non-Archimedean systems of magnitudes*, in *Archive for History of Exact Sciences*, 60 (1) (2006), pp. 1–121.

[17] Cf. Robinson A., op. cit.

[18] Robinson's nonstandard analysis demonstrated not only to be able to prove consistently all theorems from classical differential geometry by infinitesimals, but also to be applicable both to 'internal' mathematical problems (nonmetric topological spaces, complex analysis, analytic theory of polynomials, theory of exceptional values of entire functions, theory of linear spaces - including normed spaces and Hilbert space, spectral theory of compact operators, theory of topological and Lie groups,

analysis the latest progresses in algebra, mathematical logic and model theory, in particular, among the most representative one, the theory of formally-real fields of Artin and Schreier (1927), Skolem's work on non-standard models of arithmetic (1934)[19] and Jerzy Łoś' work on model theory (1955)[20].

1.2 From History to Philosophy: Some Philosophical Problems of Mathematical Analysis

The very brief examination outlined in the previous section provided us with a general understanding of the pivotal developments in history of mathematical analysis: the seven-eighteenth-century algebraization of analytical problems and the birth of modern calculus based on the controversial notion of infinitesimal, the nineteenth-century arithmetical rigorization of analysis by banishment of the latter, the researches on non-Archimedean continua leading finally to the twentieth-century rehabilitation of infinitesimals with Robinson's nonstandard analysis.

Now, gaining awareness of these historical developments is essential to start a philosophical problematization of both the particular concepts involved in mathematical analysis, which is relevant from the point of view of philosophy of mathematics, and of history of analysis itself as *rational reconstruction* in Lakatosian sense[21], which is important for history of mathematics and general philosophy of science.

In our case, this problematization shall regard in particular the Robinsonian nonstandard model of analysis as rehabilitation - according to the statements of its creator himself[22] - of the Leibnizian ideas on calculus. Analyzing this continuity between Leibniz and Robinson is indeed not only a matter of historiography, but a topic which opens several philosophical questions for example in the ontology of mathematics, in

classical function theory, etc.) and to empirical sciences (theoretical physics, in particular quantum field theory). Cf. Dauben J.W., op. cit., pp. 345-348 and ch. 9 par. '*Nonstandard Analysis And Quantum Physics*'; other examples of application of nonstandard analysis to physics in Lobry C., Sari T., *Nonstandard Analysis and Representation of Reality*, in International Journal of Control, 81 (3) (2007), pp. 519-536.

19 Cf. Skolem T. *Über die Nicht-charakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschließlich Zahlenvariablen*, in *Fundamenta Mathematicae*, 23 (1) (1934), pp. 150–161.

20 Especially his proof of the so-called *transfer principle* (see Section 2.1); Cf. Łoś J., *Quelques remarques, théorèmes et problèmes sur les classes définissables d'algèbres*, in *Mathematical interpretation of formal systems*, pp. 98-113, North-Holland Publishing Co., Amsterdam 1955.

21 About the relation between history and philosophy of science and about the history of science as rational reconstruction cf. Lakatos, I., *The Methodology of Scientific Research Programmes*, Cambridge University Press, 1978, ch. 2.

22 «It is shown in this book that Leibniz's ideas can be fully vindicated and that they lead to a novel and fruitful approach to classical Analysis and to many other branches of mathematics», Robinson A., op. cit. p. 2.

particular about the nature of infinite and infinitesimals and in epistemology of mathematics, in particular about the relation between mathematics and reality and about the problem of the truth in mathematics. Moreover, the investigation into the *preconceptions*[23] on which Leibniz and Robinson respectively based their philosophical and mathematical ideas can represent an occasion to reflect about the existence itself of this alleged continuity as well as about the important epistemological issue of theory replacement in mathematics. And it is just on the problematization of this continuity that we choose to focus here: Is Robinson's nonstandard analysis really the logically consistent version of Leibniz's calculus (assuming it is inconsistent, as the traditional history of analysis claims) or rather a completely different theory? Is there a philosophical continuity between Leibniz and Robinson? These are the central questions which we shall deal with in the following, after the examination of both Leibniz's and Robinson's mathematical and philosophical ideas.

1.3 Leibniz's Calculus: Some Historical and Technical Aspects

We close our historical examination with a more accurate overview of Leibniz's mathematical and philosophical ideas on calculus as well as of the criticisms which have been made to them, which was intentionally omitted in Section 1.1. Such an examination seems to us quite reasonable, since Leibniz constitutes, as seen above, the starting philosophical and mathematical point of Robinson's nonstandard model of analysis.

Leibniz's occupation in mathematics began roughly in 1672, when he was in Paris as diplomat and knew there, among others, the famous Dutch mathematician Christiaan Huygens. After he completed in few months the study of the most important contemporary works on mathematics, he was already able to conceive own mathematical ideas, which mostly flowed into several manuscripts and letters dating 1673-1676[24]. In this material the original bases of infinitesimal calculus are to be found, which, however, were systematized and published by the author only later, in 1684, in a short and quite obscure paper, *Nova methodus pro maximis et minimis, itemque tangentibus, qua nec irrationales quantitates moratur*[25], for the German scientific journal *Acta Eruditorum.*

[23] Ger. *Vormeinungen*, cf. Lorentzen P., *Konstruktive Wlissenschaftstheorie*, Shurkamp 1974, ph. *Wie ist Philosophie der Mathematik möglich?*, pp. 149-166, En. tr. by Pavlovic K.R., *Constructive Philosophy*, The University of Massachusetts Press, Amherst, 1987.

[24] Cf. Leibniz, G.W. *Mathematische Schriften*, Mayer U., Probst S. (ed.), Leibniz-Forschungsstelle der Akademie der Wissenschaften zu Göttingen beim Leibniz-Archiv der G.W. Leibniz Bibliothek (Hannover), Hannover, 2012, Reihe 7, vols. 3-7, 1673–1676.

[25] Leibniz, G.W., op. cit. 1684.

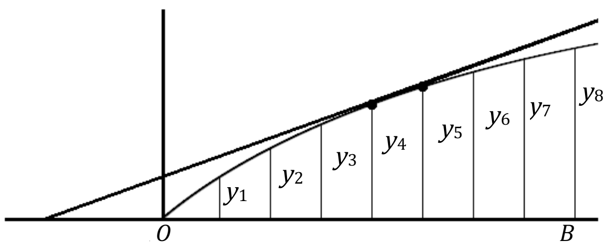
The first intuitions about infinitesimal calculus came to Leibniz within the work on the traditional geometrical problems of seventeenth-century mathematics, viz. the quadrature of curves and the tangent of a curve. In particular, Leibniz noticed that these two problems are reciprocal, i.e. the determination of quadratures and tangents are mutually inverse operations. In this respect, very important were to him the researches on number successions: He observed that the sum of the terms of a (finite or infinite) number sequence

$$b_1,\ b_2,\ b_3,\ \dots\ ,\ b_n, \tag{1.3.1}$$

where

$$b_1 = a_1 - a_2,\ b_2 = a_2 - a_3,\ b_3 = a_3 - a_4,\ \dots\ ,\ b_n = a_n - a_{n+1} \tag{1.3.2}$$

can be given by the difference $a_1 - a_{n+1}$, i.e. that the sum of the terms of a number sequence and the construction of difference sequences are mutually inverse operations. Thus, it was relatively simple for Leibniz to apply this observation to the geometry of curves and tangents and notice that (see Figure 1.3.3) to sum equidistant ordinates y_1, ... , y_n of a curve approximately correspond to the definition of its quadrature and that the difference of two consecutive ones to that of the slope of the relative tangent, inferring in this way the inverse relation between these two geometrical operations.

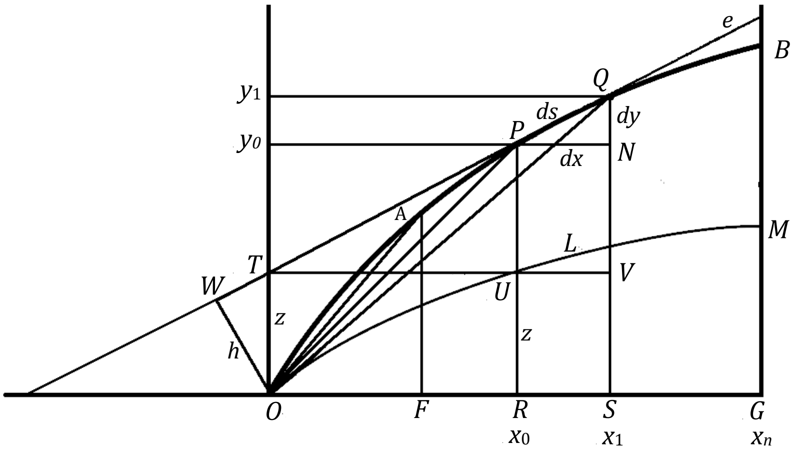


(Figure 1.3.3)

It is obvious that the smaller the distance between two ordinates is chosen, the more precise will be the definition of quadratures and tangents, so that for an *infinitely small* distance this definition would be exact.

Thus, here the main ideas of infinitesimal calculus of sums and differences were already sketched out and the important heuristic role of infinitely small quantities already acknowledged. During these ‘geometrical researches’, which took place during the years 1673-74, Leibniz also met Pascal’s use of the so-called *characteristic triangle* - a triangle with infinitesimal sides - in transformations of quadratures. While Pascal used it only for the circumference, Leibniz generalized it for all curves, which led him to the discovery of a particular transformation - the *transmutation*[26] - that allowed to calculate areas bounded by curves. We briefly discuss Leibniz's reasoning in the following.

The core idea is that every curve can be thought as composed of infinitely many infinitesimal straight segments, and thus the subtended area (its quadrature) as sum of infinitely many infinitesimal portions of area. For example, the area $OABG$ subtended to a curve OAB (see Figure 1.3.5) correspond either to the sum of infinitely many infinitesimal rectangles $RPQS$ or to the sum of the area of the triangle OBG plus the sum of infinitely many infinitesimal triangles OPQ (1.3.5):



(Figure 1.3.4)

$$OABG = \sum RPQS = \frac{OG \cdot OB}{2} + \sum OPQ, \tag{1.3.5}$$

where the area of the infinitesimal triangle OPQ is given by

$$OPQ = \frac{OW \cdot PQ}{2} = \frac{PN \cdot OT}{2} = \frac{1}{2} RUVS \tag{1.3.6}$$

since the characteristic triangle PQN associated with the point P is similar to the triangle OWT [27] and hence

$$\frac{PN}{OW} = \frac{PQ}{OT}. \tag{1.3.7}$$

[26] Cf. Leibniz. G.W., *Mathematische Schriften*, Gerhardt, C.I. (ed.), 7 vols., Eidmann, Berlin - Halle 1850-1863, v. 5, pp. 401-402; Hofmann J.E., *Die Entwicklungsgeschichte der Leibnizschen Mathematik während des Aufenthaltens in Paris 1672-1676*, De Gruyter Oldenbourg, Munich 1949, pp. 32-35; Guicciardini N., *Newton's Method and Leibniz's Calculus*, in Jahnke H.N., op cit., pp. 86-88; Bos, H.J.M., op. cit., pp. 62-65.

[27] Obtained prolonging the tangent e to the point P towards the x-axis and considering its normal OW.

For each P on OAB a corresponding U can be individuated by considering the tangent to each P and its intersection T with the y-axis. Thus, we have $RU = OT$. In this way a new curve OLM is defined, so that the quadrature (area) of the first one OAB can be given by

$$\begin{aligned} OABG &= \frac{OG \cdot OB}{2} + \sum OPQ \\ &= \frac{OG \cdot OB}{2} + \sum \frac{1}{2} OT \cdot PN \\ &= \frac{1}{2} OT \cdot PN + \frac{1}{2} \; OLMG \end{aligned} \tag{1.3.8}$$

This is exactly the Leibnizian transmutation theorem, which, applying the geometry of the characteristic triangle, makes possible to *transmute* the quadrature of a curve in that of another one, constructed from the former by use of the tangents.

Once again, the mutual connection between quadrature and definition of tangents was investigated and proven and, although the general methodological framework of this work was still the traditional, geometrical one, it anticipated the main intuitions about infinitesimal and differential calculus developed by Leibniz later, in 1675. The novelty of this work, compared to that of other contemporary or previous mathematicians, was represented, in fact, by the possibility of expressing results about quadrature, definition of tangents etc., obtained by use of infinitely small quantities, in a truly analytical and formal way, i.e. without any reference to geometry.

Indeed, going back to our example and considering the ordinate of the curve OLM, 'translated' in the modern analytical equation $z = y - x\,\frac{dy}{dx}$ for $OB = x_n$, one can also 'translate' the entire argumentation of Leibniz in modern analytical terms[28]:

$$\int_0^{x_n} y \; dx = \frac{1}{2} \int_0^{x_0} z \; dx + \frac{1}{2} x_0 y_0 \qquad \text{(1.3.9) (Transmutation rule)}$$

$$= \frac{1}{2} \int_0^{x_0} \left(y - x \frac{dy}{dx} \right) dx + \frac{1}{2} x_0 y_0$$

$$= \frac{1}{2} \int_0^{x_0} y \; dx - \frac{1}{2} \int_0^{x_0} x \frac{dy}{dx} dx + \frac{1}{2} x_0 y_0,$$

hence

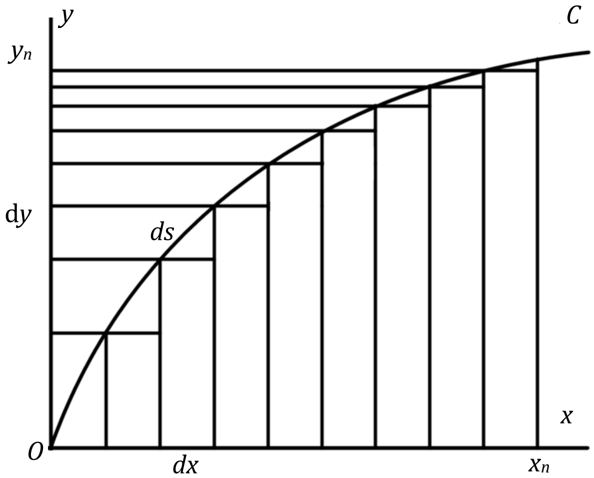
[28] Cf. Guicciardini N., op cit., p. 88; Bos, H.J.M., op. cit., p. 65.

$$\int_0^{x_n} y\ dx\ +\ \int_0^{x_0} x\ \frac{dy}{dx} = x_0 y_0\ , \tag{1.3.10}$$

equivalent to what Leibniz himself will later express[29] as a reduction formula for integration:

$$\int ydx = xy\ - \int xdy\,. \tag{1.3.11}$$

In 1675 Leibniz finally sharpened his ideas and developed those algorithms, rules and symbols of infinitesimal and differential calculus which already spread among contemporary mathematicians and scientists and are essentially the same still used today, albeit within a totally different conceptual and foundational framework (see Section 1.1). He again considered the two ideas of the characteristic triangle and the quadrature of a curve as sum of infinitely many infinitesimal rectangles in order to calculate the area subtended to a curve C in a Cartesian coordinate system (see Figure 1.3.12).



(Figure 1.3.12)

He noticed that infinitely many infinitesimal intervals[30] $dx = x_{n+1} - x_n$ on the x-axis individuate on the curve C infinitely many infinitesimal arcs $ds = s_{n+1} - s_n$ and on the y-axis infinitely many infinitesimal intervals $dy = y_{n+1} - y_n$[31], so that the resulting characteristic triangle for each point $s_n\ (x_n,\ y_n)$ on C has sides dx, ds, dy. The ratio

[29] Leibniz, G.W., *Historia et origo calculi differentialis*, Manuscript, 1714, in Gerhardt, C.I. (ed.), op. cit., v. 5, pp. 392-410, p. 408.

[30] Leibniz calls them ‘difference’ instead of today’s ‘differentials’.

[31] No function.

$\frac{dy}{dx}$ corresponds, as we know, to the slope of the tangent to C in the point s_n[32]. Considering these differential quantities Leibniz could express the area subtended to the curve as sum of infinitely many infinitesimal rectangles ydx, indicating it with a new symbol of his invention, $\int$ (ancient form for the 's', meaning '*summa*'), hence $\int ydx$. From his researches on number sequences Leibniz however knew that an infinite sequence can be expressed as difference sequence (see 1.3.2), thus he also acknowledged the mutually reciprocal relation between calculating sums and calculating differences, i.e. between integration and differentiation. In order to calculate the differences l of a given sequence from a calculus of sums $\int l$, he introduced the symbol d so that, if $\int l = ya$, where $\int l$ is the sum of all y's of a given curve and a ist quadrature (subtended area), then

$$l = ya \cdot \frac{1}{d}\,, \quad \text{expressed later more conveniently as} \quad l = d(ya). \tag{1.3.13}$$

During the years elapsed between the manuscripts of 1675 and the first important mathematical publications (1684[33] and 1686[34]), Leibniz improved his infinitesimal calculus, especially working on its formalism. As we have seen, he began to develop algorithms and rules for sums and differences which allowed to work only manipulating symbols, instead of geometrical entities, and it was really this purely analytical implementation the very novelty that determined to a greater extent the fortune of Leibniz's calculus.

1.4 Leibniz's Philosophy of Calculus and Foundational Problems

The developments of the calculus, especially in such an analytical and formalistic way, was conceived by Leibniz within the context of a more general philosophical project, which interested him since the early intellectual experiences. This project was that of the *characteristica universalis*, i.e. of a universal formal language through which all sort of philosophical, scientific and mathematical arguments could automatically be carried out and led to correct conclusions (thus, without long and tedious

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32 Many historians (see Bos 1980, Guicciardini 2003) stress that one does not have to (mis)interpret this ratio in modern terms as derivative f' of a function *f(x)* in a point x_n, i.e. as a difference quotient in modern sense. It expresses, in fact, merely a ratio between differential quantities, and the reason is obvious: Leibniz has no concept of function - which will be introduced in calculus only later by Euler and others - as well as no concept of limit, necessary for the definition of the modern notion of derivative.

33 Leibniz, G.W., op. cit. 1684. Cf. Dupont P., Roero C.S., *Leibniz 84 : il decollo enigmatico del calcolo differenziale*, Mediterranean Press, Cosenza 1991.

34 Leibniz, G.W., *De geometria recondita et analysi indivisibilium atque infinitorum*, in *Acta Eruditorum*, June 1686, in Gerhardt, C.I. (ed.), op. cit., v. 5, pp. 226-233.

discussions between philosophers). This language was thought as made up by symbols and formulas whose combinations were ruled by specific algorithms which would have ensured the correctness of the arguments themselves. From this perspective it is not surprising that, within the researches on calculus, Leibniz paid great attention to symbols, whose formal function had to be well defined and that had to be, above all, easy to manipulate: In this respect, he was looking for a *calculus*, i.e. for a handy, mechanic and 'automatic' calculating method in dealing with the old infinitesimal-geometrical problems, excluding, in this way, the complicated geometrical constructions commonly used in the past.

Leibniz also showed this pragmatic attitude towards the foundational problem of calculus. The foundational problem in mathematics did not exists at that time in the same terms of today or of the 19^{th} and 20^{th} century. Nevertheless, it was already felt the need to base such a useful, effective and powerful mathematical instrument as the calculus on solid foundations, all the more so because it dealt with infinitesimal quantities, the differentials, which seemed to exhibit self-contradictory properties when applied.

From the ontological point of view, Leibniz does not conceive infinitesimals in a platonistic way, i.e. as really existing, actual entities[35]. To him they are only fictions - but well-founded ones - without any external reference, useful to «shorten the path of the thought»[36] and facilitate mathematical proofs, just like for example imaginary numbers. This very modern formalistic, almost Hilbertian conception of infinite, and more in general of mathematics, is indeed not very surprisingly, if we think to one of the most important roots of Leibniz's philosophy, namely the universal characteristic. The Leibnizian formalistic interpretation of mathematical infinite is also consistent with his metaphysics, as shown for example by Russell[37]. Following Russell's interpretation, a metaphysical argument in this respect affirms, for example, that actual infinite, as possibility of infinite divisibility at the basis of the mathematical continuum and of the abstract concepts of time and space can exist only as ideal but not as real. For what is real is always an aggregate of parts (monads), which, even if infinitely many, are however well determinate. On the contrary, the infinite parts which compose number continuum or space and time continuum are *not* something determinate.

[35] Cf. Leibniz G.W., *Letter to Varignon*, 2 February 1702, in Gerhardt, C.I, op. cit., v. 6, pp. 91–95, Leibniz G.W., *Letter to Des Bosses*, 11 March 1706, in Gerhardt, C.I, op. cit., v. 2, p. 305.

[36] Ferraro, G., *The rise and development of the theory of series up to the early* 1820s, in *Sources and Studies in the History of Mathematics and Physical Sciences*, Springer, New York 2008, p. 35.

[37] Russell B., *A Critical Exposition of the Philosophy of Leibniz*, Cambridge University Press, Cambridge 1900, ch. 9.

Therefore, actual infinite in mathematics cannot be conceived as substance, i.e. as something real, ontologically characterized in a particular way.

However, we may ask now: What does the well-foundedness of infinitesimals consist in? Why can we use them consistently in calculus?

The technical considerations of Section 1.3 allow us to see that, in his researches on calculus, Leibniz dealt with traditional geometrical problems starting from traditional infinitary ideas, which, nevertheless, are reinterpreted by him in a quite novel sense. For example, the idea that geometrical objects can be considered as composed of infinitely many more simple ones was already present in Archimedean methodologies like the exhaustion. But, whereas in these methodologies *finite* and *indivisible* quantities were involved, in Leibniz these quantities are conceived as *infinitesimal* and are homogeneous with the objects they compose (i.e. their sum is possible and corresponds to the finite object they compose)[38]. So, a curve is composed by infinitesimal segments and no longer by finite and indivisible points, the subtended area by infinitesimal strips, not indivisible lines, and so on.

For these reasons many scholars have acknowledged in Leibniz's calculus two different infinitary methodologies[39]: A first one bounded up with a kind of Archimedean methodology, at the base of the method of exhaustion and a second, Bernoullian[40] one, purely based on the use of infinitesimal quantities. In this respect, calculus can be seen, on the one hand, as shorthand, as a simplified language for the proofs by exhaustion, if differential and infinitesimal quantities are approximated to finite ones so long as these are chosen small enough, and, on the other, as mere manipulation of non-referential, fictionally symbols, representing infinitely small or large objects.

Now, the 'interface' between A-methodology to the B-methodology, i.e. the possibility to pass from finite and assignable quantities to infinite and unassignable ones is ensured for Leibniz by a principle which thus also provides the consistent foundation

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[38] Cf. Katz M.G., Sherry D., *Leibniz's Infinitesimals: their Fictionality, their Modern Implementations, and their Foes from Berkeley to Russell and beyond*, in Erkenntnis, 78, pp. 571–625, 2013, pp. 5-7.

[39] Cf. for example Bos, H.J.M., *Differentials, higher-order differentials and the derivative in the Leibnizian calculus*, in *Archive for History of Exact Science*, 14 (1974), pp. 1-90, p. 55; Ferraro, op. cit. Katz M.G., Sherry D., op. cit.; Horváth, M., *On the attempts made by Leibniz to justify his calculus*, in *Studia Leibnitiana* 18 (1) (1986), pp. 60-71; Jesseph, Laugwitz, etc. The distinction between these two methodologies has however to be interpreted not as an ontological but only as a procedural one. The difference between ontology and procedures of calculus is underlined also by Bair. J. et al., *Interpreting the Infinitesimal Mathematics of Leibniz and Euler*, in *Journal for General Philosophy of Science*, 48 (2017), pp. 195-238, sec. 2.4. On this topic cf. also Wartofsky M., *The Relation Between Philosophy of Science and History of Science* in *Essays in Memory of Imre Lakatos*, D. Reidel, Dordrecht 1976. pp. 717–737.

[40] Leibniz's pupil, Johann Bernoulli was a strong supporter of the exclusive use of infinitesimal methodology in mathematics.

of his calculus, i.e. the possibility to deal with these infinite and infinitesimal fictional quantities without logical contradictions, applying to them the same laws of finite arithmetic. He calls such a principle *lex continuitatis*, En. *law of continuity*[41], which asserts that the rules that apply in the domain of the finite are also valid in that of the infinite[42]. This is rather a philosophical postulate, which applies for Leibniz also in physics, metaphysics, cosmology and other research fields[43].

Leibniz never offered a mathematical, formal proof of this law for what concerns its utilization in calculus and mathematics and formulated it in many manuscripts and letters, always in a different form[44]. Nevertheless, he gave many examples of its application which show how it functions. He mentioned, for instance, the case of the «parabolic ellipse with one focus at infinity»[45], i.e. the case of a parabola obtained from an ellipse by distancing infinitely its second focus from the first one. A mathematical implementation of this example in modern terms can help us to better understand Leibniz's ideas[46].

We take an ellipse E in a Cartesian coordinate system with apex at $(0, -1)$ and foci at $(0, 0)$ and $(0, H)$. Its equation will be

$$\sqrt{x^2 + y^2} + \sqrt{x^2 + (y - H)^2} = H + 2 \tag{1.4.1}$$

after squaring and moving the second radical to the left-hand side

$$2\sqrt{(x^2 + y^2)\big(x^2 + (H - y)\big)} = H^2 + 4H + 4 - (x^2 + y^2 + x^2 + (H - y)^2) \tag{1.4.2}$$

thus, after squaring again and cancelling

$$\left(y + 2 + \frac{2}{H}\right)^2 - (x^2 + y^2)\left(1 + \frac{4}{H} + \frac{4}{H^2}\right) = 0\,. \tag{1.4.3}$$

[41] Cf. Leibniz G.W., *Cum prodiisset atque increbuisset Analysis mea infinitesimalis...* , 1701, in Gerhardt, C.I (ed.), *Historia et Origo calculi differentialis a G.G. Leibnitio conscripta*, Hannover, 1846, pp. 39-50, in part. p. 40; Katz M.G., Sherry D., op. cit. sec. 4.2; Katz M.G., Sherry D., *Leibniz's Laws of Continuity and Homogeneity*, in *Notices of the American Mathematical Society*, 59 (11) 2012, 1550-1558.

[42] This formulation in particular comes from Leibniz G.W., op. cit., 1702.

[43] Cf. Breger H., *Das Kontinuum bei Leibniz* in Lamarra A. (ed.), *L'infinito in Leibniz, problemi e terminologia, Roma*, Edizioni dell'Ateneo, 1990, pp. 53-67; Russell B., op. cit. pp. 60-66;

[44] Cf. Leibniz G.W., op. cit. 1702, pp. 46-47; Jorgensen, L., *The Principle of Continuity and Leibniz's Theory of Consciousness*, in *Journal of the History of Philosophy*, 47 (2) (2009), pp. 223-248, in part. 224-229.

[45] Leibniz, G.W., op. cit. 1702, pp. 46-47. Cf. Child, J.M. (ed.), *The early mathematical manuscripts of Leibniz*, The Open Court Publishing Co., Chicago-London 1920, p. 149.

[46] Cf. Katz M.G., Sherry D., op. cit. 2013, sec. 4.5.

What we obtain with (1.4.3) is defined by Leibniz as *status transitus*[47], a *state of transition* of a certain object or quantity between finite and infinite, assignable and unassignable. So, for example, in Figures (1.3.4) and (1.3.12) the hypotenuses *ds*'s of the characteristic triangles are the unassignable *stati transiti* for the corresponding assignable[48], finite arcs of the curve. In our present example we still have an ellipse, since it satisfys the equations (1.4.1) and (1.4.3), but no longer a finite entity, as it has foci at the origin and at an infinitely distant point $(0, H)$. This infinite object is hence an intermediate, transitive state between a finite ellipse and a finite parabola. The most important thing that one has to note here is however that in these calculations the same rule of finite arithmetic (squaring inverse of radicals, algebraic identities, transferring terms to the respectively opposite equation side, etc.) also apply in dealing with infinite entities such as H: this is precisely the effect of the law of continuity. This law allows one to work genuinely within the B-methodology, i.e. whit infinite and infinitesimal entities without contradictions. What ontological nature these entities have is for Leibniz «open to question»[49], and, indeed, even worthless for practiced mathematics of infinite:

> «It will be sufficient simply to make use of them as a *tool* that has *advantages* for the purpose of the calculation, just as the algebraists retain imaginary roots with great profit»[50].

In addition to the law of continuity Leibniz introduced a second important principle to justify the procedures used in his calculus, a principle whose relevance is therefore quite obvious in regard to the foundational problem. This principle is called *lex homogeneorum transcendentalis*, En. *transcendental law of homogeneity*[51] (THL) and conceptually constitutes the inverse of the law of continuity (LC), (see Figure 1.4.4). In fact, if the law of continuity allows one to pass from the finite to the infinite, the transcendental low of homogeneity permits to return from the infinite to finite.

Finite Assignable quantities	$\xrightarrow{\text{LC}}$ $\xleftarrow[\text{TLH}]{}$	Infinite Unassignable quantities

(Figure 1.4.4)

Let us take our example of the parabola once again. The *status transitus* E_{ST} of the ellipse E into a parabola P expressed with (1.4.3) is the 'unassignable version',

[47] Cf. Leibniz G.W., op. cit. 1702, p. 42; Child, J.M. (ed.), op. cit. p. 149.
[48] Cf. Leibniz G.W. op. cit. 1684, Katz M.G., Sherry D., op. cit. 2013, sec. 5.1.
[49] Child J.M. (ed.), op. cit., p. 149. Cf. Katz M.G., Sherry D., op. cit. 2013, sec. 4.4
[50] Ibidem. Emphasis added.
[51] Cf. Leibniz G.W. *Symbolismus memorabilis...* , 1710 in Gerhardt C.I., op. cit. v. 5, pp. 377-382; Leibniz G.W., op. cit., 1684, p. 224. Cf. Katz M.G., Sherry D., op. cit. 2013, sec. 4.4.

the infinite equivalent of an assignable, finite, real parabola P_R, from which it differs only by an infinitesimal quantity $\frac{n}{H^n}$. In other terms, a P_R exists, if

$$\forall p \in P\ \exists p_r\ \forall \varepsilon : |\mathrm{p_r} - \mathrm{p}| < \varepsilon\,, \tag{1.4.5}$$

where ε is an assignable (finite) quantity greater than 0 and small at will. This means that we can obtain the real parabola P_R by approximation of the infinitesimal quantities to the finite, i.e., practically, *discarding* them in the *status transitus* (1.4.3):

$$\underbrace{\left(y + 2 + \frac{2}{H}\right)^2}_{=(y+2)^2} - (x^2 + y^2)\underbrace{\left(1 + \frac{4}{H} + \frac{4}{H^2}\right)}_{=1} = 0$$

$$(y+2)^2 - (x^2 + y^2) = 0$$

$$y_R = \frac{x_R^2}{4} - 1 \tag{1.4.6}$$

The obtained equation (1.4.6) represents the real parabola P_R in the Cartesian coordinate system, or better, the finite part of P, which is infinitely close to what Leibniz calls the *really true* parabola $y = \frac{x^2}{4} - 1$, i.e. that parabola which does not derive from an ellipse and has no focus at infinite.

This possibility to *discard* infinitesimal quantities when compared to other finite ones ($x + dx = x$) is just the effect of the transcendental law of homogeneity. This means, it provides the possibility of the reverse path from infinite to finite.

Whit the discussion of the law of homogeneity we therefore note that Leibniz has another account of the equality in addition to the classical, identity-based one ($x = x$): He conceives equality, in fact, also as equality up to an infinitesimal error, i.e. as the relation to be infinitely close, for which he makes use of the symbol $\sqcap$ (like for example in $x + dx \sqcap x$, which means, the quantity $x + dx$ is infinitely close to the quantity x, such that, for THL, they can be considered as equal). The distinction between these two kinds of equality is made by Leibniz in an important text written around 1676, which has however been remained unpublished until 1993. This text, *De quadratura arithmetica circuli ellipseos et hyperbolae cujus corollarium est trigonometria sine tabulis*[52], can be considered Leibniz's masterwork on calculus, and maybe the most important systematic text dealing with its foundational problems. Therein the laws of continuity and homogeneity take the 'concrete'

[52] Leibniz G.W., *De quadratura arithmetica circuli ellipseos et hyperbolae cujus corollarium est trigonometria sine tabulis* (1676), Knobloch E. (ed.), Vandenhoeck & Ruprecht, Gottingen 1993.

mathematical form of formal calculation rules which actually build the Leibnizian arithmetic of infinite[53] and, on the oder hand, found the infinitesimal calculus as logically consistent.

1.5 The Critics of the Calculus: Berkeley's *Analyst*

The fact that such an important text for understanding the foundational laws and concepts at the base of Leibniz's calculus was discovered only recently has maybe been responsible for its 'distorted' reception, namely as inconsistent, albeit useful, mathematical theory. Thus, criticisms and sometimes harsh judgments of Leibniz's calculus (but also of Newton's one), viz. of its foundational instances, did never miss already among contemporaries and continued until recent times with for example Kronecker, Cantor, Peano and astonishingly even Karl Marx[54].

The first and most famous thinker, who published a fierce as well as systematical critique of the calculus was the Irish philosopher and Anglican bishop Georg Berkeley with his *The Analyst; Or a Discourse Addressed to an Infidel Mathematician*[55] (1734). Berkeley, rigorous nominalist and critic of the contemporary skepticism, mechanism and deism, exasperated the modern empiricism up to a form of proto-idealism, where the existence of the matter itself as something external and independent from us is denied and made to depend exclusively on our sensations[56]. This immaterialist philosophy has passed into the annals of history synthetized in Berkeley's notorious motto «*esse est percipi*»[57]. Berkeley obviously associated to his metaphysics a suitable gnoseological counterpart: Universals do not exist, since their 'external' references - in form of perceptions - are given in any way to us, and all what we can know, i.e. all what can be in our mind in form of idea, is only the content of our perceptions, which are, as we know, the things themselves, from the ontological point of view. The necessity and objectivity of our ideas is finally guaranteed by God, who is cause of them.

The satirical vain, with which Berkeley usually expressed himself, did not fail to characterize also the work at stake here, dedicated to a no further identified *infidel mathematician*, probably Edmund Halley, who had criticized the foundations of

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53 Cf. Leibniz G.W., op. cit. 1993; Katz M.G., Sherry D., op. cit. 2013, sec. 5.2.

54 Cf. Marx K., *Математические Рукописи*, Yanovskaya S.A. (ed.), Nauka, Moscow 1968, En. tr. by Kol'man E. et al., *Mathematical manuscripts of Karl Marx*, New Park Publications Ltd., London 1983; Katz M.G., Sherry D., op. cit. 2013, sec. 6.1.

55 Cf. Berkeley G., *The Analyst; Or a Discourse Addressed to an Infidel Mathematician*, J. Tonson, London 1734.

56 Cf. Berkeley G., *A Treatise Concerning the Principles of Human Knowledge*, 1710, in Clarke D.M., *Berkeley: Philosophical Writings*, Cambridge University Press, 2008, pp. 67-149; Rossi M.M., *Introduzione a Berkeley*, Laterza, Roma-Bari 2005.

57 Berkeley G., op. cit. 2008, p. 85.

Christian religion. With arguments which apply both to Newton's and to Leibniz's[58] calculus, Berkeley claims its unfoundedness and so the unfoundedness of the blind faith in it professed by scientists and mathematicians, especially those, like Halley, who then allowed themselves to question the foundedness of the Christian faith.

Berkeley's critique of differential calculus can be conceptually summarized in two distinct parts: a metaphysical (and gnoseological, I would also say) criticism and a logical criticism[59], which are however interrelated and «cut from the same cloth»[60].

It is quite simple to imagine, on the base of the key points of Berkley's metaphysics and gnoseology explained above, what the first criticism of the calculus amounts to. Berkeley argues that all our ideas derive from the perceptions and are even the content of these perceptions. Now, perceptions are always something finite, therefore the ideas of infinite and infinitesimal cannot have a reference and are in this respect empty. Only the habit and the custom are responsible, as with causality in Hume, for the free use of these words which, like universals, are however meaningless because completely devoid of any real reference. Berkeley points also out that, especially in the Newtonian conception of calculus, contradictory ontological properties inhere in the nature of infinitesimal quantities: Infinitesimal increments, like the Newtonian moments, have an extension when they generate quantities, but seem to be extensionless entities when they are compared to finite quantities and therefore discarded. This criticism in particular seems however not to be valid for Leibniz, given his formalistic account of infinitesimals. Yet this shows that Berkeley was probably not aware of the particulars of Leibniz's foundational ideas on calculus. As we have seen, Leibniz always had from the very beginning a resolute formalistic account of infinitesimals, i.e. as fictions without any reference. On the contrary, Newton repeatedly reformulated his foundational ideas about infinitesimals, indeed never reaching a definitive and definite position about this. Moreover, Newton's calculus and ideas about infinitesimals never developed completely outside the (more or less evident) connection with physical and applicative intuitions, which obviously led for him to an ontologically more 'concrete' answer to the foundational problem, that could not have been a purely formalistic one as that of Leibniz, which had certainly a more 'rationalistic' philosophical background (see for example the project of the universal characteristic and the will to find rules of 'calculus' for traditional geometrical problems). Thus, in regard to Newton, this latter criticism by Berkeley seems to be consistent, though not for Leibniz.

[58] Leibniz and his successors are explicitly mentioned by Berkeley in section XVIII.
[59] Cf. Sherry D., op. cit.; Katz M.G., Sherry D., op. cit. 2013, sec. 6.
[60] Sherry D., op. cit., p. 460;

However, the logical counterpart of this metaphysical criticism which represents the point of transition between the two parts of Berkeley's critique, is valid for Leibniz's calculus too. Berkeley says, in fact, that the use of infinitesimals in mathematics does not follow the law of noncontradiction. Contradictory logical predicates are attributed to them at the same time, in the same theorem or demonstration. More in particular, their use in mathematics suffers for Berkeley of a *fallacia suppositionis*[61]. What the Irish bishop is objecting here is the fact that infinitesimals firstly are treated as non-zero quantities, for what concerns the performance of demonstrations and calculations but then they are made equal to zero, that means, at the same time

$$dx \neq 0 \text{ and } dx = 0 \tag{1.5.1}$$

which appears to Berkeley undoubtedly logically contradictory[62]. Therefore, he calls infinitesimals «ghosts of departed quantities»[63] and find it detrimental to base the calculus on them.

Now, having a look at the history of analysis (see Section 1.1), Berkeley's criticisms seem to have accomplished their purposes. They actually contributed to expel these contradictory objects, the infinitesimals, from analysis and mathematics and to bring the latter on the 'right' way, the way of the 'victors'[64], on which the work of Cauchy, Weierstrass, Dedekind, Cantor, etc. will develop[65].

Yet we claim, with Katz and Sherry[66], that this picture is only the result of a historiographical failure, of an a priori judgment made by mathematician and historian of mathematics throughout the centuries, which makes difficult the effective and genuine understanding of modern calculus.

First of all, the criticisms of Berkeley arise only from his particular metaphysical and gnoseological assumptions, i.e. from his exasperated empiricism and from his nominalism. Therefore, they do not highlight effective and inherent shortcomings in infinitesimal calculus, but only what has to be considered as inconsistent with these

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[61] Cf. Berkeley G., op. cit., "Query 28". A *fallacia suppositionis* corresponds to gain from a certain initial supposition certain points and then to infer the final conclusion combining these achieved points with the negation of the initial supposition, all within the same argument. An example in a predicate-logic formula: $((p \to q) \wedge \neg p) \to s$, which is obviously a wrong argument.

[62] In this case, the *fallacia suppositionis* implemented in predicate logic would be (using the formula in Footnote 61) p: $dx \neq 0$, q: $x + dx$ (in the calculation), $\neg$p: $dx = 0$, s: x (in the conclusion).

[63] Cf. Berkeley G., op. cit. sec. XXXV.

[64] Bair J. et al., *Is Mathematical History Written by the Victors?*, in *Notices Of The American Mathematical Society*, 60 (7) (2013), pp. 886-904.

[65] Cf. for example Cajori F., *Discussion of Fluxions: From Berkeley to Woodhouse*, in *American Mathematical Monthly*, 24 (4) (1917), pp. 145-154.; Boyer C., *The concepts of the calculus*, Hafner Publishing Company, New York 1949.

[66] Cf. Katz M.G., Sherry D., op. cit. 2013, secs. 6-8; Bair J. et al., op. cit.

philosophical assumptions themselves. We have already seen, in this respect, that some elements of Berkeley's metaphysical criticism lose their sense if one looks at the foundational problem of calculus from a different ontological and philosophical standpoint, for example the Leibnizian formalistic one. So, the rebuttal of infinite both as concept and as ontological realty, which would have led Berkeley to reject even modern theory of Archimedean continua and infinite as the Weierstrassian, Dedekindian, Cantorian ones, constitutes a perspectivist philosophical assumption which actually invalidates a proper understanding of the effective *pro und contra* of the Leibnizian oder Newtonian calculus, whose alleged unfoundedness is removed once this assumption itself is removed or the philosophical standpoint changed.

Furthermore, at least in the case of Leibniz's calculus, the Berkeleyan logical criticism also seems not to apply. In fact, we know from Section 1.4 that Leibniz managed to found his calculus as logically consistent - certainly not having in mind that standards of mathematical rigor and consistency required today by a mathematical theory. He did this through the laws of continuity and homogeneity and their 'procedural' implementation in the 1676 *De quadratura arithmetica*. The fact that this text, as mentioned, has been remained unknown until modern times has surely played a crucial role in bringing up Leibniz's foundational ideas on calculus as not always so homogeneous and systematic as they indeed are[67].

2 Robinson and the vindication of Leibniz's Calculus: The Problem of the Philosophical and Mathematical Continuity

Among those who, at the end, sided with the 'victors' of history of mathematics and with their historiographically acritical and careless judgment of Berkeley's critique as really effective there is none other than Abraham Robinson, who, in fact, in 1966 wrote:

> «The vigorous attack directed by Berkeley against the foundations of the Calculus in the forms then proposed is, in the first place, a *brilliant exposure of their logical inconsistencies*»[68].

Thus Robinson - like the supporters of the anti-infinitesimalist analysis and mathematics, developed especially during the 19th century by the mathematicians of the 'triumvirate' (Cantor, Dedekind, Weierstrass) - did manifestly not acknowledge the consistency of Leibniz's (and Newton's) calculus, which however, at least from the historiographical point of view, appears in part justifiable, since he could not know Leibniz's most important work on calculus, unpublished until 1993.

[67] Cf. Katz M.G., Sherry D., op. cit. 2013; Bair J. et al., op. cit. 2013.
[68] A. Robinson, op. cit., p. 280, emphasis added.

From these historiographical presuppositions he thus starts to develop a model of non-standard analysis in which, by means of modern logic and modern set theory, infinitesimals are implemented in a consistent theory of mathematical analysis.

After a brief discussion of the technical aspects of nonstandard analysis, we will proceed with the inquiry into Robinson's philosophy and then consider the problem of continuity between his theory and Leibniz's calculus as it has been presented in Section 1.2.

2.1 Main Concepts of Nonstandard Analysis

In Section 1.4 we pointed out the convergence in Leibniz of two types of infinitary methodologies, an Archimedean and a Bernoullian one, the latter different from the former essentially for the use of infinite and infinitesimal quantities. Now, to these two types of methodologies correspond two different conceptions of mathematical continuum: The A-continuum, composed of assignable, (theoretically) measurable quantities and thought as punctiform; the B-continuum, conceived as 'enriched' version of the A-continuum and containing infinite and infinitesimal quantities in addition to the A-quantities. B-quantities are unassignable and unmeasurable; Therefore the B-continuum has to be envisioned as non-punctiform. Many historians of mathematics like for example Felix Klein[69], have acknowledged a parallel development of A-mathematics and B-mathematics in history (the second one represented by the Leibnizian tradition), a development which has been lasting until the 19th century. At this point, in fact, Weierstrass offered a consistent non-infinitesimalist formulation of analysis and, above all, Dedekind and Cantor expelled infinitesimal from mathematics using the instruments of the newly formulated set theory and mathematical analysis in order to give models for a logically consistent and rigorous construction of the Archimedean continuum[70], i.e. what we today call the field $\mathbb{R}$ of the real numbers, where the Archimedean properties of real numbers apply[71].

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69 Klein F., *Elementarmathematik vom höheren Standpunkte aus*, Springer Verlag, Leipzig, 1908.

70 Both Cantor and Dedekind construct the set $\mathbb{R}$ of the real numbers as extension of $\mathbb{Q}$, that of the rational ones. In particular, Cantor conceives $\mathbb{R}$ (or more exactly $\mathbb{R}\backslash\mathbb{Q}$) as the quotient set of all equivalence classes of rational Cauchy sequences which converge to zero. $\mathbb{Q}$ is then embedded in $\mathbb{R}$ considering the stationary sequences of the rational numbers. Dedekind, on the oder hand, conceives $\mathbb{R}$ as the set of all Dedekind's cut on $\mathbb{Q}$. With the operations of addition and multiplication $\mathbb{R}$ is moreover defined as field.

71 $\forall a, b \in \mathbb{R},\ a, b > 0\ \exists n \in \mathbb{N}\ na > b$ and $\forall \varepsilon \in \mathbb{R}, \exists n \in \mathbb{N}\ \frac{1}{n} < \varepsilon$. These properties do not apply for elements of those field which are therefore called non-Archimedean fields, as for example that of the hyperreal numbers $^*\mathbb{R}$ (which we have informally defined 'B-continuum'), containing the elements of $\mathbb{R}$ plus infinitary elements.

A logically consistent and rigorous embedment of B-quantities in the A-continuum or, in other words, an extension of $\mathbb{R}$ up to involve infinitely great and infinitely small quantities were however considered as impossible by most of the mathematicians. Nevertheless, Cantor managed to 'tame' the former and include them consistently in A-mathematics through his set-theoretic means but, at the same time, believed even to be able to prove, by means of his infinitary mathematics, the impossibility to do this with infinitesimals. Thus, infinitesimals seemed to have really no longer place in mathematics, at least until Robinson and his nonstandard analysis (see Section 1.1).

All what Robinson did was to provide an extension of $\mathbb{R}$, termed already by Edwin Hewitt *field of hyperreal numbers* $^*\mathbb{R}$, in which infinitely great and infinitely small quantities were consistently contained, so that mathematical analysis could be finally founded on the same infinitesimalist grounds, on which Leibniz and Newton based their theories of calculus. 'Old' infinitesimals were therefore «vindicated»[72] and consistently rehabilitated in a rigorous mathematical theory, which could «lead to a novel and fruitful approach to classical Analysis and to many other branches of mathematics»[73].

As mentioned in Section 1.1, this extension was gained by means of the newest instruments of modern logic, modern set theory and model theory (ultrapowers, ultrafilters, ultraproducts, etc.). Unfortunately, for reasons of space, a formal exposition of the Robinsonian logical construction of $^*\mathbb{R}$ is not possible here. For a detailed and quite accessible discussion of it we recommend *An Introduction to Nonstandard Analysis* by Isaac Davis[74] in addition to Robinson's text itself.

Philosophically interesting is however the fact that the two Leibnizian foundational laws of calculus (see Section 1.4) are implemented by Robinson in his model of analysis as consistent and rigorous mathematical tools. In particular, the law of continuity becomes in nonstandard analysis the *transfer principle* and the transcendental law of homogeneity the *standard part function*.

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72 A. Robinson, op. cit., p. 2.

73 Ibidem.

74 Cf. Davis I., *An Introduction to Nonstandard Analysis*, in *The University of Chicago Mathematics VIGRE REU* Papers 2009, internet source: http://www.math.uchcago.edu/~may/VIGRE/VIGRE2009/REUPapers/Davis.pdf. Davis refers to the 'standard' Robinsonian construction of $^*\mathbb{R}$ (even if an analogous construction of this kind was already given by E. Hewitt in 1948), which defines the set of hyperreal numbers $^*\mathbb{R}$ as ultraproduct $\prod_{n \in \mathbb{N}} \mathbb{R}/U$ with U non-principal ultrafilter over $\mathbb{N}$. It is important to notice that this construction is however not the only possible one. For example, in Borovik, et al., *An integer construction of infinitesimals: Toward a theory of Eudoxus hyperreals*, in *Notre Dame Journal of Formal Logic* 53 (4) (2012), a construction by means of integers is offered.

The law of continuity heuristically states the equivalence between the rules of the domains both of finite and of infinite. This idea was mathematically implemented by Jerzy Łoś[75] in 1955, which thus formulated the *transfer principle* proving that every first-order statement is valid for the reals if and only if it is valid for the hyperreals too and vice versa[76]. This principle permits to show that the field of hyperreals $^*\mathbb{R}$ has all the properties of $\mathbb{R}$, and that one can prove theorems about $\mathbb{R}$ by first proving them in $^*\mathbb{R}$ and then transferring them back to $\mathbb{R}$, or vice versa.

On the oder hand, the *standard part function* formalizes the heuristic concept of 'discarding' infinitesimal quantities when compared to finite ones, a concept which Leibniz even introduced with his transcendental law of homogeneity. Properly speaking, the standard part function $\mathrm{st}(x)$[77], associates to every finite hyperreals x the unique real x_0 infinitely close to it, which is called *standard part* or *shadow* of x. Every real number in $^*\mathbb{R}$ has a neighborhood, called *monad*, with a collection of hyperreal numbers infinitely close to it. The hyperreals in a monad are however *limited* or *finite hyperreals*, i.e. all the elements belonging to the complement of a subset of $^*\mathbb{R}$ which contains the nonzero inverse of every infinitesimal of $^*\mathbb{R}$. Then the st-function can be formally definied:

$$\mathrm{st}\colon \begin{cases} ^*\mathrm{R}\backslash I^{-1} \to \mathrm{R} \\ x \qquad \mapsto x_0 \end{cases} \tag{2.1.1}$$

$^*\mathrm{R}\backslash I^{-1}$: complement of the set I^{-1} containing the nonzero inverse of every infinitesimal of $^*\mathbb{R}$

Thus, we can observe the effects of the standard part function considering once again the Leibnizian example of the parabola mentioned above (see Section 1.4). The informal and heuristic operation of discarding infinitesimal is now replaced by the rigorously defined standard part function. Since the standard part of every infinitesimal is 0:

$$\left(y+2+\mathrm{st}\left(\frac{2}{H}\right)\right)^2 - (x^2+y^2)\left(1+\mathrm{st}\left(\frac{4}{H}\right)+\mathrm{st}\left(\frac{4}{H^2}\right)\right) = 0$$

$$(y+2+0)^2 - (x^2+y^2)\,(1+0+0) = 0$$

$$(y+2)^2 - (x^2+y^2) = 0\ .$$

[75] Łoś J., op. cit.

[76] Davis I., op. cit., 2009, pp. 11-18.

[77] Robinson uses the notation $^\circ x$.

$$y = \frac{x^2}{4} - 1 \tag{2.1.2}$$

What we have now with (2.1.2) is an equation with finite (limited) values of $^*\mathbb{R}$ infinitely close to their respective standard part in $\mathbb{R} \subset {}^*\mathbb{R}$. To gain this latter, i.e. the equation of ‘real’ parabola we apply the standard part function to the equation (2.1.2) itself, so that

$$\mathrm{st}(y) = \mathrm{st}\left(\frac{x^2}{4} - 1\right)$$

$$y_0 = \frac{x_0^2}{4} - 1\ . \tag{2.1.3}$$

Another example of standard part function is provided by its application in finding the derivative $f'(x)$ of a function $f(x)$ (assuming it exists) in a certain point x_0. While in standard analysis the derivative is defined as limit for h tending to zero of the difference quotient (2.1.4), in nonstandard analysis it is instead defined as standard part of the difference quotient (2.1.5):

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x)}{h} \tag{2.1.4}$$

$$f'(x_0) = \mathrm{st}\left(\frac{f(x_0+h) - f(x)}{h}\right), \tag{2.1.5}$$

or, in Leibnizian notation,

$$f'(x) = \mathrm{st}\left(\frac{dy}{dx}\right) \tag{2.1.6}$$

where dx and dy are respectively the infinitesimal x-increment and the corresponding y-increment and the derivative $f'(x)$ the standard part of the infinitesimal ratio between them.

And thus, analogously, all notions of standard analysis, which are expressed in terms of limits, become expressible in nonstandard analysis in terms of standard part function.

2.2 Robinson’s Philosophy and the Problem of Continuity

We have now all theoretical means to provide a possible answer to our initial philosophical questions about the continuity between Leibnizian and nonstandard analysis. Nevertheless, a deeper investigation into Robinson's philosophical *presuppositions* can help us to analyze the problem more effectively.

In fact, Robinson alleges this continuity in the context of a historical investigation, and as pointed out by Lakatos[78], any investigation in scientific-historical matters implies a philosophical starting point: In history of science (and of mathematics) it is impossible to consider historiographical problems outside a particular philosophical perspective, which indicates *how* bare facts have to be interpreted. In other words, different philosophies of science give different theoretical models of 'science' and 'rationality', on the basis of which the historiographical activity as *rational reconstruction* can be carried out. Within such an activity of rational reconstruction there can also be the case, however, in which false philosophies of science give a distorted history of science, and this appears to Lakatos even to be the case of history of mathematics[79].

Therefore, it seems to us very important to consider Robinson's philosophical presuppositions, in order to understand to what extent they influenced his historical understanding of Leibniz's philosophical and mathematical ideas and his admitting a continuity between them and his own ideas.

These presuppositions, as mentioned, are implicitly involved in the rational reconstruction of the history of analysis and of Leibniz's calculus Robinson makes in the last chapter of his 1966-work. However, they have been explicitly formulated by him in many papers and articles[80] which thus demonstrate a deep philosophical interest of the author, developed throughout his entire career.

After the Platonic realism of the 1950s[81], Robinson accepted as definitive philosophical position a formalism close to the Hilbertian one but non coincident with it, whose most representative exposition is given in *Formalism 64*[82], maybe the most important philosophical text by Robinson. Therein he acknowledges as main characteristic points of his philosophical position, especially for what concerns the problem of foundations, the two following assumptions:

[78] Lakatos I., op. cit. 1978 (1).

[79] Lakatos I., *Cauchy and the continuum: the significance of non-standard analysis for the history and philosophy of mathematics*, in *Mathematics, Science and Epistemology*, Cambridge University Press, Cambridge 1978, pp. 43-60. Lakatos refers in particular to a distorted rational reconstruction of Cauchy's analysis.

[80] Cf. Robinson A., *Selected Papers of Abraham Robinson*, Keisler H.J. et al. (eds.), Yale University Press, New Haven 1979, v. 2 *Nonstandard Analysis and Philosophy.*

[81] Dauben J.W., *Abraham Robinson and Nonstandard Analysis: History, Philosophy, and Foundations of Mathematics*, University of Minnesota Press, Minneapolis 1988, p. 186.

[82] Cf. Robinson A., *Formalism 64*, Proceedings of the International Congress for Logic, Methodology and Philosophy of Science, Jerusalem 1964, in Robinson A., op. cit. 1979, v. 2, pp. 228-243.

> «(i) Infinite totalities do not exist in any sense of the word (i.e. either really or ideally). More precisely, any mention, or purported mention, of infinite totalities is, literally, *meaningless*.
> (ii) Nevertheless, we should continue the business of Mathematics "as usual", i.e. we should act *as if* infinite totalities really existed.»[83]

For Robinson actual infinite does not exist as it exists for example for a Platonist, but, on the other hand, it cannot be denied as concept as a nominalist would do. Rather, references to infinite totalities in a mathematical theory are possible, but they cannot be interpreted in actual sense, i.e. having external reference (concrete or not). Therefore, theories involving infinite totalities are in this sense meaningless, but not devoid of significance and hence superfluous.

In fact, infinite totalities have to be used in mathematics, as if they had a reference. In fact, the significance of a theory for Robinson does not depend on the direct interpretability of all its terms, but, almost syncategorematically, in being able to «follow its logical developments»[84]. Therefore, infinite terms can be used in mathematics independently from really having a reference. We can use them *as if* they had it. This applies for Robinson not only for mathematics, but, within a formalistic framework, also for metamathematics, where Hilbertians instead admit only finitary and directly interpretable structures. This is evident for example for the notions of consistency and completeness of recursive structures, whose definition requires the totality of natural number. Hence, in metamathematics itself infinite terms are present and, although they have to be regarded as meaningless, they can however be used, *as if* they had a meaning.

Now, if these are Robinson's philosophical presuppositions, it is quite normal to acknowledge a continuity of them with the Leibnizian foundational ideas on calculus. In fact, Leibniz, as seen in Section 1.4, supports a quite modern formalistic position in philosophy of mathematics believing in the inexistence of infinite and infinitesimals as actual entities. For Leibniz they are only ideals without external reference, useful in order to carry out more easily and intuitively mathematical proofs.

This similarity has, as such, certainly to be acknowledged. I think, however, that some points of the Leibnizian and Robinsonian philosophy of mathematics[85], if regarded independently from the problem of the philosophical foundations respectively of calculus and of nonstandard analysis, have to be considered as not completely coincident. This has as consequence the impossibility to affirm the existence of a

[83] Ibidem, p. 230. Emphasis in original.
[84] Ibidem, p. 235.
[85] We use here the terminology 'philosophy of mathematics' for Leibniz too for reasons of theoretical convenience, well aware of the fact that Leibniz never did 'philosophy of mathematics' in today's sense.

philosophical continuity between all aspects of their respective philosophies. In particular, Leibniz's philosophical ideas on mathematics are deeply influenced by his more general philosophical ideas, especially in ontology and epistemology. His philosophy of mathematics and even his truly mathematical ideas depend, in fact, on a 'mechanistical' conception of the entire human knowledge, as shown by his constant interest in projects as the *characteristica universalis* for the 'mechanization' of the entire knowledge. This thought is clearly not present as such in Robinson, both for historical and philosophical reasons. In fact, Robinson - like all scientists and philosophers of his time - was already aware of the fact that, as the latest developments in logic and in informatics showed, such a project was achievable only for a selected part of the knowledge, i.e. the formalizable one, and not *universally*, as Leibniz had in mind. In short, the original project of the *characteristica universalis* had become in part an anachronistic project. Moreover, in *Formalism 64* Robinson acknowledged on the one hand the merits of Gödel's theorems and on the other the theoretical insufficiency of the logical positivism, that is, the impossibility of an exact correspondence between mathematics and logic and even between knowledge and logic, i.e. the impossibility of a complete and consistent formalizability of both. On the other side, if Leibniz denied the existence of actual infinite in mathematics, he accepted it in other branches of his philosophy - namely, in ontology, the infinity of the monads - as even Robinson noticed[86]. Yet this is clearly not the ontological position of Robinson himself, who, close to the neopositivism, could certainly not have supported such metaphysical assumptions about reality.

For these reasons, I would speak here rather of a *similarity* between Leibniz's and Robinson's ideas on philosophy of mathematics than of a generic and broader philosophical continuity between the two authors. Robinson, in fact, draws on solely a specific idea from Leibniz - the specific interpretation of the nature of infinitesimals as well-founded fictions - ultimately neglecting the broader, organic philosophical context in which this idea is embedded, from which it derives.

The philosophical level is however not the only one that gives the possibility to discuss the problem of the continuity between Leibniz and Robinson. This continuity can be indeed problematized also at the merely mathematical level. We saw in Section 2.1 that all foundational principles of Leibniz's calculus find a consistent logical implementation in nonstandard analysis as well as infinitesimals themselves. Bos[87] points out that there is however a great difference between the Leibnizian law of continuity,

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[86] Cf. Robinson A., *The Metaphysic of the calculus*, in Lakatos I. (ed.), *Problems in Philosophy of Mathematics*, North-Holland Publishing Company, Amsterdam 1967, pp. 28-46. Cf. also Dauben J.W., op. cit. 1988, p. 182.

[87] Bos H.J.M., op. cit. 1974, p. 81-86.

law homogeneity and infinitesimals and the Robinsonian transfer principle, standard part function and hyperreals: Leibniz did never provide a formal proof of these principles, and the mathematical means required for this proof, as well as for the construction of a continuum involving at the same time reals and infinitesimals, were no way available neither to Leibniz nor to the mathematicians of the following generations; Moreover, while 'old' infinitesimals derived from and indeed were geometrical intuitions, modern hyperreals (and reals) are nothing but purely analytical entities, involved in mathematical structures (functions, derivatives, etc.) which do actually not have an equivalent in Leibnizian calculus. Hence, Robinsonian nonstandard analytical concepts and principles have to be mathematically regarded as something completely different from the Leibnizian ones.

Notwithstanding the accuracy and indeed the truth of this account by Bos of the differences between Leibnizian calculus and Robinsonian nonstandard analysis, I cannot however deny, as also Katz and Sherry do[88], that, given the genuine historical interests of Robinson, the Leibnizian mathematical ideas on calculus, or at least their most essential conceptual content, represented for him a conscious 'source of inspiration' and heuristic toolbox in formulating nonstandard analysis. The fact that Leibniz's ideas were not formally implemented (a thing that, as seen in Section 1.4 is only partly true, considering Leibniz's 1676-text *De quadratura*, which Bos could not know in 1974 yet) or the fact that they were still bounded to geometry (a thing that, as seen in Section 1.3 is not absolutely true) is not, in my opinion, a so decisive argument against the possibility that Robinson could find in them fruitful conceptual stimuli for his work. Moreover, Bos' arguing the complete extraneousness of Leibniz's calculus to nonstandard analysis through the lack of a formal proof and formal implementation of its foundational principles (that, as said, is however only partly true) does not take into account the fact that present standards of mathematical formalism cannot be properly used as parameter for the appraisal of historical mathematical theories[89], since 'mathematical formalism' corresponds to a conceptual category partly influenced by historical factors. Bos acknowledges this at least in regard to what concerns the related notion of mathematical rigor[90]. In fact, agreeing on this with him, one cannot judge Leibniz's and Robinson's analysis with the same standards of mathematical rigor of today[91], since 'mathematical rigor' amounts more to a historical rather than

[88] Cf. Katz M.G., Sherry D., op. cit. 2013.
[89] Lakatos I., op. cit. 1978 (2), sec. 7.
[90] Bos H.J.M., op. cit. 1974, p. 82.
[91] Cf. Katz M.G., Sherry D., op. cit. 2013, p. 30; Bos H.J.M., op. cit. 1974, sec. 7. This is also why one cannot claim the inconsistency of Leibniz's calculus having in mind the present standards of mathematical rigor, as indeed Robinson does.

to a logical category, which, as such, has repeatedly been reformulated throughout the time. Yet the same has also to be affirmed, in our opinion, for what concerns mathematical formalism, ultimately dependent on the standards of mathematical rigor and evolving in the time with them.

In this perspective, each mathematical theory has therefore to be considered as 'son of its time'. This however means that Robinson indeed failed in judging Leibniz's calculus as inconsistent and in considering his own theory as its 'logical purification' and vindication. It is true for Robinson too, that, from a historiographical perspective, he could not be totally aware of the consistence of Leibniz's calculus, whose most important foundation was given by the German philosopher in a work published only twenty-seven years after *Nonstandard Analysis*. Yet it is also true that Robinson retroactively used conceptual, philosophical and (meta)mathematical categories of his own time in apprising Leibniz's calculus, which is historiographically incorrect - as pointed out also by Bos[92] - and actually led him to a falsified historical appraisal of Leibniz's mathematical ideas. The continuity which Robinson alleges between him and Leibniz has therefore to be interpreted as undermined by this falsified historical appraisal as well.

Hence, after these considerations, we can finally state that, at the mathematical level too - as at the philosophical one, one cannot speak of a generic continuity between Leibnizian calculus and nonstandard analysis as Robinson himself did, but of course also not of a complete extraneousness between them, as Bos does. So, what remains to observe, is once again only a *conceptual similarity* between Leibniz's mathematical ideas and their Robinsonian implementation, a similarity that has to be acknowledged at the informal (or pre-formal) and heuristic level, but certainly not at the formal one, if one has in mind the present standards of mathematical formalism.

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