Quantum Mechanics as the Solution to a Maximization Problem on the Entropy of Measurement Events

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Abstract

This work presents a novel formulation of quantum mechanics as the solution to an entropy maximization problem constrained by quantum measurement outcomes. By treating the set of possible quantum measurement outcomes as an optimization constraint, our entropy maximization problem derives the axioms of quantum mechanics as theorems, demonstrating that the theory's mathematical structure is the least biased probability measure consistent with the observed data. This approach reduces the foundation of quantum mechanics to a single axiom, the measurement constraint, from which the full theory emerges through entropy maximization. In contrast to the conventional axiomatic approach, the framework grounds the axioms directly in empirical data, substantially restricting the interpretational landscape and ruling out interpretations inconsistent with this empirical genesis.

1 Introduction

Quantum mechanics (QM), notable for its empirical success, encounters foundational debates, often focusing on phenomena such as wavefunction collapse, the unique role of time, entropy in measurement processes, and the quantum-classical transition. The canonical formalism of QM is based on five principal axioms[1, 2]:

Axiom 1 **State Space:** Each physical system corresponds to a complex Hilbert space, with the system's state represented by a ray in this space.

Axiom 2 **Observables:** Physical observables correspond to Hermitian operators within the Hilbert space.

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Axiom 3 **Dynamics:** The time evolution of a quantum system is dictated by the Schrödinger equation, where the Hamiltonian operator signifies the system's total energy.

To bridge theory and experiment, QM introduces two additional postulates:

Axiom 4 **Measurement:** The act of measuring an observable results in the system's transition to an eigenstate of the associated operator, with the measurement value being one of the eigenvalues.

Axiom 5 **Probability Interpretation:** The likelihood of a specific measurement outcome is determined by the squared magnitude of the state vector's projection onto the relevant eigenstate.

Contrastingly, statistical mechanics (SM), the other statistical pillar of physics, derives its probability measures through entropy maximization, informed by the empirical finding that energy measurements at thermodynamic equilibrium average to a specific value ($\overline{E}$):

$$\overline{E} = \sum_{q \in \mathbb{Q}} \rho(q) E(q) \tag{1}$$

To maximize entropy while satisfying this constraint, the theory uses a Lagrange multiplier approach[3].

$$\mathcal{L}(\rho, \lambda, \beta) = \underbrace{-k_B \sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q)}_{\text{Boltzmann entropy}} + \underbrace{\lambda \left(1 - \sum_{q \in \mathbb{Q}} \rho(q) \right)}_{\text{Normalization Constraint}} + \underbrace{\beta \left(\overline{E} - \sum_{q \in \mathbb{Q}} \rho(q) E(q) \right)}_{\text{Average Energy Constraint}} \tag{2}$$

This gives rise to the well-known Gibbs measure.

$$\rho(q) = \frac{\exp\big(-\beta E(q)\big)}{\sum_{r \in \mathbb{Q}} \exp\big(-\beta E(r)\big)} \tag{3}$$

Inspired by Gibbs' methodological innovation in statistical mechanics, this study introduces a novel formulation of QM that tackles its foundational aspects through a process of systematic derivation rather than axiomatic stipulation. While fully compatible with the canonical axioms of QM, which are derived as theorems within this framework, the Prescribed Observation Problem (POP) formulation distinguishes itself by directly incorporating the relevant empirical constraints into the derivation process. The subsequent *Results* section outlines this formal mathematical procedure, emphasizing a core observation – the phase anti-constraint – as a fundamental empirical constraint, followed by entropy maximization. This approach theorematically resolves to QM's axioms, intrinsically integrating the theory's empirical basis into its foundation.

The *Discussion* section explores the implications of this derivation, drawing parallels to Gibbs' transformative impact on statistical mechanics, with the aim of clarifying and addressing enduring debates within QM. The additional empirical basis in the POP formulation allows for the promotion of axioms to theorems, increasing the stringency of the theory and permitting the ruling out of alternative interpretations. By founding theory construction on measurement outcomes and entropy maximization techniques, we propose the first fully consistent QM formulation that invalidates competing interpretations, offering a compelling resolution to the interpretational and foundational dilemmas that have long plagued QM.

2 Results

In statistical mechanics, the founding observation is that energy measurements of a thermally equilibrated system tend towards an average value. Comparatively, in quantum mechanics (QM), the founding observation involves the interplay between the systematic elimination of complex phases in measurement outcomes and the presence of interference effects in repeated measurement outcomes. To represent this observation, we introduce the *Phase Anti-Constraint:*

$$0 = \operatorname{tr} \sum_{q\in\mathbb{Q}} \rho(q) \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{4}$$

At first glance, this expression may seem to reduce to a tautology equating zero with zero, suggesting it imposes no restriction on energy measurements. However, this appearance is deceptive. Unlike a conventional constraint that limits the solution space, this expression serves as a formal device to expand it, allowing for the incorporation of complex phases into the probability measure. The expression's role in broadening, rather than restricting, the solution space leads to its designation as an "anti-constraint."

We recognize that the anti-constraint may initially appear abstract at this stage of the derivation. However, its significance will become evident upon the completion of the optimization problem. For the moment, this expression can be conceptualized as the correct expression that, when incorporated as an anti-constraint within an entropy-maximization problem, theorematically resolves into the axioms of quantum mechanics[1].

In the Prescribed Observation Problem (POP) framework, a 'prescribed observation' serves as the foundational constraint for entropy maximization. Constructing this constraint involves a comprehensive synthesis of empirical evidence, which is obtained by considering the ensemble (denoted as $\mathbb{Q}$) of all

[1]The Phase Anti-Constraint was derived through a combination of physical intuition and mathematical exploration. The wavefunction's association with a probability measure via the Born rule suggested that it could be derived from a maximum entropy principle, analogous to other probability measures in physics. The specific form of the anti-constraint was obtained by seeking a mathematical expression that, when incorporated into an entropy maximization problem, would yield the fundamental principles of quantum mechanics.

possible measurement outcomes of a quantum system. To gather this evidence, repeated measurements are performed on a collection of identically-prepared quantum systems. The founding observation then encapsulates the empirical data from the ensemble $\mathbb{Q}$ to establish the foundational constraint for entropy maximization.

Our next procedural step involves solving the corresponding Lagrange multiplier equation, mirroring the methodology employed in statistical mechanics. For that, we deploy the following Lagrange multiplier equation[2]:

$$\mathcal{L} = \underbrace{-\sum_{q\in\mathbb{Q}} \rho(q) \ln \frac{\rho(q)}{p(q)}}_{\substack{\text{Relative}\\ \text{Shannon}\\ \text{Entropy[4, 5]}}} + \underbrace{\lambda\left(1 - \sum_{q\in\mathbb{Q}} \rho(q)\right)}_{\substack{\text{Normalization}\\ \text{Constraint}}} + \underbrace{\tau\left(\operatorname{tr}\sum_{q\in\mathbb{Q}} \rho(q) \left[\begin{smallmatrix} 0 & -E(q) \\ E(q) & 0 \end{smallmatrix}\right]\right)}_{\text{Prescribed Observation}} \tag{5}$$

We solve this prescribed observation problem as follows:

$$\frac{\partial \mathcal{L}(\rho,\lambda,\tau)}{\partial \rho(q)} = -\ln \frac{\rho(q)}{p(q)} - 1 - \lambda - \tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{6}$$

$$0 = \ln \frac{\rho(q)}{p(q)} + 1 + \lambda + \tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{7}$$

$$\implies \ln \frac{\rho(q)}{p(q)} = -1 - \lambda - \tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix} \tag{8}$$

$$\implies \rho(q) = p(q) \exp(-1-\lambda) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right) \tag{9}$$

$$= \frac{1}{Z(\tau)} p(q) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(q) \\ E(q) & 0 \end{bmatrix}\right) \tag{10}$$

The partition function, is obtained as follows:

$$1 = \sum_{r\in\mathbb{Q}} p(r) \exp(-1-\lambda) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix}\right) \tag{11}$$

$$\implies \big(\exp(-1-\lambda)\big)^{-1} = \sum_{r\in\mathbb{Q}} p(r) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix}\right) \tag{12}$$

$$Z(\tau) := \sum_{r\in\mathbb{Q}} p(r) \exp\left(-\tau \operatorname{tr} \begin{bmatrix} 0 & -E(r) \\ E(r) & 0 \end{bmatrix}\right) \tag{13}$$

[2]The relative Shannon entropy includes a reference probability measure $p(q)$, often called the prior, which will represent the probability associated with the system's wavefunction in its initial state.

Finally, the probability measure is:

$$\rho(q) = \frac{p(q)\exp\left(-\tau\operatorname{tr}\begin{bmatrix}0 & -E(q)\\ E(q) & 0\end{bmatrix}\right)}{\sum_{r\in\mathbb{Q}} p(r)\exp\left(-\tau\operatorname{tr}\begin{bmatrix}0 & -E(r)\\ E(r) & 0\end{bmatrix}\right)} \tag{14}$$

Though initially unfamiliar, this form effectively establishes a comprehensive formulation of quantum mechanics, as we will demonstrate.

Upon examination, we find that phase elimination is manifestly evident in the probability measure: since the trace evaluates to zero, the probability measure simplifies to classical probabilities, aligning precisely with the Born rule's exclusion of complex phases:

$$\rho(q) = \frac{p(q)}{\sum_{r\in\mathbb{Q}} p(r)} \tag{15}$$

However, the significance of this phase elimination extends beyond this mere simplicity. As we will soon see, the partition function Z gains unitary invariance, allowing for the emergence of interference patterns and other quantum characteristics under appropriate basis changes.

We will begin by aligning our results with the conventional quantum mechanical notation. As such, we transform the representation of complex numbers from $\left[\begin{smallmatrix}a & -b\\ b & a\end{smallmatrix}\right]$ to $a+ib$. For instance, the exponential of a complex matrix is:

$$\exp\begin{bmatrix}a & -b\\ b & a\end{bmatrix} = r\begin{bmatrix}\cos(b) & -\sin(b)\\ \sin(b) & \cos(b)\end{bmatrix}, \text{ where } r = \exp a \tag{16}$$

Then, we associate the exponential trace to the complex norm using $\exp\operatorname{tr}\mathbf{M} \equiv \det\exp\mathbf{M}$:

$$\begin{aligned}\exp\operatorname{tr}\begin{bmatrix}a & -b\\ b & a\end{bmatrix} = \det\exp\begin{bmatrix}a & -b\\ b & a\end{bmatrix} &= r^2\det\begin{bmatrix}\cos(b) & -\sin(b)\\ \sin(b) & \cos(b)\end{bmatrix}, \text{ where } r = \exp a && (17)\\ &= r^2(\cos^2(b)+\sin^2(b)) && (18)\\ &= \|r(\cos(b)+i\sin(b))\| && (19)\\ &= \|r\exp(ib)\| && (20)\end{aligned}$$

Finally, substituting $\tau = t/\hbar$ analogously to $\beta = 1/(k_BT)$, and applying the complex-norm representation to both the numerator and to the denominator, consolidates the Born rule, normalization, and initial state probability into a coherent probability measure:

$$\rho(q) = \underbrace{\frac{1}{\sum_{r\in\mathbb{Q}} p(r)\|\exp\big(-itE(r)/\hbar\big)\|}}_{\text{Unitarily Invariant Partition Function}}\underbrace{\|\exp\big(-itE(q)/\hbar\big)\|}_{\text{Born Rule}}\underbrace{p(q)}_{\text{Initial State}} \tag{21}$$

We are now in a position to explore the expanded solution space that "POPs-out" of the optimization problem.

The wavefunction is delineated by decomposing the complex norm into a complex number and its conjugate. It is then visualized as a vector within a complex n-dimensional Hilbert space. The partition function acts as the inner product. This relationship is articulated as follows:

$$\sum_{r\in\mathbb{Q}} p(r)\|\exp\big(-itE(r)/\hbar\big)\| = Z = \langle\psi|\psi\rangle \tag{22}$$

where

$$\begin{bmatrix}\psi_1(t)\\ \vdots \\ \psi_n(t)\end{bmatrix} = \begin{bmatrix}\exp\big(-itE(q_1)/\hbar\big) & & \\ & \ddots & \\ & & \exp\big(-itE(q_n)/\hbar\big)\end{bmatrix}\begin{bmatrix}\psi_1(0)\\ \vdots \\ \psi_n(0)\end{bmatrix} \tag{23}$$

We clarify that $p(q)$ represents the probability associated with the initial preparation of the wavefunction, where $p(q_i) = \big\langle\psi_i(0)\big|\psi_i(0)\big\rangle$.

We also note that Z is invariant under unitary transformations.

Let us now investigate how the axioms of quantum mechanics are recovered from this result:

- The entropy maximization procedure inherently normalizes the vectors $|\psi\rangle$ with $1/Z = 1/\sqrt{\langle\psi|\psi\rangle}$. This normalization links $|\psi\rangle$ to a unit vector in Hilbert space. Furthermore, as the POP formulation of QM associates physical states with its probability measure, and the probability is defined up to a phase, we conclude that physical states map to Rays within Hilbert space. This demonstrates Axiom 1.
- In Z, an observable must satisfy:

  $$\overline{O} = \sum_{r\in\mathbb{Q}} p(r)O(r)\|\exp\big(-itE(r)/\hbar\big)\| \tag{24}$$

  Since $Z = \langle\psi|\psi\rangle$, then any self-adjoint operator satisfying the condition $\langle\mathbf{O}\psi|\phi\rangle = \langle\psi|\mathbf{O}\phi\rangle$ will equate the above equation, simply because $\langle\mathbf{O}\rangle = \langle\psi|\,\mathbf{O}\,|\psi\rangle$. This demonstrates Axiom 2.
- Upon transforming Equation 23 out of its eigenbasis through unitary operations, we find that the energy, $E(q)$, typically transforms in the manner of a Hamiltonian operator:

  $$\big|\psi(t)\big\rangle = \exp\big(-it\mathbf{H}/\hbar\big)\,\big|\psi(0)\big\rangle \tag{25}$$

  The system's dynamics emerge from differentiating the solution with respect to the Lagrange multiplier. This is manifested as:

$$\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = \frac{\partial}{\partial t}(\exp\big(-it\mathbf{H}/\hbar\big)\left|\psi(0)\right\rangle) \tag{26}$$

$$= -i\mathbf{H}/\hbar\exp\big(-it\mathbf{H}/\hbar\big)\left|\psi(0)\right\rangle \tag{27}$$

$$= -i\mathbf{H}/\hbar\left|\psi(t)\right\rangle \tag{28}$$

$$\implies \mathbf{H}\left|\psi(t)\right\rangle = i\hbar\frac{\partial}{\partial t}\left|\psi(t)\right\rangle \tag{29}$$

Which is the Schrödinger equation. This demonstrates Axiom 3.

- From Equation 23 it follows that the possible microstates $E(q)$ of the system correspond to specific eigenvalues of $\mathbf{H}$. An observation can thus be conceptualized as sampling from $\rho(q,t)$, with the post-measurement state being the occupied microstate q of $\mathbb{Q}$. Consequently, when a measurement occurs, the system invariably emerges in one of these microstates, which directly corresponds to an eigenstate of $\mathbf{H}$. Measured in the eigenbasis, the probability distribution is:

$$\rho(q,t) = \frac{1}{\langle\psi|\psi\rangle}(\psi(q,t))^{\dagger}\psi(q,t). \tag{30}$$

In scenarios where the probability measure $\rho(q,\tau)$ is expressed in a basis other than its eigenbasis, the probability $P(\lambda_i)$ of obtaining the eigenvalue λ_i is given as a projection on a eigenstate:

$$P(\lambda_i) = |\langle\lambda_i|\psi\rangle|^2 \tag{31}$$

Here, $|\langle\lambda_i|\psi\rangle|^2$ signifies the squared magnitude of the amplitude of the state $|\psi\rangle$ when projected onto the eigenstate $|\lambda_i\rangle$. As this argument hold for any observables, this demonstrates Axiom 4.

- Finally, since the probability measure (Equation 21) replicates the Born rule, Axiom 5 is also demonstrated.

Revisiting quantum mechanics with this perspective offers a coherent and unified narrative. Specifically, the phase anti-constraint (Equation 4) is sufficient to entail the foundations of quantum mechanics (Axiom 1, 2, 3, 4 and 5) through the principle of entropy maximization. Equation 4 becomes the formulation's sole axiom, and Axioms 1, 2, 3, 4, and 5 now pop out as theorems.

3 Discussion

The POP framework, drawing on the principles of entropy maximization pioneered by Josiah Willard Gibbs, introduces an innovative approach to the construction of physical theories. Distinct from conventional formulations that rely on axiomatic declarations, the POP framework derives fundamental principles as theorems entailed by a prescribed observation. This methodology presents several notable advantages:

- **Empirical Grounding:** The core theorems of the theory are directly derived from a prescribed observation, ensuring that the theoretical framework is strictly anchored in empirical reality.
- **Internal Consistency:** The derivation process itself is an essential aspect of the theory, providing insight into its genesis and ultimate justification, ensuring a high degree of internal consistency.
- **Unifying Basis:** Echoing Gibbs' contributions, the POP framework offers a coherent interpretative basis that is applicable across various physical theories, facilitating a greater unity in physics.
- **Optimal Inference:** The theoretical constructs, formulated as solutions to optimization problems, inherently represent the least biased representations achievable within the constraints of the available data.
- **Adaptability:** The framework allows for seamless theoretical adjustments in response to new empirical findings without revising its foundation. This adaptability, rooted in its use of founding observations for theory construction, ensures the framework can smoothly integrate new data, potentially making it a dynamic tool that evolves alongside scientific progress.
- **Interpretational Stringency:** The additional stringency of the POP formulation, containing its own empirical basis and derivation procedure, can be used to rule out a large class of interpretations that are inconsistent with the theorem, its basis and its derivation, providing a more definitive resolution to interpretational ambiguities.

This discussion will explore the interpretive implications of the POP approach and its potential to resolve longstanding debates in quantum mechanics.

3.1 QM and SM as inferred Solutions: A Shared Interpretive Foundation

The POP framework offers a shared interpretive foundation valid across the domains of both SM and QM.

1. <u>A Prescribed Observation as the Sole Axiom:</u>

   - **Statistical Mechanics:** The founding observation in SM is that energy measurements of a system in thermodynamic equilibrium converge to an average value ($\overline{E}$). This observation is prescribed as the core constraint that leads to the derivation of the Gibbs measure, recognized as the least biased probability measure consistent with the constraint. The theory is encapsulated in the following solution:

$$\overbrace{\underbrace{-k_B \sum_{q\in\mathbb{Q}} \rho(q)\ln\rho(q)}_{\text{Boltzmann entropy}} + \underbrace{\lambda\left(1-\sum_{q\in\mathbb{Q}}\rho(q)\right)}_{\text{Normalization Constraint}} + \beta\underbrace{\left(\overline{E}-\sum_{q\in\mathbb{Q}}\rho(q)E(q)\right)}_{\text{Prescribed Observation}}}^{\text{Prescribed Observation Problem}} \implies \underbrace{\rho(q)=\frac{\exp\bigl(-\beta E(q)\bigr)}{\sum_{r\in\mathbb{Q}}\exp\bigl(-\beta E(r)\bigr)}}_{\text{Gibbs' Measure}} \tag{32}$$

- **Quantum Mechanics:** The founding observation in the POP formulation of QM identifies the systematic elimination of complex phases and the occurrence of interference effects in measurement outcomes. This observation is prescribed as a constraint, enabling the application of entropy maximization to derive a probability measure aligned with the principles of quantum mechanics. Consequently, traditional axioms of QM are reformulated as theorems, demonstrating that QM, akin to SM, is fundamentally informed by a prescribed observation. The theory is encapsulated in the following solution:

$$\overbrace{\underbrace{-\sum_{q\in\mathbb{Q}}\rho(q)\ln\frac{\rho(q)}{p(q)}}_{\substack{\text{Relative}\\ \text{Shannon}\\ \text{Entropy}}} + \underbrace{\lambda\left(1-\sum_{q\in\mathbb{Q}}\rho(q)\right)}_{\substack{\text{Normalization}\\ \text{Constraint}}} + t/\hbar\underbrace{\left(\operatorname{tr}\sum_{q\in\mathbb{Q}}\rho(q)\begin{bmatrix}0 & -E(q)\\ E(q) & 0\end{bmatrix}\right)}_{\text{Prescribed Observation}}}^{\text{Prescribed Observation Problem}} \implies \underbrace{\rho(q)=\frac{\|\exp\bigl(-itE(q)/\hbar\bigr)\|p(q)}{\sum_{r\in\mathbb{Q}}p(r)\|\exp\bigl(-itE(r)/\hbar\bigr)\|}}_{\text{Born Rule and Unitarily Invariant Partition Function}} \tag{33}$$

2. Ontological Status of the Wavefunction:

   - **Statistical Mechanics:** In SM, the Gibbs measure serves as a predictive instrument, facilitating statistical forecasts in situations where full information about the system's state is unavailable. This role does not ascribe to it any inherent ontological significance.

   - **Quantum Mechanics:** The POP formulation of QM conceptualizes the wavefunction similarly to SM's Gibbs measure, as a construct for probabilistic forecasting. This interpretation alleviates the wavefunction from ontological responsibilities.

3. Rationalization of the Born Rule:

   - **Statistical Mechanics:** The derivation of the Gibbs measure from empirical observations affirms its theoretical solidity, eliminating any notion of arbitrariness.

   - **Quantum Mechanics:** The POP formulation of QM elucidates the Born rule as a natural outcome of entropy maximization, integrating it seamlessly into quantum theory. This reimagines the Born rule as a theorem derived from the theory's foundation.

4. Role of Entropy in Measurements

- **Statistical Mechanics:** The inherent entropy in the Gibbs measure reflects the uncertainty about a system's exact microstate configuration, symbolizing the informational limitations on complete system knowledge.

Specifically, the entropy is given as follows:

$$
\begin{aligned}
-k_B \sum_{q\in\mathbb{Q}} \rho(q) \ln \rho(q) &= -k_B \frac{1}{Z} \sum_{q\in\mathbb{Q}} \exp\bigl(-\beta E(q)\bigr) \ln \frac{1}{Z} \exp\bigl(-\beta E(q)\bigr) && (34) \\
&= k_B \frac{1}{Z} \sum_{q\in\mathbb{Q}} \exp\bigl(-\beta E(q)\bigr)\bigl(\beta E(q) + \ln Z\bigr) && (35) \\
&= \frac{1}{T}\overline{E} + k_B \ln Z && (36)
\end{aligned}
$$

- **Quantum Mechanics:** In the POP formulation of QM, entropy serves to quantify the uncertainty associated with measurement outcomes. For instance, a photon polarized as $|\psi\rangle = \alpha\,|H\rangle + \beta\,|V\rangle$, the entropy for measurements within its eigenbasis is

$$
S = -|\alpha|^2 \ln|\alpha|^2 - |\beta|^2 \ln|\beta|^2 \tag{37}
$$

5. Emergence of Time:

- **Statistical Mechanics:** In SM, temperature emerges from the collective interactions of particles occupying discrete energy states. The Lagrange multiplier $\beta = 1/k_B T$, derived from maximizing entropy while constraining the average energy, relates inversely to temperature. This highlights temperature as an intensive property, characterizing the system's overall statistical behavior. Ideal thermometers measure temperature by equilibrating with the system, reflecting the macroscopic thermal state without perturbing specific microstates. Temperature, thus, emerges as a collective property, governing the energy state distributions at thermodynamic equilibrium.

- **Quantum Mechanics:** The POP formulation of QM treats time as an emergent property, arising from the system's statistical description. Time is represented by the Lagrange multiplier $\tau = t/\hbar$, derived from entropy maximization. In this framework, time is an emergent property arising from the system's statistical configuration, akin to how temperature arises in statistical physics. Time, thus understood, governs the probabilistic evolution of quantum states through unitary transformations, paralleling the regulatory role of temperature in SM. This perspective challenges the traditional portrayal of time as an external parameter, offering a novel understanding of its nature in quantum theory.

6. Microscopic Equation of State

- **Statistical Mechanics:** Taking the total derivative of the entropy yields a macroscopic equation of state: $T\mathrm{d}S = \mathrm{d}E$. This equation quantifies thermodynamic cycles involving transitions between equilibrium states using energy (E) and temperature (T) as macroscopic variables. Such cycles typically involve: (1) The system transitioning from state (E_1, T_1) to (E_2, T_2), and (2) returning to the original state (E_1, T_1). These cycles are characterized by entropy changes, usually an increase, in line with the second law of thermodynamics.

- **Quantum Mechanics:** In the POP formulation of QM, there is no macroscopic equation of state because expectation values of "macroscopic variables," such as energy $\langle H \rangle$ and other observables $\langle O \rangle$, are unitarily invariant and do not depend on time. Instead, the Schrödinger equation acts as a microscopic equation of state, governing the temporal evolution of quantum states via unitary transformations. The Von Neumann entropy, $S(\rho) = -\operatorname{Tr}(\rho \log \rho)$, serves as another unitarily invariant macroscopic variable. The microscopic equation of state in QM allows for internal changes in quantum state configurations without affecting macroscopic variables.

3.2 Ruling Out Some Alternative Interpretations

A common misconception in the foundations of quantum mechanics is that any interpretation consistent with the mathematical formalism, i.e., the axioms, is equally valid. However, the POP framework challenges this notion by demonstrating that the axioms themselves are not fundamental postulates but rather theorems derived from a more foundational empirical basis. As such, for an interpretation to be considered valid, it must not only align with the axioms but also be consistent with this empirical genesis.

The POP framework's reformulation of quantum mechanics, grounded in inferential reasoning based on measurement outcomes, provides a stringent criterion for assessing the validity of various interpretations. By tracing the logical flow from empirical data to the mathematical structure of quantum theory, the POP approach exposes the inadequacies of interpretations that fail to consider the theory's empirical genesis.

In this light, interpretations that introduce additional ontological elements or propose mechanisms not directly derived from the foundational empirical constraints can be seen as superfluous and, in some cases, even contradictory to the core principles of quantum mechanics.

In the following items, we will critically examine several prominent interpretations of quantum mechanics in light of the POP framework's insights. By evaluating their consistency with the empirical basis and the inferential structure of quantum theory, we will demonstrate how the POP approach can effectively rule out interpretations that fail to meet these criteria, thereby providing a more solid foundation for our understanding of quantum reality.

- **Circular Fallacies**

  The POP framework exposes a fundamental fallacy in interpretations that propose wavefunction collapse, such as the Copenhagen interpretation. Interpretations within this class assume that the wavefunction exists prior to measurement and that measurement causes a collapse of the wavefunction.

This line of reasoning is circular because it fails to recognize that the wavefunction itself is inferred from the statistical regularities observed in measurement outcomes. In the POP framework, the logical flow is from measurement outcomes to the wavefunction, not the other way around. The wavefunction is a mathematical tool derived from empirical data.

By attempting to explain measurement outcomes as a consequence of the wavefunction's collapse, these interpretations are essentially trying to use the wavefunction to explain the very empirical data from which it is derived. This circular reasoning arises from treating the wavefunction as a fundamental entity that exists independently of measurements, rather than recognizing it is inferred from measurement outcomes.

The POP framework resolves this inconsistency by properly acknowledging the logical flow from measurement outcomes to the wavefunction. It treats the wavefunction as an inferred entity, derived as a consequence of the statistical regularities observed in measurement outcomes, and does not assume its existence prior to measurement. By avoiding the circular reasoning of collapse interpretations, the POP framework provides a more logically consistent and empirically grounded description of quantum phenomena.

- **Superfluous Structures**

  The POP framework exposes the problem of superfluous structures in interpretations of quantum mechanics, such as the Many-Worlds Interpretation (MWI) and Pilot Wave Theory. These interpretations introduce additional elements that are not derived from the empirical constraints of measurement outcomes and are ultimately superfluous to the description of quantum mechanics.

  The Many-Worlds Interpretation proposes that every quantum measurement splits the universe into multiple branches, each representing a different outcome. However, the POP framework demonstrates that this interpretation arises from a fundamental misunderstanding of the empirical basis of quantum mechanics. In the POP framework, measurement outcomes are sufficient to entail quantum mechanics. If the branching of multiple worlds were required for the axioms to be derived, the POP framework would not be able to derive them merely from the 'clicks' registered in this universe alone. The fact that the POP framework yields the complete foundation of quantum mechanics without invoking multiple universes within the founding empirical basis indicates that the multi-universe structure must necessarily be irrelevant to quantum mechanics.

  Similarly, Pilot Wave Theory introduces the concept of quantum potential and particle positions to explain the behavior of quantum systems. According to this interpretation, particles have well-defined positions and are guided by a quantum potential determined by the wavefunction. However, the POP framework demonstrates that these additional elements are not required to derive the foundation of quantum mechanics from the empirical constraints of measurement outcomes and are thus superfluous to the description of quantum mechanics. The POP framework can derive the complete structure of quantum mechanics, including the wavefunction and its evolution according to the Schrödinger equation, without invoking particles piloted by the wavefunction.

Both the Many-Worlds Interpretation and Pilot Wave Theory introduce unnecessary additional structures that are not required to derive the basis of quantum mechanics. The POP framework in this context illustrates the importance of including the empirical genesis of a physical theory as its foundational theorem, which automatically enforces parsimony by revealing superfluous structures.

- **Conflation of Domains**

  The Ensemble Interpretation of quantum mechanics states that the wavefunction describes an ensemble of similarly prepared systems rather than individual systems. While this interpretation correctly recognizes the importance of considering an ensemble of systems, the POP framework reveals that it mischaracterizes the role of the ensemble in the derivation of the wavefunction and its subsequent application.

  In the POP framework, an ensemble of identically prepared systems is indeed required to identify the full set of measurements $\mathbb{Q}$ that can be made on the system. By considering the outcomes of measurements performed on this ensemble, one can derive the wavefunction through the entropy maximization procedure. However, the crucial point is that the derived wavefunction applies to a single upcoming identically prepared system, not to the ensemble as a whole.

  The Ensemble Interpretation conflates the necessity of an ensemble for deriving the wavefunction with the application of the wavefunction itself. While an ensemble is required to infer the wavefunction, the wavefunction, once derived, provides a description of the probabilistic behavior of a single identically-prepared system subjected to measurement.

3.3 A Contention-Free Formulation of QM

The POP reformulates QM by prioritizing measurement outcomes and inferential techniques over traditional axiomatic declarations. This approach aims to derive the existence and properties of quantum entities directly from empirical observations rather than presupposing them through ontological assertions. By inferring quantum theory, the POP formulation seeks to mitigate theoretical contention and philosophical disputes.

- **Making Measurements Maximally Informative**

The POP framework's inference of quantum mechanics through entropy maximization reveals a profound insight: when an observer is given a sequence of measurement outcomes and is free to formulate a theory to explain these outcomes, the theory that makes the measurements maximally informative emerges as the most effective choice for understanding and predicting observable phenomena. Remarkably, this theory resolves to quantum mechanics, which has been highly successful in describing the behavior of physical systems.

To appreciate the significance of this finding, let us consider the spectrum of theories an observer could formulate. At one end of the spectrum lies a deterministic theory, where the outcomes of measurements are completely predictable, given the initial conditions and the laws governing the system. In a deterministic framework, the Shannon entropy associated with measurement outcomes is zero, indicating that no new information is gained from the measurements. In other words, a deterministic theory assigns no information content to measurement outcomes, as they are fully determined by the pre-existing state of the system.

At the other end of the spectrum lies the theory that maximizes the information content of measurements, which is precisely what quantum mechanics does, as inferred through the POP framework. Between these two extremes, there exists a continuum of theories that assign varying levels of information to measurement outcomes. However, the theory that maximizes the information content of measurements is unique and stands out as the optimal choice for the observer seeking to extract the most information from empirical data. This suggests that an observer, given the freedom to formulate a theory based on measurement outcomes, would naturally arrive at quantum mechanics as the most effective choice.

- **Empirical Evidence of Measurement Entropy**

  Quantum randomness generation[6], where outcomes are inherently unpredictable, provides compelling empirical support for our framework. Consider a standard setup: an electron, polarized along the vertical axis, is measured at a 45-degree angle. Each 45-degree angle measurement generates a random outcome with equal probability. This randomness, foundational in theory, underpins practical applications such as quantum cryptography[7], where unpredictability is essential.

  The POP formulation of QM emphasizes Shannon entropy as the key tool to quantify the information content in these random measurement outcomes. Importantly, quantifying the information within experimentally generated sequences aligns precisely with predictions stemming from this methodology, demonstrating its empirical validity for a process fundamental to the understanding and applications of quantum physics.

  Let us now compare the conventional von Neumann entropy to our measurement entropy involving Shannon entropy:

1. **Von Neumann Entropy for a Pure State**:
   Given a pure state $|\psi\rangle$, the density matrix is $\rho = |\psi\rangle\langle\psi|$. The von Neumann entropy $S(\rho)$ is:

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho) \tag{38}$$

   For pure states, $S(\rho) = 0$, reflecting a lack of statistical uncertainty about the system.

2. **Shannon Entropy for a 45° Measurement**:
   Assuming equal probability (0.5) for both outcomes after measuring an electron at a 45° angle, the Shannon entropy H is:

$$H = -0.5 \log(0.5) - 0.5 \log(0.5) = 1 \tag{39}$$

3. **Shannon Entropy for an Arbitrary Angle**:
   When measuring an electron at an arbitrary angle, the probabilities of the outcomes may be unequal, resulting in fractional bits of information. For example, if the probabilities are 0.9 and 0.1, the Shannon entropy H is:

$$H = -0.9 \log(0.9) - 0.1 \log(0.1) \approx 0.47 \tag{40}$$

4. **Shannon Entropy for a 90° Measurement on a Vertically Polarized Electron**:
   For a vertically polarized electron measured at a 90° angle, the outcome becomes deterministic rather than random. Therefore, the Shannon entropy is:

$$H = 0 \tag{41}$$

The distinction between von Neumann entropy and Shannon entropy becomes evident when quantifying the information generated in quantum randomness experiments. While von Neumann entropy assigns zero entropy to a pure state, the POP formulation of QM effectively quantifies the unpredictability inherent in scenarios like the 45-degree measurements. By explicitly incorporating Shannon entropy as the measure of measurement uncertainty, it seamlessly captures the true informational content, including fractional bits generated by measurements at different angles.

- **Automatic Mitigation of Ontological Misrepresentations**

  The canonical formulation of QM places infinite-dimensional Hilbert spaces on the same ontological footing as finite-dimensional Hilbert spaces. This assertion has led to debates among physicists and philosophers[8], with some questioning the validity of treating space as continuous[9], arguing that it may be fundamentally discrete, given that only a finite number of measurements of the wavefunction's position can be made at any given time. Others maintain that infinite-dimensional Hilbert spaces provide a fundamental and ontologically real description of the quantum wavefunction in space.

Let us now investigate how the POP formulation of QM infers infinite-dimensional Hilbert spaces. We recall that in the POP formulation of QM the construction of a prescribed observation involves performing measurements on multiple copies of identically prepared quantum systems to obtain a comprehensive set of measurements and construct the ensemble $\mathbb{Q}$. Consistently with this approach, let us now consider the extension of the entropy maximization problem from the discreet Σ to the continuum $\int$:

$$\mathcal{L} = -\lim_{n\to\infty}\left(\sum_{i=1}^{n}\rho(x_i)\ln\frac{\rho(x_i)}{p(x_i)} + \lambda\left(1-\sum_{i=1}^{n}\rho(x_i)\right) + \tau\left(\operatorname{tr}\sum_{i=1}^{n}\rho(x_i)\frac{1}{\varepsilon(x_i)}\begin{bmatrix}0 & -E(x_i)\\ E(x_i) & 0\end{bmatrix}\right)\right)\Delta x \tag{42}$$

where

- n is the number of subintervals,
- $\Delta x = (b-a)/n$ is the width of each subinterval,
- x_i is a point within the i-th subinterval $[x_{i-1}, x_i]$, often chosen to be the midpoint $(x_{i-1}+x_i)/2$.
- $1/\varepsilon(x_i)$ is a factor required to transform the energy $E(x)$ into an energy density $\mathcal{E}(x) = E(x)/\varepsilon(x)$, required for integration[3].

By using this limit, we have applied the definition of the Riemannian sum to the Lagrange equation, which yields an integral:

$$\mathcal{L} = -\int_a^b \rho(x)\ln\frac{\rho(x)}{p(x)}\,\mathrm{d}x + \lambda\left(1-\int_a^b\rho(x)\,\mathrm{d}x\right) + \tau\left(\operatorname{tr}\int_a^b\rho(x)\frac{1}{\varepsilon(x)}\begin{bmatrix}0 & -E(x)\\ E(x) & 0\end{bmatrix}\mathrm{d}x\right) \tag{43}$$

Solving this optimization problem yields a wavefunction defined in an infinite-dimensional Hilbert space and parametrized over the continuum.

[3]The presence of the term $1/\varepsilon(x)$ in the prescribed observation is a necessary feature of the continuum. It allows the conversion of the energy $E(x_i)$ to an energy density $E(x_i)/e(x_i)$, which along with $\rho(x)$ forms a trio that is invariant with respect to an integration change of basis. This is the same reason why the relative Shannon entropy (as opposed to merely the Shannon entropy) is the correct entropy to use in the continuum case. The relative Shannon entropy is invariant with respect to an integration change of basis; whereas the Shannon entropy isn't.

What does this tell us about the ontology of infinite-dimensional Hilbert space?

The subtlety is in the limiting process. By introducing the limit to define the Riemann sum and obtain the integral, we extend the ensemble size of measurement outcomes to the continuum. However, in the lab we do not perform measurements on uncountably-many identically prepared quantum systems; we only test a finite number.

Due to laboratory limitations, the prescribed observation associated to the continuum is obtained by performing a finite number of measurements on n identically prepared systems (n being finite) and then assuming, by induction, that the observed patterns hold for $n + 1$ up to infinity. With this assumption, we mathematically complete the set of discrete measurements, achieving the smoothness of the continuum, and solve the optimization problem to infer an infinite-dimensional Hilbert space.

The goal of this analysis is not to question the existence of infinite-dimensional Hilbert spaces as mathematical entities but to highlight that the POP formulation of QM assigns different ontological statuses to infinite-dimensional and finite-dimensional Hilbert spaces. Specifically, the former requires an induction assumption (encapsulated in the limit of a Riemannian sum) that the latter does not. Thus in the POP formulation of QM, the continuum cannot penetrate the same ontological layer of certainty as the one occupied by the discrete case.

By carefully distinguishing between the mathematical representation and the underlying ontology, the POP framework offers a more transparent and philosophically consistent foundation for quantum mechanics. It acknowledges the utility of infinite-dimensional Hilbert spaces as mathematical constructs while recognizing the induction assumption involved in extending discrete measurements to the continuum. The POP approach, as it embeds the genesis of the theory within its foundation, automatically acquires the tools to mitigate ontological misrepresentations.

4 Conclusion

The POP framework introduces a revolutionary approach to quantum theory construction, distinguishing itself from traditional methodologies by inferring theoretical constructs from measurement outcomes rather than declaring them via axioms. Central to this framework is the utilization of the anti-constraint allowing for the elimination of complex phases and production of interference effects, which, supported by a century of empirical validation, serves as the founding observation to infer quantum mechanics. By solving an entropy maximization problem, the POP framework successfully reconstructs the complete structure of quantum theory from the ground up (deriving Axiom 1, 2, 3, 4 and 5 as theorems), using a single axiom (Equation 4). Such an approach not only mitigates the longstanding ontological and interpretive debates that have characterized the field of quantum mechanics but also paves the way for significant advancements in our understanding of quantum foundations.

Statements and Declarations

- Competing Interests: The author declares that he has no competing financial or non-financial interests that are directly or indirectly related to the work submitted for publication.
- Data Availability Statement: No datasets were generated or analyzed during the current study.
- During the preparation of this manuscript, we utilized a Large Language Model (LLM), for assistance with spelling and grammar corrections, as well as for minor improvements to the text to enhance clarity and readability. This AI tool did not contribute to the conceptual development of the work, data analysis, interpretation of results, or the decision-making process in the research. Its use was limited to language editing and minor textual enhancements to ensure the manuscript met the required linguistic standards.

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